Gibbs measures: definition, uses and abuses

Roberto Fernández
Utrecht University

Motivation: stat mech

Stat mech à la Gibbs

Issue: to study systems with many components

Examples:

▶ *Particles in space*: Each particle characterized by a position and a velocity
▶ *Spins in a lattice* (pixels, particles): Each spin has a finite number of possible values

Stat mech approach:

▶ Look at finite “windows” (finite regions) $\Lambda$
▶ Replace detailed laws by a probabilistic description
▶ Find the asymptotic behavior for $\Lambda$ huge
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Rough probabilistic prescription

Probability weights or densities

\[ \frac{e^{-\beta H_{\Lambda}}}{Z_{\Lambda}} \]

where

- \( H_{\Lambda} \) = Hamiltonian; must be sum of local terms so that
  \( H_{\tilde{\Lambda}} - H_{\Lambda} \sim |\tilde{\Lambda} \setminus \Lambda| \) for \( \tilde{\Lambda} \subset \Lambda \)
- \( \beta \) = inverse temperature ("coolness")
- \( Z_{\Lambda} \) = partition function (normalization). Physics info:

\[ \lim_{\Lambda} \frac{1}{|\Lambda|} \log Z_{\Lambda} = \text{pressure or free energy} \]
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Set up: Finite-spin lattice systems

- **Lattice** = Countable set \( \mathbb{L} \) (e.g. \( \mathbb{L} = \mathbb{Z}^d \))
  - sites \( x \in \mathbb{L} \)
  - finite regions \( \Lambda, \Gamma \subseteq \mathbb{L} \)

- **Single-spin space** \( S \), here finite (e.g. Ising spins: \( S = \{-1, 1\} \))

- **Configuration space** \( \Omega = S^\mathbb{L} \) (A copy of \( S \) at each site)
  - Notation: \( \Omega_\Lambda := S^\Lambda \)

- **Configurations**: \( \Omega \ni \omega = (\omega_x)_{x \in \mathbb{L}} \)
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Basic kernels

Formally, finite window = finite region of an infinite system:

- Inside $\Lambda$: probability measure
- Outside $\Lambda$: fixed configuration (external condition)

That is, a family of probability measures

$$\pi(\cdot \mid \omega_{\Lambda^c})$$

or, more precisely, a kernel with two slots $\pi(\cdot\mid\cdot)$

This is a probability kernel

Need some formalization
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Need some formalization
Measure-theoretical and topological set-up

Gibbsianness: interplay between topology and measure-theory

- $S$ endowed with discrete topology and $\sigma$-algebra
- $\Omega$ endowed with the *product* topology and $\sigma$-algebra

In more detail:

- $\mathcal{F} = \sigma$-algebra generated by the cylinders
  \[
  C_{\sigma_\Lambda} = \{ \omega \in \Omega : \omega_\Lambda = \sigma_\Lambda \}
  \]
- $\mathcal{F}_\Gamma = \sigma$-algebra generated by cylinders with basis in $\Gamma \subset \mathcal{L}$
  \[
  C_{\sigma_\Lambda}, \Lambda \subset \Gamma
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- Topology also generated by the cylinders
  - cylinders are open
  - continuous functions are measurable
Topology and measure structure

**Measure-theoretical and topological set-up**

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- Topology also generated by the cylinders
  - cylinders are open
  - continuous functions are measurable
Locality and continuity

\( f \) is a local function if

- It depends only on the spins on a finite region
- \( \exists \Gamma \subseteq \mathbb{L} \) such that \( f(\omega) = f(\sigma) \) whenever \( \omega_{\Gamma} = \sigma_{\Gamma} \)
- \( \exists \Gamma \subseteq \mathbb{L} \) such that \( (f \in \mathcal{F}_\Gamma) \)

Properties:

- Local functions are continuous
- More generally: \( f \) is continuous iff, it is quasilocal

\[
\sup_{\omega \in \Omega} \sup_{\sigma \in \Omega} \left| f(\omega_{\Lambda_n} \sigma) - f(\omega) \right| \xrightarrow{n \to \infty} 0
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More generally: \( f \) is continuous iff, it is *quasilocal*

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\sup_{\omega \in \Omega} \sup_{\sigma \in \Omega} |f(\omega \wedge n \sigma) - f(\omega)| \xrightarrow{n \to \infty} 0
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Basic definition

**Probability kernels**

**Definition**
A *probability kernel* $\Psi$ from a probability space $(\mathcal{A}, \Sigma)$ to another probability space $(\mathcal{A}', \Sigma')$ is a function

$$\Psi(\cdot | \cdot) : \Sigma' \times \mathcal{A} \longrightarrow [0, 1]$$

such that

(i) $\Psi(\cdot | \omega)$ is a probability measure on $(\mathcal{A}', \Sigma')$ for each $\omega \in \mathcal{A}$;

(ii) $\Psi(\mathcal{A}'| \cdot)$ is $\Sigma$-measurable for each $\mathcal{A}' \in \Sigma'$. 
System in $\Lambda \subseteq \mathbb{L}$ described by a probability kernel

$$\pi_\Lambda(\cdot | \cdot) \text{ from } (\Omega, \mathcal{F}) \text{ to itself}$$

where

$$\pi_\Lambda(f | \omega) = \text{equilibrium value of } f \text{ when the configuration outside } \Lambda \text{ is } \omega$$
## Operations with kernels

### Composition of kernels

\( \Psi \) from \((A, \Sigma)\) to \((A', \Sigma')\) and \(\Psi'\) from \((A'\Sigma')\) to \((A'', \Sigma'')\),

\[
\left(\Psi\Psi'\right)(A''|\omega) = \int_{A'} \Psi(d\omega'|\omega) \Psi'(A''|\omega')
\]

### Linear transformations of measures

\[
\mathcal{P}(A, \Sigma) \quad \rightarrow \quad \mathcal{P}(A', \Sigma') \\
\mu \quad \quad \quad \rightarrow \quad \mu' = \mu \Psi \\
\mu'(A') = \int_A \mu(d\omega) \Psi(A'|\omega)
\]
Operations with kernels

Composition of kernels

\( \Psi \) from \((\mathcal{A}, \Sigma)\) to \((\mathcal{A}', \Sigma')\) and \(\Psi'\) from \((\mathcal{A}'\Sigma')\) to \((\mathcal{A}'', \Sigma'')\),

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Linear transformations of measures

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System in $\Lambda \subseteq \mathbb{L}$ described by a probability kernel $\pi_\Lambda(\cdot | \cdot)$

Equilibrium in $\Lambda$ iff equilibrium in every box $\Lambda' \subset \Lambda$:

$$\pi_\Lambda(f | \omega) = \pi_\Lambda(\pi_{\Lambda'}(f | \cdot) | \omega) \quad (\Lambda' \subset \Lambda \in \mathbb{L})$$
Equilibrium condition

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The notion of specification

**Specification:**
Family $\Pi = \{\pi_\Lambda : \Lambda \subseteq \mathbb{L}\}$ of prob. kern. from $(\Omega, \mathcal{F})$ to itself s.t.

(i) $\pi_\Lambda(f \mid \cdot) \in \mathcal{F}_{\Lambda^c}$ for each $\Lambda \subseteq \mathbb{L}$ and bounded measurable $f$

(ii) Each $\pi_\Lambda$ is *proper*: If $g \in \mathcal{F}_{\Lambda^c}$,

$$\pi_\Lambda(g f \mid \omega) = g(\omega) \pi_\Lambda(f \mid \omega)$$

for all $\omega \in \Omega$ and bounded measurable $f$

(iii) The family $\Pi$ is *consistent*:

$$\pi_\Lambda \pi_{\Lambda'} = \pi_\Lambda \quad \text{if } \Lambda' \subseteq \Lambda \subseteq \mathbb{L}$$
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Specifications

**The notion of specification**

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Family Π = \{π_Λ : Λ ⊆ ℒ\} of prob. kern. from (Ω, ℱ) to itself s.t.

1. \(\pi_Λ(f \mid \cdot) \in ℱ_{Λc}\) for each \(Λ \subseteq ℒ\) and bounded measurable \(f\)
2. Each \(\pi_Λ\) is proper: If \(g \in ℱ_{Λc}\),
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   \pi_Λ(g f \mid ω) = g(ω) \pi_Λ(f \mid ω)
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   for all \(ω \in Ω\) and bounded measurable \(f\)
3. The family Π is consistent:
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Consistent measures

Definition

\( \mu \) on \( \mathcal{F} \) is **consistent** with a specification \( \Pi = \{ \pi_\Lambda : \Lambda \subseteq \mathbb{L} \} \) if

\[
\mu \pi_\Lambda = \mu \quad \text{for each } \Lambda \subseteq \mathbb{L}
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(DLR equations)

Remarks

- Several consistent measures = first-order phase transition
- specification ~ system of regular conditional probabilities
  - No apriori measure: conditions for all \( \omega \) rather than a.s.
- Stat. mech.: conditional probabilities in search of measures
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Boltzmann prescription

Heuristically: $\pi_\Lambda \propto e^{-\beta H_\Lambda}$

- $\beta$ inverse temperature (to be absorbed)
- $H_\Lambda$ Hamiltonian = sum of local terms

Formally:

Interaction: family $\Phi = \{\phi_A \in F_A : A \in \mathbb{L}\}$

Example: Ising interaction

$$\phi_A(\omega) = \begin{cases} -J_{\{x,y\}} \omega_x \omega_y & \text{if } A = \{x,y\} \text{ with } |x-y| = 1 \\ -h_x \omega_x & \text{if } A = \{x\} \\ 0 & \text{otherwise} \end{cases}$$
Gibbsian specifications

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\]
**Hamiltonian** for $\Lambda \in \mathbb{L}$ with frozen external condition $\omega$

$$H_{\Lambda}^{\Phi}(\sigma_{\Lambda} \mid \omega_{\Lambda^c}) = \sum_{A \in \mathbb{L} : A \cap \Lambda \neq \emptyset} \phi_A(\sigma_{\Lambda} \omega)$$

Existence: $\Phi$ uniformly absolutely summable ($\Phi \in B_1$) if

$$\sum_{A \ni x} \| \Phi_A \|_{\infty} < \infty \quad \text{for each } x \in \mathbb{L}.$$  

**Definition**

The **Gibbsian specification** for $\Phi \in B_1$ has kernels

$$\pi_{\Lambda}^{\Phi}(C_{\sigma_{\Lambda}} \mid \omega) = \frac{e^{-H_{\Lambda}^{\Phi}(\sigma_{\Lambda} \mid \omega_{\Lambda^c})}}{\text{Norm.}}$$

A **Gibbs measures** for $\Phi$ is a measure consistent with $\Pi^{\Phi}$
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Gibbsian specifications
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Gibbsianness and its properties

**Definition**

- $\Pi$ is a **Gibbsian specification** if $\exists \Phi \in \mathcal{B}_1$ s.t. $\Pi = \Pi^\Phi$
- $\mu$ is a **Gibbs measure** if it is consistent with some $\Pi^\Phi$
Gibbsian description (1968): *equilibrium* statistical mechanics

Exploited in other settings:

- Renormalized measures
- Spin-flip evolutions
- Particle systems
- Dynamical systems
- Quenched disordered systems.
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10^6$ question: How to tell whether a measure is Gibbsian?

**Theorem (Kozlov)**

A specification is Gibbsian if, and only if, it is both

(i) **Uniformly non-null**: for each $\Lambda \in \mathbb{L}$

$$
\inf_{\sigma_{\Lambda} \in \Omega_{\Lambda}, \omega_{\Lambda^c} \in \Omega_{\Lambda^c}} \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\Lambda^c}) =: c_{\Lambda} > 0
$$

(ii) **Quasilocal** (almost Markovian): for each $\Lambda \in \Lambda$, $\sigma_{\Lambda} \in \Omega_{\Lambda}$

$$
\sup_{\omega, \eta, \tilde{\eta} \in \Omega} \left| \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\Lambda_n} \eta) - \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\Lambda_n} \tilde{\eta}) \right| \xrightarrow{n \to \infty} 0
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[C.f. Markovian:

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$$\inf_{\sigma_{\Lambda} \in \Omega_{\Lambda}, \omega_{\Lambda c} \in \Omega_{\Lambda c}} \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\Lambda c}) =: c_{\Lambda} > 0$$

** (ii) Quasilocal (almost Markovian):** for each $\Lambda \subseteq \Lambda$, $\sigma_{\Lambda} \in \Omega_{\Lambda}$

$$\sup_{\omega, \eta, \tilde{\eta} \in \Omega} \left| \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\Lambda n} \eta) - \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\Lambda n} \tilde{\eta}) \right| \xrightarrow[n \to \infty]{} 0$$

[C.f. Markovian:

$$\pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\partial r \Lambda} \eta) - \pi_{\Lambda}(C_{\sigma_{\Lambda}} \mid \omega_{\partial r \Lambda} \tilde{\eta}) = 0$$]
Non-nullness = no forbidden configuration
Quasilocal =
  Info from infinity only through intermediate fluctuations
  Experiment independent on state of Andromeda galaxy
Comments

- Non-nullness = no forbidden configuration
- Quasilocal =
  - Info from infinity only through intermediate fluctuations
  - Experiment independent on state of Andromeda galaxy
## When is a measure non-quasilocal

Notation: \( \mu_\Lambda(f \mid \omega) = E_\mu(f \mid \mathcal{F}_\Lambda^c)(\omega) \)

Key observations:

- \( \mu \) is quasilocal if consistent with *no* quasilocal specification
- Need violation at a single \( \hat{\omega} \) for a single \( \mu_\Lambda \) for a single \( f \)
- Discontinuity must be *essential* (e.g., for open neighborhoods)

Recipe: Find

- Sequence of frozen regions \( \Lambda_{N_i} \)
- “Tilting” configurations \( \eta^\pm \)
- Larger annulus \( \Lambda_{R_i} \) to define open sets

Show that

\[
\mu_\Lambda(f \mid \hat{\omega}_{N_i} \eta^+_R \setminus \Lambda_{N_i}) \text{ and } \mu_\Lambda(f \mid \hat{\omega}_{N_i} \eta^+_R \setminus \Lambda_{N_i})
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are different
Non-Gibbsianness criterion:

µ not quasilocal if there exist

- a finite region Λ (often |Λ| = 1)
- a “special” configuration ˆω
- a (quasi)local function f
- a diverging sequence of regions \((Λ_N_i)_{i≥1}\)
- some δ > 0 (independent of i)

such that for each \(i ≥ 1\) there exist

(i) larger regions \(Λ_{R_i}, R_i > N_i\)

(ii) two configurations \(η^+, η^-\) (possibly i-dependent), with

\[
\lim_{i→∞} \left| \mu_Λ(f \mid ˆω_{Λ_{N_i}} η^+_{Λ_{R_i} \setminus Λ_{N_i}} σ^+) - \mu_Λ(f \mid ˆω_{Λ_{N_i}} η^-_{Λ_{R_i} \setminus Λ_{N_i}} σ^-) \right| ≥ δ
\]

for every \(σ^± ∈ Ω\)
Non-Gibbsieness criterion:

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for every \( \sigma^\pm \in \Omega \)
Causes of non-quasilocality

**Mechanism: Hidden variables**

Quasilocality: frozen spins shield influence of distant regions

Non-quasilocality: info from afar even without fluctuations

**Mechanism?**

For transformed measures, original variables act as “hidden-variables”

- Freezing transformed vbles = conditioning of original vbles
- These conditioned variables keep some freedom to fluctuate
- For particular \( \omega \) the conditioned “hidden” system
  - exhibit a *phase transition*
  - hence, there is long-range order
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Main example: Transformations of measures

Linear stochastic transformations

A linear stochastic transformation is defined by

- An initial or *object* space $S^L$
- A transformed or *image* space $S^{L'}$
- A kernel $\tau$ from $S^L$ to $S^{L'}$ where

$$\tau(d\omega' \mid \omega) = \text{distribution of image spins when the initial spin configuration is } \omega$$

Particular cases:

- Stochastic evolutions: Image = evolved
- Renormalization transf.: Image = renormalized
Main example: Transformations of measures

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Particular cases:

- **Stochastic evolutions:** Image $=$ evolved
- **Renormalization transf.:** Image $=$ renormalized
**Block renormalization transformations**

**Definition:** Kernel from \( S^L \) to \((S')^{L'}\) of the form

\[
\tau(d\omega' \mid \omega) = \prod_{x' \in L'} \tau_{x'}(d\omega'_{x'} \mid \omega_{B_{x'}})
\]

Main examples: \( L' = L \), \( B_{x'} = \Lambda_{b-1} + bx' \)

Particular case: **Deterministic transformations**

\[
\tau_{x'}(\cdot \mid \omega_{B_{x'}}) = \delta_{T_{x'}}(\omega_{B_{x'}})(\cdot)
\]
Renormalization transformations

**Definition:** Kernel from $S^\mathbb{L}$ to $(S')^{\mathbb{L}'}$ of the form

$$
\tau(d\omega' \mid \omega) = \prod_{x' \in \mathbb{L}'} \tau_{x'}(d\omega'_{x'} \mid \omega_{B_x'})
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Main examples: $\mathbb{L}' = \mathbb{L}$, $B_{x'} = \Lambda b_{-1} + bx'$

Particular case: **Deterministic transformations**

$$
\tau_{x'}(\cdot \mid \omega_{B_{x'}}) = \delta_{T_{x'}}(\omega_{B_{x'}})(\cdot)
$$
Deterministic block RT

- **Decimation:** $S = S'$

  $$\tau_{x'}(\cdot | \omega_{B_{x'}}) = \delta_{\omega_{b_{x'}}}$$

- **Spin contractions:** $S' \subset S$, $B_{x'} = \{x'\}$
  - **Sign fields:** $S \subset \mathbb{R}$ symmetric,
    $$T_{x'}(\omega_{x'}) = \text{sign}(\omega_{x'})$$
  - **“Fuzzy” spins:** $S = \cup_{i \in I} S_i$ (partition), $S' = I$
    $$T_{x'}(\cdot | \omega_{x'}) = \sum_{i \in I} i \mathbb{1}_{\{\omega_{x'} \in S_i\}}$$
Renormalization transformations

Deterministic block RT

- Decimation: \( S = S' \)

\[
\tau_{x'}(\cdot | \omega_{B_{x'}}) = \delta_{\omega_{Bx'}}
\]

- Spin contractions: \( S' \subset S, B_{x'} = \{x'\} \neq \)
  - Sign fields: \( S \subset \mathbb{R} \) symmetric,

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- “Fuzzy” spins: \( S = \bigcup_{i \in I} S_i \) (partition), \( S' = I \)

\[
T_{x'}(\cdot | \omega_{x'}) = \sum_{i \in I} i \mathbb{1}_{\{\omega_{x'} \in S_i\}}
\]
Renormalization transformations

- **Block average:** $S' \supset S$

  $$T_{x'}(\omega_{B_{x'}}) = \frac{1}{|B_{x'}|} \sum_{y \in B_{x'}} \omega_y$$

- **Majority rule:** $S' = S = \{-1, 1\}$

  $$T_{x'}(\omega_{B_{x'}}) = \text{sign} \left[ \sum_{y \in B_{x'}} \omega_y \right]$$
Renormalization transformations

Stochastic block RT

- ** Majority with even block**: stochastic decision if
  \[
  \sum_{y \in B_{x'}} \omega_y = 0
  \]

- **p-Kadanoff transformation**: \( S = S' \)
  \[
  \tau_{x'}(d\omega'_{x'} \mid \omega_{B_{x'}}) = \frac{\exp[p \omega'_{x'} \sum_{y \in B_{x'}} \omega_y]}{\text{Norm.}} d\omega_{B_{x'}}
  \]
The renormalization issue

The question

Physicists: RT at Hamiltonian level

\[ \mu \overset{\tau}{\longrightarrow} \mu' \]
\[ \Phi \overset{\mathcal{R}}{\longrightarrow} \Phi' \]

Success led to applications to 1st-order phase transitions. Then,

\[ \{\mu_1, \cdots\} \quad \overset{\text{or}}{\longrightarrow} \quad \{\mu'_1, \cdots\} \]
\[ \Phi \quad \overset{\text{or}}{\longrightarrow} \quad \Phi' \]

or

\[ \{\mu_1, \cdots\} \quad \overset{\text{or}}{\longrightarrow} \quad \{\Phi'_1, \cdots\} \]
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The question

Physicists: RT at Hamiltonian level

\[
\begin{align*}
\mu & \xrightarrow{\tau} \mu' \\
\uparrow & \quad \downarrow \\
\Phi & \xrightarrow{\mathcal{R}} \Phi'
\end{align*}
\]

Success led to applications to 1st-order phase transitions. Then,

\[
\begin{align*}
\{\mu_1, \cdots\} & \implies \{\mu'_1, \cdots\} \quad \text{or} \quad \{\mu_1, \cdots\} \implies \{\mu'_1, \cdots\} \\
\uparrow\uparrow\uparrow & \quad \downarrow\downarrow\downarrow \\
\Phi & \quad \Phi'
\end{align*}
\]
The renormalization issue

The answer

In fact,

\[
\{\mu_1, \cdots\} \xrightarrow{\Phi} \{\mu'_1, \cdots\}
\]

\[
\{\mu_1, \cdots\} \xrightarrow{\Phi'} \{\mu'_1, \cdots\}
\]

or

\[
\{\mu_1, \cdots\} \xrightarrow{\Phi} \{\mu'_1, \cdots\}
\]

and even

\[
\mu \xrightarrow{\tau} \mu'
\]

\[
\Phi \xrightarrow{??}
\]
Israel: 2 × 2-decimation of the Ising model

- \( \hat{\omega}_{x'} = (-1)^{|x'|} \) = decorated Ising model on internal spins
- Model equivalent to an Ising model at a higher temperature
- \( \eta_{x'}^{\pm} = \pm 1 \) in an annulus chooses the “\( \pm \)”-phase

Thus,

\[
\mu' \left( \sigma'_0 \mid \hat{\omega}'_{\Lambda'_R} (+1)_{\Lambda'_{R+1}\setminus\Lambda'_R} \sigma'^+ \right) - \mu' \left( \sigma'_0 \mid \hat{\omega}'_{\Lambda'_R} (-1)'_{\Lambda'_{R+1}\setminus\Lambda'_R} \sigma'^- \right) \xrightarrow{R \to \infty} 2m(\beta')
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General case

Specification with densities proportional to

\[ e^{-H_{\Lambda}^\Phi(\sigma_\Lambda | \omega_{\Lambda^c})} \prod_{x' \in B'_\Lambda} T_{x'}\left(\omega'_{x'} \mid (\sigma_\Lambda \omega)_{B_{x'}}\right) \]

\[ = \exp\left\{ -H_{\Lambda}^\Phi(\sigma_\Lambda | \omega_{\Lambda^c}) + \sum_{x' \in B'_\Lambda} \log T_{x'}\left(\omega'_{x'} \mid (\sigma_\Lambda \omega)_{B_{x'}}\right) \right\} \]

The image \( \omega' \) acts as “fields” on the original \( \sigma_\Lambda \omega \)
\( \hat{\omega}' \) so that conditioned original spins have a phase transition.
In this way, all usual transformations lead to non-Gibbsianness (even outside the coexistence region)
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Non-Gibbsianess in spin-flip evolutions

Simulations: spin-flip dynamics converging to a target measure (Metropolis, heat-bath, Glauber)

Often: ordered initial configuration

“Unquenching”: high-$T$ dynamics applied a low-$T$ Gibbs state

Non-Gibbsianess enters into the picture
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Out and in from Gibbsianness

**Results for parallel independent updating**

\[
\begin{align*}
(h = 0) & \quad \text{Gibbs} \quad \cdots \quad \text{Non-G (NQL)} \\
0 & \quad n_1 \quad n_2 \\
(h > 0) & \quad \text{Gibbs} \quad \cdots \quad \text{Non-G (NQL)} \quad \cdots \quad \text{Gibbs} \\
0 & \quad n_1 \quad n_2 \quad n_3 \quad n_4
\end{align*}
\]
Interpretation: The key questions

Which is the most probable history of an improbable configuration?

Is the (atypical) droplet $\hat{\omega}'_\Lambda$

- *Nurture*: created by the dynamics?
- *Nature*: created initially and survived?

The history of $\hat{\omega}'_\Lambda$

- is it uniquely defined by the final configuration?
- admits competing possibilities?
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Interpretation: Tentative answers:

Short times:
  ▶ Only a few changes possible
  ▶ Everybody nature
  ▶ Only one possible history

Not-too-short times:
  ▶ System relaxes first and forms $\tilde{\omega}'_\Lambda$ at the last moment
  ▶ Everybody nurture
  ▶ Possibility of multiple histories:
    ▶ Histories start from typical configurations of different phases
    ▶ Same volume cost, different boundary cost
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Non-Gibbsianness as discontinuity

Conclusion:

Non-Gibbsianness = history with discontinuous dependence on the surrounding configuration

Scenario:

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Wake-up calls for non-Gibbsianness

Griffiths and Pearce (1978): *peculiarities* in Renormalized measures

Israel (1979): peculiarity=absence of *quasilocality*

Other examples (1987–9):

- Spin contractions (Lebowitz-Maes, Dorlas-van Enter)
- Lattice projections (Schonmann)
- Stationary measures of stochastic evolutions (Lebowitz-Schonmann)

Systematization and overview: van Enter, F. and Sokal (1993)
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State of affairs: Positive side

Extensive catalog of instances

- Renormalization transformations
- Spin-flip evolutions (simulations)
- Joint measures of disordered systems
- Intermittency in dynamical systems

Good knowledge of non-Gibbsianness mechanisms

- Physical: hidden variables, ph. transitions of restricted systems
- Mathematical: lack of quasilocality, lack of non-nullness

Clarification of conceptual issues

- Renormalization transformations are not discontinuous
- Morita approach for disordered systems redeemed
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- Joint measures of disordered systems
- Intermittency in dynamical systems

Good knowledge of non-Gibbsianess mechanisms

- Physical: hidden variables, ph. transitions of restricted systems
- Mathematical: lack of quasilocality, lack of non-nullness

Clarification of conceptual issues

- Renormalization transformations are not discontinuous
- Morita approach for disordered systems redeemed
Lack of answers to practitioners

- Calculations of critical exponents?
- Consequences for simulations or sampling schemes?
- Observable (numerical) consequence of non-Gibbsianness? (van Enter and Verbitskiy!)
State of affairs: Negative side - Homework

Lack of answers to practitioners

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