Gibbs measures: definition, uses and abuses

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(A. van Enter, A. Sokal, Ch.-Ed. Pfister, R. Kotecký, R. Schonmann, S. Shlosman, A. Toom, F. den Hollander, F. Redig, A. Le Ny, R. Dobrushin, J, Lebowitz, C. Maes, C. Külske,. . .)

Stat mech *à la* Gibbs

Issue: to study systems with many components
Examples:

- *Particles in space*: Each particle characterized by a position and a velocity
- *Spins in a lattice* (pixels, particles): Each spin has a finite number of possible values

Stat mech approach:

- Look at finite "windows" (finite regions) Λ
- Replace detailed laws by a probabilistic description
- Find the asymptotic behavior for Λ huge

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Rough probabilistic prescription

Probability weights or densities

$$\frac{\mathrm{e}^{-\beta H_\lambda}}{Z_\Lambda}$$

where

- H_Λ=Hamiltonian; must be sum of local terms so that $H_{\widetilde{\Lambda}} - H_\Lambda \sim \left|\widetilde{\Lambda} \setminus \Lambda\right|$ for $\widetilde{\Lambda} \subset \Lambda$
- β = inverse temperature ("coolness")
- Z_Λ = partition function (normailzation). Physics info:

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \log Z_\Lambda \;=\; \text{pressure or free enegy}$$

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Set up: Finite-spin lattice systems

- *Lattice* = Countable set $\mathbb{L}$ (e.g. $\mathbb{L} = \mathbb{Z}^d$)
  - sites $x \in \mathbb{L}$
  - finite regions $\Lambda, \Gamma \Subset \mathbb{L}$
- *Single-spin space* S, here finite (e.g. Ising spins: $S = \{-1, 1\}$)
- *Configuration space* $\Omega = S^{\mathbb{L}}$ (A copy of S at each site)
  - Notation: $\Omega_\Lambda := S^\Lambda$
- *Configurations:* $\Omega \ni \omega = (\omega_x)_{x\in\mathbb{L}}$
  Notation:
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Basic kernels

Formally, finite window = finite region *of an infinite system*:

- Inside Λ: probability measure
- Outside Λ: fixed configuration (external condition)

That is, a family of probability measures

$$\pi\left(\cdot \,\middle|\, \omega_{\Lambda^c} \right)$$

or, more precisely, a kernel with two slots $\pi(\ \mid\ )$

This is a *probability kernel*

Need some formalization

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Measure-theoretical and topological set-up

Gibbsianness: interplay between topology and measure-theory

- S endowed with discrete topology and σ-algebra
- Ω endowed with the *product* topology and σ-algebra

In more detail:

- $\mathcal{F}$ = σ-algebra generated by the cylinders

$$C_{\sigma_\Lambda} = \{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\}$$

- $\mathcal{F}_\Gamma$ = σ-algebra generated by cylinders with basis in $\Gamma \subset \mathbb{L}$

$$C_{\sigma_\Lambda}\ ,\ \Lambda \subset \Gamma$$

- Topology also generated by the cylinders
  - cylinders are open
  - continuous functions are measurable

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Locality and continuity

f is a *local function* if

- It depends only on the spins on a *finite* region
- $\exists\, \Gamma \Subset \mathbb{L}$ such that $f(\omega) = f(\sigma)$ whenever $\omega_\Gamma = \sigma_\Gamma$
- $\exists\, \Gamma \Subset \mathbb{L}$ such that $(f \in \mathcal{F}_\Gamma)$

Properties:

- Local functions are continous
- More generally: f is continuous iff, it is *quasilocal*

$$\sup_{\omega\in\Omega}\ \sup_{\sigma\in\Omega} \Big| f(\omega_{\Lambda_n}\sigma) - f(\omega) \Big| \xrightarrow[n\to\infty]{} 0$$

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Probability kernels

Definition

A **probability kernel** Ψ from a probability space $(\mathcal{A}, \Sigma)$ to another probability space $(\mathcal{A}', \Sigma')$ is a function

$$\Psi(\,\cdot \mid \cdot\,) : \Sigma' \times \mathcal{A} \longrightarrow [0,1]$$

such that

(i) $\Psi(\,\cdot\,|\omega)$ is a probability measure on $(\mathcal{A}', \Sigma')$ for each $\omega \in \mathcal{A}$;

(ii) $\Psi(A'|\,\cdot\,)$ is Σ-measurable for each $A' \in \Sigma'$.

Equilibrium systems in stat mech

System in $\Lambda \Subset \mathbb{L}$ described by a probability kernel

$$\pi_\Lambda(\cdot \mid \cdot) \text{ from } (\Omega, \mathcal{F}) \text{ to itself}$$

where

$$\pi_\Lambda(f \mid \omega) \;=\; \text{equilibrium value of } f \text{ when the configuration outside } \Lambda \text{ is } \omega$$

Operations with kernels

Composition of kernels

Ψ from $(\mathcal{A}, \Sigma)$ to $(\mathcal{A}', \Sigma')$ and Ψ' from $(\mathcal{A}'\Sigma')$ to $(\mathcal{A}'', \Sigma'')$,

$$\big(\Psi\Psi'\big)(A''|\omega) \;=\; \int_{\mathcal{A}'} \Psi(d\omega'|\omega)\,\Psi'(A''|\omega')$$

Linear transformations of measures

$$\begin{array}{ccc} \mathcal{P}(\mathcal{A}, \Sigma) & \longrightarrow & \mathcal{P}(\mathcal{A}', \Sigma') \\ \mu & \longmapsto & \mu' = \mu\Psi \end{array}$$

$$\mu'(A') \;=\; \int_{\mathcal{A}} \mu(d\omega)\,\Psi(A'|\omega)$$

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Equilibrium condition

System in $\Lambda \Subset \mathbb{L}$ described by a probability kernel $\pi_\Lambda(\cdot \mid \cdot)$

Equilibrium in Λ iff equilibrium in every box $\Lambda' \subset \Lambda$:

$$\pi_\Lambda(f \mid \omega) = \pi_\Lambda\Big(\pi_{\Lambda'}(f \mid \cdot) \Big| \omega\Big) \qquad (\Lambda' \subset \Lambda \Subset \mathbb{L})$$

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The notion of specification

Specification:
Family $\Pi = \{\pi_\Lambda : \Lambda \Subset \mathbb{L}\}$ of prob. kern. from $(\Omega, \mathcal{F})$ to itself s.t.

(i) $\pi_\Lambda(f \mid \cdot) \in \mathcal{F}_{\Lambda^c}$ for each $\Lambda \Subset \mathbb{L}$ and bounded measurable f

(ii) Each π_Λ is *proper*: If $g \in \mathcal{F}_{\Lambda^c}$,

$$\pi_\Lambda(g\, f \mid \omega) \;=\; g(\omega)\, \pi_\Lambda(f \mid \omega)$$

for all $\omega \in \Omega$ and bounded measurable f

(iii) The family Π is *consistent*:

$$\pi_\Lambda\, \pi_{\Lambda'} \;=\; \pi_\Lambda \qquad \text{if } \Lambda' \subset \Lambda \Subset \mathbb{L}$$

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Consistent measures

Definition

μ on $\mathcal{F}$ is **consistent** with a specification $\Pi = \{\pi_\Lambda : \Lambda \Subset \mathbb{L}\}$ if

$$\mu\,\pi_\Lambda \;=\; \mu \qquad \text{for each } \Lambda \Subset \mathbb{L}$$

(DLR equations)

Remarks

- Several consistent measures = first-order phase transition
- specification $\sim$ system of regular conditional probabilities
  - No apriori measure: conditions for *all* ω rather than a.s.
- Stat. mech.: conditional probabilities in search of measures

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Boltzmann prescription

Heuristically: $\pi_\Lambda \;\propto\; \mathrm{e}^{-\beta H_\Lambda}$

- β *inverse temperature* (to be absorbed)
- H_Λ Hamiltonian = sum of local terms

Formally:

Interaction: family $\Phi = \{\phi_A \in \mathcal{F}_A : A \Subset \mathbb{L}\}$

Example: Ising interaction

$$\phi_A(\omega) \;=\; \begin{cases} -J_{\{x,y\}}\,\omega_x\omega_y & \text{if } A=\{x,y\} \text{ with } |x-y|=1 \\ -h_x\,\omega_x & \text{if } A=\{x\} \\ 0 & \text{otherwise} \end{cases}$$

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Hamiltonian for $\Lambda \Subset \mathbb{L}$ with frozen external condition ω

$$H^{\Phi}_{\Lambda}(\sigma_\Lambda \mid \omega_{\Lambda^c}) \;=\; \sum_{A \Subset \mathbb{L}: A\cap\Lambda\neq\emptyset} \phi_A(\sigma_\Lambda \omega)$$

Existence: Φ **uniformly absolutely summable** ($\Phi \in \mathcal{B}_1$) if

$$\sum_{A\ni x} \|\Phi_A\|_\infty \;<\; \infty \quad \text{for each } x \in \mathbb{L}\,.$$

Definition

The **Gibbsian specification** for $\Phi \in \mathcal{B}_1$ has kernels

$$\pi^{\Phi}_{\Lambda}(C_{\sigma_\Lambda} \mid \omega) \;=\; \frac{e^{-H^{\Phi}_{\Lambda}(\sigma_\Lambda \mid \omega_{\Lambda^c})}}{\text{Norm.}}$$

A **Gibbs measures** for Φ is a measure consistent with Π^{Φ}

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Gibbsianness and its properties

Definition

- Π is a **Gibbsian specification** if $\exists\ \Phi \in \mathcal{B}_1$ s.t. $\Pi = \Pi^{\Phi}$
- μ is a **Gibbs measure** if it is consistent with some Π^{Φ}

Gibbsian description (1968): *equilibrium* statistical mechanics

Exploited in other settings:

- Renormalized measures
- Spin-flip evolutions
- Particle systems
- Dynamical systems
- Quenched disordered systems.

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10^6\$ question: How to tell whether a measure is Gibbsian?

Theorem (Kozlov)

A specification is Gibbsian if, and only if, it is both

(i) **Uniformly non-null**: *for each* $\Lambda \Subset \mathbb{L}$

$$\inf_{\sigma_\Lambda \in \Omega_\Lambda, \omega_{\Lambda^c} \in \Omega_{\Lambda^c}} \pi_\Lambda(C_{\sigma_\Lambda} \mid \omega_{\Lambda^c}) =: c_\Lambda > 0$$

(ii) **Quasilocal** *(almost Markovian): for each* $\Lambda \Subset \Lambda$, $\sigma_\Lambda \in \Omega_\Lambda$

$$\sup_{\omega,\eta,\widetilde{\eta} \in \Omega} \Big| \pi_\Lambda(C_{\sigma_\Lambda} \mid \omega_{\Lambda_n}\eta) - \pi_\Lambda(C_{\sigma_\Lambda} \mid \omega_{\Lambda_n}\widetilde{\eta}) \Big| \xrightarrow[n\to\infty]{} 0$$

[C.f. **Markovian**:

$$\pi_\Lambda(C_{\sigma_\Lambda} \mid \omega_{\partial_r\Lambda}\eta) - \pi_\Lambda(C_{\sigma_\Lambda} \mid \omega_{\partial_r\Lambda}\widetilde{\eta}) = 0]$$

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[C.f. **Markovian**:

$$\pi_\Lambda\big(C_{\sigma_\Lambda} \mid \omega_{\partial_r \Lambda} \eta\big) - \pi_\Lambda\big(C_{\sigma_\Lambda} \mid \omega_{\partial_r \Lambda} \widetilde{\eta}\big) \;=\; 0$$]

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Notation: $\mu_\Lambda(f \mid \omega) = E_\mu(f \mid \mathcal{F}_{\Lambda^c})(\omega)$

Key observations:

- μ is quasilocal if consistent with *no* quasilocal specification
- Need violation at a *single* $\widehat{\omega}$ for a *single* μ_Λ for a *single* f
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Recipe: Find

- Sequence of frozen regions Λ_{N_i}
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Non-Gibbsianness criterion:

μ not quasilocal if there exist

- a finite region Λ (often $|\Lambda| = 1$)
- a "special" configuration $\widehat{\omega}$
- a (quasi)local function f
- a diverging sequence of regions $\big(\Lambda_{N_i}\big)_{i\geq 1}$
- some $\delta > 0$ (independent of i)

such that for each $i \geq 1$ there exist

(i) larger regions Λ_{R_i}, $R_i > N_i$

(ii) two configurations η^+, η^- (possibly i-dependent), with

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Mechanism: Hidden variables

Quasilocality: frozen spins shield influence of distant regions

Non-quasilocality: info from afar even without fluctuations

Mechanism?

For transformed measures,

original variables act as "hidden-variables"

- Freezing transformed vbles = conditioning of original vbles
- These conditioned variables keep some freedom to fluctuate
- For particular ω the conditioned "hidden" system
  - exhibit a *phase transition*
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Linear stochastic transformations

A linear stochastic transformation is defined by

- An initial or *object* space $S^{\mathbb{L}}$
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- A kernel τ from $S^{\mathbb{L}}$ to $S'^{\mathbb{L}'}$ where

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Particular cases:

- **Stochastic evolutions:** Image = evolved
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Definition: Kernel from $S^{\mathbb{L}}$ to $(S')^{\mathbb{L}'}$ of the form

$$\tau(d\omega' \mid \omega) \;=\; \prod_{x' \in \mathbb{L}'} \tau_{x'}(d\omega'_{x'} \mid \omega_{B_{x'}})$$

Main examples: $\mathbb{L}' = \mathbb{L}$, $B_{x'} = \Lambda_{b-1} + bx'$

Particular case: **Deterministic transformations**

$$\tau_{x'}(\,\cdot \mid \omega_{B_{x'}}) = \delta_{T_{x'}(\omega_{B_{x'}})}(\,\cdot\,)$$

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Deterministic block RT

- *Decimation:* $S = S'$

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- *Spin contractions:* $S' \underset{\neq}{\subset} S$, $B_{x'} = \{x'\}$
  - *Sign fields:* $S \subset \mathbb{R}$ symmetric,

$$T_{x'}(\omega_{x'}) = \mathrm{sign}(\omega_{x'})$$

  - *"Fuzzy" spins:* $S = \cup_{i \in I} S_i$ (partition), $S' = I$

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- *Block average:* $S' \supsetneq S$

$$T_{x'}(\omega_{B_{x'}}) = \frac{1}{|B_{x'}|} \sum_{y \in B_{x'}} \omega_y$$

- *Majority rule:* $S' = S = \{-1, 1\}$

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Stochastic block RT

- *Majority with even block*: stochastic decision if

$$\sum_{y \in B_{x'}} \omega_y \;=\; 0$$

- *p-Kadanoff transformation: $S = S'$*

$$\tau_{x'}(d\omega'_{x'} \mid \omega_{B_{x'}}) \;=\; \frac{\exp\left[p\, \omega'_{x'} \sum_{y \in B_{x'}} \omega_y\right]}{\text{Norm.}}\, d\omega_{B_{x'}}$$

The question

Physicists: RT at Hamiltonian level

$$\begin{array}{ccc} \mu & \stackrel{\mathcal{T}}{\longrightarrow} & \mu' \\ \uparrow & & \downarrow \\ \Phi & \stackrel{\mathcal{R}}{\longrightarrow} & \Phi' \end{array}$$

Success led to applications to 1st-order phase transitions. Then,

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Israel: 2×2-decimation of the Ising model

- $\widehat{\omega}'_{x'} = (-1)^{|x'|}$ = decorated Ising model on *internal* spins
- Model equivalent to an Ising model at a higher temperature
- $\eta'^{\pm}_{x'} = \pm 1$ in an annulus chooses the "$\pm$"-phase

Thus,

$$\mu'\Big(\sigma'_0 \,\Big|\, \widehat{\omega}'_{\Lambda'_R}\,(+1)_{\Lambda'_{R+1}\setminus\Lambda'_R}\,\sigma'^{+}\Big) - \mu'\Big(\sigma'_0 \,\Big|\, \widehat{\omega}'_{\Lambda'_R}\,(-1)'^{-}_{\Lambda'_{R+1}\setminus\Lambda'_R}\,\sigma'^{-}\Big)$$

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The image ω' acts as “fields” on the original $\sigma_\Lambda\omega$

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Non-Gibbsianness in spin-flip evolutions

Simulations: spin-flip dynamics converging to a target measure (Metropolis, heath-bath, Glauber)

Often: ordered initial configuration

"Unquenching": high-T dynamics applied a low-T Gibbs state

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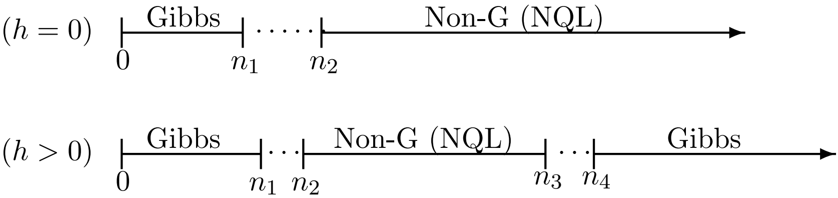
Results for parallel independent updating



Interpretation: The key questions

Which is the most probable history of an improbable configuration?

Is the (atypical) droplet $\widehat{\omega}'_\Lambda$

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- Everybody nature
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Not-too-short times:

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Non-Gibbsianness = history with discontinuous dependence on the surrounding configuration

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- Spin-flip evolutions (simulations)
- Joint measures of disordered systems
- Intermittency in dynamical systems

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- Joint measures of disordered systems
- Intermittency in dynamical systems

Good knowledge of non-Gibbsianness mechanisms

- Physical: hidden variables, ph. transitions of restricted systems
- Mathematical: lack of quasilocality, lack of non-nullness

Clarification of conceptual issues

- Renormalization transformations are not discontinuous
- Morita approach for disordered systems redeemed

State of affairs: Negative side - Homework

Lack of answers to practitioners

- Calculations of critical exponents?
- Consequences for simulations or sampling schemes?
- Observable (numerical) consequence of non-Gibbsianness? (van Enter and Verbitskiy!)

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