Analytic families of Harish-Chandra modules

Vincent van der Noort

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Motto

“The trouble with the representation theory of real reductive Lie groups is that the objects you’re studying are not representations of real reductive Lie groups.”
Bill Casselman

Lie groups

- A Lie group G is a group which is at the same time a smooth manifold such that the multiplication map

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- Every Lie group has a Lie algebra.

Representation theory

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- A representation of a Lie group G on a topological vector space V is a continuous homomorphism π of G into the group $\mathrm{GL}(V)$ of invertible linear transformations of V such that the action map

$$(g, v) \mapsto \pi(g)v \colon G \times V \to V$$

is continuous.

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- The group of symmetries of the object act on these spaces
- Cut the middle man!

Questions in representation theory

1. Irreducible representations:
   - What do they look like?
   - Can we find them all?
   - How can larger representations be understood in terms of irreducible ones?
2. What is the relationship between Lie algebra representations and Lie group representations?

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For compact groups the answers to these questions are well known for a long time.

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- Remarkable: the set of irreducible representations has a complex structure.

Analytic families of representations

Let G be a Lie group and Ω be a complex manifold. By an analytic family of G-representations (V, π) we understand a Fréchet space V and a continuous map

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2. For every $g \in G$ and $v \in V$ the map

$$\zeta \mapsto \pi_\zeta(g)v : \Omega \to V$$

is holomorphic.

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2. What is the relation between families of $\mathfrak{g}$-representations and families of G-representations?

The Harish-Chandra class of groups

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- Examples: closed connected subgroups of $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$ that are invariant under taking conjugate transpose such as $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SU}(p,q)$, $\mathrm{SO}(p,q)$ etc.

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- Every group G in Harish-Chandra class has a maximal compact subgroup K, unique up to conjugacy, which in the above example may be taken to be the intersection with $O(n)$ (in the real case) or $U(n)$ (in the complex case).

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- Every group G in Harish-Chandra class has a maximal compact subgroup K, unique up to conjugacy, which in the above example may be taken to be the intersection with $O(n)$ (in the real case) or $U(n)$ (in the complex case).
- A finite dimensional representation of $\mathfrak{g}$ globalizes to a representation of G on the same space if and only if its restriction to $\mathfrak{k}$ globalizes to a representation of K.
- However, when G is non-compact and non-abelian, 'most' (irreducible) representations are infinite dimensional.

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- In that case there is a natural action of $\mathfrak{g}$ on V_K. We call V_K the $(\mathfrak{g}, K)$-module of V.
- For admissible representations, questions concerning irreducibility can be studied on the level of $(\mathfrak{g}, K)$-modules.

Admissible $(\mathfrak{g}, K)$-modules

In general a $(\mathfrak{g}, K)$-module (V, π) for G is a simultaneous representation π of $\mathfrak{g}$ and K on a vector space V satisfying $V_K = V$ and certain compatibility conditions suggesting that it could come from a G-representation as above.

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   Casselman–Wallach theorem, 1989: Every finitely generated admissible $(\mathfrak{g}, K)$-module appears as the $(\mathfrak{g}, K)$-module of a unique smooth Fréchet representation of moderate growth.

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3. Does there exist a Casselman-Wallach theorem for families?

Irreducibility and finite generation

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- If for some $\zeta_0 \in \Omega$ the module (V, π_{ζ_0}) is irreducible then (V, π_ζ) is irreducible for every $\zeta \in \Omega$ outside a locally finite union of zero sets of globally defined analytic functions. (Thesis, Thm. 3.3.9.)

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- We will call families of this type generically irreducible.

Subrepresentation theorem for families

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- Let (V, π) be a holomorphic family of Harish-Chandra modules for a real rank one group G parametrized by a one dimensional parameter space Ω. Then for every $\zeta_0 \in \Omega$ there are a neighborhood Ω_0 of ζ_0 and a family of finite dimensional P-representations (F, σ) parametrized by Ω_0 such that the restriction of the family (V, π) to Ω_0 embeds holomorphically into the family $\mathrm{ind}_P^G(\sigma)$ of induced representations. (Thesis, Thm. 5.514)

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- Cor.: The K-finite matrix coefficients of such a family are real analytic as function on $\Omega \times G$. (Thesis, Thm. 5.6.2).

Globalization of one parameter families

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- Let (V, π) be a generically irreducible family of Harish-Chandra modules for a real rank one group G, parametrized by an open subset $\Omega \subset \mathbb{C}$. Let $\zeta_0 \in \Omega$. Then there exists an open neighborhood $U \ni 0$ in $\mathbb{C}$ and a positive integer N such that the family $\tilde{\pi}$ defined by

$$\tilde{\pi}_z := \pi_{\zeta_0 + z^N} \qquad (z \in U)$$

  globalizes to a family of smooth Fréchet representations of G of moderate growth, parametrized by U. (Thesis, Thm. 6.3.18)

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- In case the infinitesimal character of the family depends holomorphically on the parameter, there is no need to pass to a cover. (Thesis, Thm. 6.4.4.)