β-expansions and multiple tilings

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Introduction

Let $\beta > 1$ and $A = \{a_0, \ldots, a_m\}$ a set of real numbers with $a_0 < a_1 < \ldots < a_m$. Expressions of the form

$$x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n},$$

with $b_n \in A$ for all $n \geq 1$, are called β-expansions with arbitrary digits. This gives numbers in the interval $[\frac{a_0}{\beta - 1}, \frac{a_m}{\beta - 1}]$.

β is called the base, A is the digit set and elements of A are called digits.

Allowable digit sets

If, for a given $\beta > 1$, a set of real numbers $A = \{a_0, \ldots, a_m\}$ satisfies

(i) $a_0 < \ldots < a_m$,

(ii) $\max_{1 \le j \le m}(a_j - a_{j-1}) \le \dfrac{a_m - a_0}{\beta - 1}$,

it is called an allowable digit set. Then

- every $x \in \left[\dfrac{a_0}{\beta - 1}, \dfrac{a_m}{\beta - 1}\right]$ has a β-expansion with digits in A. (Pedicini, 2005)
- the minimal amount of digits in A is $\lceil \beta \rceil$.

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Outline

- Introduce a class of transformations that generate β-expansions.
- Characterize the set of digit sequences given by such a transformation.
- For specific β's (Pisot units) give a construction of a natural extension for the transformation.
- From the natural extension, get an absolutely continuous invariant measure.
- Under a further assumption, construct a multiple tiling of a Euclidean space and give an example that shows that the Pisot conjecture does not hold in this more general setting.

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Transformations

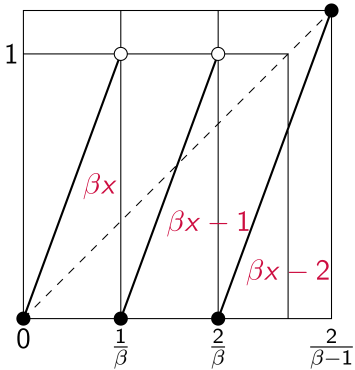
For each $\beta > 1$ and allowable digit set $A = \{a_0, \dots, a_m\}$ there exist transformations that generate β-expansions with digits in A by iteration.

Example: Classic β-expansions

Consider a non-integer $\beta > 1$ and digit set $A = \{0, 1, \dots, \lfloor \beta \rfloor\}$. This gives 'classic' β-expansions for all $x \in \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$. A transformation that generates these is

$$Tx = \begin{cases} \beta x \pmod 1, & \text{if } x \in [0, 1), \\ \beta x - \lfloor \beta \rfloor, & \text{if } x \in \left[1, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]. \end{cases}$$

The classic β-expansions



This is the classic greedy β-transformation.

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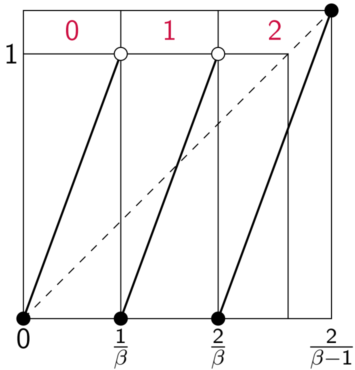

Assign a digit to each interval.

Make a digit sequence by setting

$$b_1(x) = \begin{cases} j, & \text{if } x \in \left[\frac{j}{\beta}, \frac{j+1}{\beta-1}\right], \\ & \text{for } 0 \le j \le \lfloor\beta\rfloor, \\ \lfloor\beta\rfloor, & \text{if } x \in \left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right]. \end{cases}$$

and $b_n(x) = b_1(T^{n-1}x)$ for $n \ge 1$. Then we have $Tx = \beta x - b_1$ and $T^2x = \beta Tx - b_2$, etc.

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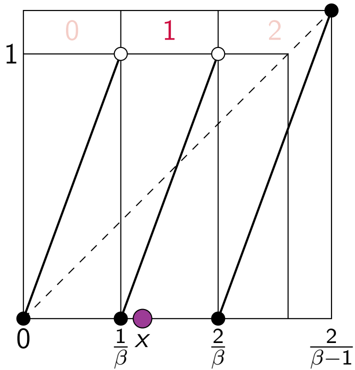

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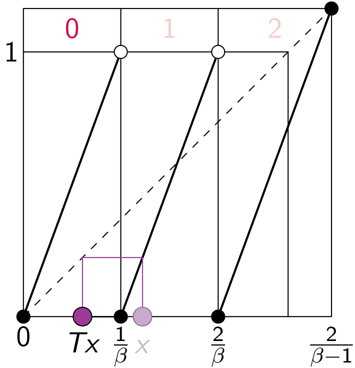

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The classic β-expansions

If $Tx = \beta x - b_1(x)$, then $x = \frac{b_1}{\beta} + \frac{Tx}{\beta}$. Iterating this, we get after n steps,

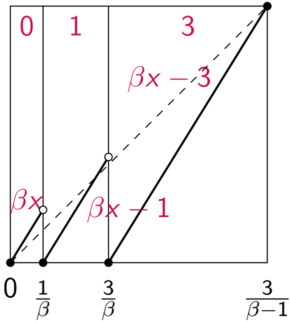
$$x = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \frac{T^2 x}{\beta^2} = \cdots = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_n}{\beta^n} + \frac{T^n x}{\beta^n}.$$

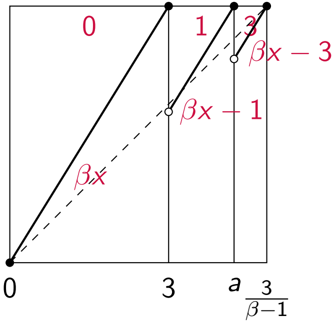
Since $T^n x \in \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right)$ for all n, this converges and gives

$$x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}.$$

Other transformations: greedy and lazy

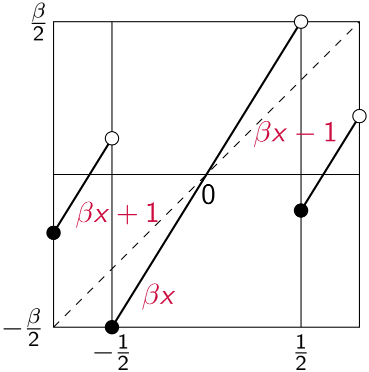
Take β to be the golden mean and $A = \{0, 1, 3\}$. These are the greedy and lazy β-transformations with digits in A. [Dajani & K., 2007]





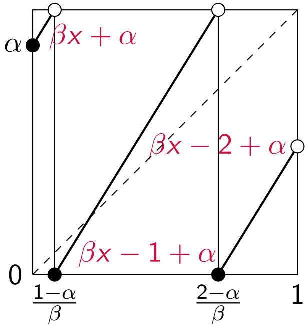
Other transformations: the minimal weight transformation

Take β to be the golden mean and $A = \{-1, 0, 1\}$. This is a minimal weight transformation, i.e. if an x has a finite β-expansion, then the expansion generated by this transformation has the highest number of 0's. [Frougny & Steiner, 2009]



Other transformations: the linear mod 1 transformation

Take $\beta > 1$ and $0 \le \alpha < 1$. Suppose $n < \beta + \alpha \le n + 1$. The linear mod 1 transformation below ($Tx = \beta x + \alpha \pmod 1$) gives β-expansions with digits in $\{j - \alpha : 0 \le j \le n\}$.



The class of transformations

Given a real number $\beta > 1$ and a digit set $A = \{a_0, \ldots, a_m\}$, we consider the class of transformations that have the following properties.

- For each digit in the digit set a_i, there is a bounded interval X_i and if $i \neq j$, then $X_i \cap X_j = \emptyset$.
- On the interval X_i the transformation is given by $Tx = \beta x - a_i$.
- If $X = \bigcup_{i:a_i \in A} X_i$, then $TX = X$.

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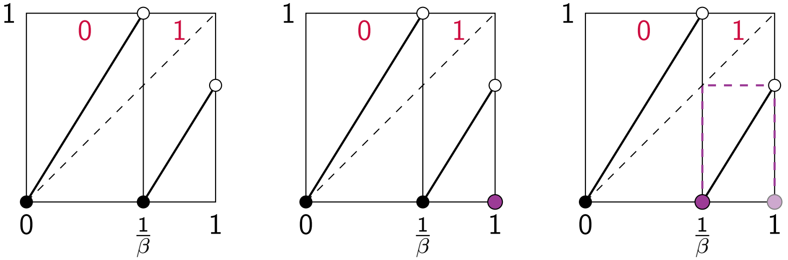
Admissible sequences: The golden mean

We can characterise the digit sequences generated by a transformation. For the classic greedy β-transformation with β the golden mean, we have the following.



0 can be followed by 0 or 1, but 1 is always followed by 0. This is given by the orbit of 1. Hence, T produces precisely the sequences from the set $\{u_1 u_2 \cdots \mid u_n u_{n+1} \neq 11,\ n \geq 1\}$.

The set of admissible sequences

Given a transformation T for a $\beta > 1$ and digit set A, we call a sequence $u_1 u_2 \cdots \in A^{\mathbb{N}}$ admissible for T if there is an $x \in X$ such that $u_1 u_2 \cdots = b(x)$.

A two-sided sequence $\cdots u_{-1} u_0 u_1 \cdots$ is called admissible if for each $n \in \mathbb{Z}$ there is an $x \in X$, such that $u_n u_{n+1} \cdots = b(x)$.

Notation: $\mathcal{S}$ is the set of two sided admissible sequences.

Relation between expansions and sequences

To a β-transformation on an interval, there corresponds a shift-transformation on a set of digit sequences.

$$\begin{array}{rcccl} & T & & & \\ x & = & \sum_{n=1}^{\infty} \frac{b_n}{\beta^n} & \Rightarrow & b(x) = b_1 b_2 \cdots . \\ Tx & = & \sum_{n=1}^{\infty} \frac{b_{n+1}}{\beta^n} & \Rightarrow & b(Tx) = b_2 b_3 \cdots . \end{array}$$

T is not invertible: after applying T to x we 'lose' the first digit b_1.

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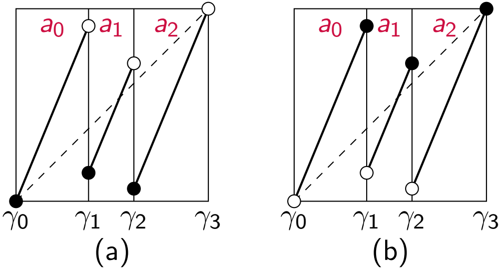
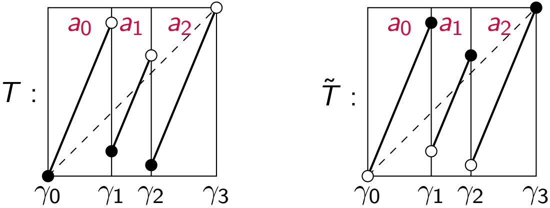


Let $b(x)$ be a digit sequence given by (a) and $\tilde{b}(x)$ the one given by (b). Then we have the following characterization in terms of the sequences $b(\gamma_j)$ and $\tilde{b}(\gamma_j)$.

Admissible sequences



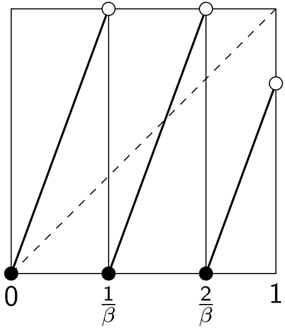
Admissible sequences

A sequence $u_1 u_2 \cdots \in \{a_0, \ldots, a_m\}^{\mathbb{N}}$ is generated by T iff for each $n \geq 1$, if $u_n = a_j$, then

$$b(\gamma_j) \preceq u_n u_{n+1} \cdots \prec \tilde{b}(\gamma_{j+1}),$$

where $\preceq$ denotes the lexicographical ordering.

The classic admissible sequences



In the case of the classic greedy β-transformation, only the orbit of 1 is important. This gives the Parry condition.

Theorem (Parry, 1960)

Let $\tilde{b}(1)$ be the expansion of 1 generated by T. Then a sequence $u_1 u_2 \cdots \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ is generated by T iff for each $n \geq 1$,

$$u_n u_{n+1} \cdots \prec \tilde{b}(1).$$

Invariant measures

The classic greedy β-transformation T_β has the following properties.

- It has an invariant measure that is equivalent to the Lebesgue measure on the unit interval $[0, 1)$. (Rényi, 1957)
- The density function is given by

$$h_c : [0,1) \to [0,1) : x \mapsto \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T^n 1)}(x),$$

  where $F(\beta)$ is a normalizing constant. (Gel'fond, 1959, and Parry, 1960)

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Natural extensions

A way to find an invariant measure is by studying the natural extension of the dynamical system.

Consider the non-invertible system $(X, \mathcal{B}, T)$, where $\mathcal{B}$ is the Lebesgue σ-algebra on X. Then a version of the natural extension of $(X, \mathcal{B}, T)$ is an invertible system $(\hat{X}, \hat{\mathcal{B}}, \hat{T})$, such that

- There is a map $\pi : \hat{X} \to X$ that is surjective, measurable and such that $\pi \circ \hat{T} = T \circ \pi$.
- This system is the smallest in the sense of σ-algebras: $\bigvee_{n \geq 0} \hat{T}^n(\pi^{-1}(\mathcal{B})) = \hat{\mathcal{B}}$.

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Pisot β's

To be able to say more, we assume that the real number $\beta > 1$ has some additional properties. Numbers with all these properties are called Pisot units.

- β is an algebraic unit: it is a root of a minimal polynomial of the form $x^d - c_1 x^{d-1} - \cdots - c_d$, with $c_i \in \mathbb{Z}$ for all i and $c_d \in \{-1, 1\}$.
- Denote all the other roots of the polynomial $x^d - c_1 x^{d-1} - \ldots - c_d$ by β_j, then $|\beta_j| < 1$ for all j.

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Contracting and expanding eigenspaces

Let $\beta > 1$ be a Pisot unit with minimal polynomial $x^d - c_1 x^{d-1} - \cdots - c_d$. Let $\beta_2, \ldots, \beta_d$ be the Galois conjugates of β. Consider the matrix M:

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

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Eigenvalues: $\beta_1 = \beta, \beta_2, \ldots, \beta_d$.
Eigenvectors: $\mathbf{v}_1, \ldots, \mathbf{v}_d$.
$|\det M| = 1$.

Let H be the hyperplane of $\mathbb{R}^d$ which is spanned by the real and imaginary parts of $\mathbf{v}_2, \ldots, \mathbf{v}_d$.

Consider the space $H + \mathbb{R}\mathbf{v}_1$. M is expanding by a factor β in the direction of $\mathbf{v}_1$ and contracting by a factor $1/\beta$ on H.

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Example: the golden mean

Take $\beta > 1$ such that $\beta^2 - \beta - 1 = 0$. Then $\frac{1}{\beta} = \beta - 1$. So, β is an algebraic unit. Then

$$\Big(-\frac{1}{\beta}\Big)^2 - \Big(-\frac{1}{\beta}\Big) - 1 = (1-\beta)^2 + (\beta - 1) - 1 = \beta^2 - \beta - 1 = 0.$$

Hence, $\beta_2 = -\frac{1}{\beta}$ and β is a Pisot unit. We have

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} \beta \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\beta} \\ 1 \end{pmatrix}.$$

Then $H = \mathbb{R}\mathbf{v}_2$ and $\mathbb{R}^2$ is spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Example: the tribonacci number

Take $\beta > 1$ such that $\beta^3 - \beta^2 - \beta - 1 = 0$. Since $\frac{1}{\beta} = \beta^2 - \beta - 1$, β is an algebraic unit. We have $\beta_2 \in \mathbb{C}$ and $\beta_3 = \overline{\beta}_2$. Also,

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} \beta^2 \\ \beta \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \beta_2^2 \\ \beta_2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} \beta_3^2 \\ \beta_3 \\ 1 \end{pmatrix}.$$

Since $\mathbf{v}_3 = \overline{\mathbf{v}}_2$, H is spanned by $\Re(\mathbf{v}_2)$ and $\Im(\mathbf{v}_2)$.

Set-up

We now have the following set-up:

- $\beta > 1$ is a Pisot unit.
- This gives a matrix M with eigenvalues $\beta = \beta_1, \beta_2, \ldots, \beta_d$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$.
- The space H is the hyperplane of $\mathbb{R}^d$ spanned by the real and imaginary parts of $\mathbf{v}_2, \ldots, \mathbf{v}_d$. Every point in $\mathbf{x} \in \mathbb{R}^d$ can be written as $x\mathbf{v}_1 - \sum_{j=2}^{d} y_j \mathbf{v}_j$ with $x \in \mathbb{R}$ and $y_j \in \mathbb{C}$, $2 \leq j \leq d$.
- The transformation $T : X \to X$ is given by β, an allowable digit set $A \subset \mathbb{Q}(\beta)$ and a finite union of bounded intervals X. For each digit $a_i \in A$, let $X_i \subseteq X$ be the interval on which T is given by $Tx = \beta x - a_i$.
- For $x \in X$, $b(x)$ denotes de digit sequence of x generated by T. The transformation T specifies a set of two-sided admissible sequences

$$\mathcal{S} = \{\cdots u_{-1} u_0 u_1 u_2 \cdots \mid \forall n \in \mathbb{Z}\, \exists x \in X : u_n u_{n+1} \cdots = b(x)\}.$$

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The natural extension space

We define the natural extension space by mapping the admissible sequences into $\mathbb{R}^d$.

For $1 \leq j \leq d$, let $\Gamma_j : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j)$ be given by $\Gamma_j(\beta) = \beta_j$ and $\Gamma_j(q) = q$ for $q \in \mathbb{Q}$.

Let $w \cdot u = \cdots w_{-1} w_0 u_1 u_2 \cdots \in \mathcal{S}$. The map ϕ maps the sequence w into H:

$$\phi(w) = \phi(\cdots w_{-1} w_0) = \sum_{j=2}^{d} \sum_{n=0}^{\infty} w_{-n} \beta_j^n \mathbf{v}_j.$$

Then ψ maps $w \cdot u$ into $\mathbb{R}^d$:

$$\psi(w \cdot u) = \psi(\cdots w_{-1} w_0 u_1 u_2 \cdots) = \sum_{n=1}^{\infty} \frac{u_n}{\beta^n} \mathbf{v}_1 - \phi(w).$$

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The natural extension transformation

For the natural extension transformation $\hat{T} : \hat{X} \to \hat{X}$ we want the following.

- $\hat{T}$ is a.e. invertible.
- $\hat{T}$ preserves the dynamics of T.
- $\hat{T}$ is invariant wrt the Lebesgue measure.

Partition $\hat{X} = \bigcup_{i \in I} \hat{X}_i$ with $\hat{X}_i = \{\psi(w \cdot u) \mid u_1 = a_i\}$. For $\mathbf{x} \in \hat{X}$, write $\mathbf{x} = x\mathbf{v}_1 - \sum_{j=2}^{d} y_j \mathbf{v}_j$. If $\mathbf{x} \in \hat{X}_i$, take

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An invariant measure for T

The Lebesgue measure on λ^d is invariant for $\hat{T}$. Let $\pi : \hat{X} \to X$ be given by $\pi\big(x\mathbf{v}_1 - \sum_{j=2}^d y_j \mathbf{v}_j\big) = x$.

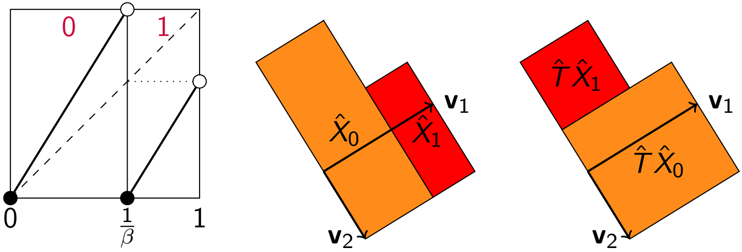
Define the measure μ on $(X, \mathcal{L})$ by $\mu(E) = (\lambda^d \circ \pi^{-1})(E)$ for each $E \in \mathcal{L}$. Then μ is invariant for T.

Note: Since $\hat{T}$ is only invertible a.e. we need to remove the right sets of measure zero everywhere.

An example: the golden mean

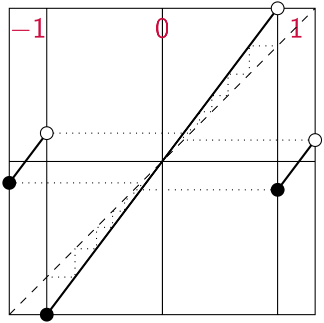
Take β to be the golden mean and T the classic greedy β-transformation, then $A = \{0, 1\}$. Recall that

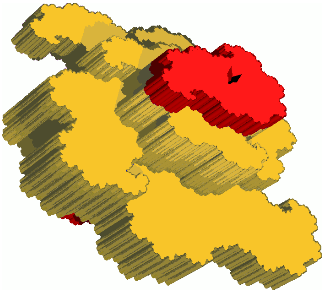
$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} \beta \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\beta} \\ 1 \end{pmatrix}.$$



An example: the smallest Pisot number

Let β be the real solution of $x^3 - x - 1 = 0$. This is the smallest Pisot number. Take $A = \{-1, 0, 1\}$, $X_{-1} = \Big[-\frac{\beta^7}{\beta^8-1}, -\frac{\beta^6}{\beta^8-1}\Big)$, $X_0 = \Big[-\frac{\beta^6}{\beta^8-1}, \frac{\beta^6}{\beta^8-1}\Big)$ and $X_1 = \Big[\frac{\beta^6}{\beta^8-1}, \frac{\beta^7}{\beta^8-1}\Big)$. Then T is a minimal weight transformation.





An example: the tribonacci number

Let β be the tribonacci number. Take $A = \{-1, 0, 1\}$,
$X_{-1} = \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1} \right)$, $X_0 = \left[-\frac{1}{\beta+1}, \frac{1}{\beta+1} \right)$ and $X_1 = \left[\frac{1}{\beta+1}, \frac{\beta}{\beta+1} \right)$.
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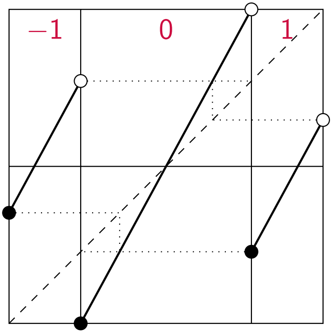
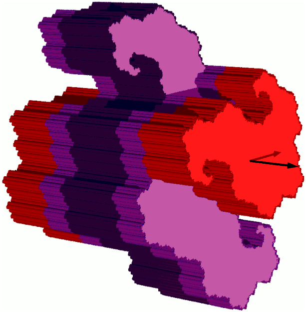




Multiple tilings

Under a certain condition we can say more about the invariant measure for T given by its natural extension. Moreover, this condition allows us to construct a tiling of the space H. A tiling is the following.

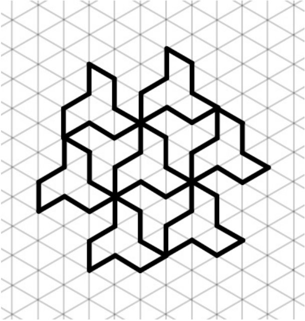
Start with a finite set of prototiles in H, compact sets that are the closure of their interior. $\mathcal{D} = \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$.

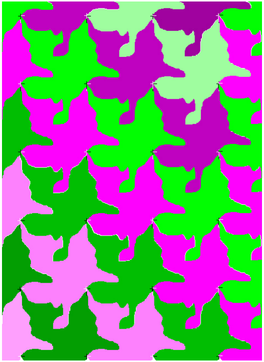
A tile is a translation of a prototile. $\mathcal{T}_x = \mathcal{D}_i + \mathbf{v}_x$ for some vector $\mathbf{v}_x \in H$.

Let $\mathrm{M} \geq 1$. A multiple tiling of degree M of H is a covering of H such that almost all elements are in exactly M different tiles.

A tiling of H is a multiple tiling of degree 1.

Tilings



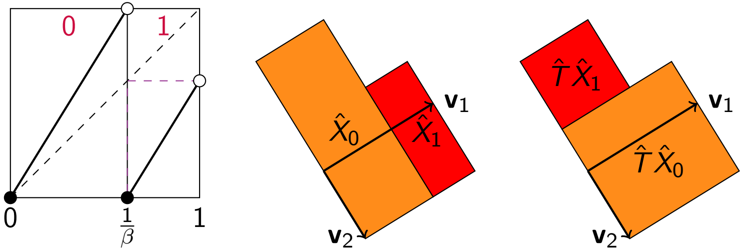


A finite set of prototiles

For each $x \in X$, let $\mathcal{D}_x = \{\phi(w) \mid w \cdot b(x) \in \mathcal{S}\}$. Then each set $\mathcal{D}_x$ is compact and

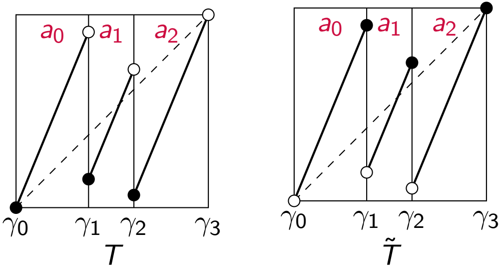
$$\hat{X} = \bigcup_{x \in X} x\mathbf{v}_1 + \mathcal{D}_x.$$

We will construct a multiple tiling of H with $\{\mathcal{D}_x \mid x \in \mathbb{Z}[\beta] \cap X\}$ as the set of prototiles. Therefore, we would like to have only finitely many different sets $\mathcal{D}_x$.



A finite set of prototiles

Recall the transformation $\tilde{T}$:



For γ_i, let n_i be the minimal k such that $T^k\gamma_i = \tilde{T}^k\gamma_i$ with $n_i = \infty$ if this doesn't happen.

A finite set of prototiles

Suppose that $A = \{a_0, \ldots, a_m\}$.

Theorem

If the set

$$\mathcal{V} = \bigcup_{i=0}^{m+1} \{\gamma_i\} \cup \bigcup_{1 \le k < n_i, \gamma_i \in X, i \neq 0} \{T^k \gamma_i, \tilde{T}^k \gamma_i\}$$

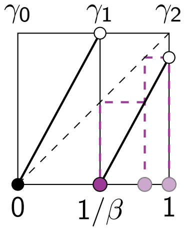
is finite, then there are only finitely many different sets $\mathcal{D}_x$, $x \in X$.

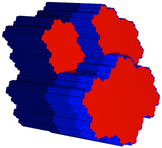
Under this condition the density of the invariant measure for T, $\mu = \lambda^d \circ \pi^{-1}$ is a finite sum of indicator functions. Let $\mathcal{D}_1, \ldots, \mathcal{D}_\kappa$ be these sets and $X'_1, \ldots, X'_\kappa$ the corresponding subsets of X. Then

$$\mu([s,t)) = c \int_{[s,t)} \sum_{k=1}^{\kappa} \lambda^{d-1}(\mathcal{D}_k) 1_{X'_k} d\lambda.$$

An example: the classic greedy tribonacci-transformation

$$\mathcal{V} = \bigcup_{i=0}^{m+1} \{\gamma_i\} \cup \bigcup_{1 \le k < n_i, \gamma_i \in X, i \neq 0} \{T^k \gamma_i, \tilde{T}^k \gamma_i\}$$





- $\bigcup_{i=0}^{2} \{\gamma_i\} = \{0, \frac{1}{\beta}, 1\}$.
- $\{\gamma_i \in X \mid i \neq 0\} = \{1/\beta\}$.
- $T^k(\frac{1}{\beta}) = 0$ for all $k \geq 1$.
  $\tilde{T}(\frac{1}{\beta}) = 1$, $\tilde{T}^2(\frac{1}{\beta}) = \beta - 1$, $\tilde{T}^3(\frac{1}{\beta}) = \frac{1}{\beta}$.
  So, $n_1 = \infty$, but γ_1 is periodic for $\tilde{T}$.
- $\mathcal{V} = \{0, \frac{1}{\beta}, \beta - 1, 1\}$ is a finite set.
- This gives 3 different prototiles $\mathcal{D}_x$.

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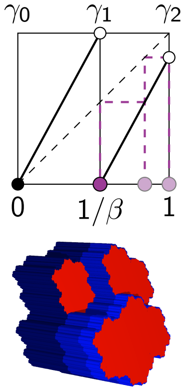

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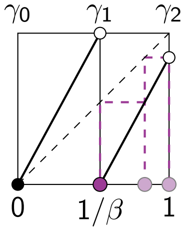

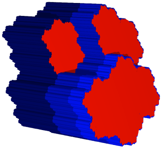

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  So, $n_1 = \infty$, but γ_1 is periodic for $\tilde{T}$.
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An example: the classic greedy tribonacci-transformation

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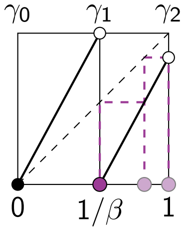

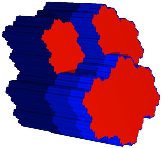

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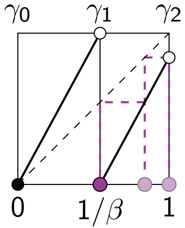

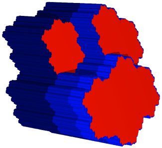

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The translation vectors

Suppose that $\mathcal{V}$ is a finite set. The set $\{\mathcal{D}_x \mid x \in X\}$ is finite and is the set of prototiles. We now want a set of translation vectors, so that

- all the translates of $\mathcal{D}_x$ together cover the whole space H, and
- there is an $\mathrm{M} \geq 1$ such that a.e. point in H is in exactly M translates of prototiles.

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So the set of translation vectors must be big enough, but not too big.

The multiple tiling

Define the function $\Phi : \mathbb{Q}(\beta) \to H$ by $\Phi(x) = \sum_{j=2}^{d} \Gamma_j(x)\mathbf{v}_j$.

For $x \in \mathbb{Z}[\beta] \cap X$, define the tiles $\mathcal{T}_x = \Phi(x) + \mathcal{D}_x$.

Theorem

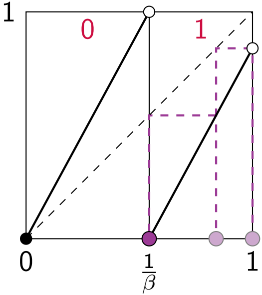
There is an $\mathrm{M} \geq 1$, such that the set $\{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a multiple tiling of degree M of H.

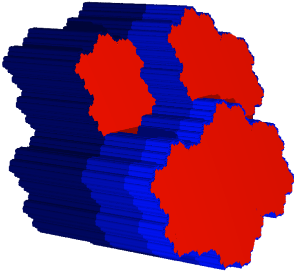
Pisot conjecture (Akiyama, 2002 and Sidorov, 2003)

If β is a Pisot number and T is the classic greedy β-transformation, then this construction gives a tiling of the space H.

An example: the Rauzy tiling (Rauzy, 1982)

Let β be the tribonacci number and T the classic greedy β-transformation.





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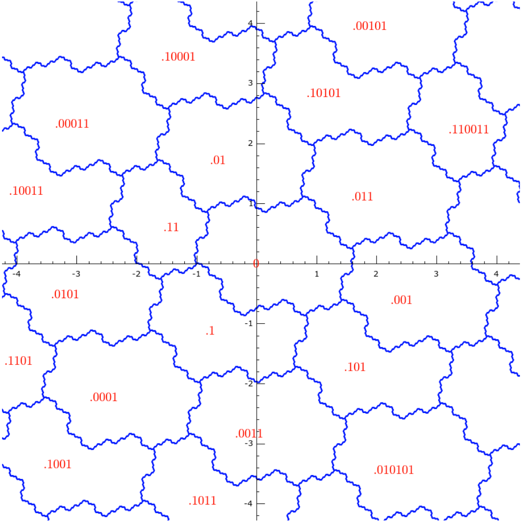
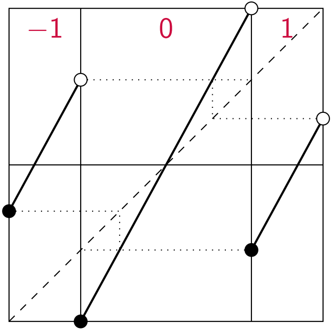
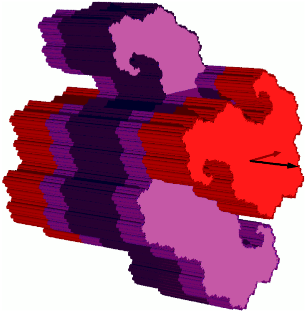


A tiling: the tribonacci number

Let β be the tribonacci number. Take $A = \{-1, 0, 1\}$,
$X_{-1} = \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right)$, $X_0 = \left[-\frac{1}{\beta+1}, \frac{1}{\beta+1}\right)$ and $X_1 = \left[\frac{1}{\beta+1}, \frac{\beta}{\beta+1}\right)$.
Then T is a minimal weight transformation.





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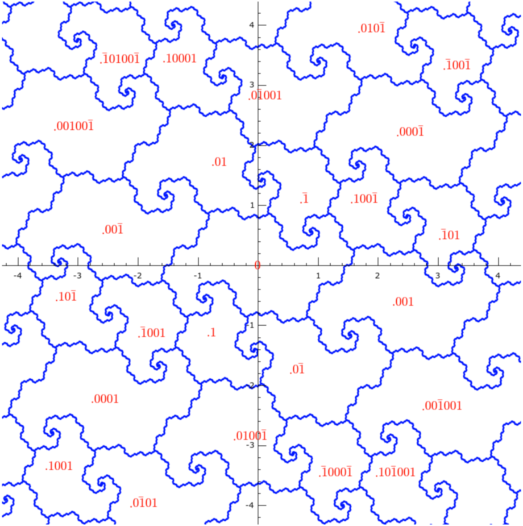
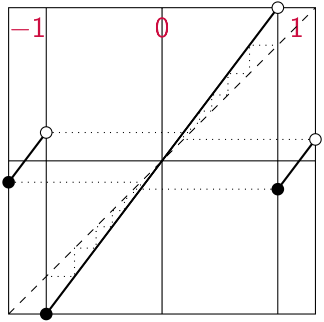
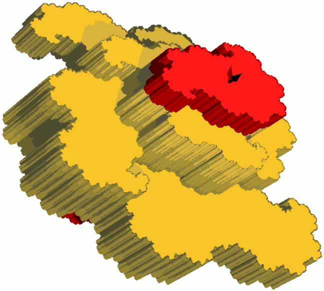


A tiling: the smallest Pisot number

Let β be the real solution of $x^3 - x - 1 = 0$. This is the smallest Pisot number. Take $A = \{-1, 0, 1\}$, $X_{-1} = \left[-\frac{\beta^7}{\beta^8-1}, -\frac{\beta^6}{\beta^8-1}\right)$, $X_0 = \left[-\frac{\beta^6}{\beta^8-1}, \frac{\beta^6}{\beta^8-1}\right)$ and $X_1 = \left[\frac{\beta^6}{\beta^8-1}, \frac{\beta^7}{\beta^8-1}\right)$. Then T is a minimal weight transformation.





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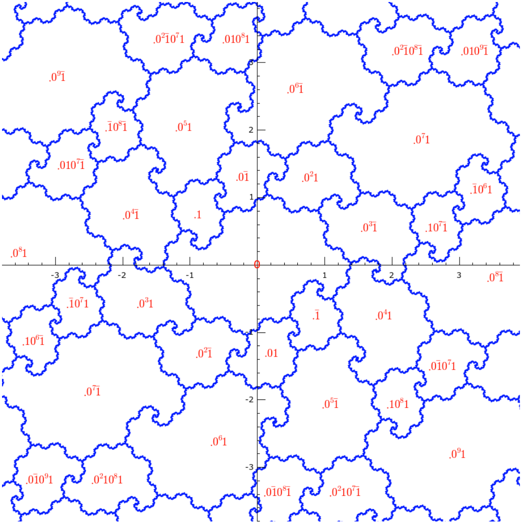

A double tiling: the tribonacci number

Let β be the tribonacci number. Take $A = \{-1, 0, 1\}$,
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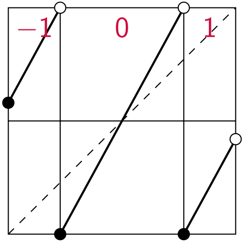

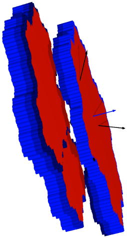

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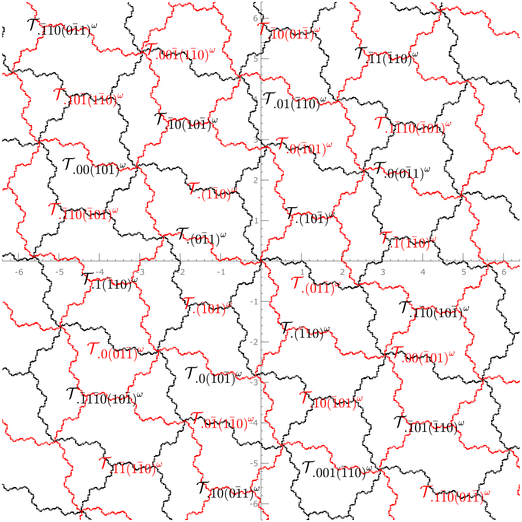

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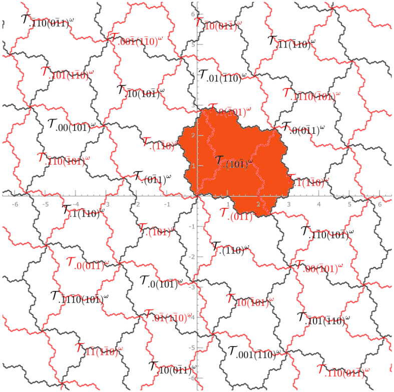

Consider 2 tiles $\mathcal{T}_{1-\frac{1}{\beta}}$ and $\mathcal{T}_{\frac{1}{\beta^3}}$.

Take a point $\mathbf{y}$ from the yellow ball in $\mathcal{T}_{1-\frac{1}{\beta}}$.

Then $\mathbf{y} = \Phi(1-\frac{1}{\beta}) + \phi(w)$ for some w such that $w \cdot b(1-\frac{1}{\beta}) \in \mathcal{S}$.

Show that there is a sequence w', such that $w' \cdot b(\frac{1}{\beta^3})$ and $\mathbf{y} = \Phi(\frac{1}{\beta^3}) + \phi(w')$.

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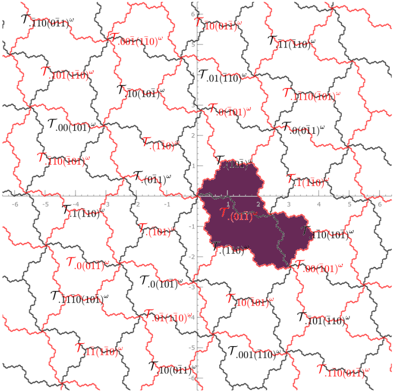

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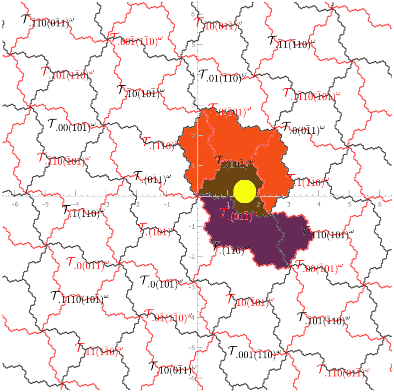

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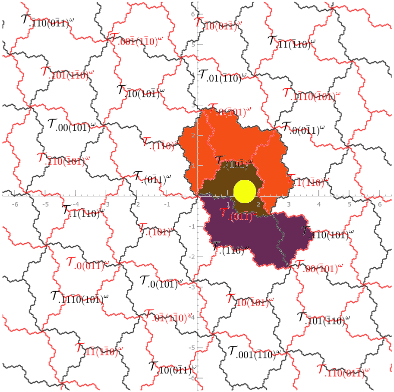

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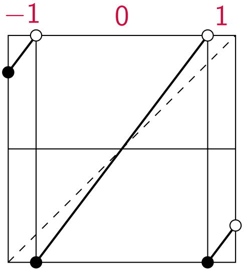

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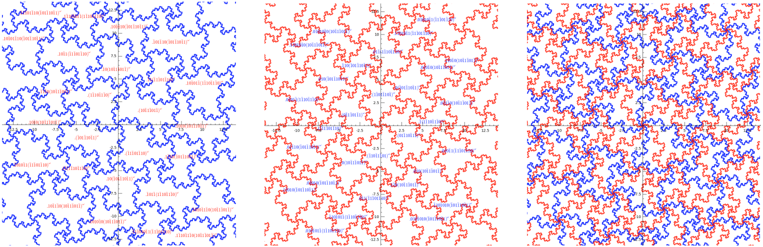

Properties of the multiple tiling

- The tiles that contain the origin are precisely the tiles $\mathcal{T}_x$ for which x has a purely periodic digit sequence $b(x)$.
- The multiple tiling is self-replicating.
- The multiple tiling is quasi-periodic: $\forall r > 0\ \exists R$ such that for all $\mathbf{y}, \mathbf{y}' \in H$, the local configuration that we see in the ball $B(\mathbf{y}, r)$ also occurs in the ball $B(\mathbf{y}', R)$.
- If the multiple tiling is a tiling, then the closure of the natural extension domain gives a tiling of the torus.
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