Colouring random geometric graphs

Tobias Müller

March 2007

We construct a random geometric graph G_n by picking n vertices $X_1, \dots, X_n \in [0,1]^d$ uniformly at random and adding an edge $X_iX_j \in E(G_n)$ if $\|X_1 - X_2\| < r$, where $r > 0$ is a predetermined parameter.

A k-colouring of a graph G is a map $c : V(G) \to \{1, \dots, k\}$ such that $c(v) \neq c(w)$ whenever $vw \in E(G)$, and the chromatic number $\chi(G)$ of G is the least k for which G admits a k-colouring.

We will consider the behaviour of the chromatic number $\chi(G_n)$ as n tends to infinity where $r = r(n)$ is allowed to vary with n. Earlier work by McDiarmid (on the two-dimensional case) and Penrose (general dimension) showed there is a dramatic difference in the behaviour of $\chi(G_n)$ between the case when $nr^d \ll \ln n$ and the case when $nr^d \gg \ln n$ (it is natural to describe the various cases in terms of the quantity nr^d, as the expected number of neighbours of a vertex is proportional to nr^d). Neither author considered the chromatic number in the "phase change" when $nr^d = \Theta(\ln n)$ and both posed the behaviour in this range as an open problem. In this talk we will determine constants $c(t)$ such that

$$\lim_{n\to\infty} \frac{\chi(G_n)}{nr^d} = c(t) \text{ a.s.},$$

if $nr^d \sim t \ln n$.

A clique in a graph G is a subset $C \subseteq V(G)$ of the vertex-set with the property that $vw \in E(G)$ for all $v \neq w \in C$; and the clique number, denoted by $\omega(G)$ is the largest cardinality of a clique in G. Note that $\chi(G) \geq \omega(G)$ for all G.

(If time permits) we will see that there is a "sharp threshold" r_0 of the form $r_0(n) = (\frac{t_0 \ln n}{n})^{\frac{1}{d}}$ for some constant $t_0 > 0$ such that

$$\frac{\chi(G_n)}{\omega(G_n)} \to 1 \text{ a.s.},$$

if $r \leq r_0$ and

$$\liminf_{n\to\infty} \frac{\chi(G_n)}{\omega(G_n)} > 1 \text{ a.s.},$$

if $r > (1+\varepsilon)r_0$ for some $\varepsilon > 0$.

The results generalise to the case of an arbitrary probability distribution with a bounded density function and an arbitrary norm on $\mathbb{R}^d$.

(This is joint work with Colin McDiarmid)