Ergodic Solenoids

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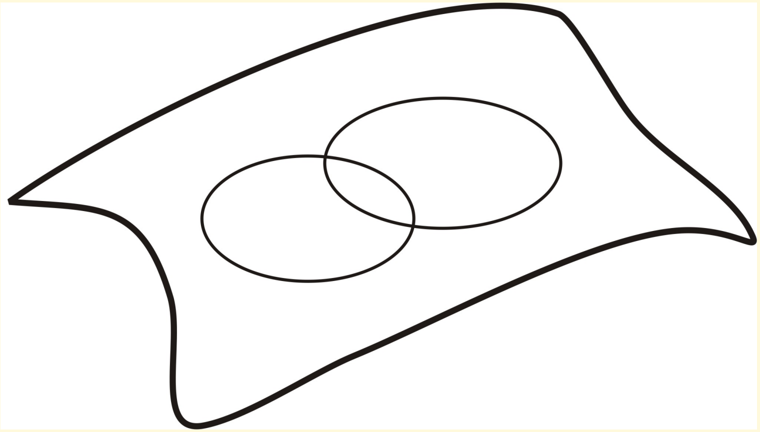
1. Manifolds and homology

Manifolds are the main object of interest in (differential) geometry. A manifold is obtained by pasting together open subsets of $\mathbb{R}^n$ (charts).

They are the abstraction of the idea of a *geometrical space*.



The main goals of geometry can be summarised as†:

- Classify (smooth) manifolds.
- Understand geometric (underlying) structures on a manifold: riemannian metrics, complex structures, . . .
- Understand subojects (submanifolds).
- Understand global (topological) properties of manifolds.

† **Disclaimer:** this is a personal vision.

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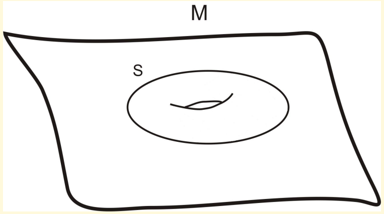
Suppose that $S \hookrightarrow M$ is a k-dimensional submanifold. Then we can move S inside M (isotopy, homotopy, cobordism), and consider the equivalence classes of these objects.



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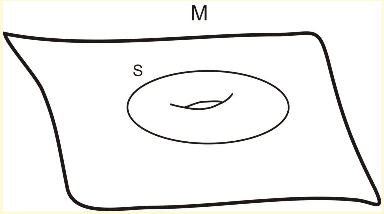

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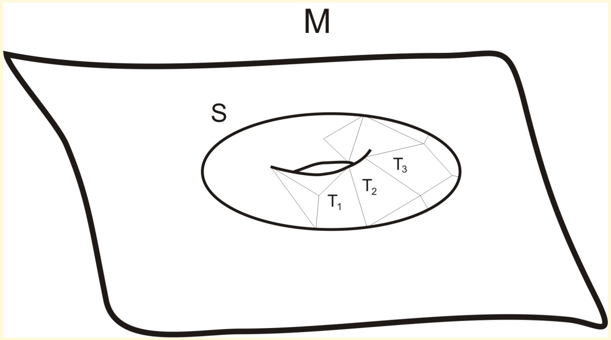
In 1896, Poincaré introduced the concept of homology. This is a more tractable and computable invariant.

Triangulate $S = \sum T_i, \quad T_i : [0,1]^k \to M.$



Chains: $C_k(M) = \{\sum n_i T_i \; ; \; T_i : [0,1]^k \to M, n_i \in \mathbb{Z}\}$

There is a well-defined *boundary map* ∂ for chains.

Cycles: $Z_k(M) = \{a = \sum n_i T_i \; ; \; \partial(a) = 0\}$.

Boundaries: $B_k(M) = \{\partial(\sum m_j T'_j) \; ; \; \sum m_j T'_j \in C_{k+1}(M)\}$.

The homology is defined as

$$H_k(M,\mathbb{Z}) = \frac{Z_k(M)}{B_k(M)} .$$

If S is an oriented compact submanifold, then there is a well-defined element

$$[S] \in H_k(M,\mathbb{Z}).$$

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If we take coefficients $n_i \in \mathbb{R}$, we obtain $H_k(M, \mathbb{R})$.

De Rham (1931) observed a *duality* given by integration. If S is an oriented submanifold, then there is a map

$$\begin{aligned} S : \Omega^k(M) &\to \mathbb{R}, \qquad (*) \\ \alpha &\mapsto \langle S, \alpha \rangle , \end{aligned}$$

which sends any k-form $\alpha = \sum f_{\mathbf{i}}\, dy_{i_1} \wedge \ldots \wedge dy_{i_k}$ to

$$\langle S, \alpha \rangle = \int_S \alpha = \sum n_i \int_{T_i} \alpha \, .$$

The spaces of k-forms, with the *exterior differential*, $(\oplus_k \Omega^k(M), d)$, form a differential algebra.

$$Z^k(M) = \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)),$$

$$B^k(M) = \operatorname{img}(d : \Omega^{k-1}(M) \to \Omega^k(M)).$$

The *De Rham cohomology* is defined as:

$$H^k(M, \mathbb{R}) = \frac{Z^k(M)}{B^k(M)}.$$

Stokes theorem says that

$$\int_S d\beta = 0,$$

so $(*)$ gives a map

$$\begin{aligned} [S] : H^k(M,\mathbb{R}) &\to \mathbb{R}, \\ [\alpha] &\mapsto \langle [S],[\alpha]\rangle = \langle S,\alpha\rangle \,. \end{aligned}$$

This produces a duality $H_k(M,\mathbb{R}) \otimes H^k(M,\mathbb{R}) \to \mathbb{R}$. So

$$[S] = \text{“} \int_S \text{”} \in \mathrm{Hom}(H^k(M,\mathbb{R}),\mathbb{R}) \cong H_k(M,\mathbb{R}).$$

$(*)$ is called the *integration current* along S.

2. Realization of homology classes

Question: Let M be a smooth compact manifold of dimension n.
Let $a \in H_k(M,\mathbb{Z})$, $0 < k < n$.
Does there exist a submanifold $S \subset M$ such that $[S] = a$?

Thom (1953, Fields medal) gave the answer. He transformed the problem to a dual problem: the existence of a map,

$$f : M \longrightarrow U ,$$

where U is a universal space, $U_0 \subset U$, and $S = f^{-1}(U_0)$ is the sought submanifold. Then he applied transversality to the map f.

- It is not always possible to get such submanifold S.
  Required the existence of $f : M \to U$ such that $f^*[U_0] = a$. This is a topological obstruction.
- There are positive answers. E.g., there exists $m \gg 0$ such that for $m\,a \in H_k(M)$ there exists $S \subset M$, $[S] = m\,a$.
- If $n-k$ is odd, then $U \sim S^{n-k}$, $U_0 = pt$, so we get $S = f^{-1}(pt)$, with trivial normal bundle (and $[S] = m\,a$, $m \gg 0$).

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3. Digression: The complex case

Let M be a compact complex manifold.
$M \subset \mathbb{P}^N_{\mathbb{C}}$, $M = Z(f_1, \ldots, f_t)$, f_i polynomials.

There is a *Hodge decomposition* $H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)$,
Dually, $H_k(M) = \bigoplus_{p+q=k} H_{p,q}(M)$,

If $S \subset M$ is a complex submanifold, then $[S] \in H_{k,k}(M)$.

Hodge Conjecture (1950):

Given $a \in H_{k,k}(M) \cap H_{2k}(M, \mathbb{Z})$.
Does there exist $m \gg 0$ and $S \subset M$ a complex submanifold, s.t. $[S] = m\,a$?

(Note: Atiyah–Hirzebruch gave a counterexample if we ask for $m = 1$ above).

4. Geometric representatives of non-integer homology classes

Let M be a compact manifold.

Let $a \in H_k(M, \mathbb{R})$ be a *real* homology class.

We look for geometric representatives of a.

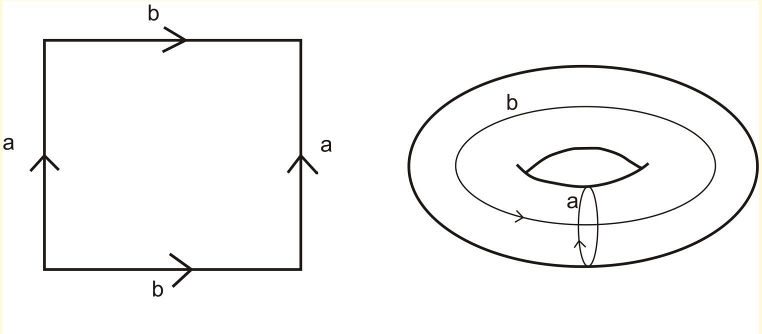
We aim for a (smooth) sub-object $S \hookrightarrow M$ defining an integration current s.t. $[S] = a$.

Note: by definition, we can take a chain with real coefficients

$$a = \sum \lambda_i C_i \, .$$

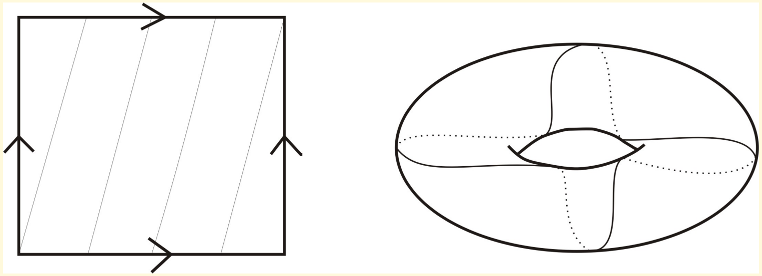
But this may be non-smoothable (with corners). Moreover, the λ_i's are weights for each of the faces C_i, and this is non-geometric data.

Look at the case of the **torus** $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.



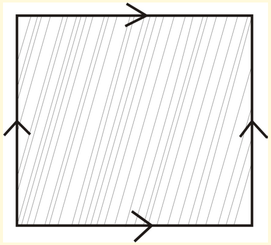
$$H_1(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2.$$

For a homology class $\alpha = p_1 e_1 + p_2 e_2$, we can consider $\alpha' = e_1 + \frac{p_2}{p_1} e_2$. This is represented by (the image of) the line $y = \frac{p_2}{p_1} x$.

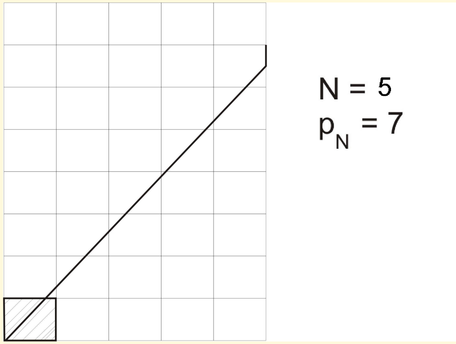


This winds p_1-times around the e_1-direction, and p_2-times around the e_2-direction.

If we consider $\lambda \in \mathbb{R} - \mathbb{Q}$, then the line $y = \lambda x$ is dense in the torus, because it never closes.



Consider long portions of this curve: Take $N \gg 0$, and let $l_N = \{(x, \lambda x); x \in [0, N]\}$.



This is approximately $[l_N] \sim N e_1 + p_N e_2$, where $\frac{p_N}{N} \sim \lambda$. Therefore

$$\frac{1}{N}[l_N] = [e_1 + \frac{p_N}{N} e_2] \longrightarrow [e_1 + \lambda e_2] \in H_1(\mathbb{T}^2, \mathbb{R}).$$

Let M be a compact manifold. A parametrized curve

$$c : \mathbb{R} \longrightarrow M$$

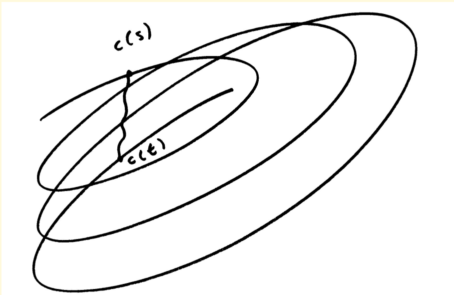
defines (in good cases) a real homology class as follows.

For each pair of points $p, q \in M$, choose a *short* arc $\gamma_{p,q}$ from p to q (say, with bounded length).

The loop

$$c_{s,t} := c([s,t]) * \gamma_{c(t),c(s)}$$

defines a homology class $[c_{s,t}] \in H_1(M, \mathbb{Z})$.



If the limit

$$[c] = \lim_{\substack{t\to+\infty \\ s\to-\infty}} \frac{[c_{s,t}]}{t-s}$$

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Close

Quit

exists, we have that c defines a real 1-cycle

$$[c] \in H_1(M,\mathbb{R}) = \mathbb{R} \otimes H_1(M,\mathbb{Z}).$$

Actually c defines an integration current

$$c : \Omega^1(M) \longrightarrow \mathbb{R}.$$

For any 1-form $\alpha \in \Omega^1(M)$,

$$\begin{aligned} \langle [c_{s,t}], \alpha \rangle &= \int_{c_{s,t}} \alpha = \int_{c([s,t])} \alpha + O(1) = \\ &= \int_s^t \alpha(c'(u))\, du + O(1)\,. \end{aligned}$$

So there is a well-defined limit

$$\langle [c], \alpha \rangle = \lim_{\substack{t\to+\infty \\ s\to-\infty}} \frac{1}{t-s} \int_s^t \alpha(c'(u)) du\,.$$

This corresponds to integrating along $c([s,t])$, normalizing, and then taking the limit.

This generalizes Schwartzman (1957) definition of real 1-cycles. These real cycles can be called *Schwartzman cycles*.

Schwartzman k-dimensional cycles

We can extend the definition to higher dimensions $k > 1$.

Let M be a smooth compact manifold.

S smooth (Riemannian) complete non-compact manifold, $x_0 \in S$.

A smooth map $f : S \to M$ defines a Schwartzman k-cycle

$$[f] \in H_k(M, \mathbb{R})$$

if for any k-form $\omega \in \Omega^k(M)$, the limit

$$\langle [f], \omega \rangle = \lim_{R \to +\infty} \frac{1}{\mathrm{Vol}_S(B_R(x_0))} \int_{B_R(x_0)} f^* \omega$$

exists, where $B_R(x_0)$ is the ball of radius R around x_0 in S.

Alternatively, if we can cap off the submanifolds with boundary $f(B_R(x_0))$ with a small cap C_R, $S_R = f(B_R(x_0)) \cup C_R$, then we can define

$$[f] = \lim_{R \to +\infty} \frac{[S_R]}{\mathrm{Vol}_S(B_R(x_0))} \in H_k(M, \mathbb{R}) .$$

This happens, for instance, when there is a trapping region, that is, a ball $B \subset M$, and a sequence $R_n \to +\infty$, s.t. $f(\partial B_{R_n}(x_0)) \subset B$.

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Close
Quit

5. Solenoids

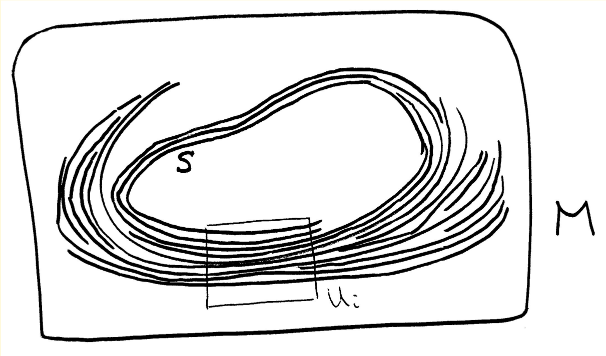
(Pérez-Marco, Muñoz, math.DG/0702501).

A k-dimensional solenoid S is a compact Hausdorff topological space with an atlas of flow-boxes (U_i) such that

$$U_i \cong D^k \times T_i \, .$$

The (local) leaves are $L_y = D^k \times \{y\}$.
The (local) transversals are T_i.

The following is an embedded solenoid $f : S \hookrightarrow M$.



An immersed solenoid $f : S \to M$ can define an integration current as follows:

$$S : \Omega^k(M) \longrightarrow \mathbb{R}.$$

Let $\alpha \in \Omega^k(M)$. Fix a covering (U_i) of S, with $U_i \cap S = D^k \times T_i$, (ρ_i) a partition of unity subordinated to (U_i). Then first set

$$\int_S \alpha = \sum_i \int_{U_i \cap S} (\rho_i \, \alpha).$$

Let $\alpha_i = \rho_i \, \alpha$, which is supported in one flow-box. We integrate α_i along the local leaves

$$y \mapsto \int_{D^k \times \{y\}} \alpha_i(x, y) \, dx,$$

and then we integrate in the y-direction. For this we need a measure $\mu_{T_i}(y)$ on T_i.

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A transversal measure is a collection of measures $\mu = (\mu_{T_i})$ for each transversal T_i. They should be invariant by *holonomies*. The holonomies are the maps from one transversal to another, by travelling along the leaves.

Let $f : S \hookrightarrow M$ be an oriented solenoid in M, with a transversal measure μ. Then there is an integration current

$$\begin{aligned} (S,\mu) : \Omega^k(M) &\longrightarrow \mathbb{R} \\ \alpha &\mapsto \sum_i \int_{T_i} \left(\int_{D^k \times \{y\}} \rho_i \alpha \right) \, d\mu_{T_i}(y) \, . \end{aligned}$$

This defines a *real* homology class

$$[S,\mu] \in \mathrm{Hom}(H^k(M,\mathbb{R}),\mathbb{R}) = H_k(M,\mathbb{R})$$

by duality.

This generalizes a construction of Ruelle-Sullivan (1975).

If S is embedded without compact leaves, then $[S,\mu]^2 = 0$. So to represent a homology class $a \in H_k(M,\mathbb{R})$ with $a^2 \neq 0$, we need an immersed solenoid with self-intersections.

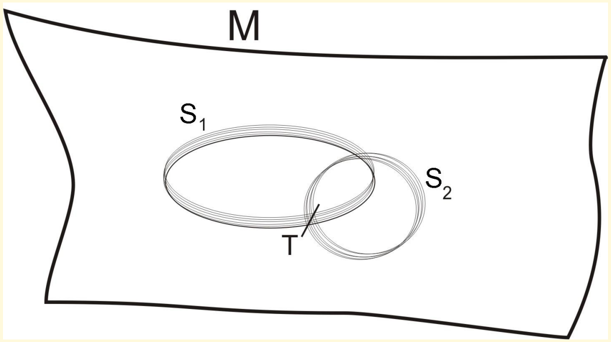
So we may call $[S,\mu]$ a generalized Ruelle-Sullivan cycle.

6. Ergodicity

Let M be a smooth manifold. Suppose that $e_1, e_2 \in H_k(M, \mathbb{Z})$ are represented by submanifolds $S_1, S_2 \subset M$. We want to represent

$$a = \lambda_1 e_1 + \lambda_2 e_2 \in H_k(M, \mathbb{R}).$$

Considering parallel copies of S_1 and S_2, we have two solenoids $(\tilde{S}_1 = S_1 \times T_1, \mu_1)$, $(\tilde{S}_2 = S_2 \times T_2, \mu_2)$. We have chosen the total measures $\mu_i(T_i) = \lambda_i$. Consider $(S, \mu) = (\tilde{S}_1, \mu_1) \sqcup (\tilde{S}_2, \mu_2)$.



Then a leaf of S represents either e_1 or e_2, but does not represent a.

We want a to be represented by *any* leaf of S.

For this, we need mixing in the holonomy. Let T be a global transversal (in the example above, arrange $T = T_1 = T_2$). Let

$$h : T \to T$$

be the holonomy map (assume there is only one holonomy). The transversal measure μ_T is invariant by h.

- S is minimal if all leaves are dense (the sets $\{h^n(x_0); n \in \mathbb{Z}\}$ are dense in T).
- S is ergodic if almost all leaves represent the same homology class (for any h-invariant $A \subset T$, either $\mu_T(A)$ is zero or total measure).
- S is uniquely ergodic if there exists a unique (up to positive scalar multiples) transversal measure μ_T (with full support).

Note that a *uniquely ergodic solenoid* S is a purely geometric entity, as it "contains" as much information as the measured solenoid (S, μ). In general a measured solenoid (S, μ) has a geometric datum, S, and a non-geometric datum (weight), μ.

Full Screen

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Theorem (PM-M):

Let S be a *uniquely ergodic* oriented k-solenoid, and let $f : S \to M$ be an immersion. (If $k > 1$ we assume that there is a trapping region.)

Then for each leaf $l \subset S$ we have that $f|_l : l \to M$ is a Schwartzman k-cycle, and

$$[f|_l] = [S, \mu] \in H_k(M, \mathbb{R})\,.$$

(The proof is an application of Birkhoff's ergodic theorem.)

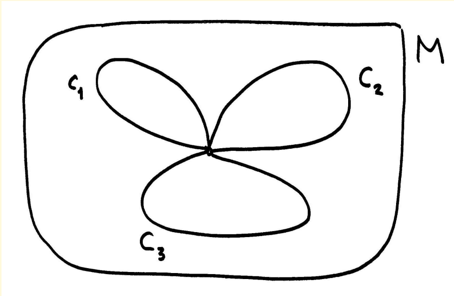
7. Realization of real homology classes

Theorem (PM-M):

Let M be a compact smooth manifold, $a \in H_k(M, \mathbb{R})$. Then there exists a uniquely ergodic oriented immersed solenoid (S, f) representing a. (If $k > 1$ then S has a trapping region.)

Proof: Assume $k = 1$ for simplicity (for the pictures).

Take loops $C_1, \ldots, C_{b_1} \subset M$ which form a basis of $H_1(M, \mathbb{Z})$ (and share a base point).

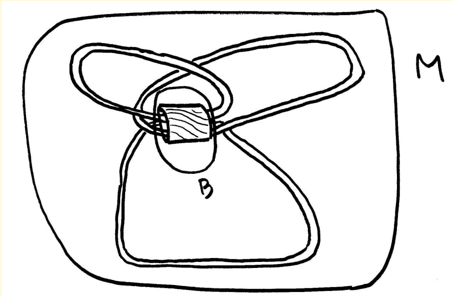


There are real numbers $\lambda_1, \ldots, \lambda_r > 0$ such that

$$a = \lambda_1 C_1 + \cdots + \lambda_r C_r$$

(switching orientations and reordering the cycles if necessary). By dividing by $\sum \lambda_i$, we can assume that $\sum \lambda_i = 1$.

Thicken the loops to bands $[0,1] \times T_i$, where each $[0,1] \times \{y\}$ is homotopic to C_i. Now we need to introduce a *mixing* inside a ball B around the base point.



For this, we arrange the bands T_i in circular order forming a S^1. We want to get a solenoid with a transversal $T = S^1$. The holonomy will be

$$h : T \to T\,.$$

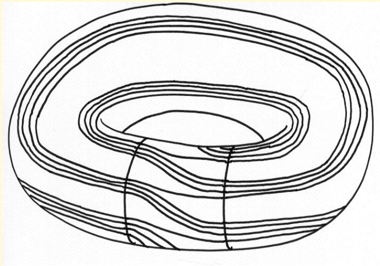
Fact: the rotation $r_\theta : S^1 \to S^1$ of irrational angle θ is uniquely ergodic.

To get the smoothness of the solenoid, we need to use a transversal $T \subset S^1$ with "holes".

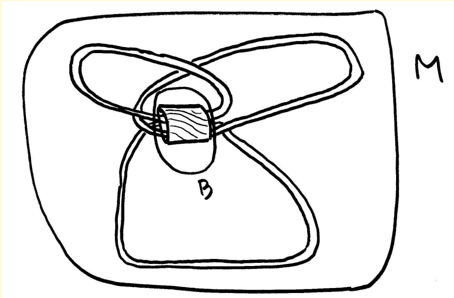
We substitute the map r_θ by a Denjoy example. This is a map $h : S^1 \to S^1$ with the following properties:

- h is of class $C^{2-\epsilon}$.
- h leaves invariant a Cantor set $K \subset S^1$.
- h has rotation angle θ, i.e.
  $r_\theta^n(x) = x + n\,\theta$
  $h^n(x) = x + n\,\theta + o(n)$.
- h is uniquely ergodic, with a unique invariant measure μ_K supported exactly on the Cantor set K.

The suspension Σ_h of the map $h|_K : K \to K$ is:



Then we plug the middle part of Σ_h inside B, in such a way that the holes where the bands coalesce or split are outside K.



The resulting solenoid S is uniquely ergodic and $[S, \mu] = a$.

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8. Digression: Solenoidal Hodge Conjecture

Solenoidal Hodge Conjecture:

Let M be a compact complex manifold.
Let $a \in H_{k,k}(M) \cap H_{2k}(M, \mathbb{R})$.
Then a is represented by a complex immersed solenoid.

This is weaker that the usual Hodge Conjecture.

This removes the arithmetic issues, which are due to how the lattice $H_{2k}(M, \mathbb{Z}) \subset H_{2k}(M, \mathbb{R})$ intersects the Hodge pieces $H_{p,q}(M) \subset H_{2k}(M)$.