Cubic Surfaces

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IWI-RuG & DIAMANT

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(Utrecht maths colloquium)

starting point: Marcel van der Vlugt (Leiden),

"An application of elementary algebraic geometry in coding theory",

Nieuw Archief voor Wiskunde (4th series, vol. 14, 1996)

same result using elliptic curves by G. van der Geer and M. van der Vlugt (1994) and elementary proof by coding theorists G. L. Feng, K. K. Tzeng, and V. K. Wei (1992)

Problem: fix $m \geq 1$, put $n := 2^m - 1$ and enumerate

$$\mathbb{F}_{2^m} = \{0, \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

The binary code

$$\mathsf{BCH}(m) := \left\{(a_1, a_2, \ldots, a_n) \in \mathbb{F}_2^n \,;\, \sum a_i \alpha_i = 0 = \sum a_i \alpha_i^3\right\}.$$

Alternative: put $\mathbb{F}_{2^m} = \{0\} \cup \{\alpha^\ell \,;\, \ell \in \mathbb{Z}\}$.

Then

$$\mathsf{BCH}(m) = \left\{f = \sum a_i x^i \in \mathbb{F}_2[x]/(x^n - 1) \,;\, f(\alpha) = 0 = f(\alpha^3)\right\}.$$

Easy: every $v \neq 0$ in BCH(m) has at least 5 coordinates 1.

How many v have precisely 5 coordinates 1?

Idea: if $x_1, x_2, x_3, x_4, x_5 \in \mathbb{F}_{2^m}$ are the powers of α which give v, then

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0. \end{cases}$$

So the problem boils down to: count the points with coordinates in $\mathbb{F}_{2^m}$ on the projective surface S defined here.

S defines a (smooth) cubic surface.

It was first studied by Alfred Clebsch:

Mathematische Annalen Vol. 4 (1871), see page 331.

Let $P := (1 : \zeta_1 : \zeta_2 : \zeta_3 : \zeta_4) \in S$ and $\bar{P} := (1 : \zeta_1^4 : \zeta_2^4 : \zeta_3^4 : \zeta_4^4) \in S$, in which the ζ_j are the four different primitive 5th roots of unity (in $\mathbb{F}_{16}$).

The line containing P and $\bar{P}$ is contained in S. Permuting the primitive roots of 1 gives 12 such lines in S (in fact, two sets of 6 pairwise disjoint lines).

There are more straight lines contained in S: any system $x_i + x_j = 0 = x_k + x_\ell$ with $1 \leq i, j, k, \ell \leq 5$ pairwise different, also yields a line in S. In total, this yields another 15 lines.

Using 4 of the rational lines and a pair of conjugate ones, it is possible to obtain a set of 6 pairwise disjoint lines (which as a set is defined over $\mathbb{F}_2$).

One now contracts this set of 6 lines to 6 points. This can be done over $\mathbb{F}_2$. The image of S under this turns out to be (over $\mathbb{F}_2$) isomorphic to the projective plane.

Since counting on the projective plane is trivial, this allows one to count on S as well, and hence to determine the desired number of vectors v in BCH(m).

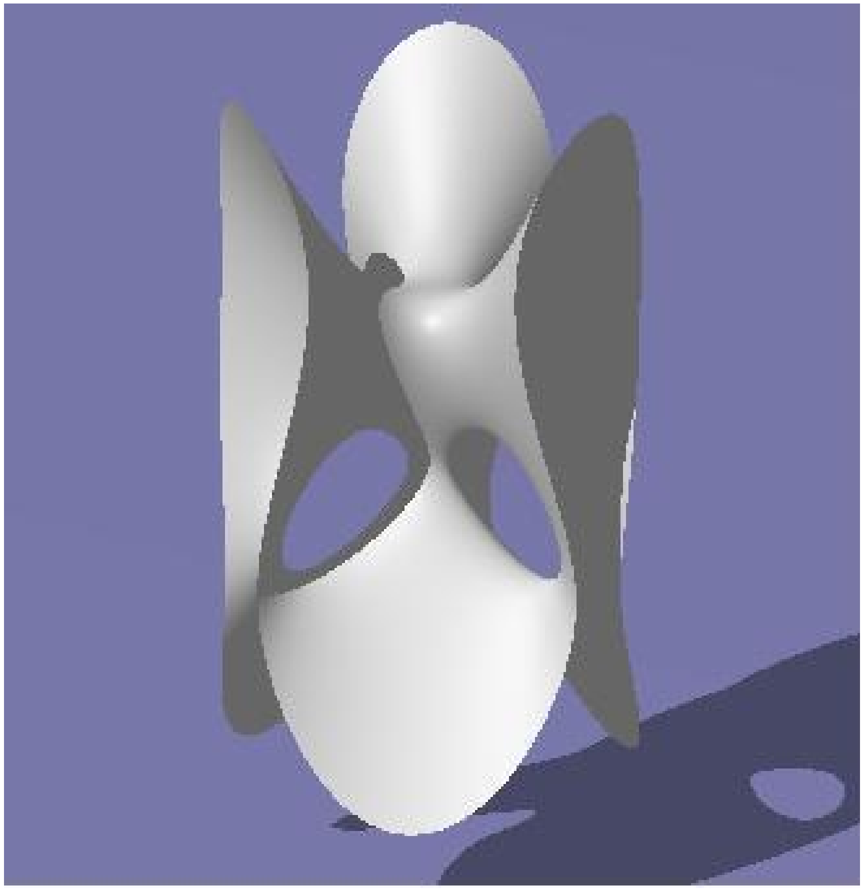
Alfred Clebsch (1833–1872)

student Adolf Weiler (Göttingen) instructed by Clebsch made a plaster model of the real surface given by the same equations.

More precisely: take (x, y, z) such that

$$\begin{aligned} -\sqrt{6}x + y + \sqrt{2}z &= 3x_1 + 3 \\ \sqrt{6}x + y + \sqrt{2}z &= 3x_2 + 3 \\ y - \sqrt{8}z &= 3 + x_3 \\ y &= x_4 - 3 \\ -2y &= x_5 \end{aligned}$$

yields in x, y, z coordinates:

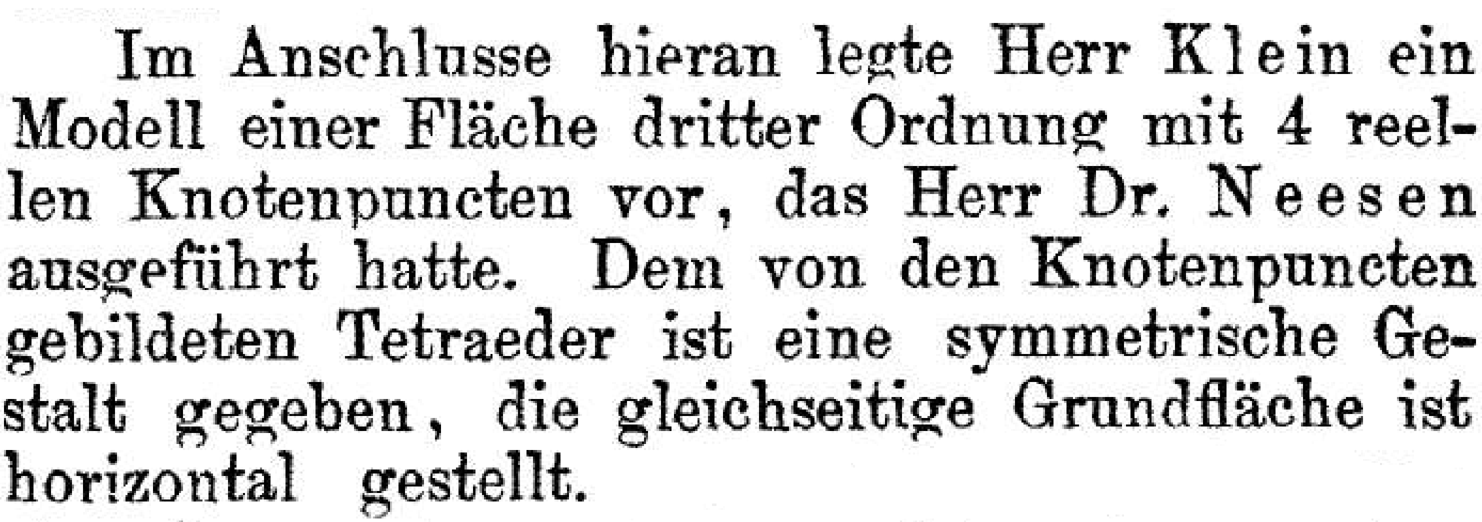


Clebsch's diagonal surface

1872, Sitzungsbericht der Königlichen Gesellschaft der Wissenschaften zu Göttingen, August 3:

Hr. Clebsch legte zwei Modelle vor, welche Hr. stud. Weiler hierselbst dargestellt hatte, und welche sich auf eine besondere Classe von Flächen dritter Ordnung beziehen. In No. 12 der Nachrichten von 1871 wurde als Diagonalfläche diejenige Fläche dritter Ordnung bezeichnet, welche durch die Diagonalen aller Vierseite geht, die auf den Ebenen eines Pentaeders jedesmal von den vier andern ausgeschnitten werden. Das eine der beiden Modelle stellte die 27 Geraden dieser Fläche dar, das andere die Fläche selbst, ein Gypsmodell, auf welchen die 27 Geraden gezeichnet waren. Die Geraden der Fläche theilen sich in 15 und 12, von denen erstere die oben angegebenen Diagonalen sind, während die 12 andern eine durch sie bestimmte Doppelsechs bilden.

Same journal, 1872, August 3:


Im Anschlusse hieran legte Herr Klein ein Modell einer Fläche dritter Ordnung mit 4 reellen Knotenpuncten vor, das Herr Dr. Neesen ausgeführt hatte. Dem von den Knotenpuncten gebildeten Tetraeder ist eine symmetrische Gestalt gegeben, die gleichseitige Grundfläche ist horizontal gestellt.

A reproduction is since 2009 used for the Compositio Prize

Also 1872, 3 months later,
Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen:

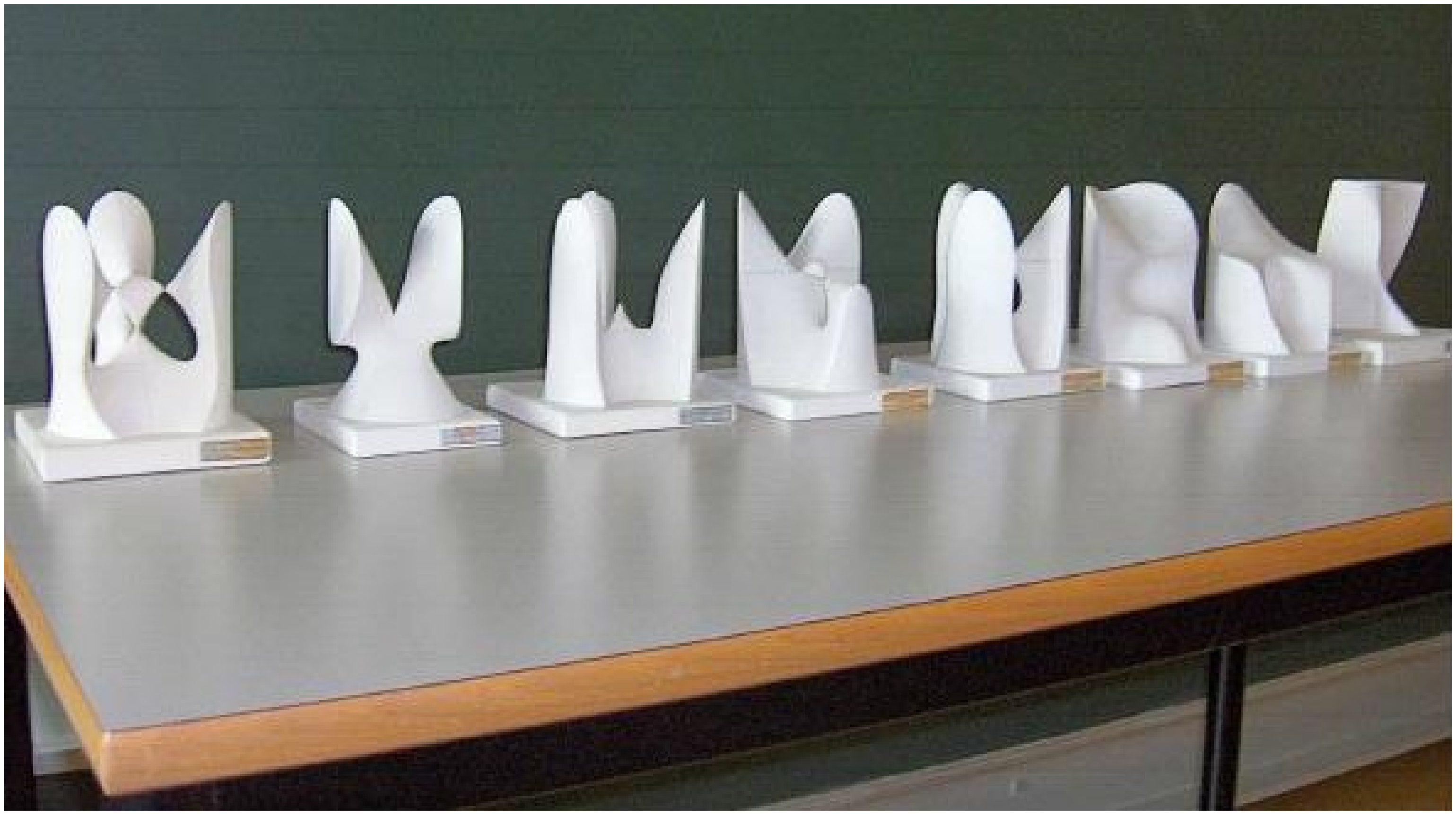
Von ihren ordentlichen Mitgliedern verlor die Societät in diesem Jahre durch den Tod:

Alfred Clebsch, seit 1864 Correspondent, seit 1868 Mitglied der mathematischen Classe. Er starb am 7. November d. J.; er war geboren am 19. Januar 1833.

Reproductions of plaster and string and wire models were made and sold (and advertized) by L. Brill in Halle (brother of Alexander) and later by M. Schilling in Leipzig.

The mathematics departments in Amsterdam, Leiden, Utrecht and Groningen still own many such models.

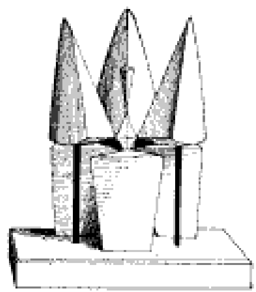
In Groningen this is due to P.H. Schoute (1842 - 1910)



Pieter Hendrik Schoute (1842–1910)

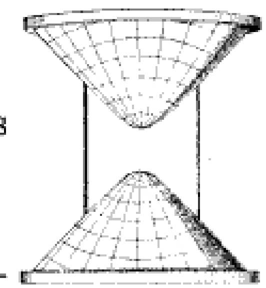
Models for the Higher Mathematical Instruction

PUBLISHED BY L. BRILL IN DARMSTADT (GERMANY).



MODELS

Of Plaster, constructed of Silk Threads in Brass Frames, of Wire, Sheet-Brass, etc.



16 SERIES.

The models of seven of those series are constructed after the originals in the Mathematical Institute of the Royal Polytechnicum in Munich, under the direction of Prof. Dr. BRILL, Prof. Dr. KLEIN and Prof. Dr. DYCK. Other series of Prof. Dr. KUMMER in Berlin, Prof. Dr. NEOVIUS in Helsingfors, Prof. Dr. RODENBERG in Hannover, Prof. Dr. ROHN in Dresden, Dr. SCHLEGEL in Hagen, Prof. Dr. WIENER in Karlsruhe, Privat-Docent Dr. WIENER in Halle, etc.

Excepting two series, all the models can be obtained separately. An explanatory text accompanies most of them. The prices are exclusive of packing and transportation.

Prospectus furnished, if desired, gratis and postpaid. Of the whole 217 numbers of the collection, 158 are of plaster, 19 are constructed of silk threads, 40 of wire, etc. They refer to almost all the departments of mathematical knowledge: synthetical and analytical geometry, theory of curvature, mathematical physics, theory of functions, etc.

July 1890, American Journal of Mathematics

DECLARATIE van den ondergeteekende
Oscar Keij, instrumentmaker
te Groningen
wegens het onderstaande geleverd of verricht ten behoeve van de Wiskundige modellenverzameling
der Rijksuniversiteit te Groningen. in het jaar 1900 [illegible]

Datum.	OMSCHRIJVING.	Eenheidsprijs.	Bedrag.
1900	rubriek E		
Octob. 7	10 modellen van [illegible]	15	f 150 —
Novemb. 3	15 draadmodellen van oppervlakken	18	270 —
" 24	20 gipsmodellen van oppervlakken [illegible]	10	200 —
Decemb. 1	20 d° d° [illegible]	10	200 —
" 5	1 Kopermodel van een schroefvlak		60 —
" "	1 d° " " schroefvormig oppervlak		105 —
" "	1 d° " " [illegible]		75 —
	Te samen		f 1060 —

De

De ondergeteekende verklaart deze declaratie deugdelijk en onvergolden tot een bedrag van éénduizend zestig Gulden.

Groningen, den 6 Maart 190[illegible].

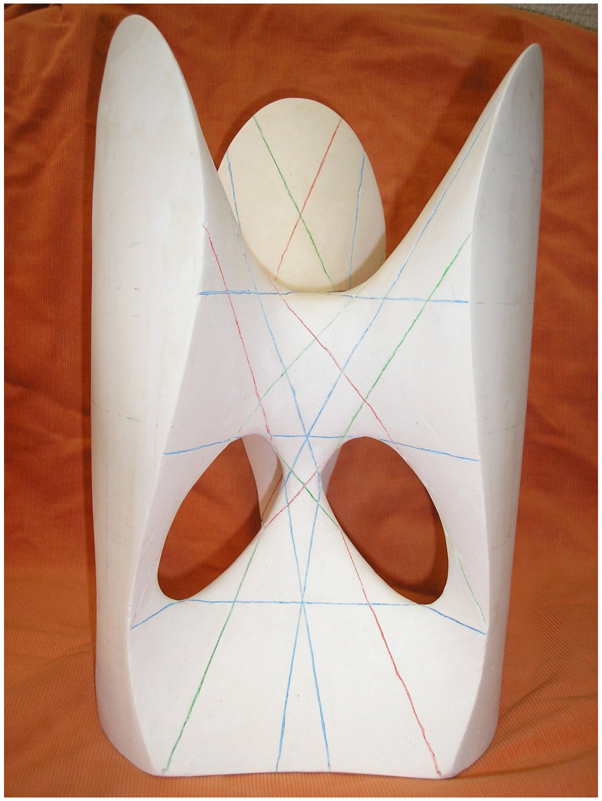
Oscar Keij

De ondergeteekende Hoogleeraar Directeur van de Wiskundige modellenverzameling verklaart, dat het in deze declaratie vermelde is geschied op zijn mondelingen last ten behoeve van genoemde inrichting, dat de oplevering daarvan behoorlijk heeft plaats gehad en keurt mitsdien deze declaratie goed tot een bedrag van éénduizend zestig Gulden.

Groningen, den 6 Maart 190[illegible].

P. H. Schoute.

Question: is one of the original Clebsch diagonal surface plaster reproductions still present in a Dutch institute?



(a modern one, Groningen)

Duco van Straten and Stephan Endraß had many copies made using a 3D printer in Mainz

Heinrich Heine Universität, Düsseldorf
(1999, Claudia Weber & Ulrich Forster, ceramics)

Special property of Clebsch's diagonal surface S: each of the planes $x_j + x_k = 0$ intersects S in a union of three concurrent lines.

Point of intersection of three such lines is called Eckardt point.
The Clebsch surface is the unique cubic with 10 such points.
The Fermat cubic $x^3 + y^3 + z^3 + w^3 = 0$ is the unique one with 18 such points.
All other cubic surfaces have $0, 1, 2, 3, 4, 6$ or 9 such points, as Beniamino Segre proved in 1942.

F.E. Eckardt: highschool teacher in Chemnitz. He has exactly one paper, published after he died: Mathematische Annalen Vol. 10 (1876)

In Utrecht, Tu Nguyen finished his PhD thesis in 2000 concerning Eckardt points.

Math. Annalen 10 (1876) p. 227

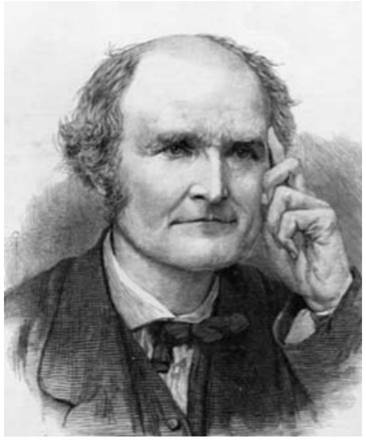
Ueber diejenigen Flächen dritten Grades, auf denen sich drei gerade Linien in einem Punkte schneiden*).

Von F. E. Eckardt †.

*) Wir entnehmen diese letzte Arbeit des verstorbenen Verfassers dem Osterprogramm 1875 der Realschule I. Ordnung zu Chemnitz. Die Red.

The starting point

1849, Arthur Cayley & George Salmon



Theorem. *a (smooth, projective) cubic surface over* $\mathbb{C}$ *contains precisely 27 lines*

1866, Alfred Clebsch

Theorem. *any smooth cubic surface over* $\mathbb{C}$ *is the closure of* $\varphi(\mathbb{P}^2(\mathbb{C}) - \{p_1, \ldots, p_6\})$ *for certain* $p_1, \ldots, p_6 \in \mathbb{P}^2$*, with* $\varphi(p) = (f_1(p) : f_2(p) : f_3(p) : f_4(p))$ *bijective, and*

$\sum \mathbb{C} f_j = \{f \in \mathbb{C}[x, y, z]$ *homog. of degree* 3 & $f(p_n) = 0, n = 1, \ldots, 6\}$.

example: Clebsch diagonal surface

$$
\begin{aligned}
x_1 &= (b-c)(ab+ac-c^2) \\
x_2 &= ac^2+bc^2-a^3-c^3 \\
x_3 &= a(c^2-ac-b^2) \\
x_4 &= c(a^2-ac+bc-b^2) \\
x_5 &= -v-w-x-y.
\end{aligned}
$$

modern proof: choose skew lines $\ell, m \subset S$

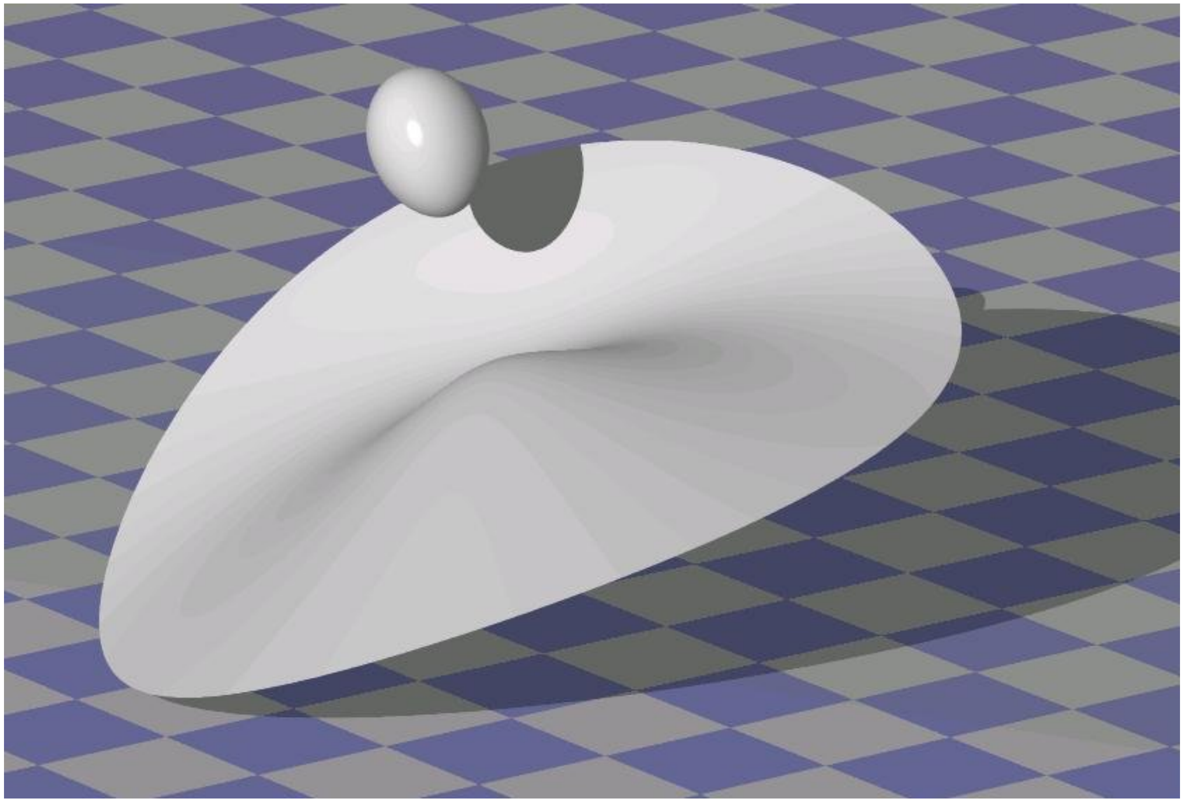
consider $S \to \ell \times m$: $P \mapsto (Q, R)$ where P, Q, R are collinear

there are five lines in S meeting ℓ as well as m, they have a point as image (but: every line in S meets at least one of the given five; so adapt:)

select one of these five image points (Q, R), '*blow up*' $\ell \times m$ in (Q, R) and '*blow down*' the lines $Q \times m$ and $\ell \times R$. result: $\mathbb{P}^2$, and inverse is $\mathbb{P}^2 \to S$ given by cubic polynomials.

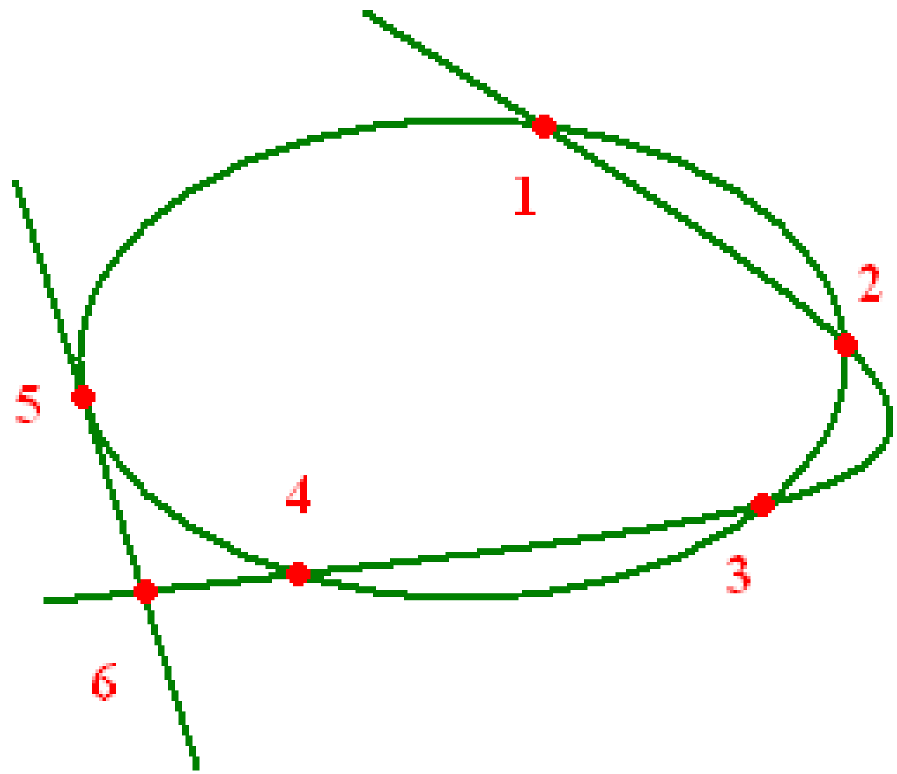
(with Irene Polo, CAGD vol. 26 (2009))

Theorem: *this is possible over $\mathbb{R}$ if and only if the real surface given by the cubic equation is connected.*



$$x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + y^2z - 6z^2 + 11z - 6 = 0$$

Following Silhol & Kollár, all non-connected cubics are obtained as follows:



Take two conics $f = 0$ and $g = 0$ intersecting in $1, 2, 3, 4$.
Take 5 on $f = 0$ and 6 on the tangent line to $f = 0$ at 5.
If no three of $1, 2, \ldots 6$ are on a line, let S_0 be the cubic surface corresponding to these points.

Define a rational involution τ on $\mathbb{P}^2$ by the rule:
$\{P, \tau(P)\}$ are the intersection points of the line through P and 6, with the conic through P and $1, 2, 3, 4$.

Then $S := S_0 \otimes \mathbb{C}/(\tau \otimes c)$ (with $c =$ complex conjugation) defines a cubic surface over $\mathbb{R}$ having two real connected components.

(with Irene Polo, Canad. Math. Bull. vol. 51 (2008))

What about cubic surfaces over $\mathbb{Q}$ (or, over number fields)?

Thm. (Swinnerton-Dyer, 1970). *A smooth cubic surface over* $\mathbb{Q}$ *is over* $\mathbb{Q}$ *birational to* $\mathbb{P}^2$*, if and only if it contains a rational point, and it contains a Galois-stable set of* 2 *or* 3 *or* 6 *pairwise skew straight lines.*

(Michigan Math. Journal vol. 7, 1970)

N.B.: in the case of a Galois-stable set of two skew lines, there is automatically a rational point: fix any rational point P in $\mathbb{P}^3$. Then $\mathbb{P}^2$ can be identified with the planes in $\mathbb{P}^3$ containing P.

Any such plane meets each of the two skew lines in a point, and the two points obtained this way determine a line. The third point of intersection of this line with the cubic surface is rational in case the plane we started with is rational.

In fact, this argument constructs a birational map from $\mathbb{P}^2$ to the cubic surface.

In the case of a Galois-stable set of 6 pairwise skew lines, the map contracting (“blowing down”) these 6 lines is birational and since the image has a rational point, it is actually isomorphic (over $\mathbb{Q}$) to $\mathbb{P}^2$.

So in this case we are in the situation of Clebsch’s theorem; in particular, the surface can be parametrized using polynomials of degree 3.

An example: $x^3 + y^3 + z^3 + w^3 = 0$. Here a Galois-stable set of 6 pairwise skew lines exists. So a parametrization over $\mathbb{Q}$ using degree 3 polynomials exists.

In this case Leonhard Euler (1707 - 1783) found a parametrization, however, it uses polynomials of degree 4 . . .

So it is possible to improve Euler's result!

parametrizations using cubic polynomials were recently found algebraically by Noam Elkies and geometrically (using the modern proof of Clebsch's theorem above) by Irene Polo and me:

$$x = -a^3 - 2a^2c + 3a^2b + 12abc - 3ab^2 - 4ac^2 + 6b^2c + 12bc^2 + 9b^3$$
$$y = a^3 + 2a^2c + 3a^2b + 12abc + 3ab^2 + 4ac^2 - 6b^2c + 12bc^2 + 9b^3$$
$$z = -8c^3 - 8ac^2 - 9b^3 - a^3 - 3a^2b - 3ab^2 - 4a^2c - 12b^2c$$
$$w = 8c^3 + 8ac^2 - 9b^3 + a^3 - 3a^2b + 3ab^2 + 4a^2c + 12b^2c.$$

Interest in cubic surfaces increased enormously because of applications in Computer Aided Geometric Design. Idea: use patches of cubic surfaces to approximate a surface in 3-space. Store the equation, and for displaying the surface, calculate a parametrization.

Fast calculation of parametrizations (and of equations), and low degrees in the polynomials giving the parametrization, means improved efficiency.

Recent work of C.L. Bajaj, T.G. Berry, R.L. Holt, L. Gonzalez-Vega, S. Lodha, A.N. Netravali, M. Paluszny, R.R. Patterson, T.W. Sederberg, J.P. Snively, J. Warren and others.

Simple practical Maple algorithm for finding the 27 lines: part of René Pannekoek's master's thesis (2009):

```
with(PolynomialTools):with(Groebner):
# select a cubic surface
F := x^3 + y^3 + z^3 + w^3 :
# search for lines of the form (1:t:a+ct:b+dt)
rule:={x=1, y=t, z=a+c*t, w=b+d*t}:
F1a:= expand(subs(rule,F)):
F2a:= collect(F1a,t):
vect:=CoefficientVector(F2a,t):
F3a:=[vect[1],vect[2],vect[3],vect[4]]:
G1a:=gbasis(F3a,plex(a,b,c,d)):
sol:=solve({seq(G1a[i],i=1...4)},{a,b,c,d});
```

If S is not birational to $\mathbb{P}^2$, then a "next best" is that S might be '*unirational*': there is a rational map $\mathbb{P}^2 \to S$ of positive degree (not necessarily degree 1).

Thm. (proven in Manin's book "Cubic Forms", 1972) *A smooth, cubic surface over $\mathbb{Q}$ is unirational if and only if it contains a rational point.*

Idea: take the rational point. If it is on a line contained in the surface, the proof is easy. If not, then the tangent plane to the surface in the point intersects the surface in a rational curve (a singular cubic). Now repeat this idea for all points on this rational curve.

Using this and Swinnerton-Dyer's result, one finds counter examples to "Lüroth's problem for transcendence degree 2" (work of Iskovskikh and Manin, 1971).

Explicit construction of smooth cubic surfaces containing a Galois-stable set of 6 pairwise skew lines, but not containing any rational point.

This is work with René Pannekoek, to appear in the Journal of Symbolic Computation.

Idea: change $\mathbb{P}^2$ into a variety over $\mathbb{Q}$ which is birational to it over an extension field, but not over $\mathbb{Q}$. Then use Clebsch's method to "blow up" a Galois orbit consisting of 6 points on this "Severi-Brauer" surface, resulting in a cubic surface as desired.

Explicit example:

$$
\begin{gathered}
-22751x^3 - 103032x^2y + 492480xy^2 - 373248y^3 + \\
+908712x^2z - 2612736xyz + 2612736y^2z - 2387737268z^3 + \\
+210066063732z^2w - 61927476156zw^2 + 60162954012w^3 = 0.
\end{gathered}
$$

Variant of Manin's unirationality result, with possibly a lower degree:

Thm. (Pannekoek, master's thesis, 2009) *Let* S *be a smooth cubic surface containing a rational point and a Galois-stable orbit of three pairwise disjoint lines.*

Then the following construction shows that S *is unirational over the base field.*

Construction: Let $p \in S$ be a rational point. Consider the $\mathbb{P}^2$ of planes $P \subset \mathbb{P}^3$ containing p.

For almost all such P one has: $P \cap S$ is a smooth plane cubic curve C containing p as a rational point. The three given lines yield, when intersected with P, three points in C.

Add these points in the group law on C with unit element p. The result is a point $q \in C \subset S$.

The assignment $P \mapsto q$ yields a rational map of positive degree, defined over the base field.