Groupoids, KK-theory and boundary actions

Bram Mesland

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Definition 1. Let $\Gamma \subset \text{Isom}^+(\mathbb{H}^{n+1})$ be a finitely generated discrete group. The *limit set* of Γ is the set of accumulation points of a given orbit:

$$\Lambda_\Gamma = \overline{\{p\gamma : \gamma \in \Gamma\}} \setminus \mathbb{H}^{n+1} \subset S^n.$$

Theorem 2. *Γ acts properly discontinuously on $S^n \setminus \Lambda$ and with dense orbits on Λ.*

Definition 3. Fix $p \in \mathbb{H}^{n+1}$, and for each $x \in \Lambda$ choose a sequence $p_n \to x$. The *Busemann function* is defined as

$$h_\gamma(x) := \lim_{n\to\infty} d(p, p_n) - d(p\gamma, p_n).$$

Definition 4. The *Poincaré series* of Γ is the series

$$\sum_{\gamma\in\Gamma} e^{-sd(p,p\gamma)}.$$

The *critical exponent* of Γ is the extended real number

$$\delta_\Gamma := \inf\{s \in \mathbb{R} : \sum_{\gamma\in\Gamma} e^{-sd(p,p\gamma)} < \infty\}.$$

Theorem 5 (Sullivan). *Let Γ be of finite critical exponent. Then for each $p \in \mathbb{H}^{n+1}$ there exists a meausre μ_p on Λ such that*

$$\frac{d\gamma\mu_p}{d\mu_p}(x) = e^{\delta_\Gamma h_\gamma(x)}.$$

Definition 6. The *crossed product algebra* $C(\Lambda) \rtimes \Gamma$ is a completion of the algebra $C_c(\Lambda \rtimes \Gamma)$ with *convolution product*:

$$f * g(x,\gamma) = \sum_{\delta\in\Gamma} f(x,\delta)g(x\delta,\delta^{-1}\gamma).$$

The same formula represents $C_c(\Lambda\times\Gamma)$ on $L^2(\Lambda\rtimes\Gamma)$ and $C(\Lambda) \rtimes \Gamma$ is the completion of $C_c(\Lambda \rtimes \Gamma)$ in this representation. It carries a natural involution given by $f^*(x,\gamma) := \overline{f(x\gamma,\gamma^{-1})}$.

Definition 7. A groupoid is a small category $\mathcal{G}$ in which all morphisms are invertible. It is *locally compact Hausdorff* if the morphism set of $\mathcal{G}$ carries such a topology and the domain and range maps $r, d : \mathcal{G} \to \mathcal{G}^{(0)}$ are continuous. It is *étale* if these maps are local homeomorphisms.

Example 8. $\Lambda \rtimes \Gamma$ is a groupoid with objects Λ and morphisms $\Lambda \times \Gamma$.

$$d(x, \gamma) = x\gamma, \quad r(x, \gamma) = x, \quad (x, \gamma)(x\gamma, \delta) = (x, \gamma\delta).$$

Definition 9. A C^*-*algebra* is a norm closed *-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Theorem 10 (Gelfand-Naimark)**.** *Any commutative unital C^*-algebra is isomorphic to $C(X)$ for X some locally compact Hausdorff space.*

Definition 11. Let B be a C^*-algebra. A *right C^*-B-module* is a complex vector space $\mathcal{E}$ which is also a right B-module, equipped with a bilinear pairing

$$\begin{aligned} \mathcal{E} \times \mathcal{E} &\to B \\ (e_1, e_2) &\mapsto \langle e_1, e_2 \rangle, \end{aligned}$$

such that

- $\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle^*$,
- $\langle e_1, e_2 b \rangle = \langle e_1, e_2 \rangle b$,
- $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0 \Leftrightarrow e = 0$,
- $\mathcal{E}$ is complete in the norm $\|e\|^2 := \|\langle e, e \rangle\|$.

We use the notation $\mathcal{E} \leftrightharpoons B$ to indicate this structure. $\mathcal{E}$ is *full* if $\langle \mathcal{E}, \mathcal{E} \rangle$ is dense in B.

Definition 12.

$$\mathrm{End}^*_B(\mathcal{E}) := \{T : \mathcal{E} \to \mathcal{E} : \exists T^* : \mathcal{E} \to \mathcal{E}\};$$

$$\mathbb{K}_B(\mathcal{E}) := \overline{\mathcal{E}\tilde{\otimes}_B\mathcal{E}},$$

where the action of $\mathcal{E}\tilde{\otimes}_B\mathcal{E}$ on $\mathcal{E}$ is given by $e \otimes e'(f) = e\langle e', f\rangle$. Two C^*-algebras A, B are *Morita equivalent* if there exists a full C^*-module $\mathcal{E} \leftrightharpoons B$ such that $A \cong \mathbb{K}_B(\mathcal{E})$.

Theorem 13. *Two commutative C^*-algebras are Morita equivalent if and only if they are isomorphic.*

Theorem 14. *Let Γ act properly discontinuously on a space X. Then $C(X) \rtimes \Gamma$ and $C(X/\Gamma)$ are Morita equivalent.*

Definition 15. Let $\mathcal{E}, \mathcal{F}$ be C^*-B-modules. A densely defined closed operator $D : \mathfrak{Dom} D \to \mathcal{F}$ is called *regular* if

- D^* is densely defined in $\mathcal{F}$

- $1 + D^*D$ has dense range.

Definition 16. Let $A \to \mathcal{E} \leftrightarrows B$ be a graded bimodule and $D : \mathfrak{Dom} D \to \mathcal{E}$ an odd regular operator. $(\mathcal{E}, D)$ is an KK-*cycle* for (A, B) if, for all $a \in \mathcal{A}$, a dense subalgebra of A

- $[D, a]$ extends to an adjointable operator in $\text{End}^*_B(\mathcal{E})$

- $a(1 + D^*D)^{-\frac{1}{2}} \in \mathbb{K}_B(\mathcal{E})$.

Example 17. $B = \mathbb{C}$, $A = C(M)$, M a compact spin manifold. Take D to be the Dirac operator in $L^2(S)$, where S is the spinor bundle on M. Then the metric on M is given by the formula

$$d(x, y) = \sup\{|f(x) - f(y)| : \|[D, f]\| \leq 1\}.$$

Theorem 18. *Let $c : \Lambda \rtimes \Gamma \to \mathbb{R}$ be a cocycle. The operator $D_c : C_c(\Lambda \rtimes \Gamma) \to C_c(\Lambda \rtimes \Gamma)$ defined by*

$$D_c f(x, \gamma) = c(x, \gamma) f(x, \gamma),$$

is a $C_c(\mathcal{H})$-linear derivation, where $\mathcal{H} = \ker c$. It extends to a selfadjoint regular operator on the C^*-module completion $\mathcal{E}_c$ of $C_c(\Lambda \rtimes \Gamma)$ with respect to the inner product

$$\langle f, g \rangle := \rho(f^* * g),$$

where $\rho : C_c(\mathcal{G}) \to C_c(\mathcal{H})$ is the restriction map. If the map $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is closed then D_c has compact resolvent, and $(\mathcal{E}_c, D_C)$ is a KK-cycle for $(C(\Lambda) \rtimes \Gamma, C^*(\mathcal{H}))$.

Definition 19. Let $(\mathcal{E}, D)$ be a KK-cycle for (A, B). The *bounded transform* of D is the operator $F = \mathfrak{b}(D) := D(1 + D^*D)^{-\frac{1}{2}}$. It satisfies

$$a(F - F^*), a(F^2 - 1), [F, a] \in \mathbb{K}_B(\mathcal{E}).$$

The pair $(\mathcal{E}, F)$ is called a *Kasparov module*. Two Kasparov modules $(\mathcal{E}', F')$ and $(\mathcal{E}'', F'')$ are *homotopic* if there is a Kasparov module $(\mathcal{E}, F)$ for $(A, B\tilde{\otimes}C([0,1]))$ such that $(\mathcal{E}_0, F_0) = (\mathcal{E}', F')$ and $(\mathcal{E}_1, F_1) = (\mathcal{E}'', F'')$. Two KK-cycles are called *equivalent* if their bounded transforms are homotopic, and the set of equivalence classes of KK-cycles for (A, B) is denoted $KK_*(A, B)$.

Theorem 20. *$KK_*(A,B)$ is an abelian group.*

KK-theory comes equipped with the structure of a category by the intricate Kasparov product

$$KK_*(A,B) \otimes KK_*(B,C) \to KK_*(A,C),$$

where A, B and C are C^*-algebras. It unifies K-theory and K-homology in the sense that there are natural isomorphisms

$$KK_*(\mathbb{C},A) \cong K_*(A), \quad KK_*(A,\mathbb{C}) \cong K^*(A).$$

The Patterson-Sullivan measure induces a cocycle

$$c : \Lambda \rtimes \Gamma \to \mathbb{R}$$
$$(x, \gamma) \mapsto \ln \frac{d\gamma\mu_p}{d\mu}(x\gamma).$$

This cocycle is closed for Kleinian groups of the first kind. Therefore it induces a map $K_1(C(\Lambda \rtimes \Gamma) \to K_0(C^*(\mathcal{H}))$. The groupoid $\mathcal{H}$ is unimodular, so integration over the unit space induces a trace $C^*(\mathcal{H}) \to \mathbb{C}$, which gives a map $K_0(C^*(\mathcal{H})) \to \mathbb{C}$.

Thus, for such groups we get a homomorphism $K_1(C(\Lambda) \rtimes \Gamma)) \to \mathbb{C}$. We can let the basepoint p vary to obtain a more general result.

Theorem 21. *There is a globalized index map*

$$\mathrm{Ind}_\mu : KK_1^M(C_0(M), C_M^*(\Lambda \rtimes \Gamma)) \to C(M).$$

This map is compatible with the maps Ind_{μ_p} : $K_1(C^*(\Lambda \rtimes \Gamma)) \to \mathbb{C}$ *in the sense that the diagram*

$$\begin{array}{ccc} KK_1^M(C_0(M), C_M^*(\Lambda \rtimes \Gamma)) & \xrightarrow{\mathrm{Ind}_\mu} & C(M) \\ \downarrow e_m & & \downarrow e_m \\ K_1(C^*(\Lambda \rtimes \Gamma)) & \xrightarrow{\mathrm{Ind}_{\mu_m}} & \mathbb{C}, \end{array}$$

commutes.