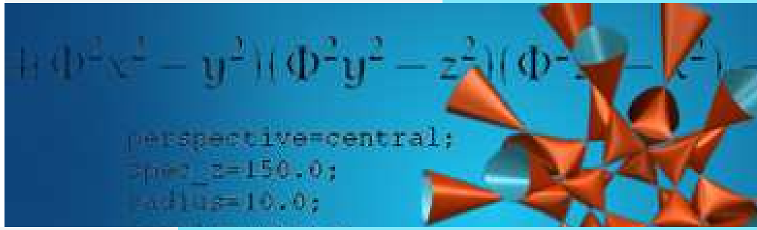
Philipps Universität Marburg

Old and new on the exceptional Lie group G_2



Prof. Dr. habil. Ilka Agricola
Philipps-Universität Marburg

May 2010, Utrecht University
Literature: Notices of the AMS 55 (2008), 922-929

"Moreover, we hereby obtain a direct definition of our 14-dimensional simple group $[G_2]$ which is as elegant as one can wish for."

Friedrich Engel, 1900.

"Zudem ist hiermit eine direkte Definition unsrer vierzehngliedrigen einfachen Gruppe gegeben, die an Eleganz nichts zu wünschen übrig lässt."

Friedrich Engel, 1900.

Friedrich Engel in the note to his talk at the Royal Saxonian Academy of Sciences on June 11, 1900.

In this talk:

- History of the discovery and realisation of G_2
- Role & life of Engel's Ph. D. student Walter Reichel
- Significance for modern differential geometry

1880-1885: simple complex Lie algebras $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ were well-known; Lie and Engel knew about $\mathfrak{sp}(n, \mathbb{C})$, but nothing was published

In 1884, Wilhelm Killing starts a correspondence with Felix Klein, Sophus Lie and, most importantly, Friedrich Engel

> Killing's ultimate goal: Classification of all real space forms, which requires knowing all simple real Lie algebras

April 1886: Killing conjectures that $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ are the *only* simple complex Lie algebras (though Engel had told him that more simple algebras could occur as isotropy groups)

March 1887: Killing discovers the root system of G_2 and claims that it should have a 5-dimensional realisation

October 1887: Killing obtains the full classification, prepares a paper after strong encouragements by Engel

Wilhelm Killing (1847–1923)

- 1872 thesis in Berlin on 'Flächenbündel 2. Ordnung' (advisor: K. Weierstraß)
- 1882–1892 teacher, later principal at the Lyceum Hosianum in Braunsberg (East Prussia)
- 1884 *Programmschrift* [Studium der Raumformen über ihre infinitesimalen Bewegungen]
- 1892–1919 professor in Münster (rector 18897-98)
- W. Killing, *Die Zusammensetzung der stetigen endlichen Transformationsgruppen*, Math. Ann. 33 (1889), 1-48.

Satz (W. Killing, 1887). The only complex simple Lie algebras are $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C})$ as well as five exceptional Lie algebras,

$$\mathfrak{g}_2 \ := \mathfrak{g}_2^{14},\ \mathfrak{f}_4^{52},\ \mathfrak{e}_6^{78},\ \mathfrak{e}_7^{133}, \mathfrak{e}_8^{248}.$$

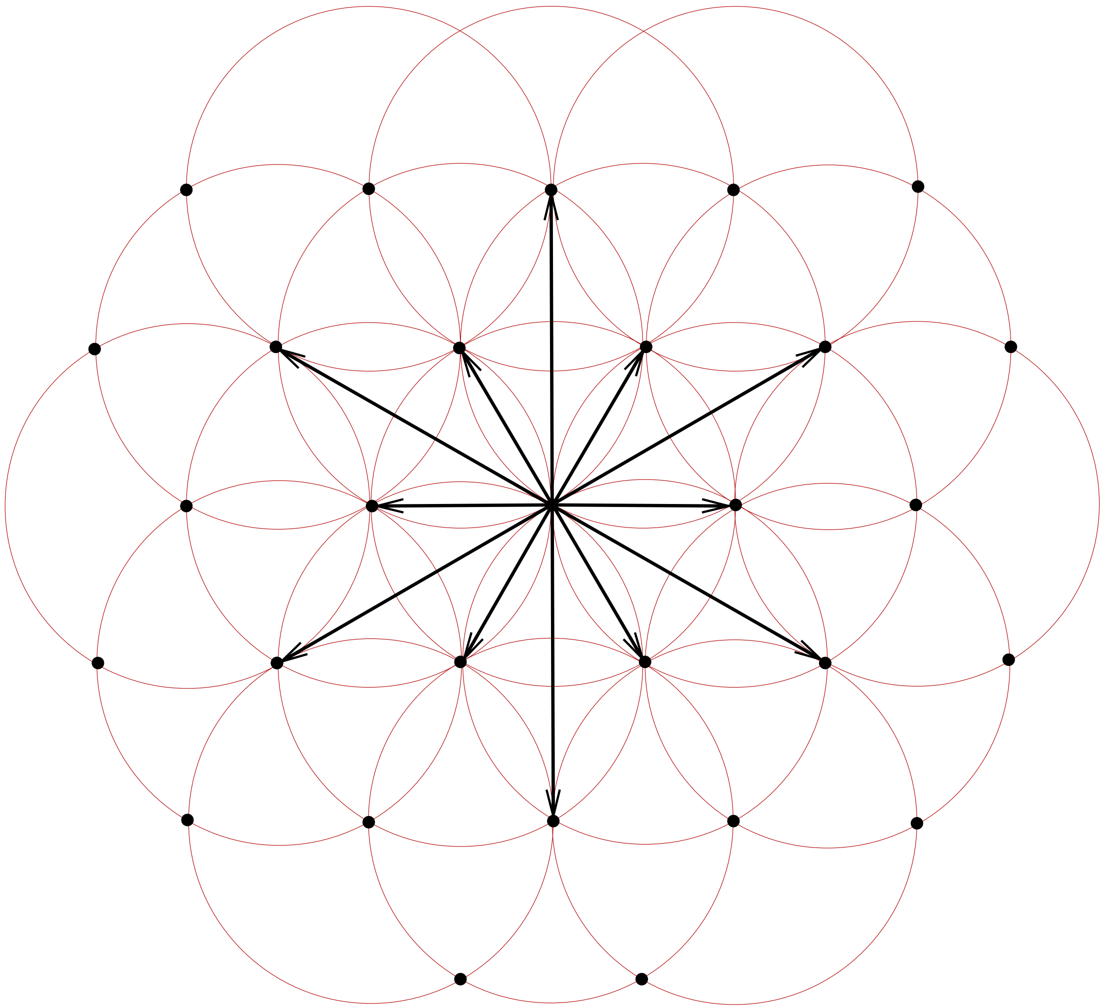
(upper index: dimension, lower index: rank)

Killing's proof contains some gaps and mistakes. In his thesis (1894), Élie Cartan gave a completely revised and polished presentation of the classification, which has therefore become the standard reference for the result.

Notations:

- G_2, $\mathfrak{g}_2$: complex Lie group resp. Lie algebra
- G_2^c, $\mathfrak{g}_2^c$: real *compact* form of G_2, $\mathfrak{g}_2$
- G_2^*, $\mathfrak{g}_2^*$: real *non compact* form of G_2, $\mathfrak{g}_2$

Root system of $\mathfrak{g}_2$



(only root system in which the angle $\pi/6$ appears between two roots)

Cartan's thesis

Last section: derives from weight lattice the lowest dimensional irreducible representation of each simple complex Lie algebra

Result. $\mathfrak{g}_2$ admits an irreducible representation on $\mathbb{C}^7$, and it has a $\mathfrak{g}_2$-invariant scalar product

$$\beta := x_0^2 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Interpreted as a *real* scalar product, it has signature $(4,3)$: Cartan's representation restricts to an irred. $\mathfrak{g}_2^*$ representation inside $\mathfrak{so}(4,3)$.

Problem: direct construction of $\mathfrak{g}_2$ and its real forms $\mathfrak{g}_2^*, \mathfrak{g}_2^c$?

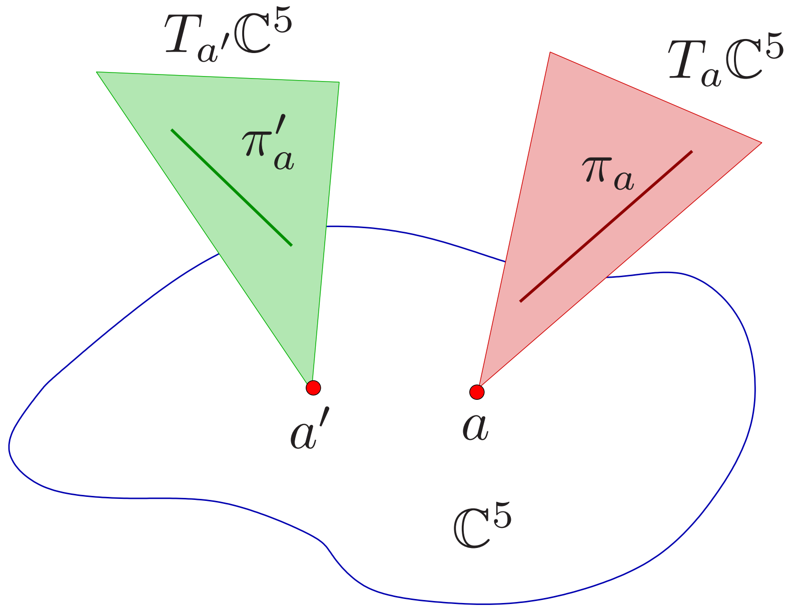
First step: Engel & Cartan, 1893

In 1893, Engel & Cartan publish simultaneously a note in C. R. Acad. Sc. Paris. They give the following construction:

Consider $\mathbb{C}^5$ and the 2-planes $\pi_a \subset T_a\mathbb{C}^5$ defined by

$$dx_3 = x_1\, dx_2 - x_2\, dx_1,$$
$$dx_4 = x_2\, dx_3 - x_3\, dx_2,$$
$$dx_5 = x_3\, dx_1 - x_1\, dx_3.$$

The 14 vector fields whose (local) flows map the planes π_a to each other satisfy the commutator relations of $\mathfrak{g}_2$!



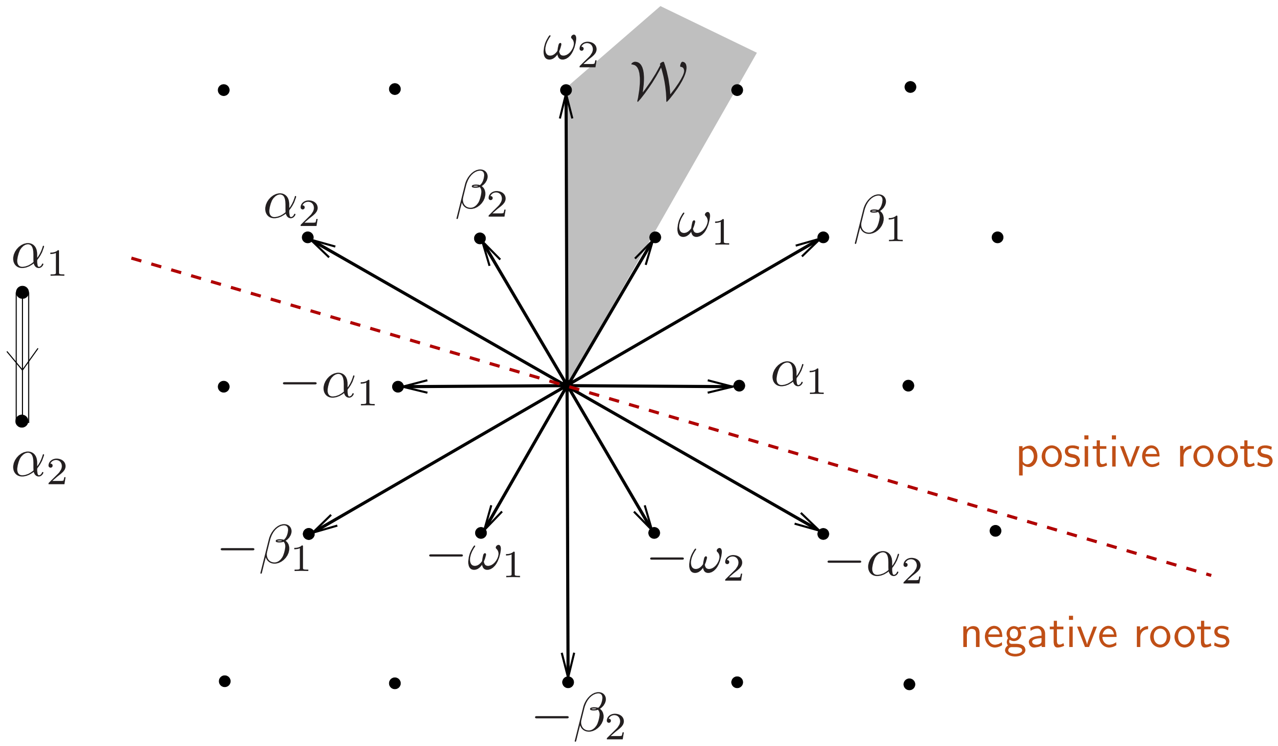
Both give a second, non equivalent realisation of $\mathfrak{g}_2$:

- Engel: through a contact transformation from the first
- Cartan: as symmetries of solution space of the 2nd order pde's ($f = f(x, y)$)

$$f_{xx} = \frac{4}{3}(f_{yy})^3, \;\; f_{xy} = (f_{yy})^2.$$

Root system of $\mathfrak{g}_2$ (II)

For a modern interpretation of the Cartan/Engel result, we need:

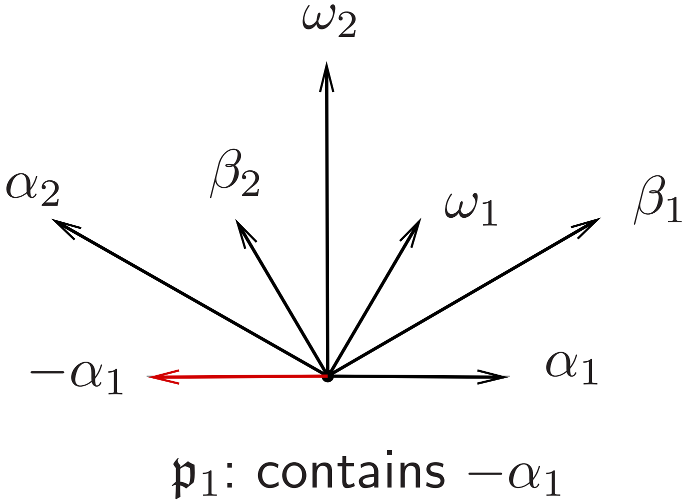


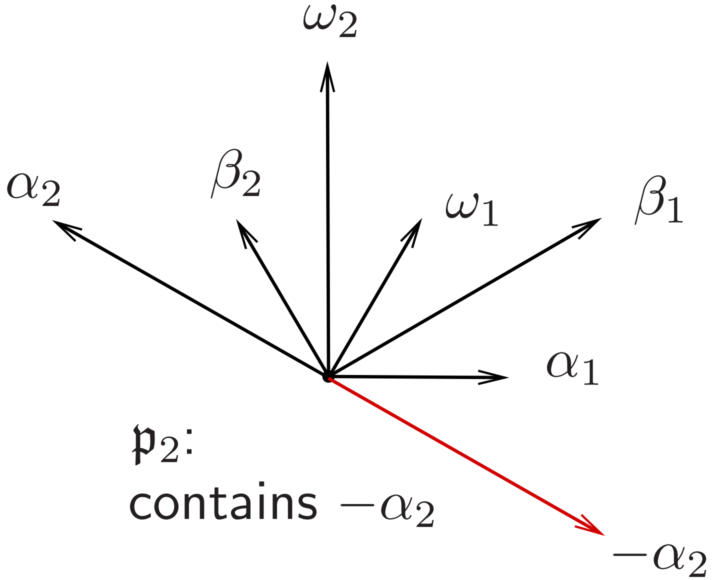
$\alpha_{1,2}$: simple roots

$\omega_{1,2}$: fundamental weights (ω_1: 7-dim. rep., ω_2: adjoint rep.)

$\mathcal{W}$: Weyl chamber = cone spanned by ω_1, ω_2

Parabolic subalgebras of $\mathfrak{g}_2$





Every parabolic subalgebra contains *all positive roots* and (eventually) some negative simple roots:

$\mathfrak{p}_1 = \mathfrak{h} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1}$ [9-dimensional]

$\mathfrak{p}_2 = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_2}$ [9-dimensional]

$\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\beta_2} \oplus \mathfrak{g}_{\omega_2} \oplus \mathfrak{g}_{\omega_1} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{\alpha_1}$ [8-dim. Borel alg.]

Modern interpretation

The complex Lie group G_2 has two maximal parabolic subgroups P_1 and P_2 (with Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$)

$\Rightarrow G_2$ acts on the two 5-dimensional compact homogeneous spaces

- $M_1^5 := G_2/P_1 = \overline{G \cdot [v_{\omega_1}]} \subset \mathbb{P}(\mathbb{C}^7) = \mathbb{CP}^6$: a quadric
- $M_2^5 := G_2/P_2 = \overline{G \cdot [v_{\omega_2}]} \subset \mathbb{P}(\mathbb{C}^{14}) = \mathbb{CP}^{13}$ 'adjoint homogeneous variety'

where v_{ω_1}, v_{ω_2} are h. w. vectors of the reps. with highest weight ω_1, ω_2.

Cartan and Engel described the action of $\mathfrak{g}_2$ on some open subsets of M_i^5.

Real situation: To $P_i \subset G_2$ corresponds a real subgroup $P_i^* \subset G_2^*$, hence the split form G_2^* has two real compact 5-dimensional homogeneous spaces on which it acts.

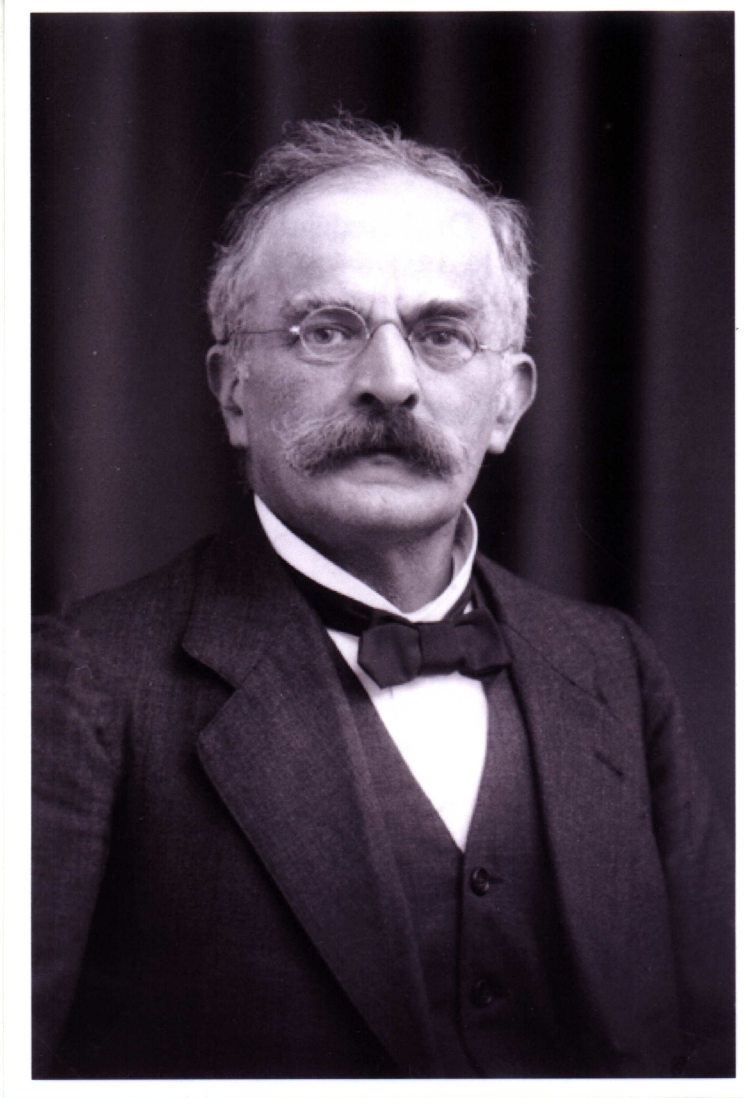
However, G_2^c has no 9-dim. subgroups! (max. subgroup: 8-dim. $\mathrm{SU}(3) \subset G_2$)

Q: Direct realisation of G_2^c ?

Élie Cartan (1869–1951)

- 1894 thesis at ENS (Paris), *Sur la structure des groupes de transformations finis et continus*.
- 1894–1912 maître de conférences in Montpellier, Nancy, Lyon, Paris
- 1912-1940 Professor in Paris
- É. Cartan, *Sur la structure des groupes simples finis et continus*, C. R. Acad. Sc. 116 (1893), 784-786.
- É. Cartan, *Nombres complexes*, Encyclop. Sc. Math. 15, 1908, 329-468.
- É. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Éc. Norm. 27 (1910), 109-192.

Friedrich Engel (1861–1941)



- 1883 thesis in Leipzig on contact transformations
- 1885–1904 Privatdozent in Leipzig
- 1904–1913 Professor in Greifswald, since 1913 in Gießen
- F. Engel, *Sur un groupe simple à quatorze paramètres*, C. R. Acad. Sc. 116 (1893), 786-788.
- F. Engel, *Ein neues, dem linearen Complexe analoges Gebilde*, Leipz. Ber. 52 (1900), 63-76, 220-239.
- editor of the complete works of S. Lie and H. Grassmann

G_2 and 3-forms in 7 variables

Non-degenerate 2-forms are at the base of symplectic geometry and define the Lie groups $\mathrm{Sp}(n, \mathbb{C})$.

Q: Is there a geometry based on 3-forms ?

• Generic 3-forms (i. e. with dense $\mathrm{GL}(n, \mathbb{C})$ orbit inside $\Lambda^3\mathbb{C}^n$) exist only for $n \leq 8$.

• To do geometry, we need existence of a compatible inner product, i. e. we want for generic $\omega \in \Lambda^3\mathbb{C}^n$

$$G_\omega := \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \omega = g^*\omega\} \subset \mathrm{SO}(n, \mathbb{C}).$$

This implies (dimension count!) $n = 7, 8$.

And indeed: for $n = 7$: $G_\omega = G_2$, for $n = 8$: $G_\omega = \mathrm{SL}(3, \mathbb{C})$.

In fact, Engel had had this idea already in 1886. From a letter to Killing (8.4.1886):

"There seem to be relatively few simple groups. Thus first of all, the two types mentioned by you [$\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C}]$. If I am not mistaken, the group of a linear complex in space of $2n - 1$ dimensions $(n > 1)$ with $(2n+1)2n/2$ parameters [$\mathrm{Sp}(n, \mathbb{C})$] is distinct from these. In 3-fold space [$\mathbb{CP}^3$] this group [$\mathrm{Sp}(4, \mathbb{C})$] is isomorphic to that [$\mathrm{SO}(5, \mathbb{C})$] of a surface of second degree in 4-fold space. I do not know whether a similar proposition holds in 5-fold space. The projective group of 4-fold space [$\mathbb{CP}^4$] that leaves invariant a trilinear expression of the form

$$\sum_{ijk}^{1\ldots5} a_{ijk} \begin{vmatrix} x_i & y_i & z_i \\ x_k & y_k & z_k \\ x_j & y_j & z_j \end{vmatrix} = 0$$

will probably also be simple. This group has 15 parameters, the corresponding group in 5-fold space has 16, in 6-fold space [$\mathbb{CP}^6$] has 14, in 7-fold space [$\mathbb{CP}^7$] has 8 parameters. In 8-fold space there is no such group. These numbers are already interesting. Are the corresponding groups simple? Probably this is worth investigating. But also Lie, who long ago thought about similar things, has not yet done so."

Thm (Engel, 1900). A generic complex 3-form has precisely one $\mathrm{GL}(7,\mathbb{C})$ orbit. One such 3-form is

$$\omega_0 := (e_1e_4 + e_2e_5 + e_3e_6)e_7 - 2e_1e_2e_3 + 2e_4e_5e_6.$$

Every generic complex 3-form $\omega \in \Lambda^3(\mathbb{C}^7)^*$ satisfies:

1) The isotropy group G_ω is isomorphic to the simple group G_2;

2) ω defines a non degenerate symmetric BLF β_ω, which is cubic in the coefficients of ω and the quadric M_1^5 is its isotropic cone in $\mathbb{CP}^6$. In particular, G_ω is contained in some $\mathrm{SO}(7,\mathbb{C})$.

3) There exists a G_2-invariant polynomial $\lambda_\omega \neq 0$, which is of degree 7 in the coefficients of ω.

„Zudem ist hiermit eine direkte Definition unsrer vierzehngliedrigen einfachen Gruppe gegeben, die an Eleganz nichts zu wünschen übrig lässt.“
F. Engel, 1900

In modern notation: Set $V = \mathbb{C}^7$. Then

$$\beta_\omega : V \times V \to \Lambda^7 V^*, \quad \beta_\omega(X,Y) := (X \lrcorner \omega) \wedge (Y \lrcorner \omega) \wedge \omega$$

is a symmetric BLF with values in the 1-dim. space $\Lambda^7(\mathbb{C}^7)^*$ [R. Bryant, 1987]

Hence β_ω defines a map $K_\omega : V \to V^* \otimes \Lambda^7 V^*$, and

$$\det K_\omega \in (\Lambda^7 V)^* \otimes \Lambda^7 (V^* \otimes \Lambda^7 V^*) = \Lambda^9 (\Lambda^7 V^*).$$

Assume V is oriented $\Rightarrow$ fix an element $(\det K_\omega)^{1/9} \in \Lambda^7 V^*$ and set

$g_\omega := \frac{\beta_\omega}{(\det K_\omega)^{1/9}}$: this is a true scalar product, and $g_\omega = g_{-\omega}$.

$\det g_\omega := \lambda_\omega^3$ for an element of 'order' 7 in ω

$$\boxed{\lambda_\omega \neq 0 \Leftrightarrow \omega \text{ is generic} \Leftrightarrow g_\omega \text{ is nondegenerate}}$$

This allows a more concise description of the 2nd homogeneous space G_2/P_2:

Consider

$G^7_0(2,7) = \{\pi^2 \subset \mathbb{C}^7 \,:\, \beta_\omega\big|_{\pi^2} = 0\} \subset G^{10}(2,7) \subset \mathbb{P}(\Lambda^2\mathbb{C}^7)$ (Plücker emb.)

Then $G_2/P_2 = \{\pi^2 \subset G^7_0(2,7) \,:\, \pi^2 \lrcorner\, \omega = 0\}$

On the other hand, we know that

$G_2/P_2 = \overline{G \cdot [v_{\omega_2}]} \subset \mathbb{P}(\mathfrak{g}^2) \subset \mathbb{P}(\Lambda^2 V)$ (because $\Lambda^2 V = \mathfrak{g}_2 \oplus V$)

$\longrightarrow$ turns out: $G_2/P_2 = G^{10}(2,7) \cap \mathbb{P}(\mathfrak{g}^2)$ inside $\mathbb{P}(\Lambda^2 V)$

[Landsberg-Manivel, 2002/04]

Facts:

- G_2/P_2 has degree 18

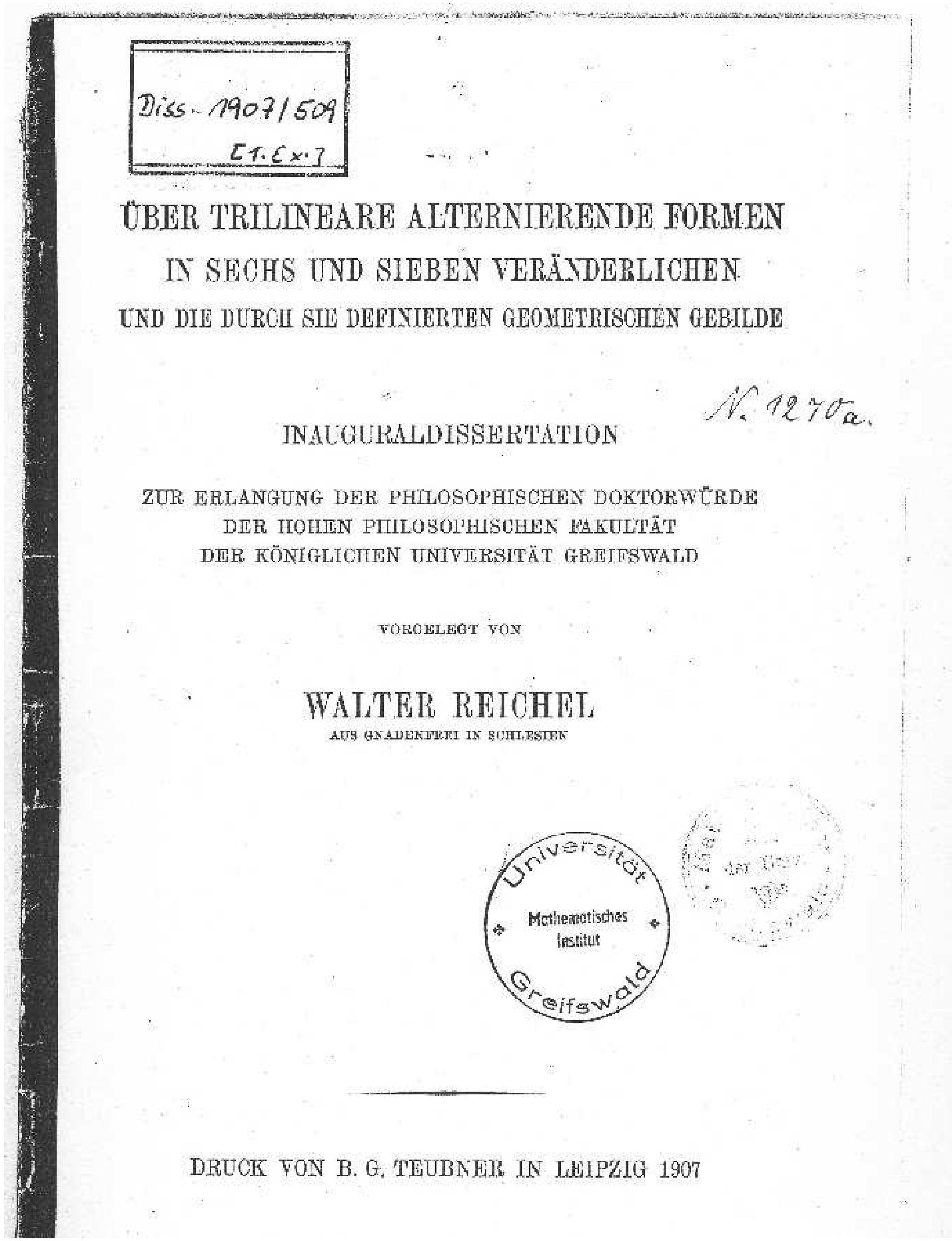
- a smooth complete intersection of G_2/P_2 with 3 hyperplanes is a K3 surface of genus 10. [Borcea, Mukai]

Walter Reichel's thesis (Greifswald, 1907)

- complete system of invariants for complex 3-forms in 6 und 7 variables through Study's symbolic method
- normal forms for 3-forms under $\mathrm{GL}(6,\mathbb{C}), \mathrm{GL}(7,\mathbb{C})$.

$n = 7$: vanishing of λ_ω for non generic 3-forms and rank of β_ω play a decisice role

- Lie-Algebra $\mathfrak{g}_\omega$ for any 3-form ω expressed in terms of its coefficients



ÜBER TRILINEARE ALTERNIERENDE FORMEN

IN SECHS UND SIEBEN VERÄNDERLICHEN

UND DIE DURCH SIE DEFINIERTEN GEOMETRISCHEN GEBILDE

INAUGURALDISSERTATION

ZUR ERLANGUNG DER PHILOSOPHISCHEN DOKTORWÜRDE
DER HOHEN PHILOSOPHISCHEN FAKULTÄT
DER KÖNIGLICHEN UNIVERSITÄT GREIFSWALD

VORGELEGT VON

WALTER REICHEL
AUS GNADENFREI IN SCHLESIEN

DRUCK VON B. G. TEUBNER IN LEIPZIG 1907

Over $\mathbb{R}$, there are two $\mathrm{GL}(7, \mathbb{R})$ orbits of generic 3-forms!

$\Rightarrow$ Reichel's formulas allow to compute the isotropy Lie group on both orbits, and indeed:

- one isotropy group is G_2^*, and the scalar product β_ω has signature $(4, 3)$
- the other isotropy group is G_2^c, and the scalar product β_ω is positive definite.

Hence, Walter Reichel's thesis establishes for the first time a geometric realisation of G_2^c – in fact, the one which explains its importance in modern geometry (and maybe physics).

– more on Walter Reichel to follow later.

First, let us discuss the scientific impact of Reichel's work!

Authors who cite Walter Reichel:

- 1931, Schouten: normal forms of 3-forms on $\mathbb{C}^7$ without invariant theory
- 1935, Gurevich: normal forms of 3-forms on $\mathbb{C}^8$

  . . .

- 1978, Elashvili & Vinberg: normal forms of 3-forms on $\mathbb{C}^9$
- Since 1987: More than 100 citations

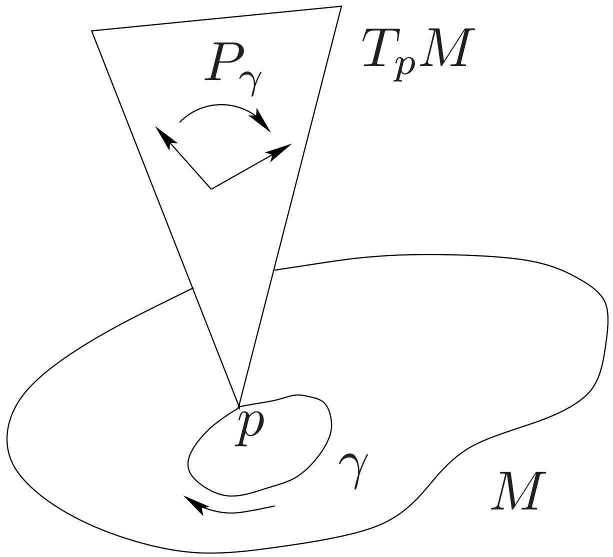
G_2^c and the octonians:

- 1908 and 1914, É. Cartan: observes that $G_2^c \cong \mathrm{Aut}\mathbb{O}$
- this approach becomes popular by the work of Hans Freudenthal (after 1951)

In fact, the 3-form approach and the the octonian picture are equivalent
(a third equivalent description is through 'vector cross products')

[see J. Baez, 2002, for a modern account]

Holonomy group of a connection ∇



- γ: closed path through $p \in M$,
  $P_\gamma : T_pM \to T_pM$ parallel transport
- P_γ isometry $\Leftrightarrow$: ∇ metric
- $C_0(p)$: null-homotopic γ's

  $$\mathrm{Hol}_0(M;\nabla) := \{P_\gamma \mid \gamma \in C_0(p)\} \subset \mathrm{SO}(n)$$

Thm (Berger [& Simons], ≥ 1955). The reduced holonomy $\mathrm{Hol}_0(M;\nabla^g)$ of the LC connection ∇^g is either that of a symmetric space or

$\mathrm{Sp}(n)\mathrm{Sp}(1)$ [qK], $\mathrm{U}(n)$ [K], $\mathrm{SU}(n)$ [CY], $\mathrm{Sp}(n)$ [hK], G_2^c, $\mathrm{Spin}(7)$, [$\mathrm{Spin}(9)$]

— will henceforth be called ***'integrable or parallel geometries'***.

These are the possible holonomy groups: for some classes ($\mathrm{SU}(n), G_2^c, \mathrm{Spin}(7)\ldots$), no examples were known!

However, Berger missed that [Bonan, 1966]

- manifolds with holonomy G_2^c have a ∇^g-parallel 3-form,
- manifolds with holonomy $\mathrm{Spin}(7)$ have a ∇^g-parallel 4-form,
- and, in consequence, both have to be Ricci-flat.

Weak holonomy (A. Gray, 1971):

Idea: Enlarge the successful holonomy concept to wider classes of manifolds (contact manifolds, almost Hermitian manifolds etc.)

Dfn. ***'nearly parallel G_2^c-manifold':*** has structure group G_2^c, but 3-form ω is not parallel, but rather satisfies

$$d\omega = \lambda * \omega \text{ for some } \lambda \neq 0.$$

Fernandez-Gray, 1982: Show that there are 4 basic classes of manifolds with G_2^c-structure and construct first examples:

$S^7 = \mathrm{Spin}(7)/G_2^c$, $\mathrm{SU}(3)/S^1$ (Aloff-Wallach spaces), extensions of Heisenberg groups. . .

Progress in the parallel G_2^c case:

• 1987-89, R. Bryant and S. Salamon: local complete metrics with Riemannian holonomy G_2^c

• 1996, D. Joyce: existence of compact Riemannian 7-dimensional manifolds with Riemannian holonomy G_2^c

Today's general philosophy:

Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection ∇ with torsion that preserves the geometric structure!*

$$\text{torsion: } T(X,Y,Z) \ := \ g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) \ = \ g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

- representation theory yields

- a clear answer *which* G-structures admit such a connection; if existent, it's unique and called the *'characteristic connection'*

- a *classification scheme* for G-structures with characteristic connection: $T_x \in \Lambda^3(T_x M) \stackrel{G}{=} V_1 \oplus \ldots \oplus V_p$

- study Dirac operator $\not{D}$ of the metric connection with torsion $T/3$: *'characteristic Dirac operator'* (generalizes the Dolbeault operator, Kostant's cubic Dirac operator) [IA, since 2003]

7-dimensional G_2-manifold [Friedrich-Ivanov, 2002]

$\exists$ char. connection $\nabla \Leftrightarrow \exists$ VF β s. t. $\delta\omega = -\beta \lrcorner \omega$, torsion:

$$T = -*\, d\omega - \tfrac{1}{6}(d\omega, *\omega)\omega + *(\beta \wedge \omega)$$

- $\nabla\omega^3 = 0$, at least on spinor field with $\boxed{\nabla\psi = 0 \text{ and } \mathrm{Hol}_0(\nabla) \subset G_2 \subset \mathrm{SO}(7)}$

This last property comes not as a surprise:

Alternative description: $G_2 = \{A \in \mathrm{Spin}(7) \mid A\psi = \psi\}$

$\Rightarrow$ explains physicists' interest in G_2^c

- more recently: **superstring theory:**

torsion $\cong$ field, ∇-parallel spinor $\cong$ supersymmetry transformation.

duality: $T = 0$: 'vacuum solutions' of superstring theory $\longrightarrow$ algebraic geometry (K3 surfaces, Calabi-Yau manifolds. . .)

$T \neq 0$: 'non vacuum solutions' of superstring theory $\longrightarrow$ differential geometry, connections with torsion

Thm. Let M be a *compact*, Ricci-flat manifold from Berger's list, $\psi \neq 0$ a ∇-parallel spinor for some $T \in \Lambda^3(M)$ s. t. $\langle dT \cdot \psi, \psi \rangle \leq 0$. Then $T = 0$, i. e. *only* ∇^g can have parallel spinors.

– Physics interpretation: vacuum solutions are 'rigid' –

Different situation if M^n is not compact:

Consider solvmanifolds $Y^7 = N \times \mathbb{R}$, $\mathfrak{n}$: nilpotent 6-dim. Lie algebra ($\neq \mathfrak{h}_3 \oplus \mathfrak{h}_3$) $\Rightarrow$

1) N carries "half flat" $\mathrm{SU}(3)$ structure,

2) Y carries a G_2 structure (ω, g) with characteristic torsion $\neq 0$,

3) Y carries – after a conformal change of the metric – an *integrable* G_2 structure $(\tilde{\omega}, \tilde{g})$. In particular, $\tilde{g}$ is Ricci flat und admits (at least) one LC-parallel spinor.

[Gibbons, Lü, Pope, Stelle (2002): described such a metric in local coordinates]

[Heber (1998): noncompact Einstein manifolds]

[Chiossi, Fino (2004): classification of all such solvmnfds (6 cases)]

[Hitchin (2001): existence of conformal change 3)]

Thm. For $\mathfrak{n} \cong (0, 0, e_{15}, e_{25}, 0, e_{12})$, there exists on $(Y, \tilde{\omega}, \tilde{g})$ a 1-parametric family $(T_h, \psi_h) \in \Lambda^3(Y) \times S(Y)$ s. t. every connection ∇^h with torsion T_h satisfies:

$$\nabla^h \psi_h \;=\; 0.$$

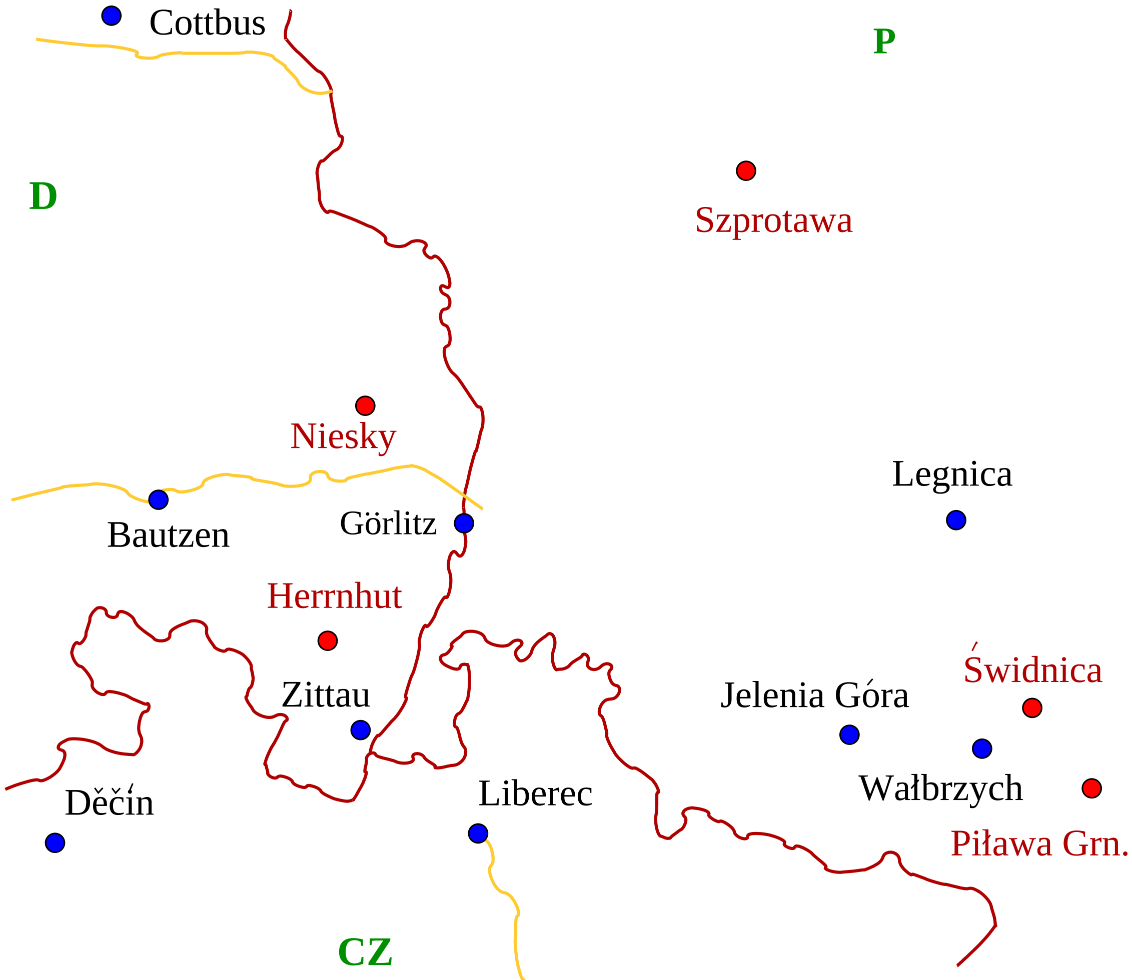
For $h = 1$: $T_h = 0, \nabla^h = \nabla^g$ und ψ_h coincides with the LC-parallel spinor.

[Agricola, Chiossi, Fino, 2006]

Only example of a Riemannian mnfd carrying a Ricci-flat integrable and a non-integrable geometry!

Walter Reichel (1883-1918)

- born 3.11.1883 in Gnadenfrei (Silesia, now Piława Górna/PL) as son of the deacon of the *Moravian Church*
- primary school at home, 4 years at the 'Pädagogium' in Niesky, then 3 years at the Gymnasium in Schweidnitz (now Świdnica/PL)
- 1902–1906: studies mathematics, physics, and philosophy in Greifswald, Leipzig, Halle, and again Greifswald



Moravian Church (Unitas Fratrum): emerged in the 15th ct. from the Bohemian Reformation Movement around Jan Hus (1369-1415), and was renewed in the early 18th Century in Herrnhut, where the management of its European branch and its Archive are still hosted today.

He listened to lectures by

- Friedrich Engel and Theodor Vahlen (in Greifswald)

- Carl Neumann (in Leipzig), who gave its name to the Neumann boundary condition and founded the *Mathematischen Annalen* together with Alfred Clebsch

- Georg Cantor und Felix Bernstein (in Halle), to whom we owe the foundations of logic / set theory and the Cantor-Bernstein-Schröder Theorem in logic

- the theoretical physicist Gustav Mie (in Greifswald), who made important contributions to electromagnetism and general relativity

- the experimental physicist Friedrich Ernst Dorn (in Halle), who discovered the gas Radon in 1900

In addition: philosophy, chemistry, zoology and art history.

July 1907: passed teacher's examination 'with distinction' in pure and applied mathematics, physics and philosophical propaedeutics".

- teacher in training at the Reformrealgymnasium in Görlitz

- Summer 1908: teacher at Realprogymnasium zu Sprottau (now Szprotawa/PL)

- April 1914: teacher at Oberrealschule i. E. Schweidnitz (now Świdnica/PL)

- marries 1909 his wife Gertrud, born Müller (1889-1956)

- publishes two articles on high school mathematics

November 1914

„Dem lieben Krieger, der im fernen Feindesland uns Haus und Herd beschirmen hilft herzliche Weihnachtsgrüße! – Schweidnitz, 24.11.14“

With the beginning of the First World War, he was drafted (high school files show that teachers were drafted without exceptions).

Walter Reichel died in France on March 30, 1918.

Children: three sons (born 1910, 1913, 1916) und a daughter Irmtraut (born 11.3.1918), married Schiller. Irmtraut Schiller lives in Bremen and has three children.

After the first World War, the Reichel widow moved with her children to Niesky, where she was supported by the Moravian Church. For many years, she accomodated pupils of the 'Pädagogium' who did not live in the boarding school's dormitories.

The Old Pädagogium in Niesky

The Old Pädagogium in Niesky (now public library), built in 1741 as first parish house of the newly founded community in Niesky. Since 1760, it was used as a advanced boarding school.

In the 19th ct., the building became to small, and a New Pädagogium was built nearby. It was completely destroyed during WW II.

The 'God's acre' in Niesky

Left: men, right: women.

rechts: Johann Raschke / geb. d. 14. März 1702 / in Lichtenau i. Böhmen / heimgegangen / d. 4. August 1762 / Er leitete als Vorsteher / der Gemeine den Anbau / von Niesky

The memorial stone on the ‘God’s acre’

Detail of the inscription on the memorial stone

Name und date of death of Walter Reichel are in the 2nd row from below; the stone is damaged and repaired just above his name.

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Many thanks go to: Irmtraut Schiller & family,

Archiv der Universität Greifswald,

Unitätsarchiv der Brüdergemeine in Herrnhut,

Forschungsbibliothek Gotha der Universität Erfurt,

Antiquariat Wilfried Melchior in Spreewaldheide.