Random geometric graphs

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RANDOM GRAPH MODELS

- The classical Erdös–Rényi model $G(n, p)$

  (Bollobás 1984/2001. Janson, Łuczak and Rucinski 2000)

- Random geometric graph $G(\mathcal{B}_n, r)$

  (Penrose 2003)

  $\mathcal{B}_n = \{X_1, \ldots, X_n\}$, X_i independent uniform random in $[0, 1]^d$.

  Motivation: sensor networks; spatial/multivariate statistics; analysis of algorithms; alternative to $G(n, p)$

DEGREE DISTRIBUTIONS

Let D_i be the degree of vertex X_i in G. Then for $r_n \to 0$,

$$\mathbf{E}[D_1] \sim n\pi_d r_n^d \quad \text{in} \quad G(n, r_n)$$

If $r_n \sim an^{-1/d}$ then $D_1 \sim$ Poisson. [$G(n, a/n)$ similar]

Let Δ_i denote the number of triangles including X_i.

If $r_n \sim an^{-1/d}$, then $\mathbf{E}\Delta_1 \sim c$

whereas for $G(n, a/n)$, $\mathbf{E}\Delta_1 \sim c'/n$.

THE GIANT COMPONENT for $G(n,p)$

Let $L_i(G)$ be the ith largest component size in G. Given $\lambda > 0$,

$$\begin{aligned} n^{-1}L_1(G(n,\lambda/n)) \quad &\to_p \quad \phi(\lambda) \quad \text{as} \quad n \to \infty \\ n^{-1}L_2(G(n,\lambda/n)) \quad &\to_p \quad 0 \end{aligned}$$

where ϕ is continuous and

$$\begin{aligned} \phi(\lambda) = 0 \qquad & \lambda \le 1 \\ \phi(\lambda) > 0 \qquad & \lambda > 1 \end{aligned}$$

$\phi(\lambda)$ is the survival probability for a Poisson(λ) branching process.

THE GIANT COMPONENT for $G(\mathcal{B}_n, r)$.

Given $\lambda > 0$, as $n \to \infty$,

$$n^{-1} L_1(G(n, \lambda n^{-1/d})) \to \theta(\lambda)$$
$$n^{-1} L_2(G(n, \lambda n^{-1/d})) \to 0$$

where for some $\lambda_c \in (0, \infty)$, and

$$\theta(\lambda) = 0 \qquad \lambda < \lambda_c$$
$$\theta(\lambda) > 0 \qquad \lambda > \lambda_c$$

$\theta(\lambda)$ is the continuum percolation probability for a Poisson point process $\mathcal{H}_\lambda$ of intensity λ in $\mathbf{R}^d$, and is continuous on $\lambda \neq \lambda_c$

MORE PERCOLATION: $\mathcal{H}_\lambda$: Homogeneous Poisson process in $\mathbf{R}^d$

$\theta(\lambda) = P[\mathbf{0}$ lies in an infinite component of $G(\mathcal{H}_\lambda \cup \{\mathbf{0}\}, 1)$]

$$\lambda_c = \sup\{\lambda : \theta(\lambda) = 0\} \in (0, \infty), \quad \text{for} \quad d \geq 2.$$

$\theta(\lambda)$ is right continuous at 0, but $\theta(\lambda_c)$ is known to be zero only for $d = 2$ or d large.

For both $G(n, p)$ and $G(\mathcal{B}_n, r)$, the giant component result can be guessed but needs work to prove. Uniqueness of the infinite component for percolation (i.e. for $G(\mathcal{H}_\lambda, 1)$) is a key result.

CONNECTIVITY

Let $K(G)$ be the event that G is connected. If $n\pi_d r_n^d = c \ln n$,

$$P[K(G(\mathcal{B}_n, r_n))] \to 1 \quad \text{if} \quad c > 1$$

$$P[K(G(\mathcal{B}_n, r_n))] \to 0 \quad \text{if} \quad c < 1$$

Idea of proof:

$$P[D_1 = 0] \sim \exp(-n\pi_d r_n^d) \sim n^{-c}$$

so if $N_0 := \sum_{i=1}^n \mathbf{1}\{D_i = 0\}$, the number of vertices of degree 0,

$$\mathbf{E}[N_0] = nP[D_1 = 0] = n^{1-c}$$

so $P[N_0 > 0] \to 0$ if $c > 1$, and suggests $P[N_0 > 0] \to 1$ if $c < 1$.

$\{N_0 = 0\}$ is clearly necessary for $K(G)$, and turns out to be asymptotically sufficient.

In fact, if $R_n(A) := \min\{r : G(\mathcal{B}_n, r) \in A\}$ for a given increasing graph property A, then

$$P[R_n(\{N_0 = 0\}) = R_n(K)] \to 1.$$

Combined with [Dette and Henze (1989, 1990)] this gives

$$P[n\pi R_n(K) - \log n < t] \to \exp(-e^{-t}), \quad t \in \mathbf{R}$$

[MP 1997; Gupta and Kumar 1998, MP 1998, Hsing and Rootzen 2005, Gupta and Iyer 200?]

CENTRAL AND LOCAL LIMIT THEOREMS

Fix λ and suppose $nr_n^d = \lambda$. Let M_n be the number of components of $G(\mathcal{B}_n, r_n)$. Let $\phi(t)$ be the standard normal pdf, and $\Phi(t) = \int_{-\infty}^t \phi(t) dt$. Then there is a constant $\sigma > 0$ such that for $t \in \mathbf{R}$,

$$P\left[\frac{M_n - \mathbf{E}M_n}{\sigma\sqrt{n}} \leq t\right] \to \Phi(t)$$

and also (work in progress, with Y. Peres)

$$\sup_{j \in \mathbf{N}} \left\{ n^{1/2} P[M_n = j] - \sigma^{-1} \phi\left(\frac{j - \mathbf{E}M_n}{\sigma\sqrt{n}}\right) \right\} \to 0.$$

CENTRAL LIMIT THEOREM FOR LARGEST COMPONENT

Fix $\lambda > \lambda_c$ and suppose $nr_n^d = \lambda$. Let $L_1(G_n)$ be the size of the largest component of $G_n = G(\mathcal{B}_n, r_n)$. Then there is a constant $\sigma > 0$ such that for $t \in \mathbf{R}$,

$$P\left[\frac{L_1(G_n) - \mathbf{E}L_1(G_n)}{\sigma\sqrt{n}} \leq t\right] \to \Phi(t)$$

(Local limit theorem should also hold).

CLTs also hold for the number of edges, number of triangles etc, number of isolated points, isolated edges, etc. both when nr_n^d is constant or decays like a small negative power of n.

IDEA OF PROOF OF CLTS: STABILIZATION

Poissonization: let $\mathcal{P}_n = \mathcal{B}_{N_n}$ where $N_n \sim \text{Poisson}(n)$. Then

$$G'_n := G(\mathcal{P}_n; (\lambda/n)^{1/d}) \sim G(\mathcal{H} \cap Q_n; \lambda^{1/d})$$

where $\mathcal{H} = \mathcal{H}_1$ is a homogeneous Poisson point process of unit intensity on $\mathbf{R}^d$ and Q_n is a cube of volume n.

So both $K(G'_n)$ and $L_1(G'_n)$ are of the form $F(\mathcal{H} \cap Q_n)$ with F defined on finite point sets, and translation invariant.

In both cases, can show F is *stabilising*, i.e. local changes have only local effects F

STABILIZATION

Let B_r be the ball of radius r around the origin. Say F is *stabilizing* if there are almost surely finite random variables R, Δ such that

$$F((\mathcal{H} \cap Q) \cup \{\mathbf{0}\}) - F(\mathcal{H} \cap Q) = \Delta$$

for all cubes Q with $B_R \subset Q$.

General result: if F stabilises and satisfies a 4th moments condition, then $F(\mathcal{H} \cap Q_n)$ satisfies a CLT.

Δ is sometimes called the (limiting) *add one cost.*

WHY DOES K STABILIZE?

Let $\rho > 0$, and suppose $F(\mathcal{S})$ the number of components of $G(\mathcal{S}, \rho)$.

Then $F(\mathcal{S} \cup \mathbf{0}) - F(\mathcal{S})$ is the number of distinct components near $\mathbf{0}$, minus 1.

This is bounded by a constant (kissing number).

If $B_{r-\rho}$ contains all finite components of $G(\mathcal{H}, \rho)$ then $F(\mathcal{H} \cap Q) - F(Q)$ is the same for all Q with $B_r \subset Q$.

Moments condition harder here than for K.

WHY DOES L_1 STABILIZE?

Let $\rho > 0$, and suppose $F(\mathcal{S}) = L_1(G(\mathcal{S}, \rho))$ (supercritical).

Adding $\mathbf{0}$ may cause some extra finite components to be joined to the infinite component of $G(\mathcal{H}, \rho)$.

Let Δ be the the total size of these added components, plus 1.

Claim: if r is big enough, and $B_r \subset Q$, then

$$F((\mathcal{H} \cap Q) \cup \{\mathbf{0}\}) - F(\mathcal{H} \cap Q) = \Delta.$$

WHY STABILIZATION IMPLY A CLT?

Let $Y_n = F(\mathcal{H} \cap Q_n)$. Divide Q_n into unit cubes, list lexicographically as $C_{n,1}, \ldots, C_{n,n}$. Let $\mathcal{F}_i$ be σ-field generated by $\mathcal{H} \cap \cup_{j=1}^{i} C_{n,j}$. Let

$$D_i = \mathbf{E}[Y_n | \mathcal{F}_i] - \mathbf{E}[Y_n | \mathcal{F}_{i-1}] = \mathbf{E}[Y_n - Y_{n,i} | \mathcal{F}_i]$$

where $Y_{n,i}$ is obtained by replacing $\mathcal{H} \cap C_{n,i}$ by an independent copy. CLT for martingale differences (McLeish 1974) requires

$$n^{-1} \sum_{i=1}^{n} D_i \to \sigma^2$$

for some constant σ^2. This follows from the Ergodic Theorem, and the stabilization and moments conditions.

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