Research Proposal

September 29, 2006

1 Project

1.1 Project title

Generating Functions and Poisson manifolds

1.2 Project Acronym

GFP

1.3 Principal Investigator

Prof. I. Moerdijk

2 Summary

2.1 English

There is a certain correspondence between the formalism of Classical Mechanics and the formalism of Quantum mechanics. It is known as the Correspondence Principle (CP). In practice, physicists take the motion equation of a classical system, and proceed to some mysterious substitutions to get the motion equation of the associated quantum system. This correspondence is not completely coherent nor completely understood from a mathematical point of view. In Mathematics, one usually formalizes a correspondence either with the notion of mapping between spaces or with the notion of functor between categories. The attempts to conceptualize the CP as a mapping have lead to the algebraic notion of star product, which is an associative deformation of the usual product of functions on a manifold. In turn, a star product

induces a Poisson structure on M which may be seen as a non linear equivalent of a Lie algebra. On the other hand, the attempts to understand the CP as a functor have lead to the geometric notion of symplectic groupoid, which is the non-linear analogue of the Lie group integrating a Lie algebra.

In 1997, M. Kontsevich in [23] gave a universal formula of a star product associated to a given Poisson structure. Recently, it was proved in [5] that the semi-classical part of Kontsevich star product also provides a universal generating function inducing the (local) symplectic groupoid integrating the Poisson structure. The aim of this research proposal is to investigate the consequences of having a universal generating function in Poisson geometry. The applications concern the integration, foliation and quantization theory of Poisson manifolds.

2.2 Nederlandse samenvatting voor leken

De Hamilton mechanica en quantum mechanica hebben aanleiding gegeven tot de studie van verschillende types meetkundige objecten, namelijk die van symplectische varieteiten en Poisson varieteiten. Deze objecten behoren tot het domein van de differentiaalmeetkunde. Een belangrijke component van de structuur van een Poisson varieteit is een operatie op de gladde functies op de varieteit, die de formele eigenschappen heeft van een infinitesimaal restant van een associatief product op deze functies. Een van de centrale vragen in het vakgebied is of zo'n infinitesimale strucuur ook daadwerkelijk restant is van een globaal product van functies op de varieteit. Deze vraag staat bekend als het deformatie-quantisatie probleem, en werd aan het eind van de negentiger jaren positief beantwoord door Kontsevich, die voor speciale gevallen ook een expliciete formule voor dit product vond.

Poisson varieteiten blijken een centrale positie in te nemen, daar zij een rol spelen in uiteenlopende onderwerpen zoals de theory van Lie algebras, foliatietheorie, de theorie van symplectische groupoiden, en in het probleem van deformatie-quantisatie. Voor een gegeven Poisson varieteit is het verband tussen een mogelijke bijbehorende ("integrerende") symplectische groupoide en een expliciet associatief product (à la Kontsevich) een groot mysterie. Uit werk van met name Cattaneo en Felder is nu een nieuwe structuur opgekomen, die van een universele genererende functie, die zowel met de symplectische groupoide als met dit associatieve product te maken heeft, en een mogelijke link tussen de twee vormt. Bovendien blijkt de universele genererende functie veel gelijkenis te vertonen met ingredienten van de Connes-Kreimer Hopf algebra die ten grondslag ligt aan de normalisatie van quantumveldentheorie. Al met al voldoende reden om zulke genererende functies grondig te onderzoeken.

3 Classification

Mathematics

53D17 53D55 58H05 18D50

4 Composition of the Research Team

- Prof I. Moerdijk (Utrecht University, tenured): algebraic topology, in particular theory of operads
- Dr M. Crainic (Utrecht University, tenured): Poisson geometry, theory of Lie groupoids
- Postdoc to be appointed. When the application is succesful, we will employ Dr B. Dherin (presently at Geneva University, fixed term), whose specialty is symplectic geometry, groupoid theory, operad theory.
- C. Arias Abad (Utrecht University PhD student), Lie groupoids
- A. Lukacs (Utrecht University,PhD student), Operad theory

5 Research school

The proposed research will be part of the MRI research.

6 Description of the Proposed Research

6.1 Motivation

Poisson manifolds may be regarded as generalizations of Lie algebras in the sense that any Lie algebra possesses on its dual a linear Poisson structure and any linear Poisson manifold induces on its dual a Lie algebra. As for Lie Algebras, Poisson manifolds can be studied from the perspective of **integration** theory (see [6], [14], [11]) – in this context symplectic groupoids play the role of the integrating Lie group – and **foliation** theory – Poisson manifolds carry a natural generalization of the co-adjoint orbit foliation on the dual of a Lie algebra. On the other hand, Poisson manifolds are also generalizations of symplectic manifolds which constitute the natural framework of classical dynamics. The Poisson structure allows us to associate to any function on the manifold – the Hamiltonian – an equation of motion:

the Hamilton–Poisson system. This makes Poisson manifolds relevant for **quantization** theory as symplectic manifolds are (see [23], [7], [8]).

Recently, the notion of generating functions of Poisson structures on an open subset U of $\mathbb{R}^d$ was introduced in [5]. These generating functions induce a Poisson structure on U together with the (local) symplectic groupoid integrating it. There exists a universal generating function which constitutes a non-linear equivalent of the famous Baker-Campbell-Hausdorff formula (see [5]). Generating functions seem to be at the crossroad of integration, foliation and quantization theory of Poisson manifolds. The goal of this project is to investigate some consequences of having such a universal generating function in Poisson geometry from these three points of view.

6.2 People and Places

This project is to be conducted at Utrecht University in the group of Moerdijk and Crainic. Earlier work done by this group which is of relevance for this project includes [12], [13], [14], [16], [25],[26]. Moreover, we expect to collaborate closely with Weinstein (Berkeley University), Cattaneo (Zürich University), Felder (ETH Zürich), and Hairer (Geneva University).

6.3 Objectives

Integration Theory

Let (M, α) be a Poisson manifold. To each local chart U, we may assign a generating function via the universal formula $S(\alpha)$ described in [5]. This formula is however not covariant and thus fails to give a global generating function. Nevertheless, in the case of analytical Poisson structures, it has been shown in [17] that the local symplectic groupoids generated by $S(\alpha)$ on the local charts coincide with the ones constructed by Maslov and Karasev in [21]. The Karasev–Maslov symplectic groupoids are symplectomorphic on overlapping charts and thus generate a global object, the local symplectic groupoid of (M, α). Moreover, a concept of equivalence for generating functions of Poisson structures was defined in [4]. It was also proved that equivalent generating functions induce isomorphic (local/formal) symplectic groupoids.

Objective 1: In the spirit of Maslov and Karasev globalization, we propose to show that the universal generating function formula produces equivalent generating functions on overlapping charts, providing thus a local/formal symplectic groupoid integrating (M, α).

The local/formal symplectic groupoid on a Poisson manifold M may also be realized by considering the phase space of the Poisson σ-model considered by Cattaneo and Felder in [6], or the construction of Crainic and Fernandes in [14]. In both cases, elements of the symplectic groupoid are given by equivalence classes of paths staying in the symplectic leaves of the Poisson manifold. The groupoid structure is induced by the concatenation of paths. "Short" paths induce the local/formal symplectic groupoid.

Objective 2: We plan to investigate how these local/formal symplectic groupoids relate with the ones induced by the universal generating function. This may lead to other characterizations of the universal generating function.

Quantization theory

In parallel and independently, Karasev in [20], Weinstein in [28] and Zakrzewski in [30] initiated a geometric program for quantizing Poisson manifolds via symplectic groupoids. The first step, as explained in [29], is to deform the trivial groupoid structure of the contangent bundle T^*M of a Poisson manifold into a family $G_\epsilon(M)$ of symplectic groupoids over M such that $G_\epsilon(M)$ tends to T^*M as ϵ goes to zero and such that $G_\epsilon(M)$ is symplectomorphic to T^*M. The next step is to quantize, via geometric quantization, the Lagrangian submanifolds $L_f := \{(x, df(x)) : \quad x \in M\}$ of T^*M associated to functions $f \in C^\infty(M)$. The geometric analog of a deformation quantization would then be $Q(L_f) \star_\epsilon Q(L_g) := Q(L_f \bullet_\epsilon L_g)$ where Q stands for the quantization via geometric methods (see [1]) and $\bullet_\epsilon$ is the product of Lagrangian submanifolds in $G_\epsilon(M)$. This program was never completed. In [5], it was shown that for any Poisson structure α on $U \subset \mathbb{R}^d$, there exists a formal universal generating function $S_\epsilon(\alpha)$. It turns out that the formula for $S_\epsilon(\alpha)$ is exactly the tree–level part of Kontsevich star product. Moreover, $S_\epsilon(\alpha)$ induces a structure of (local/formal) symplectic groupoid $G_\epsilon(U)$ whose limit is T^*U as ϵ goes to zero. This is the first step of Weinstein's program in the local/formal setting.

Objective 3: We propose to investigate the second step of Weinstein's program to get back a star product on the ring of smooth functions $C^\infty(U)$ on U from the symplectic groupoid generated by $S_\epsilon(\alpha)$.

Kontsevich deduced the formula for his star product on $\mathbb{R}^d$ from very general relations between the DGLA[1], $T^\bullet(\mathbb{R}^d)$, of multi-vector fields on $\mathbb{R}^d$ and the DGLA, $D^\bullet(\mathbb{R}^d)$, of multi-differential operators on $C^\infty(\mathbb{R}^d)$. In fact, he provided the explicit formula of an L_∞-quasi-isomorphism between $T^\bullet(\mathbb{R}^d)$ and $D^\bullet(\mathbb{R}^d)$. Such is the content of his formality Theorem (see [23]). On the other hand, the proof of the existence of a universal generating function $S_\epsilon(\alpha)$ is direct. Using the language of operad theory, it is shown in [4] how we may transport most of the structures involved in the Kontsevich formality theorem (which is stated in the vector space category) to the extended symplectic category. In this setting, the symplectic manifold $T^*\mathbb{R}^d$ plays the role of the vector space $C^\infty(\mathbb{R}^d)$ and a solution of the Maurer-Cartan is precisely a deformation of the generating function of the trivial symplectic groupoid $T^*\mathbb{R}^d$ over $\mathbb{R}^d$.

Objective 4: We aimed at stating and proving a "symplectic" version of the Kontsevich formality theorem using operad theory in the spirit of [4].

6.4 Related and future work

Foliation theory

Recent developments in the numerics of ODE focus on finding numerical integrators respecting geometric invariants which are also preserved by the exact flow of the differential equation (see [19]). In the context of Poisson geometry, we look for Runge–Kutta methods approximating Hamilton–Poisson systems whose corresponding integrator (called Poisson integrator) respects the natural foliation. It is noticeable that, in the current state of research, there is no universal Runge–Kutta method generating Poisson integrators independently from the Poisson structure at hand (see [22] for a review). At best, there are methods which give Poisson integrators in the Lie algebra case. The main idea here is to use the symplectic groupoid structure of the cotangent bundle T^*G of the Lie group G integrating the Lie algebra $\mathcal{G}$. The key point is that, in this case, the symplectic groupoid structure may be given in term of explicit tractable formulas. For general Poisson structure on open subset of $\mathbb{R}^d$, the universal generating function provides now the same kind of explicit description. It sounds reasonnable to think that the use of generating function in this context may lead to new discovery in the field of Poisson integrators. This aspect of the proposal may be related to the research of the group of Prof. E. Hairer at the Geneva University who is a leading actor in this field ([19]).

[1] Differential Graded Lie Algebra

In [10], Connes and Moscovici have computed the local index formula for transversal hypoelliptic operators for foliated spaces. They accomplished this by using a certain Hopf algebra describing the formal diffeomorphisms of the transverse geometry. At the same time, Kreimer in [24] discovered a Hopf algebra structure on the rooted trees that may be extracted from the Feynman graphs of perturbative expansions in QFT. Shortly after, Connes and Kreimer realized that these Hopf algebras were intimately linked, the two describing the same group of formal diffeomorphisms (see [9]). In [2], Brouder noticed that the Hopf algebra of renormalization was in fact the algebra of characters on the Butcher group which describes the composition of Runge–Kutta methods in numerical analysis (see [3]). From a conceptual point of view, the discoveries of Connes and Moscovici should be as relevant for Poisson manifolds as they are for foliated spaces. Similarly, there must be a correspondent to the discovery of Kreimer in Poisson theory as a Poisson manifold comes with an associated topological QFT, the Poisson σ-model (see [6]). Surprisingly, the proof of the existence of the universal generating function for Poisson structures α on open subsets of $\mathbb{R}^d$ (see [5]) has a lot in common with the combinatorial techniques developed by Connes and Kreimer [9]. It seems extremely interesting to understand the combinatorial techniques used in the existence proof of the generating function in the light of Connes-Kreimer-Moscovici. This may lead to new connections between Poisson geometry, Index Theorem and deformation quantization. These questions are very closed to the work of Nest and Tsygan [27] or Fedosov [18] on Index Theorems.

7 Project planning

Objective 1 and 2 are going to be carried out first. Technically, they are easier. Conceptually, Objective 1 attempts to fix the application field of generating functions: general Poisson manifolds ; Objective 2 relates the generating function approach with previously known approachs. This understanding will be a good start to tackle the two last objectives which are the most ambitious ones.

An approximative schedule may be the following: Objective 1 and 2 seems to be within reach in the first year of the project if accepted. Objective 3 and 4 may retain the attention of the team for the two last years. The relevant techniques for their resolution may well be related to the algebraic and combinatoric methods appearing, at the same time, in Connes–Kreimer renormalization and Connes-Moscovici index theorem. Working on these methods in the resolution of Objective 3 and Objective 4 may, as a by product, prepare for subsequent research proposals in the direction of Index theorems and theirs relations to Poisson Geometry and Deformation Quantization.

8 Expected use of instrumentation

1 Unix workstation

9 Literature

9.1 Bibliography

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