Random geometric graphs

Mathew D. Penrose

University of Bath, UK

Universiteit Utrecht
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RANDOM GRAPH MODELS

- The classical Erdős–Rényi model $G(n, p)$

- Random geometric graph $G(\mathcal{B}_n, r)$
  (Penrose 2003)
  \[ \mathcal{B}_n = \{X_1, \ldots, X_n\}, \text{ } X_i \text{ independent uniform random in } [0, 1]^d. \]

**Motivation:** sensor networks; spatial/multivariate statistics; analysis of algorithms; alternative to $G(n, p)$
DEGREE DISTRIBUTIONS

Let $D_i$ be the degree of vertex $X_i$ in $G$. Then for $r_n \to 0$,

$$\mathbb{E}[D_1] \sim n\pi d r_n^d \quad \text{in} \quad G(n, r_n)$$

If $r_n \sim an^{-1/d}$ then $D_1 \sim \text{Poisson}$. \quad [G(n, a/n) \text{ similar}]

Let $\Delta_i$ denote the number of triangles including $X_i$.

If $r_n \sim an^{-1/d}$, then $\mathbb{E}\Delta_1 \sim c$

whereas for $G(n, a/n)$, $\mathbb{E}\Delta_1 \sim c'/n$. 
THE GIANT COMPONENT for $G(n,p)$

Let $L_i(G)$ be the $i$th largest component size in $G$. Given $\lambda > 0$,

$$n^{-1}L_1(G(n, \lambda/n)) \rightarrow_p \phi(\lambda) \text{ as } n \rightarrow \infty$$

$$n^{-1}L_2(G(n, \lambda/n)) \rightarrow_p 0$$

where $\phi$ is continuous and

$$\phi(\lambda) = 0 \quad \lambda \leq 1$$

$$\phi(\lambda) > 0 \quad \lambda > 1$$

$\phi(\lambda)$ is the survival probability for a Poisson($\lambda$) branching process.
THE GIANT COMPONENT for $G(\mathcal{B}_n, r)$.

Given $\lambda > 0$, as $n \to \infty$,

$$ n^{-1}L_1(G(n, \lambda n^{-1/d})) \to \theta(\lambda) $$

$$ n^{-1}L_2(G(n, \lambda n^{-1/d})) \to 0 $$

where for some $\lambda_c \in (0, \infty)$, and

$$ \theta(\lambda) = 0 \quad \lambda < \lambda_c $$

$$ \theta(\lambda) > 0 \quad \lambda > \lambda_c $$

$\theta(\lambda)$ is the continuum percolation probability for a Poisson point process $\mathcal{H}_\lambda$ of intensity $\lambda$ in $\mathbb{R}^d$, and is continuous on $\lambda \neq \lambda_c$.
MORE PERCOLATION: $\mathcal{H}_\lambda$: Homogeneous Poisson process in $\mathbb{R}^d$

$\theta(\lambda) = P[0 \text{ lies in an infinite component of } G(\mathcal{H}_\lambda \cup \{0\}, 1)]$

$$\lambda_c = \sup\{\lambda : \theta(\lambda) = 0\} \in (0, \infty), \quad \text{for } d \geq 2.$$ 

$\theta(\lambda)$ is right continuous at 0, but $\theta(\lambda_c)$ is known to be zero only for $d = 2$ or $d$ large.

For both $G(n, p)$ and $G(\mathcal{B}_n, r)$, the giant component result can be guessed but needs work to prove. Uniqueness of the infinite component for percolation (i.e. for $G(\mathcal{H}_\lambda, 1)$) is a key result.
CONNECTIVITY

Let $K(G)$ be the event that $G$ is connected. If $n \pi_d r_n^d = c \ln n$,

$$P[K(G(\mathcal{B}_n, r_n))] \to 1 \quad \text{if} \quad c > 1$$
$$P[K(G(\mathcal{B}_n, r_n))] \to 0 \quad \text{if} \quad c < 1$$

Idea of proof:

$$P[D_1 = 0] \sim \exp(-n \pi_d r_n^d) \sim n^{-c}$$

so if $N_0 := \sum_{i=1}^n 1\{D_i = 0\}$, the number of vertices of degree 0,

$$\mathbb{E}[N_0] = n P[D_1 = 0] = n^{1-c}$$

so $P[N_0 > 0] \to 0$ if $c > 1$, and suggests $P[N_0 > 0] \to 1$ if $c < 1$. 

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\{N_0 = 0\} is clearly necessary for \(K(G)\), and turns out to be asymptotically sufficient.

In fact, if \(R_n(A) := \min\{r : G(B_n, r) \in A\}\) for a given increasing graph property \(A\), then

\[ P[R_n(\{N_0 = 0\}) = R_n(K)] \rightarrow 1. \]

Combined with [Dette and Henze (1989, 1990)] this gives

\[ P[n\pi R_n(K) - \log n < t] \rightarrow \exp(-e^{-t}), \quad t \in \mathbb{R} \]

CENTRAL AND LOCAL LIMIT THEOREMS

Fix $\lambda$ and suppose $nr_n^d = \lambda$. Let $M_n$ be the number of components of $G(B_n, r_n)$. Let $\phi(t)$ be the standard normal pdf, and $\Phi(t) = \int_{-\infty}^{t} \phi(t)dt$. Then there is a constant $\sigma > 0$ such that for $t \in \mathbb{R}$,

$$P \left[ \frac{M_n - \mathbb{E}M_n}{\sigma \sqrt{n}} \leq t \right] \rightarrow \Phi(t)$$

and also (work in progress, with Y. Peres)

$$\sup_{j \in \mathbb{N}} \left\{ n^{1/2} P[M_n = j] - \sigma^{-1} \phi \left( \frac{j - \mathbb{E}M_n}{\sigma \sqrt{n}} \right) \right\} \rightarrow 0.$$
CENTRAL LIMIT THEOREM FOR LARGEST COMPONENT

Fix $\lambda > \lambda_c$ and suppose $nr_n^d = \lambda$. Let $L_1(G_n)$ be the size of the largest component of $G_n = G(B_n, r_n)$. Then there is a constant $\sigma > 0$ such that for $t \in \mathbb{R}$,

$$P \left[ \frac{L_1(G_n) - \mathbb{E}L_1(G_n)}{\sigma \sqrt{n}} \leq t \right] \to \Phi(t)$$

(Local limit theorem should also hold).

CLTs also hold for the number of edges, number of triangles etc, number of isolated points, isolated edges, etc. both when $nr_n^d$ is constant or decays like a small negative power of $n$. 

IDEA OF PROOF OF CLTS: STABILIZATION

Poissonization: let $\mathcal{P}_n = \mathcal{B}_{N_n}$ where $N_n \sim \text{Poisson}(n)$. Then

$$G'_n := G(\mathcal{P}_n; (\lambda/n)^{1/d}) \sim G(\mathcal{H} \cap Q_n; \lambda^{1/d})$$

where $\mathcal{H} = \mathcal{H}_1$ is a homogeneous Poisson point process of unit intensity on $\mathbb{R}^d$ and $Q_n$ is a cube of volume $n$.

So both $K(G'_n)$ and $L_1(G'_n)$ are of the form $F(\mathcal{H} \cap Q_n)$ with $F$ defined on finite point sets, and translation invariant.

In both cases, can show $F$ is stabilising, i.e. local changes have only local effects $F$
STABILIZATION

Let $B_r$ be the ball of radius $r$ around the origin. Say $F$ is stabilizing if there are almost surely finite random variables $R, \Delta$ such that

$$F((H \cap Q) \cup \{0\}) - F(H \cap Q) = \Delta$$

for all cubes $Q$ with $B_R \subset Q$.

General result: if $F$ stabilises and satisfies a 4th moments condition, then $F(H \cap Q_n)$ satisfies a CLT.

$\Delta$ is sometimes called the (limiting) add one cost.
WHY DOES K STABILIZE?

Let $\rho > 0$, and suppose $F(S)$ the number of components of $G(S, \rho)$.

Then $F(S \cup 0) - F(S)$ is the number of distinct components near 0, minus 1.

This is bounded by a constant (kissing number).

If $B_{r-\rho}$ contains all finite components of $G(\mathcal{H}, \rho)$ then $F(\mathcal{H} \cap Q) - F(Q)$ is the same for all $Q$ with $B_r \subset Q$.

Moments condition harder here than for $K$. 
WHY DOES $L_1$ STABILIZE?

Let $\rho > 0$, and suppose $F(S) = L_1(G(S, \rho))$ (supercritical).

Adding 0 may cause some extra finite components to be joined to the infinite component of $G(\mathcal{H}, \rho)$.

Let $\Delta$ be the total size of these added components, plus 1.

Claim: if $r$ is big enough, and $B_r \subset Q$, then

$$F((\mathcal{H} \cap Q) \cup \{0\}) - F(\mathcal{H} \cap Q) = \Delta.$$
WHY STABILIZATION IMPLY A CLT?

Let $Y_n = F(\mathcal{H} \cap Q_n)$. Divide $Q_n$ into unit cubes, list lexicographically as $C_{n,1}, \ldots, C_{n,n}$. Let $\mathcal{F}_i$ be $\sigma$-field generated by $\mathcal{H} \cap \bigcup_{j=1}^i C_{n,j}$. Let

$$D_i = \mathbb{E}[Y_n|\mathcal{F}_i] - \mathbb{E}[Y_n|\mathcal{F}_{i-1}] = \mathbb{E}[Y_n - Y_{n,i}|\mathcal{F}_i]$$

where $Y_{n,i}$ is obtained by replacing $\mathcal{H} \cap C_{n,i}$ by an independent copy.

CLT for martingale differences (McLeish 1974) requires

$$n^{-1} \sum_{i=1}^n D_i \rightarrow \sigma^2$$

for some constant $\sigma^2$. This follows from the Ergodic Theorem, and the stabilization and moments conditions.
References


