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More on equilibria in competitive markets with externalities and a continuum of agents

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Abstract

Equilibrium existence results are presented for competitive markets with externalities in price and consumption and a measure space of agents. These unify and extend [Balder, E.J., 2003. Existence of competitive equilibria in economies with a measure space of consumers and consumption externalities. Preprint, in press, electronically available at http://www.math.uu.nl/publications/preprints/1294.ps.gz; Balder, E.J., 2005. More about equilibrium distributions for competitive markets with externalities. Working paper, Department of Economics, University of Illinois.] and generalize the main existence results by Aumann [Aumann, R.J., 1964. Markets with a continuum of traders. Econometrica 32, 39–50], Schmeidler [Schmeidler, D., 1969. Competitive equilibria in markets with a continuum of traders and incomplete preferences. Econometrica 37, 578–585; Schmeidler, D., 1973. Equilibrium points of nonatomic games. Journal of Statististical Physics 7, 295–300.], Greenberg et al. [Greenberg, J., Shitovitz, B., Wieczorek, A., 1979. Existence of equilibria in atomless production economies with price dependent preferences. Journal of Mathematical Economics 6, 31–41.], Yamazaki [Yamazaki, A., 1978. An equilibrium existence theorem without convexity assumptions. Econometrica 46, 541–555.], Noguchi [Noguchi, M., 2005. Interdependent preferences with a continuum of agents. Working paper, Meijo University, 2001. Journal of Mathematical Economics 41, 665–686.], Cornet [Cornet, B., Topuzu, M., 2005. Existence of equilibria for economies with externalities and a measure space of consumers. Economic Theory 26, 397–421.], and Noguchi and Zame [Noguchi, M. and Zame, W.R., 2004. Equilibrium distributions with externalities. Preprint.].

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1. Introduction

Aumann's model of a pure exchange economy with a continuum of agents plays an important role in economic theory (Aumann, 1964, 1966). Extensive discussions of it can be found in Ellickson (1993) and Hildenbrand (1974). In Aumann (1964), Aumann proved a very elegant core equivalence theorem and in Aumann (1966) he presented the first competitive (i.e., Walrasian) equilibrium existence result for such a model.

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Compared to the general equilibrium literature for economies with a finite number of players (e.g., see Ellickson, 1993, 7.3), a remarkable shortcoming of models such as found in Aumann (1964, 1966); Ellickson (1993); Hildenbrand (1974), and Schmeidler (1969) and the related literature, is that they lack price and consumption externalities. This is all the more remarkable since continuum game theory, the strand in the game theory literature that was generated by Aumann's work, does exhibit externalities, as any description of game-like situations would be obliged to do. The absence of price externalities in Aumann's original model was remedied by Greenberg, Shitovitz and Wieczorek in Greenberg et al. (1979). Unlike Aumann (1966), their main existence result is for free disposal competitive equilibria and compact consumption sets. In Balder (1999a) the present author extended the main existence result of Greenberg et al. (1979) so as to apply to both price and consumption externalities. At the same time, Balder (1999a) demonstrated that such existence results could also be applied to the well-known model of Schmeidler for games with a continuum of players (Khan, 1984; Schmeidler, 1973). In some respects Balder (1999a) is very general. For instance, both prices and consumptions can be infinite-dimensional, which is quite in line with the continuum game literature that followed Schmeidler (1973) (e.g., see Khan, 1984). However, just as in Greenberg et al. (1979), the main existence result of Balder (1999a) requires the feasible consumption sets to be compact. Thus, it cannot be applied directly to Aumann's model, even though it was also shown in Balder (1999a) and Greenberg et al. (1979) that this can be done indirectly, namely by substantially simplifying the model and using a well-known truncation procedure. But in this simplification and the associated purification step the consumption externality is lost. For this reason, two new competitive equilibrium existence results, applicable to an Aumann-type model that has both price and consumption externalities, were presented in Balder (2003; Theorems 2.2 and 2.3). They apply directly to a situation where feasible consumption sets are finite-dimensional, but need not be compact. One of these holds under additional strong monotonicity conditions for the preferences; it does not need free disposal but follows from the other result, which does allow for free disposal. As in Balder (1995a, 1999a), the approach in Balder (2003) depends on the use of *relaxation* in the sense of Young. Here the essential idea is to take the original economy $\mathcal{E}$ as the starting point for the formulation of a more general *relaxed economy* $\mathcal{RE}$. In the latter consumers are allowed to have their consumption choices to be mixed, i.e., randomized. As explained in Remark 3.8, by using the *ws*-topology a completely equivalent description of such relaxation can be given in terms of Walrasian equilibrium distributions over pairs of consumers' characteristics and consumption choices in the sense of Hildenbrand (1974) (however, Hildenbrand, 1974 uses a less general topology for the distributions, which does not take advantage of the fact that all marginal distributions of agents' characteristics must be the same). So, by itself, relaxation is not a new idea in economics. However, what gives the approach and results in (Balder, 2003; in particular its Theorem 3.1) their generality and power is the fact that we exploit a very suitable body of knowledge, often referred to as *Young measure theory*, that was developed over the past decades. We briefly review it in Section 2. By using this theory, the existence of a competitive equilibrium for $\mathcal{RE}$ turns out to be a much easier – and mathematically more fundamental – question to address than the original existence question for $\mathcal{E}$. In this paper such competitive equilibria for $\mathcal{RE}$ will be called *relaxed competitive equilibria*. Subsequently, by using additional purification, relaxed competitive equilibria for $\mathcal{RE}$ can be turned into ordinary competitive equilibria for the original economy $\mathcal{E}$. In such a way, the previously mentioned existence Theorems 2.2 and 2.3 in Balder (2003) were obtained from its Theorem 3.1. In an independent development, Cornet and Topuzu (2003, 2005), and Topuzu (2004) gave existence results, based on similar monotonicity conditions as used in Theorem 2.2 of Balder (2003). They did so for possibly unordered preference relations (in Balder, 2003 only ordered preference relations, i.e., utility functions, were used—however, see Remark 3.5 below). In turn, they were inspired by work of Noguchi (2005), whose main existence result they extended (see also Haller, 1993 for antecedents). The results in Cornet and Topuzu (2003) and Topuzu (2004) did not include Aumann's original result, because they require additional convexity. However, by means of convexification arguments and an abstract purification condition (i.e., Cornet and Topuzu, 2005, [Assumption EC], which is analyzed in Section 5.2 below), Cornet and Topuzu obtained in Cornet and Topuzu (2005, Theorem 2) an existence result that includes the existence results of Aumann (1966) and Schmeidler (1969).

In the meantime, it had been demonstrated in Balder et al. (2004) that the relaxation approach of Balder (2003) could also be applied to the models of Cornet and Topuzu (2003) and Topuzu (2004) to develop a second version of the above-mentioned relaxed existence result in Balder (2003, Theorem 3.1(ii)), which is already geared towards using the monotonicity assumptions to begin with, and by replacing non-atomicity purification, used in Balder (2003), with convexity purification. The connections thus shown in Balder et al. (2004) were imperfect, because Cornet and Topuzu (2003, 2005), and Topuzu (2004) use unordered preference relations, whereas Balder (2003) and Balder et al. (2004) only consider ordered preference relations, i.e., preference relations that are generated by utility functions. Subsequently, the working paper Balder (2005) demonstrated that the approach to existence problems under monotonicity assumptions, as taken in Balder (2003); Balder et al. (2004), and Cornet and Topuzu (2005), could easily be extended to general preference relations; this was done in order to demonstrate that the main equilibrium distribution existence result of Noguchi and Zame (2004) essentially follows from (and is generalized by) the principal relaxed existence results in Balder (2003) and Balder et al. (2004).

The purpose of the present paper is to place the approach and previously mentioned results on a higher platform of generality, which unifies them and in particular includes the aforementioned existence results in Balder (2003, Theorems 2.1 and 3.1(i)) that do not need monotonicity. It turns out that this can be done at a level high enough to yield, as specializations, the main existence results by Aumann (1964); Schmeidler (1969, 1973); Greenberg et al. (1979); Yamazaki (1978); Noguchi (2005); Cornet and Topuzu (2005) and Noguchi and Zame (2004). Thus, our program is as follows. In Section 3, after introducing ingredients that will also play a role in the ordinary economy $\mathcal{E}$, we formulate the relaxed economy $\mathcal{RE}$ and two principal existence results for that economy (Theorems 3.1 and 3.2). This order of presentation emphasizes the central place that the existence of relaxed competitive equilibria deserves by the great generality and versatility of the subject. This goes considerably beyond subsequent specializations to the ordinary economy $\mathcal{E}$.[1] Even though our presentation is at a rather high level of generality, the important Example 3.1 below, as well as Proposition 3.1, should give a clear sense of orientation. In Section 4 the two main relaxed existence results are transformed, via the key Proposition 4.1 about purification, into two existence results for ordinary competitive equilibria, namely Theorems 4.1 and 4.2. An independent and apparently new development, in the form of Corollaries 4.1 and 4.2, complements this. It requires no purification. In Section 5 the question is addressed under which conditions the purification Proposition 4.1 is valid. Next to giving new existence results, this section demonstrates that the main existence results in Aumann (1966); Balder (2003, 2005); Balder et al. (2004); Cornet and Topuzu (2003, 2005); Greenberg et al. (1979); Noguchi and Zame (2004), and Schmeidler (1969, 1973) all follow from Theorems 4.1 and 4.2.

2. Mathematical preliminaries

In this section we recapitulate, in simplified form (namely by using $\mathbb{R}^l$ instead of a metrizable Suslin space), some important results for the narrow topology for transition probabilities (alias Young measures). For more material on Young measure theory the reader is referred to Balder (1988, 2000a, 2000b, 2001); general background material about measure theory, topology and multifunctions can be found in Billingsley (1968); Castaing and Valadier (1977); Choquet (1969), and Neveu (1965).

Let $(T, \mathcal{T}, \mu)$ be a finite measure space and let l be a given dimension. Vectors in $\mathbb{R}^l$ will be denoted as $x = (x^1, \dots, x^l)$ and their Euclidean norm as $|x|$. For any x and $\bar{x}$ in $\mathbb{R}^l$ we shall write $\bar{x} \geq x$ if $\bar{x}^i \geq x^i$ for $i = 1, \dots, l$. As usual, $\mathbb{R}^l_+$ will denote the nonnegative orthant in $\mathbb{R}^l$ and $\mathbb{R}^l_{++} \subset \mathbb{R}^l_+$ the strictly positive orthant. By $\mathrm{Prob}(\mathbb{R}^l)$ we denote the set of all probability measures on $\mathbb{R}^l$, where $\mathbb{R}^l$ is equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{R}^l)$. Recall that the *s* upport of a probability measure $\nu \in \mathrm{Prob}(\mathbb{R}^l)$ is defined as the smallest closed set $F \subset \mathbb{R}^l$ such that $\nu(F) = 1$; this set is denoted by $\mathrm{supp}\,\nu$. For $a \in \mathbb{R}^l$ the *Dirac probability* (alias *point probability*) entirely concentrated at a is denoted by ϵ_a. In this setup we recall the following key notions and results:

(i) The classical *weak topology* on $\mathrm{Prob}(\mathbb{R}^l)$ is the coarsest topology on $\mathrm{Prob}(\mathbb{R}^l)$ for which the mapping $\nu \mapsto \int_{\mathbb{R}^l} c \, \mathrm{d}\nu$ is continuous for every bounded and continuous function $c : \mathbb{R}^l \to \mathbb{R}$; cf. (Billingsley, 1968).

(ii) A *transition probability* from $(T, \mathcal{T})$ into $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$ is a function $\delta : T \to \mathrm{Prob}(\mathbb{R}^l)$ such that $t \mapsto \delta(t)(B)$ is $\mathcal{T}$-measurable for every $B \in \mathcal{B}(\mathbb{R}^l)$ (Neveu, 1965, III). The set of all such transition probabilities is denoted by $\mathcal{R}(T; \mathbb{R}^l)$. Incidentally, by Bertsekas and Shreve (1978, Proposition 7.25) one has for any $\delta : T \to \mathrm{Prob}(\mathbb{R}^l)$ that $\delta \in \mathcal{R}(T; \mathbb{R}^l)$ if and only if δ is measurable with respect to $\mathcal{T}$ and the Borel σ-algebra on $\mathrm{Prob}(\mathbb{R}^l)$ that corresponds to the above-mentioned weak topology. A transition probability $\delta \in \mathcal{R}(T; \mathbb{R}^l)$ is said to be *D* irac if there exists a function $f : T \to \mathbb{R}^l$ such that $\delta(t) = \epsilon_{f(t)}$ for every $t \in T$ (this function f is then automatically measurable, because of the above measurability hypothesis for δ); in this case we denote δ as ϵ_f.

[1] A different order of presentation in Balder (2003) appears to have created serious confusion in Noguchi and Zame (2004, 2006) about the generality of results such as Balder (2003, Theorem 3.1, Remark 3.2).

(iii) A *normal integrand* on $T \times \mathbb{R}^l$ is a $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable function $g : T \times \mathbb{R}^l \to \mathbb{R}$ such that $g(t, \cdot)$ is lower semi-continuous on $\mathbb{R}^l$ for every $t \in T$. The function g is said to be *integrably bounded from below* if there exists an integrable $\phi : T \to \mathbb{R}$ such that $g(t, x) \geq \phi(t)$ for all $(t, x) \in T \times \mathbb{R}^l$. The set of all normal integrands on $T \times \mathbb{R}^l$ that are integrably bounded from below is denoted by $\mathcal{G}^{\mathrm{bb}}(T; \mathbb{R}^l)$. Let $\mathcal{H}^{\mathrm{bb}}(T; \mathbb{R}^l)$ be the set of all $h \in \mathcal{G}^{\mathrm{bb}}(T; \mathbb{R}^l)$ for which $h(t, \cdot)$ is inf-compact on $\mathbb{R}^l$ (i.e., the set $\{x \in \mathbb{R}^l : h(t, x) \leq \beta\}$ is compact for every $\beta \in \mathbb{R}$) for every $t \in T$.

(iv) By the theory involving Fubini's theorem (Neveu, 1965, III.2) one has that for every $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable function $g : T \times \mathbb{R}^l \to (-\infty, +\infty]$ that is integrably bounded from below the integral:

$$I_g(\delta) := \int_T \left[\int_{\mathbb{R}^l} g(t, x)\delta(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t) \tag{2.1}$$

is well-defined in $(-\infty, +\infty]$; in particular, (Neveu, 1965, III.2) shows that $t \mapsto \int_{\mathbb{R}^l} g(t, x)\delta(t)(\mathrm{d}x)$ is a $\mathcal{T}$-measurable function. Moreover, every $\delta \in \mathcal{R}(T; \mathbb{R}^l)$ induces a finite product measure on $(T \times \mathbb{R}^l, \mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l))$, which we denote by $\mu \otimes \delta$ and for which we have $I_g(\delta) = \int_{T \times \mathbb{R}^l} g d(\mu \otimes \delta)$ for all such g.

(v) The *narrow topology* on $\mathcal{R}(T, \mathbb{R}^l)$ is the coarsest topology for which all mappings $\delta \mapsto \int_A [\int_{\mathbb{R}^l} c(x)\delta(t)(\mathrm{d}x)]\mu(\mathrm{d}t)$ are continuous for every $A \in \mathcal{T}$ and every bounded continuous function $c : \mathbb{R}^l \to \mathbb{R}$. This generalizes the usual weak topology on $\mathrm{Prob}(\mathbb{R}^l)$, discussed in (i) above, as is seen by taking T to be a singleton. Several useful and equivalent characterizations of the narrow topology are available (Balder, 1988, 2000a, 2000b) (see Balder, 2001 for the corresponding equivalences for the ws-topology, discussed in (xi) below). For instance, the narrow topology on $\mathcal{R}(T, \mathbb{R}^l)$ can equivalently be defined as the coarsest topology for which all functionals $\delta \mapsto I_g(\delta)$, $g \in \mathcal{G}^{\mathrm{bb}}(T; \mathbb{R}^l)$, are lower semi-continuous. Another equivalence of this kind is as follows Balder (2003, Theorem 4.13). Denote by $\hat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ the usual Alexandrov compactification of the set of natural numbers (this is a compact metric space). Then $\{\delta_k\}_k := \{\delta_k\}_{k \in \mathbb{N}}$ converges narrowly to δ_0 in $\mathcal{R}(T; \mathbb{R}^l)$ if and only if

$$\liminf_k \int_T \left[\int_{\mathbb{R}^l} \tilde{g}(t, k, x)\delta_k(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t) \geq \int_T \left[\int_{\mathbb{R}^l} \tilde{g}(t, \infty, x)\delta_0(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t) \tag{2.2}$$

for every $\tilde{g} \in \mathcal{G}^{\mathrm{bb}}(T; \hat{\mathbb{N}} \times \mathbb{R}^l)$.

(vi) The existence of a semi-metric ρ for the narrow topology on $\mathcal{R}(T; \mathbb{R}^l)$ allows us to concentrate on sequential topological arguments; such a semi-metric exists when the measure space $(T, \mathcal{T}, \mu)$ is separable (Balder, 2000b, Theorem 4.6). Moreover, in that situation $\mathcal{R}(T; \mathbb{R}^l)$ is also separable for the narrow topology. The latter fact follows for instance by application of Theorems 4.6 and 4.10 of Balder (2000b) to the Alexandrov compactification of $\mathbb{R}^l$ (which is a Suslin space) and by the fact that compact semi-metrizable spaces are separable.

(vii) An important property of sequential narrow convergence in $\mathcal{R}(T; \mathbb{R}^l)$, in terms of the Kuratowski limes superior of the pointwise support sets, is as follows Balder (2000b, Theorem 4.12). For every sequence $\{\delta_k\}_k$ that converges narrowly to δ_0 in $\mathcal{R}(T; \mathbb{R}^l)$ the inclusion:

$$\operatorname{supp} \delta_0(t) \subset \mathrm{Ls}_k \operatorname{supp} \delta_k(t) \quad \text{for a.e. } t \text{ in } T$$

holds. Here the *Kuratowski limes superior* of a sequence $\{E_k\}_k$ of subsets of $\mathbb{R}^l$ is the set $\mathrm{Ls}_k\, E_k$ of all $x \in \mathbb{R}^l$ for which there exists a subsequence $\{E_{k_j}\}_j$ and corresponding points x_{k_j} in E_{k_j} such that $\{x_{k_j}\}_j$ converges to x.

(viii) To obtain narrowly convergent (sub)sequences in $\mathcal{R}(T; \mathbb{R}^l)$, the following narrow compactness criterion is essential; this is Prohorov's theorem for transition probabilities—see Theorem 4.10 in Balder (2000b). A sequence $\{\delta_k\}_k$ in $\mathcal{R}(T; \mathbb{R}^l)$ is relatively sequentially compact for the narrow topology if there exists $h \in \mathcal{H}^{\mathrm{bb}}(T; \mathbb{R}^l)$ for which $\sup_k I_h(\delta_k) < +\infty$.

(ix) A transition probability $\delta \in \mathcal{R}(T; \mathbb{R}^l)$ is said to be *integrable* if

$$\int_T \left[\int_{\mathbb{R}^l} |x|\delta(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t) < +\infty.$$

By $\mathcal{R}^1(T;\mathbb{R}^l) \subset \mathcal{R}(T;\mathbb{R}^l)$ we denote the set of all such integrable transition probabilities. Associated to every $\delta \in \mathcal{R}^1(T;\mathbb{R}^l)$ is its barycentric arycentric function $\mathrm{bar}\,\delta \in \mathcal{L}^1(T;\mathbb{R}^l)$. Here $\mathcal{L}^1(T;\mathbb{R}^l)$ stands for the set of all integrable functions from T into $\mathbb{R}^l$. Apart from a null set (i.e., a set of measure zero), it is uniquely determined by δ. Namely, the above integrability property of δ implies that for a.e. t in T the integrals $\int_{\mathbb{R}^l} |x^i|\delta(t)(\mathrm{d}x)$, $i = 1, \dots, l$, are finite. Call the exceptional null set involved in this statement N; then for every $t \notin N$ the usual barycenter (i.e., expectation) of the probability measure $\delta(t)$ is defined as the following vector:

$$\mathrm{bar}\,\delta(t) := \left(\int_{\mathbb{R}^l} x^1\, \delta(t)(\mathrm{d}x), \dots, \int_{\mathbb{R}^l} x^l\, \delta(t)(\mathrm{d}x)\right)$$

and this defines a measurable function on $T \setminus N$. On N one can then choose the function $\mathrm{bar}\,\delta$ arbitrarily, albeit measurably. Then by Pfanzagl (1974, Lemma) one has in addition:

$$\mathrm{bar}\,\delta(t) \in \mathrm{co}\,\mathrm{supp}\,\delta(t) \quad \text{for a.e. } t \text{ in } T, \tag{2.3}$$

where “co” stands for convex hull. In fact, this argument can be extended; see Balder (1995b, Theorem 8.2) and the proof of Balder (2000b, Theorem 5.3). Let $\{g_1, \dots, g_r\}$ be a finite collection of $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable functions and let $\delta \in \mathcal{R}(T;\mathbb{R}^l)$ be such that

$$\int_T \left[\int_{\mathbb{R}^l} |g_j(t, x)|\delta(t)(\mathrm{d}x)\right] \mu(\mathrm{d}t) < +\infty, \; j = 1, \dots, r. \tag{2.4}$$

Then use of Carathéodory’s theorem and standard measurable selection results gives that there exist $\mathcal{T}$-measurable functions $\alpha_1, \dots, \alpha_{r+1} : T \to [0, 1]$, with $\sum_i \alpha_i = 1$, and $\mathcal{T}$-measurable functions $f_1, \dots, f_{r+1} : T \to \mathbb{R}^l$ such that for a.e. t in T both

$$f_1(t), \dots, f_{r+1}(t) \in \mathrm{supp}\,\delta(t)$$

and

$$\int_{\mathbb{R}^l} g_j(t, x)\delta(t)(\mathrm{d}x) = \sum_{i=1}^{r+1} \alpha_i(t) g_j(t, f_i(t)), \; j = 1, \dots, r.$$

If the measure space $(T, \mathcal{T}, \mu)$ is non-atomic, this result can immediately be developed into the following result (Lyapunov’s theorem for Young measures Balder (2000b, Theorem 5.3)): under (2.4) there exists a $\mathcal{T}$-measurable function $f : T \to \mathbb{R}^l$ such that $f(t) \in \mathrm{supp}\,\delta(t)$ for a.e. t in T and

$$\int_T \left[\int_{\mathbb{R}^l} g_j(t, x)\delta(t)(\mathrm{d}x)\right] \mu(\mathrm{d}t) = \int_T g_j(t, f(t))\mu(\mathrm{d}t), \; j = 1, \dots, r. \tag{2.5}$$

(x) An immediate consequence of (v) is as follows. Equip $\mathcal{L}^1(T;\mathbb{R}^l)$, with the weak topology $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$. Then the mapping $\delta \mapsto \mathrm{bar}\,\delta$ into $\mathcal{L}^1(T;\mathbb{R}^l)$ is continuous on any subset $\mathcal{R}_0$ of $\mathcal{R}(T;\mathbb{R}^l)$ for which there exists a function $\psi \in \mathcal{L}^1(T;\mathbb{R})$ such that the inclusion:

$$\mathrm{supp}\,\delta(t) \subset \{x \in \mathbb{R}^l : |x| \le \psi(t)\} \quad \text{for a.e. } t \text{ in } T \tag{2.6}$$

holds for every $\delta \in \mathcal{R}_0$. To show this, let $b \in \mathcal{L}^\infty(T;\mathbb{R}^l)$ be arbitrary. Then $g_b(t, x) := \min(-\|b\|_\infty \psi(t), b(t)x)$ defines a function $g_b \in \mathcal{G}^{\mathrm{bb}}(T;\mathbb{R}^l)$ for which

$$I_{g_b}(\delta) = \int_T \left[\int_{\mathbb{R}^l} b(t)x\delta(t)(\mathrm{d}x)\right] \mu(\mathrm{d}t) = \int_T b\,\mathrm{bar}\,\delta\,\mathrm{d}\mu \quad \text{for every } \delta \in \mathcal{R}_0,$$

because of (2.6) and the Cauchy–Schwarz inequality. By definition of the narrow topology, this shows that $\delta \mapsto \int_T b \,\mathrm{bar}\, \delta$ is lower semi-continuous on $\mathcal{R}_0$. Repetition of this argument for $-b$ instead of b yields continuity of that same mapping. Hence $\delta \mapsto \mathrm{bar}\, \delta$ is as claimed. See Balder (1999b) and Balder (2002) for more on such continuities.

(xi) The *ws-topology* on $\mathrm{Prob}(T \times \mathbb{R}^l)$ is is the coarsest topology for which all mappings $\pi \mapsto \int_{A\times\mathbb{R}^l} c(x)\pi(d(t,x))$ are continuous for every $A \in \mathcal{T}$ and every bounded continuous function $c : \mathbb{R}^l \to \mathbb{R}$. Suppose that $\mu(T) = 1$. Then we have

$$\{\mu \otimes \delta : \delta \in \mathcal{R}(T;\mathbb{R}^l)\} = \mathrm{Prob}_\mu(T \times \mathbb{R}^l),$$

where $\mathrm{Prob}_\mu(T \times \mathbb{R}^l)$ is defined as the set of all $\pi \in \mathrm{Prob}(T \times \mathbb{R}^l)$ whose marginal on T is equal to μ. This identity depends upon an important result about disintegration: for every $\pi \in \mathrm{Prob}(T \times \mathbb{R}^l)$ there exists $\delta_\pi \in \mathcal{R}(T;\mathbb{R}^l)$, unique modulo null sets, such that $\pi = \mu \otimes \delta_\pi$. The ws-topology on $\mathrm{Prob}_\mu(T \times \mathbb{R}^l) \subset \mathrm{Prob}(T \times \mathbb{R}^l)$ is homeomorphic with the narrow topology on the quotient of $\mathcal{R}(T;\mathbb{R}^l)$, taken with respect to the standard a.e.-equivalence relation. We refer to Balder (2001) for details.

3. Existence of relaxed competitive equilibria

In this section we present a model of a relaxed economy:

$$\mathcal{RE} := \langle T, \{(X_t, M_t, e_t, \omega(t))\}_{t\in T}\rangle,$$

all of whose components will be introduced below. As we explained already in Section 1, allowing the consumption choices to be mixed for $\mathcal{RE}$, yields powerful relaxed equilibrium existence results. Ordinary equilibrium results for an ordinary economy $\mathcal{E}$, to be presented later, will then follow readily. A completely equivalent description of such relaxation can be given in terms of Walrasian equilibrium distributions over pairs of consumers' characteristics and consumption choices; see Remark 3.8 below.

Let $(T, \mathcal{T}, \mu)$ be a finite, separable and complete measure space[2] of *consumers* and let l be the number of commodities. For any $t \in T$ a set $X_t \subset \mathbb{R}^l_+$ is given; this forms consumer t's *feasible consumption set*.[3] An (ordinary) *consumption profile* in this model is a function $f : T \to \mathbb{R}^l_+$ that is measurable with respect to $\mathcal{T}$ and $\mathcal{B}(\mathbb{R}^l)$, with $f(t) \in X_t$ for a.e. t in T. By $\mathcal{L}^1_X$ we denote the set of all ordinary consumption profiles that are integrable over T. Also, for any $T_0 \in \mathcal{T}$ we denote by $\mathcal{L}^1_X(T_0)$ the set of all integrable functions $f : T_0 \to \mathbb{R}^l_+$ with $f(t) \in X_t$ for a.e. t in T_0. Let $\omega : T \to \mathbb{R}^l_+$ be a given function; for any $t \in T$ the commodity bundle $\omega(t) \in \mathbb{R}^l_+$ forms the *initial endowment* of consumer t.

Assumption 3.1. $D := \{(t, x) \in T \times \mathbb{R}^l : x \in X_t\}$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable.

The associated trace σ-algebra $D \cap (\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l))$ on D will be denoted as $\mathcal{D}$.

Assumption 3.2. $X_t \subset \mathbb{R}^l_+$ is closed for every $t \in T$.

Assumption 3.3. ω is integrable and $\omega(t) \in X_t$ for every $t \in T$.

We shall frequently work with the scalar function $\tilde{\omega} := \sum_{i=1}^l \omega^i$.

In the remainder of this paper the above three assumptions are always supposed to hold. Observe that Assumption 3.2 allows for possible non-convexity of the feasible consumption sets. The set of all normalized *price vectors* is $P := \{p \in \mathbb{R}^l_+ : \sum_{i=1}^l p^i = 1\}$. The *budget set* of consumer t in T under the price $p \in P$ is given by

$$B_t(p) := \{x \in X_t : px \leq p\omega(t)\};$$

[2] As explained in Balder (1999b), in the end one can drop the completeness hypothesis altogether. The separability hypothesis can be made superfluous by strengthening the measurability conditions slightly.

[3] Extension to a situation where $t \mapsto X_t$ is integrably bounded from below is straightforward: by suitable translation arguments it can be reduced to the situation considered here.

observe that this set always contains $\omega(t)$. For later use we also define the following set (possibly empty):

$$B_t^0(p) := \{x \in X_t : px < p\omega(t)\}.$$

Of course, we always have cl $B_t^0(p) \subset B_t(p)$. The converse inclusion will be of considerable importance:

Remark 3.1. For every $t \in T$ and $p \in P$ we have $(a) \Rightarrow (b) \Leftrightarrow (c)$, where

(a) $B_t(p) = \mathrm{cl} B_t^0(p)$,
(b) $B_t^0(p)$ is non-empty,
(c) $\inf_{x \in X_t} px < p\omega(t)$.

Moreover, if X_t is convex, then also $(b) \Rightarrow (a)$, and the extra convexity is essential for this implication, as simple examples show (e.g., see Example 3.3 below). Finally, we observe that for every $t \in T$ we have $(A) \Rightarrow [(a)$ for all $p \in P]$, where

(A) X_t is convex and $\omega(t) \in \mathrm{int}\, X_t$.

Here int X_t stands for the interior of the set X_t.

We define a *mixed consumption profile* to be a transition probability $\delta \in \mathcal{R}(T; \mathbb{R}^l)$ with supp $\delta(t) \subset X_t$ for a.e. t in T. The set of all mixed consumption profiles is denoted by $\mathcal{R}_X$, and we denote by $\mathcal{R}_X^1 := \mathcal{R}_X \cap \mathcal{R}^1(T; \mathbb{R}^l)$ the collection of all mixed consumption profiles that are integrable transition probabilities in the sense of Section 2(ix). The following consequence of the definition of $\mathcal{R}_X^1$ is obvious: $\delta \in \mathcal{R}_X^1$ is a Dirac transition probability (see Section 2(ii)) if and only if there exists $f \in \mathcal{L}_X^1$ such that $\delta = \epsilon_f$. Consequently, Assumption 3.3 guarantees that ϵ_ω belongs to $\mathcal{R}_X^1$.

In what follows any reference to topological properties in $\mathcal{R}(T; \mathbb{R}^l)$ and its subsets will always understood to be with respect to the narrow topology on $\mathcal{R}(T; \mathbb{R}^l)$, discussed in Section 2.

Let Y be a semi-metric and separable space, called the externality space. For every $t \in T$ let $e_t : P \times \mathcal{R}_X^1 \to Y$ be given; it is called the *(relaxed) externality mapping* of consumer t. For every $t \in T$ let $M_t : P \times Y \to 2^{X_t}$ be a given *demand multifunction*, such that

$$M_t(p, y) \subset B_t(p) \quad \text{for every } t \in T,\ p \in P \text{ and } y \in Y.$$

The interpretation of consumer t's externality mapping e_t is that it forms a certain "statistic" about the price p and the mixed consumption profile δ; however, see also Remark 3.6. Further, the set $M_t(p, e_t(p, \delta))$ can be thought of as consumer t's demand, i.e., the set of all her/his "most desirable" consumption bundles in $B_t(p)$, given the price vector p and the mixed consumption profile δ. The following example describes an important specialization of the present model; we shall refer to it as the *preference-based* model. It concerns a situation that would be rather classical (Aumann, 1966; Hildenbrand, 1974; Schmeidler, 1969), if it were not for our current use of price and mixed consumption externalities and our insistence on keeping X_t non-convex whenever possible.

Example 3.1. For every $t \in T$, $y \in Y$, let $P_{t,y} : X_t \to 2^{X_t}$ be an abstract strict preference relation. Thus, for every $t \in T$, $x \in X_t$ and $y \in Y$ the set $P_{t,y}(x) \subset X_t$ represents the (possibly empty) collection of all consumption bundles that are strictly preferred to x by consumer t, given the externality value $y \in Y$. Take

$$M_t(p, y) := \{x \in B_t(p) : P_{t,y}(x) \cap B_t(p) = \emptyset\}$$

to define the set – possibly empty – of all $P_{t,y}$-maximal elements in the budget set $B_t(p)$. These strict preference relations are said to be *ordered* if for every $t \in T$ there exists $U_t : X_t \times Y \to [-\infty, +\infty]$, the associated *utility function* of consumer t, such that $P_{t,y}(x)$ is the set of all $x' \in X_t$ with $U_t(x', y) > U_t(x, y)$. In that special case $M_t(p, y) = \mathrm{argmax}_{x \in B_t(p)} U_t(x, y)$ holds for every $t \in T$, $p \in P$ and $y \in Y$.

Assumption 3.4.

(i) $t \mapsto e_t(p, \delta)$ is $\mathcal{T}$-measurable for every $p \in P$ and $\delta \in \mathcal{R}^1_X$,
(ii) $e_t : P \times \mathcal{R}^1_X \to Y$ is continuous for every $t \in T$.

Of course, Assumption 3.4 holds trivially in the extreme case where each mapping $(p, \delta) \mapsto e_t(p, \delta)$ is constant (whence non-informative) for each t, and it holds also when each $(p, \delta) \mapsto e_t(p, \delta)$ is continuous in p and constant in δ. A more complex situation where Assumption 3.4 is fulfilled is Example 3.2 below.

Assumption 3.5. For every $p \in P$ and $y \in Y$ the following hold:

(i) $\{(t, x) \in D : x \in M_t(p, y)\}$ is $\mathcal{D}$-measurable,
(ii) $\mathrm{Ls}_k\, M_t(p_k, y_k) \subset M_t(p, y)$ for every $t \in T$ and every sequence $\{(p_k, y_k)\}_k \subset P \times Y$ converging to (p, y).

Recall here that the notion of limes superior of sets was explained in Section 2(vii).

Assumption 3.6. $M_t(p, y)$ is non-empty for every $t \in T$, $p \in P \cap \mathbb{R}^l_{++}$ and $y \in Y$.

Remark 3.2. We emphasize that Assumption 3.6 allows $M_t(p, y) = \emptyset$ for some $t \in T$, $p \in P \setminus \mathbb{R}^l_{++}$ and $y \in Y$; then Assumption 3.5(ii) just means that the set $\mathrm{Ls}_k M_t(p_k, y_k)$ is empty for every sequence $\{(p_k, y_k)\}_k \subset P \times Y$ converging to such (p, y). As a very simple illustration, take $l := 2$, $X_t := \mathbb{R}^2_+$ and $Y := \{0\}$. If we set

$$M_t(p, 0) := \begin{cases} \left\{ \left(\dfrac{p\omega(t)}{2p^1}, \dfrac{p\omega(t)}{2p^2} \right) \right\} & \text{if } p = (p^1, p^2) \in \mathbb{R}^l_{++}, \\ \emptyset & \text{otherwise,} \end{cases}$$

then $M_t(p, 0) = \mathrm{argmax}_{(x^1, x^2) \in B_t(p)} x^1 x^2$, so this is a very well-known situation: in terms of Example 3.1, we deal with ordered preferences that are described by a simple Cobb–Douglas utility function. Here Assumption 3.5(ii) is clearly fulfilled, because for any sequence $\{p_k\}_k$ in P the identity $\mathrm{Ls}_k\, M_t(p_k, 0) = M_t(p, 0)$ is obvious if $p_k \to p \in \mathbb{R}^l_{++}$, and if $p_k \to p \in P \setminus \mathbb{R}^l_{++}$ (i.e., either $p^1_k \to 0$ or $p^2_k \to 0$) then it is easy to see that $\mathrm{Ls}_k\, M_t(p_k, 0)$ is empty.

Definition 3.1. A *relaxed competitive equilibrium with free disposal* for the above relaxed economy $\mathcal{RE}$ is a pair (p_*, δ_*) in $P \times \mathcal{R}^1_X$ such that

(i) $\mathrm{supp}\, \delta_*(t) \subset M_t(p_*, e_t(p_*, \delta_*))$ for a.e. t in T,
(ii) $\int_T \mathrm{bar}\, \delta_*\, \mathrm{d}\mu \leq \int_T \omega\, \mathrm{d}\mu$.

The set of all such pairs is denoted by $\boldsymbol{CE}_1(\mathcal{RE})$.

Here $\mathrm{bar}\, \delta_*$, the barycentric function of the transition probability δ_*, is as defined in Section 2(ix). Our first relaxed competitive equilibrium result is as follows. The first subsection in Section 6 is devoted to its proof.

Theorem 3.1. *Under Assumptions 3.1–3.6 there exists an equilibrium pair in* $\boldsymbol{CE}_1(\mathcal{RE})$.

Remark 3.4 below discusses a slight extension of this result, which incorporates dispersion in the sense of Yamazaki (1978). As Remark 3.5 explains, Theorem 3.1 is actually equivalent to Theorem 3.1(i) and Remark 3.2 of Balder (2003). Remark 3.7 shows that Theorem 3.1 also generalizes Schmeidler's existence result in Schmeidler (1973, Theorem 1). Moreover, Remark 3.8 shows that the above theorem, as well as Theorem 3.2 below, have an equivalent reformulation in terms of Walrasian equilibrium distributions.

Our second central equilibrium existence result for the relaxed economy $\mathcal{RE}$, to be presented next, works with weakened versions of Assumptions 3.4 and 3.5; it compensates for this by strengthening Assumptions 3.2, 3.3 and 3.6. As shown in Proposition 3.1 below, assumptions of this type hold under classical conditions of monotonicity and nonsatiation for the preferences $P_{t,y}$ in the preference-based model of Example 3.1. The strengthened versions of Assumptions 3.2 and 3.3 are as follows, where we write $\omega^{-1}(0) := \{t \in T : \omega(t) = 0\}$.

Assumption 3.2$'$. $X_t \subset \mathbb{R}^l_+$ is closed for every $t \in T$ and convex for every $t \in T \setminus \omega^{-1}(0)$.

Assumption 3.3$'$. The function $\omega : T \to \mathbb{R}^l_+$ is integrable with $\omega(t) \in X_t$ for every $t \in T$ and there exists $\check{f} \in \mathcal{L}^1_X$ such that $\int_T (\omega - \check{f})\, \mathrm{d}\mu \in \mathbb{R}^l_{++}$ and $\omega(t) - \check{f}(t) \in \mathbb{R}^l_+ \setminus \{0\}$ for all $t \in T \setminus \omega^{-1}(0)$.

We note in particular that Assumptions 3.2$'$ and 3.3$'$ hold under the following standard conditions, which go back to Aumann (1966): $X_t := \mathbb{R}^l_+$ for all $t \in T$ and $\int_T \omega \in \mathbb{R}^l_{++}$. Next, we introduce less stringent replacements of Assumptions 3.4 and 3.5. For $m \in \mathbb{N}$ let $\mathcal{R}_X(m)$ be defined by

$$\mathcal{R}_X(m) := \{\delta \in \mathcal{R}(T; \mathbb{R}^l) : \delta(t)(X^m_t) = 1 \quad \text{for a.e. } t \text{ in } T\}.$$

Here $t \mapsto X^m_t$ is defined as the integrably bounded multifunction with values $X^m_t := \{x \in X_t : \sum_i x^i \leq m\tilde{\omega}(t)\}$. Then $\mathcal{R}_X(m) \subset \mathcal{R}^1_X$ by Assumptions 3.3 or 3.3$'$. Just like $\mathcal{R}^1(T; \mathbb{R}^l)$ and $\mathcal{R}^1_X$, the spaces $\mathcal{R}_X(m)$ inherit the narrow topology from the ambient space $\mathcal{R}(T; \mathbb{R}^l)$. Observe that $\mathcal{R}_X(m)$ is non-empty for every $m \in \mathbb{N}$, for under Assumptions 3.3 or 3.3$'$ it contains ϵ_ω.

Assumption 3.4$'$.

(i) $t \mapsto e_t(p, \delta)$ is $\mathcal{T}$-measurable for every $p \in P$ and $\delta \in \mathcal{R}^1_X$,
(ii) e_t is continuous on $P \times \mathcal{R}_X(m)$ for every $t \in T$ and $m \in \mathbb{N}$,
(iii) for every $t \in T$ and every sequence $\{p_k, \delta_k\}_k$ in $P \times \mathcal{R}^1_X$, such that $\sup_k \int_T |\text{bar}\, \delta_k|\, \mathrm{d}\mu < +\infty$, the sequence $\{e_t(p_k, \delta_k)\}_k \subset Y$ is relatively compact.

Assumption 3.5$'$. For every $p \in P$ and $y \in Y$ the following hold:

(i) $\{(t, x) \in D : x \in M_t(p, y)\}$ is $\mathcal{D}$-measurable,
(ii) $\mathrm{Ls}_k\, M_t(p_k, y_k) \subset M_t(p, y)$ for every $t \in T$ with $B_t(p) = \mathrm{cl} B^0_t(p)$ and for every sequence $\{(p_k, y_k)\}_k \subset P \times Y$ converging to (p, y).

Assumption 3.4$'$ is weaker than Assumption 3.4. This follows from the narrow compactness criterion presented in Section 2(viii) by using $h \in \mathcal{H}^{\text{bb}}(T; \mathbb{R}^l)$, defined by $h(t, x) := \sum_i x^i$ if $x \in \mathbb{R}^l_+$ and $h(t, x) := +\infty$ if $x \in \mathbb{R}^l \setminus \mathbb{R}^l_+$. Also, Assumption 3.5$'$ is evidently weaker than Assumption 3.5. Our next assumption is stronger than its counterpart Assumption 3.6. It involves the budget plane of consumer t in T under the price $p \in P$, which is defined by

$$S_t(p) : \{x \in X_t : px = p\omega(t)\}.$$

Assumption 3.6$'$. For every $t \in T$, $p \in P$ and $y \in Y$

(i) If $p \in \mathbb{R}^l_{++}$ then $M_t(p, y)$ is non-empty and $M_t(p, y) \subset S_t(p)$,
(ii) If $p \notin \mathbb{R}^l_{++}$ then $M_t(p, y)$ is empty.

Definition 3.2. A *relaxed competitive equilibrium (without free disposal)* for the above relaxed economy $\mathcal{RE}$ is a pair (p_*, δ_*) in $P \times \mathcal{R}^1_X$ such that

(i) $\text{supp}\, \delta_*(t) \subset M_t(p_*, e_t(p_*, \delta_*))$ for a.e. t in T,
(ii) $\int_T \text{bar}\, \delta_*\, \mathrm{d}\mu = \int_T \omega\, \mathrm{d}\mu$.

The set of all such pairs is denoted by $\boldsymbol{CE}_2(\mathcal{RE})$.

Theorem 3.2. *Under* Assumptions 3.1 and 3.2$'$–3.6$'$ *there exists an equilibrium pair* (p_*, δ_*) *in* $\boldsymbol{CE}_2(\mathcal{RE})$, *with* $p_* \in \mathbb{R}^l_{++}$ *and* $\delta_* \in \cup_m \mathcal{R}_X(m)$.

In its present form, which extends Balder (2003, Theorem 3.1(ii), Remark 3.2), this theorem was given in Balder (2005). We shall prove it in Section 6. An equivalent reformulation of this result for Walrasian equilibrium distributions is also possible; it goes along the lines sketched in Remark 3.8. In doing so, one achieves a substantial generalization of the main existence result in Noguchi and Zame (2004); this has been worked out in detail in Balder (2005). The model of Noguchi and Zame (2004) was further extended in Noguchi and Zame (2006) to include production. As suggested by footnote 5 in Noguchi and Zame (2006) (see also our remarks about production in Section 5.1), it would appear that the results presented here can be extended so as to include models with production as well.

We shall now formulate sufficient conditions for the non-basic Assumptions 3.4, 3.4′, 3.5, 3.5′, 3.6 and 3.6′ to hold. The following Example 3.2 deals with sufficient conditions for the former two. In the context of the preference-based model of Example 3.1, the latter four assumptions are dealt with in Proposition 3.1 below.

Example 3.2.

(a) Let $\{\ell_1, \dots, \ell_s\}$ be a collection of $\mathcal{T}\otimes\mathcal{D}$-measurable functions $\ell_j : T\times D \to \mathbb{R}$, such that for every j and $t\in T$

$$\ell_j(t,\tau,\cdot) \text{ is continuous on } X_\tau \quad \text{for every } \tau\in T.$$

and there exists $\phi_{j,t}\in\mathcal{L}^1(T;\mathbb{R})$ with

$$\sup_{x\in X_\tau} |\ell_j(t,\tau,x)| \le \phi_{j,t}(\tau) \quad \text{for every } \tau\in T. \tag{3.1}$$

Then $e_t : P\times\mathcal{R}^1_X \to \mathbb{R}^s$, defined by $e_t(p,\delta) := (p, I_t(\delta))$ satisfies Assumption 3.4, with

$$I_t(\delta) := \left(\int_T \left[\int_{\mathbb{R}^l} \ell_1(t,\tau,x)\delta(\tau)(\mathrm{d}x)\right]\mu(\mathrm{d}\tau), \dots, \int_T \left[\int_{\mathbb{R}^l} \ell_s(t,\tau,x)\delta(\tau)(\mathrm{d}x)\right]\mu(\mathrm{d}\tau)\right)$$

More precisely, Assumption 3.4(i) holds by (Neveu, 1965, III.2). The validity of Assumption 3.4(ii) is seen as follows: fix j and t and define $g_{j,t}$ by setting $g_{j,t}(\tau,x) := \ell_j(t,\tau,x)$ if $(\tau,x)\in D$ and $g_{j,t}(\tau,x) := +\infty$ if $(\tau,x)\notin D$. Then $g_{j,t}\in\mathcal{G}^{\mathrm{bb}}(T;\mathbb{R}^l)$, so $\delta\mapsto\int_T[\int^l_{\mathbb{R}}\ell_j(t,\tau,x)\delta(\tau)(\mathrm{d}x)]\mu(\mathrm{d}t)$ is lower semi-continuous on $\mathcal{R}(T;\mathbb{R}^l)$ by Section 2(v). By replacing ℓ_j with $-\ell_j$ in the preceding argument and by the definition of $\mathcal{R}^1_X$, it follows that $\delta\mapsto I_t(\delta)$ is continuous on $\mathcal{R}^1_X$, and so is e_t.

(b) If in (a) above one replaces (3.1) by the weaker condition that for every t and j there exist $\phi_{j,t}\in\mathcal{L}^1(T;\mathbb{R})$ and $k_{j,t}\in\mathbb{R}_+$ with

$$|\ell_j(t,\tau,x)| \le \phi_{j,t}(\tau) + k_{j,t}|x| \quad \text{for all } (\tau,x)\in D,$$

then Assumption 3.4′ still holds for $e_t(p,\delta) := (p, I_t(\delta))$. Indeed, for every $t\in T$ and $m\in\mathbb{N}$ the mapping I_t is continuous on $\mathcal{R}_X(m)$ by (2.6): use $\psi(\tau) := \phi_{j,t}(\tau) + k_{j,t}m\tilde{\omega}(\tau)$. So the validity of part (ii) of Assumption 3.4′ follows as above. Part (iii) holds by the fact that if any sequence $\{\delta_k\}_k$ in $\mathcal{R}^1_X$ has $\sup_k \int_T |\text{bar}\,\delta_k|\mathrm{d}\mu < +\infty$, then $\{I_t(\delta_k)\}_k$ is a bounded (whence relatively compact) sequence in $\mathbb{R}^s$, because of the preceding inequality. Of course, the validity of Assumption 3.4′(i) is as easy to check as in (a) above.

Finally, observe that, if desired, extra dependence of the ℓ_j on the price vectors $p\in P$ can also be included. Namely, if $p_k\to p_\infty$ in P, we can use (2.2) in Section 2(v) with $\tilde{g}_{t,j}(\tau,k,x) := \ell_j(t,\tau,x,p_k)$ in the argument given in part (a).

Proposition 3.1. *Consider the following conditions*:

(i) $\{(t,x,x') : t\in T, x\in X_t, x'\in P_{t,y}(x)\}$ *is* $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^l)\otimes\mathcal{B}(\mathbb{R}^l)$*-measurable for every* $y\in Y$,

(ii) *$x \notin \mathrm{co}\, P_{t,y}(x)$ for every $t \in T$, $x \in X_t$ and $y \in Y$,*
(iii) *$P_{t,y}(x)$ is open for every $t \in T$, $x \in X_t$ and $y \in Y$,*
(iv) *$\{(x', y) \in X_t \times Y : x \in P_{t,y}(x')\}$ is open for every $t \in T$ and $x \in X_t$,*
(v) *$B_t(p) = \mathrm{cl} B_t^0(p)$ for every $t \in T$ and $p \in P$,*
(vi) *for every $t \in T$, $x \in X_t$ and $\epsilon > 0$ there exists $x' \in P_{t,y}(x)$ with $|x' - x| < \epsilon$,*
(vii) *for every $t \in T$, $p \in P \setminus \mathbb{R}^l_{++}$ and $x \in B_t(p)$ there exists $x' \in B_t(p)$ with $x' \in P_{t,y}(x)$.*

Then

(a) Assumptions 3.5 and 3.6 *are met in* Example 3.1 *if conditions (i) to (v) hold.*
(b) Assumptions 3.5′ and 3.6′ *are met in* Example 3.1 *if conditions (i) to (iv) and (vi) to (vii) hold.*

Above $\mathrm{co}\, P_{t,y}(x)$ stands for the convex hull of $P_{t,y}(x)$ and open in X_t is meant in the obvious relative sense. Observe that condition (vi) amounts to local nonsatiation. Certainly conditions (vi) and (vii) will hold if $X_t = \mathbb{R}^l_+$ and if $x' \geq x$, $x' \neq x$, implies $x' \in P_{t,y}(x)$ (strong monotonicity). However, it is clear that these two conditions can hold in non-monotone situations as well. Condition (v) is the only condition in Proposition 3.1 that is not phrased in terms of the original ingredients of the model, but already Remark 3.1 gave sufficient conditions for it.

Proof.

(a) Step 1 To verify Assumption 3.5(i), we fix $p \in P$ and $y \in Y$ arbitrarily and prove that

$$E := \{(t, x) \in D : x \in B_t(p) \quad \text{and} \quad P_{t,y}(x) \cap B_t(p) = \emptyset\}$$

belongs to $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$. First, because of Assumption 3.1 and measurability of $(t, x) \mapsto p(x - \omega(t))$, the graph of the multifunction $t \mapsto B_t(p)$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable. Therefore, by Castaing and Valadier (1977, III.30) there exists an at most countable sequence of measurable functions $b_k : T \to \mathbb{R}^l_+$ with $\mathrm{cl}\{b_k(t)\}_k = B_t(p)$ for every $t \in T$. Now the complement of E is equal to the union of three sets: $E_1 := (T \times \mathbb{R}^l) \setminus D$, $E_2 := \{(t, x) \in D : px > p\omega(t)\}$ and $E_3 := \{(t, x) \in D : P_{t,y}(x) \cap B_t(p) \neq \emptyset\}$. Of these, the first two are obviously measurable by our basic assumptions. By condition (iii) and the above, E_3 is the union of the sets $E_3^k := \{(t, x) \in D : b_k(t) \in P_{t,y}(x)\}$, each of which is measurable by condition (i). So E itself is measurable.

Step 2 To verify Assumption 3.5(ii), we again fix $p \in P$ and $y \in Y$. Let $(p_k, y_k, x_k) \to (p, y, x)$ with $x_k \in M_t(p_k, y_k)$ for all $k \in \mathbb{N}$. We claim that $x \in M_t(p, y)$. If that were false, then $B_t(p) \cap P_{t,y}(x)$ would be non-empty. By conditions (iii) and (v), the set $B_t^0(p) \cap P_{t,y}(x)$ would also be non-empty and contain some bundle $\tilde{x}$. Because this means $p\tilde{x} < p\omega(t)$, it would imply $p_k\tilde{x} < p_k\omega(t)$ for k large enough. Hence, by $x_k \in M_t(p_k, y_k)$ one would have $\tilde{x} \notin P_{t,y_k}(x_k)$ for those k. So for large enough k the pair (x_k, y_k) would belong to the set $\{(x', y) \in X_t \times Y : \tilde{x} \notin P_{t,y}(x')\}$, which is closed by condition (iv). In the limit this would give $\tilde{x} \notin P_{t,y}(x)$, which is in contradiction to the above. This proves the claim, so Assumption 3.5(ii) holds.

Step 3 For $p \in \mathbb{R}^l_{++}$ the set $B_t(p)$ is convex and compact, so non-emptiness of $M_t(p, y)$ follows from conditions (ii) and (iv) by well-known existence results for abstract maximal points: apply Border (1985, Theorem 7.2) or Yannelis and Prabhakar (1983). This proves (a).

(b) The validity of Assumption 3.5′ is proven just as for Assumption 3.5. As for Assumption 3.6′(i), non-emptiness follows as in step 3 above and it is easy and standard to prove the inclusion $M_t(p, y) \subset S_t(p)$ from the local nonsatiation condition (vi). Finally, condition (vii) obviously implies that Assumption 3.6′(ii) holds. □

Remark 3.3. Proposition 3.1 remains valid if the irreflexivity condition (ii) is replaced by the following acyclicity condition for each $P_{t,y}$, $t \in T$, $y \in Y$:

(ii') $x_N \notin P_{t,y}(x_1)$ for every finite collection $\{x_1, \ldots, x_N\}$ in X_t with

$x_{i+1} \in P_{t,y}(x_i)$, $i = 1, \ldots, N-1$.

This time, in step 3 of the proof one invokes (Border, 1985, Theorem 7.12).

Example 3.3. The following examples show that condition (v) cannot be omitted from Proposition 3.1, because then existence, as we have it in Theorem 3.1, can fail. In both examples we choose $T := [0, 1]$, equipped with the Lebesgue σ-algebra and measure, $l := 2$, and we choose trivial externality mappings, i.e., $Y := \{0\}$. The example in (a) formalizes the idea behind Fig. 4 in Yamazaki (1978); example (b) is new, but undoubtedly similar examples exist in literature.

(a) For every $t \in T$ let $X_t := X$, where $X := \{(x^1, x^2) : x^1 \geq 1 \text{ or } x^2 \geq 4\}$ is non-convex. Let $\omega(t) := (2, 2)$ for all $t \in T$. As is easy to see, for $p = (1/2, 1/2)$ condition (v) does not hold, because $(0, 4) \in B_t(p) \setminus \mathrm{cl} B_t^0(p)$. Consider the ordered strict preference relation $P_{t,0} : X_t \to 2^{X_t}$ that is associated to the utility functions $U_t(x^1, x^2) := 7x^1 + 2(x^2)^2$. Then $M_t(p, 0) = \mathrm{argmax}_{x \in B_t(p)} U_t(x)$, as observed in Example 3.1. It is not very hard to check that the following holds:

$$M_t((p^1, p^2), 0) = \begin{cases} \{(0, 2/p^2)\} & \text{if } 1/2 \leq p^1 < 1, \\ \{(2/p^1, 0)\} & \text{if } 0 < p^1 < 1/2, \\ \emptyset & \text{if } p^1 = 0 \text{ or } 1. \end{cases}$$

For one, this violates Assumption 3.5, but, actually, we can see here that $\boldsymbol{CE}_1(\mathcal{RE})$ is empty. For if there were a relaxed equilibrium pair (p_*, δ_*) in $\boldsymbol{CE}_1(\mathcal{RE})$, this would imply $0 < p_*^1 < 1$ and, because the sets $M_t(p_*, 0)$ are singletons, either $\delta_*(t) = \epsilon_{(0,2/p_*^2)}$ must hold for a.e. t or $\delta_*(t) = \epsilon_{(2/p_*^1,0)}$ for a.e. t (depending on whether $p_*^1 \in [1/2, 1)$ or $p_*^1 \in (0, 1/2)$). In either case the constraint $\int_T \mathrm{bar}\, \delta_* \leq \int_T \omega$ cannot hold. Here condition (v) of Proposition 3.1 fails, due to the non-convexity of X. This also illustrates concretely why the implication $(b) \Rightarrow (a)$ in Remark 3.1 need not hold in the absence of convexity of the sets X_t.

(b) Even more illustrative is the following example of the failure of condition (v) in Proposition 3.1, because it involves convex sets X_t. For $t \in T$ let $X_t := X$, where X is the convex set given by

$$X := \{(x^1, x^2) \in (0, 1) \times \mathbb{R} : x^2 \geq 1/x^1\} \cup \{(x^1, x^2) \in [1, +\infty) \times \mathbb{R}_+ : x^2 \geq 2 - x^1\}.$$

Let $\omega(t) := (1, 1)$ for every $t \in T$ and let every consumer t use the ordered strict preference relation $P_{t,0} : X_t \to 2^{X_t}$ that is associated to the utility function $U_t(x^1, x^2) := \max(x^1, x^2)$. For $M_t(p, 0) = \mathrm{argmax}_{x \in B_t(p)} U_t(x)$ this gives directly:

$$M_t((p^1, p^2), 0) = \begin{cases} \{(1/p^1, 0)\} & \text{if } 0 < p^1 \leq 1/2, \\ \{(p^2/p^1, p^1/p^2)\} & \text{if } 1/2 < p^1 < 1, \\ \emptyset & \text{if } p^1 = 0 \text{ or } 1. \end{cases}$$

Similar to what happened in the example in part (a), this violates Assumption 3.5. Also in this part (b) we have $\boldsymbol{CE}_1(\mathcal{RE}) = \emptyset$, as can easily be seen.

Remark 3.4. Theorem 3.1 can be extended as follows to deal with dispersion in the sense of Yamazaki (1978). Consider the following weakened version of part (ii) of Assumption 3.5: (ii^{mod}) For every $p \in P$ there is a null set N_p such that for every $t \in T \setminus N_p$ and $y \in Y$

$$\mathrm{Ls}_k\, M_t(p_k, y_k) \subset M_t(p, y) \quad \text{for every sequence } \{(p_k, y_k)\}_k \subset P \times Y \text{ converging to } (p, y).$$

Then it is easy to verify that the entire proof of Theorem 3.1– and in particular the proof of Lemma 6.2– goes through if one replaces part (ii) of Assumption 3.5 by the above modification (ii^{mod}). Yamazaki's dispersion condition (Yamazaki, 1978, p. 545) requires that for every $p \in P$ the image measure of μ under the scalar mapping $t \mapsto p\omega(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Now in the situation where the X_t are identically equal to some closed subset of $\mathbb{R}^l_+$, (Yamazaki, 1978, Corollary 1) states the following: if the dispersion condition holds, then for every $p \in P$ there exists a μ-null set N_p such that

$$B_t(p) = \mathrm{cl}\, B^0_t(p) \quad \text{for every } t \in T \setminus N_p.$$

Via condition (v) in Proposition 3.1 this guarantees (ii^{mod}). This extension involves a possibly uncountable collection of exceptional null sets N_p, $p \in P$, which causes it to be relevant. This contrasts with the introduction of obvious extensions that only involve at most countably many exceptional null sets. These are not incorporated in this paper (to accomodate for them, one simply redefines the measure space by excluding the union of those null sets – because all main results are stated modulo null sets this works directly).

Remark 3.5. Under Assumptions 3.4(ii), 3.5(ii) and 3.6 the set:

$$G_t := \{(x, p, \delta) \in X_t \times P \times \mathcal{R}^1_X : x \in M_t(p, e_t(p, \delta))\}$$

is closed and non-empty for every $t \in T$. In the spirit of Shafer and Sonnenschein (1975), we define for $(t, x) \in D$, $p \in P$ and $\delta \in \mathcal{R}^1_X$:

$$U_t(x, p, \delta) := -\inf_{(x', p', \delta') \in G_t} \rho'((x', p', \delta'), (x, p, \delta)),$$

where ρ' is an arbitrary semi-metric on $\mathbb{R}^l_+ \times P \times \mathcal{R}^1_X$ (see Section 2(vi)). Observe that U_t is clearly continuous on $X_t \times P \times \mathcal{R}^1_X$ for every $t \in T$. Because G_t is closed, we have

$$\mathrm{argmax}_{x \in B_t(p)} U_t(x, p, \delta) = M_t(p, e_t(p, \delta)) \text{ if } M_t(p, e_t(p, \delta)) \neq \emptyset,$$

for then $\sup_{x \in B_t(p)} U_t(x, p, \delta) = 0$, as can easily be seen. Because of this identity, Assumptions 3.4(i) and 3.5(i) imply that the graph of the the multifunction $t \mapsto \mathrm{argmax}_{x \in B_t(p)} U_t(x, p, y)$ belongs to $\mathcal{D}$ for every $p \in P \cap \mathbb{R}^l_{++}$ and $\delta \in \mathcal{R}^1_X$ (this is the simplifying condition observed on Balder (2003, p. 17)). So from Theorem 3.1(i) and Remark 3.2 in Balder (2003) we conclude that there exist $p_* \in P$ and $\delta_* \in \mathcal{R}^1_X$ such that $\mathrm{supp}\, \delta_*(t) \subset \mathrm{argmax}_{x \in B_t(p_*)} U_t(x, p_*, \delta_*) = M_t(p_*, e_t(p_*, \delta_*))$ a.e. and $\int_T \mathrm{bar}\, \delta_* \leq \int_T \omega$. This proves that Theorem 3.1 follows from Theorem 3.1(i) in Balder (2003). Conversely, Theorem 3.1(i) in Balder (2003) follows from Theorem 3.1 by setting $Y := P \times \mathcal{R}^1_X$, $e_t :=$ identity mapping on $P \times \mathcal{R}^1_X$ and $M_t(p, y) := \mathrm{argmax}_{x \in B_t(p)} U_t(x, y)$.

Remark 3.6. The *canonical* mixed externality mappings are defined by choosing $Y := P \times \mathcal{R}^1_X$ and $e_t := i_t :=$ identity mapping on $P \times \mathcal{R}^1_X$ (semi-metrizability and separability of $P \times \mathcal{R}^1_X$ follow from Section 2(vi), continuity and measurability as in Assumptions 3.4 or 4.1$'$ are trivial). The i_t, $t \in T$, are called canonical because for any given M_t and e_t one can always redefine as follows $M'_t(p, \delta) := M_t(p, e_t(p, \delta))$. In this sense the i_t, $t \in T$, form the "best possible" mixed externality mappings, and we could have presented this entire section in terms of them without sacrificing any generality (Remark 3.2 in Balder, 2003 actually points this out). However, for the corresponding *ordinary* externality mappings, to be presented in Section 4, the situation is quite different and that explains our present choice of common notation.

Remark 3.7. As observed already in Balder (2003), by trivializing the role of the budget constraint in models where $t \mapsto X_t$ is integrably bounded, Theorem 3.1 also generalizes Schmeidler's existence result in Schmeidler (1973, Theorem 2) for Nash equilibria in non-atomic games, as well as related results. For if we suppose that there exists an integrable function $\psi : T \to \mathbb{R}$ such that $X_t \subset \{x \in \mathbb{R}^l : |x| \leq \psi(t)\}$ for every $t \in T$, then setting $\omega(t) := (\psi(t), \ldots, \psi(t))$ causes $px \leq p\omega(t)$ to hold vacuously for every $t \in T$, $x \in X_t$ and $p \in P$. The other details of the substitutions can be gleaned from the preference-based Example 3.1 and Proposition 3.1. See Balder (1995a) and Balder (2003) for some additional technical details. However, the additional integrable boundedness assumption causes the values of $t \mapsto X_t$ to be compact in such models. It is well-known that much more general existence results can then be realized: e.g., see Balder (1995a) and Balder (2002), Khan (1984) for Nash equilibria and Balder (1999a) for both Nash and Walrasian equilibria.

Remark 3.8. Suppose without loss of generality that $(T, \mathcal{T}, \mu)$ is a probability space, i.e., suppose that $\mu(T) = 1$. Then Theorems 3.1 and 3.2 can be restated, completely equivalently, in terms of Walrasian equilibrium distributions. We only sketch this conversion. This time, T represents an abstract space of *agents' characteristics*. Such characteristics are pairs of initial endowments and preference relations, and, in contrast to what we do here, in the traditional literature they are topologized *ab initio* in order to formulate sufficient conditions for existence that end up being stronger than the ones considered here (Hildenbrand, 1974). This is because the standard literature does not exploit the narrow topology or the ws-topology. Let $\mathrm{Prob}^1_\mu(D)$ be the set of all $\pi \in \mathrm{Prob}_\mu(D)$ for which $\int_D |x|\pi(d(t,x)) < +\infty$; let it be equipped with the ws-topology, discussed in Section 2(xi). Observe that Definition 3.1 is *equivalent* to stating the following about $\pi_* := \mu \otimes \delta_* \in \mathrm{Prob}^1_\mu(D)$, which causes π_* to be an extension of the notion of *Walrasian equilibrium distribution* in the sense of (Hildenbrand, 1974, p. 158): there exists $p_* \in P$ such that

$$\pi_*(\{(t,x) \in D : x \in \tilde{M}_t(p_*, \pi_*)\}) = 1 \quad \text{and} \quad \int_D x\, \pi_*(d(t,x)) \leq \int_T \omega(t)\mu(\mathrm{d}t).$$

Here we define $\tilde{M}_t(p, \pi) := M_t(p, \tilde{e}_t(p, \pi))$, with $\tilde{e}_t(p, \pi) := e_t(p, \delta_\pi)$. Consult Section 2(xi) for the meaning of δ_π. This definition of $\tilde{e}_t$ is unambiguous because of Assumption 3.4(ii) and the fact that the semi-metric ρ of Section 2(vi) gives $\rho(\delta, \delta') = 0$ if and only if $\delta(t) = \delta'(t)$ for a.e. t. Because of the homeomorphism pointed out in Section 2(xi), it is now straightforward to give an equivalent reformulation of Assumption 3.4 in terms of the above externality mappings $\tilde{e}_t$:

(i) $t \mapsto \tilde{e}_t(p, \pi)$ is $\mathcal{T}$-measurable for every $p \in P$ and $\pi \in \mathrm{Prob}^1_\mu(D)$,
(ii) $\tilde{e}_t : P \times \mathrm{Prob}^1_\mu(D) \to Y$ is continuous for every $t \in T$,

Note here that, conversely, the earlier externality mappings can be regained from the new ones via $e_t(p, \delta) := \tilde{e}_t(p, \mu \otimes \delta)$. So if one replaces Assumption 3.4 by the one above, then it can be seen that Theorem 3.1 is converted into an equivalent result that ensures the existence of a Walrasian equilibrium distribution in the sense used above. Finally, what was said in Remark 3.6 also applies here: a very general equivalent description of the model discussed in this remark is obtained by setting $Y := P \times \mathrm{Prob}^1_\mu(D)$ and by using the canonical externality $\tilde{e}_t(p, \pi) := (p, \pi)$ for all $t \in T$, $p \in P$ and $\pi \in \mathrm{Prob}^1_\mu(D)$.

4. Existence of ordinary competitive equilibria

In this section we study an ordinary pure exchange economy:

$$\mathcal{E} := \langle T, \{(X_t, M_t, d_t, \omega(t))\}_{t \in T} \rangle.$$

Here $(T, \mathcal{T}, \mu)$, X_t, ω and M_t are exactly as introduced in the previous section. However, instead of the the mixed externality mappings e_t, $t \in T$, the economy $\mathcal{E}$ works with the following objects: for every $t \in T$ let $d_t : P \times \mathcal{L}^1_X \to Y$ be a given *(ordinary) externality mapping* for consumer t. Recall that $\mathcal{L}^1_X$, introduced in Section 3, is the set of all ordinary integrable consumption profiles. Then $M_t(p, d_t(p, f)) \subset B_t(p)$ represents consumer t's *demand*, i.e., the set of all her/his "most desirable" ordinary consumption bundles in $B_t(p)$, given the price vector p and the ordinary consumption profile f.

Definition 4.1. An *(ordinary) competitive equilibrium with free disposal* for the ordinary economy $\mathcal{E}$ is a pair (p_*, f_*) in $P \times \mathcal{L}^1_X$ such that

(i) $f_*(t) \in M_t(p_*, d_t(p_*, f_*))$ for a.e. t in T,
(ii) $\int_T f_* \,\mathrm{d}\mu \leq \int_T \omega \,\mathrm{d}\mu$.

The set of all such pairs is denoted by $\boldsymbol{CE}_1(\mathcal{E})$.

Definition 4.2. An *(ordinary) competitive equilibrium (without free disposal)* for the ordinary economy $\mathcal{E}$ is a pair (p_*, f_*) in $P \times \mathcal{L}^1_X$ such that

(i) $f_*(t) \in M_t(p_*, d_t(p_*, f_*))$ for a.e. t in T,
(ii) $\int_T f_* \, d\mu = \int_T \omega \, d\mu$.

The set of all such pairs is denoted by $\boldsymbol{CE}_2(\mathcal{E})$.

Below we shall formulate sufficient conditions for the existence of ordinary competitive equilibria. We shall say that the mixed externality mappings $e_t : P \times \mathcal{R}^1_X \to Y$, $t \in T$, are *natural extensions* of the ordinary externality mappings d_t, $t \in T$, if for every $p \in P$ and $f \in \mathcal{L}^1_X$:

$$e_t(p, \epsilon_f) = d_t(p, f) \quad \text{for every } t \in T. \tag{4.1}$$

We refer to what was said about Dirac transition probabilities in Section 3, following the definition of $\mathcal{R}^1_X$. Observe that if one identifies each function $f \in \mathcal{L}^1_X$ with its Dirac counterpart $\epsilon_f \in \mathcal{R}^1_X$, similar to the well-known practice in game theory to consider ordinary strategies as special mixed strategies, then the validity of (4.1) guarantees the following pleasant state of affairs: for every $p_* \in P$, $f_* \in \mathcal{L}^1_X$ and $k = 1, 2$:

$$(p_*, \epsilon_{f_*}) \in \boldsymbol{CE}_k(\mathcal{RE}) \quad \text{if and only if} \, (p_*, f_*) \in \boldsymbol{CE}_k(\mathcal{E}). \tag{4.2}$$

Here $\mathcal{RE}$ stands for the relaxed economy $\langle T, \{(X_t, M_t, e_t, \omega(t))\}_{t \in T}\rangle$ that is associated with (4.1). As a rule, our results do not need (4.1) to hold, although Corollaries 4.1 and 4.2 are exceptions. Yet all ordinary externality mappings in Section 5 turn out to have natural extensions in the sense of (4.1). To show that to have existence of ordinary competitive equilibria is much harder than existence of relaxed competitive equilibria, consider the following example, which is taken from Balder (2003).

Example 4.1. Consider $T := [0, 1]$, equipped with the Lebesgue σ-algebra and measure. Let $l := 1$, $X_t := [0, 2]$ and $\omega(t) := 2$ for all t. In this case $P = \{1\}$. Then $[0, 2]$ is the budget set for each agent t. Consider for $Y := \mathbb{R}$ the ordinary externality mappings $d_t(f) := \int_0^t f(\tau) d\tau$ and for every $(t, y) \in T \times Y$ let $P_{t,y} : X_t \to 2^{X_t}$ be the ordered strict preference relation associated to the utility function $U_t(x, y) := |x - 1 + t - y|$. Then $M_t(1, y) = \text{argmax}_{x \in B_{t,1}} U_t(x, y)$ is as follows:

$$M_t(1, y) = \begin{cases} \{2\} & \text{if } t > y, \\ \{0, 2\} & \text{if } t = y, \\ \{0\} & \text{if } t < y. \end{cases}$$

As we shall demonstrate explicitly, in this situation (1) $\boldsymbol{CE}_1(\mathcal{E}) = \emptyset$, but (2)$\boldsymbol{CE}_1(\mathcal{RE}) \neq \emptyset$, provided that we define $e_t(\delta) := \int_0^t \text{bar}\, \delta(\tau) d\tau$ for the associated relaxed economy (note that in this way we obtain a natural extension in the sense of (4.1)). To begin with (1), suppose that we had a free disposal competitive equilibrium $(1, f_*)$ in $\boldsymbol{CE}_1\mathcal{E}$. Then for a.e. t in $[0, 1]$ the fact that $f_*(t) \in M_t(1, f_*)$ would give $f_*(t) = 0$ if $\int_0^t f_* > t$ and $f_*(t) = 2$ if $\int_0^t f_* < t$. Define $\psi := f_* - 1$ and let $\Psi(t) := \int_0^t \psi$. Then the previous lines imply that $\Psi(t)\psi(t) \leq 0$ for a.e. t in $[0, 1]$ (note that if $\int_0^t f_* = t$, then $\Psi(t) = 0$). So the absolutely continuous function Ψ^2 has a non-positive derivative. Hence, by $\Psi(0) = 0$ this implies $(\Psi(t))^2 = 0$ for every $t \in [0, 1]$. But then $\Psi(t) = 0$ for all t, which implies $\psi(t) = 0$ for a.e. t. So it follows that $f_*(t) = 1$ for a.e. t in $[0, 1]$, which is clearly nonsensical. This proves (1).

As for (2), non-emptiness of $\boldsymbol{CE}_1(\mathcal{RE})$ is already predicted by Theorem 3.1, all assumptions of which are easily seen to hold. However, we can also produce a concrete relaxed equilibrium pair, namely $p_* := 1$ and $\delta_*(t) := ((1/2)\epsilon_0) + ((1/2)\epsilon_2)$ for all $t \in [0, 1]$ (i.e., for every $t \in T$ the probability measure $\delta_*(t)$ is concentrated in 0 and 2 with equal probabilities). We claim that $(1, \delta_*)$ belongs to $\boldsymbol{CE}_1(\mathcal{RE})$. Note first that $\text{bar}\, \delta_*(t) = 1$ for all $t \in T$. This yields $\int_T \text{bar}\, \delta_* = 1 \leq 2 = \int_T \omega$ and $e_t(1, \delta_*) = \int_0^t 1 = t$ for every $t \in T$. Hence, $M_t(1, e_t(1, \delta_*)) = M_t(1, t) = \{0, 2\}$, so indeed this gives $\text{supp}\, \delta_*(t) \subset M_t(1, e_t(1, \delta_*))$ for every $t \in T$. The claim has been proven.

Notwithstanding the above counterexample, there are situations, requiring no purification whatsoever[4], in which existence of a mixed competitive equilibrium pair (p_*, δ_*) in $\boldsymbol{CE}_k(\mathcal{RE})$ directly yields existence of a pair (p_*, f_*) in

[4] However, see Remark 4.2 below.

$\boldsymbol{CE}_k(\mathcal{E})$, $k = 1, 2$. Such situations are characterized by (4.1) and the following assumption about the uniqueness of demand:

Assumption 4.1. $M_t(p, y)$ has at most one element for every $t \in T$, $p \in P$ and $y \in Y$.

In the preference-based model of Example 3.1, this assumption holds if the feasible consumption sets X_t are convex and the preference relations $P_{t,y}$ are strictly (*alias* strongly) convex on X_t for every $t \in T$ and $y \in Y$ (a special case of the latter is when the $P_{t,y}$ are ordered and the associated utility functions $U_t(\cdot, y)$ are strictly quasi-concave on X_t). This shows that Assumption 4.1 is frequently satisfied; for instance, in the example used in Remark 3.2 this is so. Observe that Example 4.1 demonstrates that Assumption 4.1 cannot be removed from the next corollary.

Corollary 4.1. *Suppose that the ordinary ordinary externality mappings d_t, $t \in T$, have as natural extensions, in the sense of* (4.1), *mixed externality mappings e_t, $t \in T$. Then, under the* Assumptions 3.1–3.6 and 4.1, *there exists an equilibrium pair in $\boldsymbol{CE}_1(\mathcal{E})$.*

Proof. The economy $\mathcal{RE}$ meets all assumptions of Theorem 3.1, so there exists a mixed equilibrium pair $(p_*, \delta_*) \in \boldsymbol{CE}_1(\mathcal{RE})$. Then it follows from Assumption 4.1 that for a.e. t in T the support $\operatorname{supp} \delta_*(t)$ is in fact a singleton. Redefine $\delta_*(t) := \epsilon_{\omega(t)}$ on the exceptional null set involved in the previous statement. Then $\delta_* \in \mathcal{R}^1_X$ is a Dirac transition probability and the redefined pair (p_*, δ_*) still belongs to $\boldsymbol{CE}_1(\mathcal{RE})$. So by what was stated about Dirac transition probabilities in Section 3, following the definition of $\mathcal{R}^1_X$, there exists a function $f_* \in \mathcal{L}^1_X$ such that $\delta_* = \epsilon_{f_*}$. Then it follows by (4.2) that (p_*, f_*) belongs to $\boldsymbol{CE}_1(\mathcal{E})$. □

Corollary 4.2. *Suppose that the ordinary ordinary externality mappings d_t, $t \in T$, have as natural extensions, in the sense of* (4.1), *mixed externality mappings e_t, $t \in T$. Then, under the* Assumptions 3.1, 3.2′, 3.3′, 3.4′, 3.5′, 3.6′ and 4.1, *there exists an equilibrium pair in $\boldsymbol{CE}_2(\mathcal{E})$.*

The proof is an obvious modification of the previous one and will be omitted. It would seem that even in the classical situation with trivial externalities (i.e., with $Y = \{0\}$) the above two corollaries are new. Because they also follow from Theorems 4.1 and 4.2 below (see Remark 4.2), this also attests to the generality achieved in those two theorems.

Example 4.2. Consider $T := [0, 1]$, equipped with the Lebesgue σ-algebra and measure and let $l := 2$. Let $X_t := \mathbb{R}^2_+$ for every t and let $\alpha^1, \alpha^2 : T \to \mathbb{R}_{++}$ be measurable functions. For $t \in T$ let $d_t : P \times \mathcal{L}^1_X$ be given by $d_t(p, f) := \int_t^1 f^1(\tau)\mathrm{d}\tau$. Let consumer t's strict preference relations $P_{t,y}$ be ordered and associated to the Cobb–Douglas utility function $U_t(x, p, y) := (x^1)^{\alpha^1(t)+y}(x^2)^{\alpha^2(t)}$. By Example 3.1, the multifunction M_t is then given by $M_t(p, y) = \operatorname{argmax}_{x \in X_t} U_t(x, p, y)$, and this gives:

$$M_t(p, y) := \begin{cases} \left\{ \left(\dfrac{\alpha^1(t) + y}{\alpha^1(t) + \alpha^2(t) + y} \dfrac{p\omega(t)}{p^1}, \dfrac{\alpha^2(t)}{\alpha^1(t) + \alpha^2(t) + y} \dfrac{p\omega(t)}{p^2} \right) \right\} & \text{if } p=(p^1, p^2) \in \mathbb{R}^l_{++}, \\ \emptyset & \text{otherwise,} \end{cases}$$

as easy calculations show. In this example the effect of the consumption externality is as follows. Consumer t's demand $M_t(p, e_t(p, \delta))$ is influenced by his/her "fellow consumers on the right", i.e., those in the interval $(t, 1]$, and it can be observed that their aggregate consumption of commodity 1 tends to increase t's own preference for that commodity. We claim that all conditions of Corollary 4.2 hold. To begin with, Assumption 4.1 is seen to hold by the above. To comply with (4.1), we define:

$$e_t(p, \delta) := \int_t^1 [\int_{\mathbb{R}^2_+} x^1 \delta^1(\tau)]\mathrm{d}\tau = \int_t^1 (\operatorname{bar} \delta(\tau))^1 \, \mathrm{d}\tau.$$

Observe that this can also be written as $e_t(p, \delta) = I_t(\delta)$, where I_t is as in Example 3.2, with $s := 1$ and $\ell_1 : T \times D \to \mathbb{R}$ defined by $\ell_1(t, \tau, x) := x^1$ if $\tau > t$ and $\ell_1(t, \tau, x) := 0$ if $\tau \leq t$. The validity of Assumption 3.4′ follows now from Example 3.2(b). It is easy to see that Assumptions 3.5′ and 3.6′ hold as well, either by direct inspection of the formula for demand given above or by invoking Proposition 3.1. The Assumptions 3.1 and 3.2′ hold obviously, and for $\omega \in \mathcal{L}^1_X$ we demand $\int_0^1 \omega \in \mathbb{R}^2_{++}$. Thus, all assumptions of Corollary 4.2 are fulfilled. It follows that there exists $(p_*, f_*) \in P \times \mathcal{L}^1([0, 1])$ such that $(p_*, f_*) \in \boldsymbol{CE}_2(\mathcal{E})$. That is to say, we have $(p^1_*, p^2_*) \in \mathbb{R}^2_{++}$ and for a.e. t in T:

$$f^1_*(t) = \frac{\alpha^1(t) + \int_t^1 f^1_*}{\alpha^1(t) + \alpha^2(t) + \int_t^1 f^1_*} \frac{p_*\omega(t)}{p^1_*}, \qquad f^2_*(t) = \frac{\alpha^2(t)}{\alpha^1(t) + \alpha^2(t) + \int_t^1 f^1_*} \frac{p_*\omega(t)}{p^2_*}.$$

This holds together with $\int_0^1 f_* = \int_0^1 \omega$.

Next, we turn to more delicate situations, which require a mix of purification by non-atomicity and convexity. Let T^{pa} be the purely atomic part of the measure space $(T, \mathcal{T}, \mu)$; it consists of the union of all non-null atoms A_i in $(T, \mathcal{T}, \mu)$, of which there are (essentially, i.e., modulo null sets) at most countably many. Then $T^{\text{na}} := T \setminus T^{\text{pa}}$, provided with the usual restrictions of $\mathcal{T}$ and μ, forms a non-atomic measure space. We follow the setup and notation of Balder (2002) and denote by $\hat{T} \subset T^{\text{na}}$ a fixed, possibly empty set in $\mathcal{T}$; we denote its complement $T \setminus \hat{T}$ by $\bar{T}$. Associated to this, we write $\hat{D} := D \cap (\hat{T} \times \mathbb{R}^l)$, $\hat{\mathcal{D}} := \hat{D} \cap (\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l))$ $\bar{D} := D \cap (\bar{T} \times \mathbb{R}^l)$ and $\bar{\mathcal{D}} := \bar{D} \cap (\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l))$. Note that in much of the related literature (e.g., Aumann, 1966; Balder, 2003; Greenberg et al., 1979; Schmeidler, 1973) one requires that the whole space $(T, \mathcal{T}, \mu)$ is non-atomic; this case can of course be handled by choosing $\hat{T} := T$ and $\bar{T} := \emptyset$.

Assumption 4.2. $M_t(p, y)$ is convex for every $t \in \bar{T}$, $p \in P$ and $y \in Y$.

We suppose that this assumption is in force from now on, together with the basic Assumptions 3.1–3.3 of the previous section. In the preference-based model of Example 3.1, Assumption 4.2 holds if the sets X_t are convex and the preference relations $P_{t,y}$ are convex on X_t for every $t \in T$ and $y \in Y$ (a special case of this is found when the $P_{t,y}$ are ordered preferences and the associated utility functions $U_t(\cdot, y)$ are quasi-concave on X_t).

Following Balder (2003) and completely analogous to similar developments for continuum games in Balder (1995a, 1999b, 2002, in press), the next proposition provides a framework for purification. Its part (a) is elementary and its part (b) is an immediate consequence of Lyapunov's theorem for Young measures in Section 2(ix). It holds for both $k = 1$ and 2.

Proposition 4.1.

(a) *Let (p, δ) in $\boldsymbol{CE}_k(\mathcal{RE})$ be arbitrary. If there exists $f \in \mathcal{L}^1_X$ such that*
  (i) $f(t) = \text{bar}\, \delta(t)$ *for a.e. t in $\bar{T}$,*
  (ii) $f(t) \in \text{supp}\, \delta(t)$ *for a.e. t in $\hat{T}$,*
  (iii) $\int_{\hat{T}} f \, d\mu = \int_{\hat{T}} \text{bar}\, \delta \, d\mu$,
  (iv) $e_t(p, \delta) = d_t(p, f)$ *for a.e. t in T,*
  then $(p, f) \in \boldsymbol{CE}_k(\mathcal{E})$.

(b) *By Lyapunov's theorem for Young measures, for every (p, δ) in $\boldsymbol{CE}_k(\mathcal{RE})$ there exists $f \in \mathcal{L}^1_X$ for which the above conditions (i), (ii) and (iii) hold.*

Proof.

(a) On $\bar{T}$, property (2.3) and Assumption 4.2 turn (i) into $f(t) \in M_t(p, e_t(p, \delta)) \subset X_t$ a.e., and on on $\hat{T}$ one has $f(t) \in \text{supp}\, \delta(t) \subset M_t(p, e_t(p, \delta)) \subset X_t$ a.e. by (ii). Hence, (iv) turns this into $f(t) \in M_t(p, d_t(p, f))$ a.e. on T. Finally, by $\int_T \text{bar}\, \delta = \int_{\hat{T}} \text{bar}\, \delta + \int_{\bar{T}} f$, condition (iii) easily causes $\int_T f \leq \int_T \omega$ or $\int_T f = \int_T \omega$, as the case may be.

(b) On $\bar{T}$, property (2.3) and Assumption 4.2 imply (i), and on $\hat{T} \subset T^{\text{na}}$ an application of Lyapunov's theorem for Young measures (Section 2(ix)) to $g_i(t, x) := x^i$, $i = 1, \ldots, l$, immediately gives (ii) and (iii).

The finding expressed in part (b) of the above proposition explains why Balder (2002, Step 4, p. 465) just concerns combining Lyapunov's theorem for Young measures with condition (iv). Of course, the validity of condition (iv) will have to be studied more closely below. □

Theorem 4.1. *Under* Assumptions 3.1–3.6 and 4.2 *there exists an equilibrium pair in* $\boldsymbol{CE}_1(\mathcal{E})$, *provided that conditions (i) to (iv) in* Proposition 4.1 *are fulfilled.*

Here conditions (i) to (iv) in Proposition 4.1 are said to be fulfilled if for every $(p, \delta) \in \boldsymbol{CE}_k(\mathcal{RE})$ there exists $f \in \mathcal{L}^1_X$ that meets conditions (i) to (iv) in Proposition 4.1.

Proof. All assumptions needed for an application of Theorem 3.1 are met. Hence, there exists an equilibrium pair (p_*, δ_*) in $\boldsymbol{CE}_1(\mathcal{RE})$. Let f_* be associated to $(p, \delta) := (p_*, \delta_*)$ as in Proposition 4.1(a). Then $(p_*, f_*) \in \boldsymbol{CE}_1(\mathcal{E})$ by that proposition. □

Theorem 4.2. *Under* Assumptions 3.1, 3.2′, 3.3′, 3.4′, 3.5′, 3.6′ and 4.2 *there exists an equilibrium pair in* $\boldsymbol{CE}_2(\mathcal{E})$, *provided that conditions (i) to (iv) in* Proposition 4.1 *are fulfilled.*

The proof is an obvious modification of the previous proof, so it is omitted.

Remark 4.1. As noted before, all assumptions of Theorem 3.1 hold in the simple Example 4.1 above. However, those of Theorem 4.1 do not. The reason for this lies in the fact that the four conditions of Proposition 4.1 are not fulfilled. To begin with, note that the non-convex nature of the sets $M_t(p, y)$ in Example 4.1 already forces $\bar{T} = \emptyset$ in Example 4.1, because of Assumption 4.2. Now in Example 4.1 we showed $(p_*, \delta_*) \in \boldsymbol{CE}_1(\mathcal{RE})$, with $p_* := 1$ and $\delta_* := ((1/2)\epsilon_0) + ((1/2)\epsilon_2)$. If all four conditions in Proposition 4.1 were to be fulfilled, there should exist $f \in \mathcal{L}^1_X$ with $\int_0^t$ bar $\delta_* = \int_0^t f$ for all $t \in T$, i.e., with $f(t) = 1$ for a.e. t in T (here we apply condition (iv)). But this is in conflict with what condition (ii) of Proposition 4.1 requires. See Balder (2003) for a similar counterexample with $\boldsymbol{CE}_2(\mathcal{E}) = \emptyset$ and $\boldsymbol{CE}_2(\mathcal{RE}) \neq \emptyset$.

Remark 4.2. Corollaries 4.1 and 4.2, derived as direct consequences of Theorems 3.1 and 3.2, can also be be viewed as corollaries of Theorems 4.1 and 4.2. Namely, for $\bar{T} := T$ it is obvious that Assumption 4.2 is fulfilled under Assumption 4.1. So to obtain Corollary 4.1 from Theorem 4.1 it is enough to show that the four conditions of Proposition 4.1 are fulfilled. Fix any $(p, \delta) \in \boldsymbol{CE}_1(\mathcal{RE})$. Then conditions (ii) and (iii) hold trivially by $\hat{T} = \emptyset$. Also, just as in the proof of Corollary 4.1, it follows directly from Assumption 4.1 that there exists $f \in \mathcal{L}^1_X$ with $\delta(t) = \epsilon_f(t)$ a.e. Clearly, condition (i) of Proposition 4.1 is then already satisfied. Now $e_t(p, \delta) = e_t(p, \epsilon_f)$ for every $t \in T$, for we have $\rho(\epsilon_f, \delta) = 0$ and $e_t(p, \cdot)$ is continuous for ρ by Assumption 3.4(ii). Here ρ is the semi-metric of Section 2(vi). Hence, (4.1) gives $e_t(p, \delta) = d_t(p, f)$ for every $t \in T$ and this satisfies condition (iv) of Proposition 4.1.

5. Special cases

We shall study some special cases for the ordinary externality mappings d_t, $t \in T$. In each case we provide corresponding mixed externality mappings e_t, $t \in T$ in such a way that Theorems 4.1 and 4.2 are applicable.

5.1. No externalities or price externalities only

The situation with no externalities is found in the classical preference-based model of Aumann (1966) and its extension to non-complete preferences by Schmeidler (1969). In that model one has $d_t(p, f) = 0$ for all $p \in P$, $f \in \mathcal{L}^1_X$. A model with only price externalities, viz. $d_t(p, f) = p$ for all $p \in P$ and $f \in \mathcal{L}^1_X$, was studied in the paper by Greenberg et al. (1979). In all three papers $(T, \mathcal{T}, \mu)$ is non-atomic, so we choose $\hat{T} = T$ and $\bar{T} = \emptyset$. Of course, the corresponding mixed externality mappings to be used in our results are respectively $e_t(p, \delta) := 0$, with $Y := \{0\}$, and $e_t(p, \delta) := p$ with $Y := P$. These choices trivially satisfy Assumptions 3.4′ and 3.4, respectively. The four conditions of Proposition 4.1 are also fulfilled: condition (iv) is now trivial and the other conditions (i) to (iii) hold by Proposition 4.1(b). As for Aumann (1966); Schmeidler (1969), we note that Assumptions 3.2′and 3.3′ are fulfilled by an earlier comment following the introduction of those assumptions. Using Proposition 3.1, it is easy to see that the other assumptions of Theorem 4.2 hold in Aumann (1966); Schmeidler (1969).

As for Greenberg et al. (1979), which works with utility functions and also considers production, we observe that the trick explained on Balder (1999a, p. 37) to incorporate production as an additional decision variable in the utility function (this is partly based on footnote 3 above), continues to be applicable in the present paper. Therefore, using Proposition 3.1, it can be seen that in Greenberg et al. (1979, Theorem, p. 33) all the assumptions of Theorem 4.1

are satisfied. Thus, Theorems 4.1 and 4.2 generalize the main results of Aumann (1966); Greenberg et al. (1979), and Schmeidler (1969).

In Remark 3.4 we already showed how Theorem 3.1 continues to hold under a slight modification of Assumption 3.5(ii). It is not hard to verify that the same holds for Theorem 4.1, so that theorem also implies the main result of Yamazaki (1978).

5.2. Price-consumption externalities of "convex" type

A situation with both price and consumption externalities is found in the paper by Cornet and Topuzu (2005), who call their consumption externalities "convex" (the apostrophes being included). In (Cornet and Topuzu, 2005, Theorem 3) they show that their work generalizes earlier results of Noguchi (2005), who works with externality mappings of the kind:

$$d_t(p, f) = \left(\int_{C^1_{t,p}} f \, d\mu, \ldots, \int_{C^N_{t,p}} f \, d\mu \right).$$

Here $C^1_{t,p}, \ldots, C^N_{t,p}$ denote the finitely many reference coalitions of consumer t. Incidentally, the competitive equilibrium existence results in Cornet and Topuzu (2005) and Noguchi (2005) are of a type similar to Theorem 3.2, but there is no analogue of Theorem 3.1; hence the main existence result of Cornet and Topuzu (2005, Theorem 2) does not apply directly to continuum games, but it does so when their monotonicity condition M is traded in for extra compactness conditions for the feasible consumption sets X_t and free disposal is allowed—see Cornet and Topuzu (2005, Theorem 4). As shown by Example 5.1 below, "convex" externalities, as used in Theorems 2, 4 of Cornet and Topuzu (2005), are rather special. That is so because Cornet and Topuzu (2005, Assumption EC) imposes the following very strong conditions on the mappings d_t, $t \in T$: for every $p \in P$, every finite collection $\{f_1, \ldots, f_N\}$ in $\mathcal{L}^1_X$ and every $f_0 \in \mathcal{L}^1_X$ with $f_0(t) \in \text{co}\{f_i(t)\}_{i=1}^N$ for a.e. t in $\hat{T}$, there must exist $\hat{f} \in \mathcal{L}^1_X$ such that

(1) $\hat{f}(t) = f_0(t)$ for a.e. t in $\bar{T}$,
(2) $\hat{f}(t) \in \{f_i(t)\}_{i=1}^N$ for a.e. t in $\hat{T}$,
(3) $\int_T \hat{f} \, d\mu = \int_T f_0 \, d\mu$,
(4) $d_t(p, \hat{f}) = d_t(p, f_0)$ for a.e. t in T.

First, we show that this implies that the four conditions of Proposition 4.1 are fulfilled, provided that we define the associated mixed externality mappings e_t, $t \in T$, as follows (Balder, 2005; Balder et al., 2004):

$$e_t(p, \delta) := d_t(p, \text{bar}\, \delta), \ p \in P, \ \delta \in \mathcal{R}^1_X, \tag{5.1}$$

This definition is both meaningful and unambiguous. It is meaningful, because in Cornet and Topuzu (2005) the sets X_t are identically equal to $\mathbb{R}^l_+$. Hence they are convex, which implies that the barycentric function $\text{bar}\, \delta$ belongs to $\mathcal{L}^1_X$ for every $\delta \in \mathcal{R}^1_X$. The definition is also unambiguous, because the weak continuity of $d_t(p, \cdot)$, adopted in Cornet and Topuzu (2005) and stated below, causes $d_t(p, f) = d_t(p, f')$ for all $t \in T$ and $p \in P$, whenever f and f' differ only on a null set. To see that Cornet and Topuzu (2005, Assumption EC) implies that the four conditions of Proposition 4.1 are fulfilled, fix any $(p, \delta) \in \boldsymbol{CE}_k(\mathcal{RE})$. Observe that by (2.3), together with Carathéodory's theorem and standard measurable selection results (see the proof of Balder, 2000b, Theorem 5.3; for the details), there exist measurable functions $\alpha_1, \ldots, \alpha_{l+1} : \hat{T} \to [0, 1]$, with $\sum_i \alpha_i = 1$, and measurable functions $f_1, \ldots, f_{l+1} : \hat{T} \to \mathbb{R}^l_+$ such that for a.e. t in $\hat{T}$:

$$\text{bar}\, \delta(t) = \sum_{i=1}^{l+1} \alpha_i(t) f_i(t) \quad \text{and } f_i(t) \in \text{supp}\, \delta(t) \subset X_t \text{ for all } i.$$

By non-negativity of the f_i and integrability of $\text{bar}\, \delta$ on $\hat{T}$, this immediately implies the integrability of $f_1, \ldots, f_{l+1}$ on $\hat{T}$. So if we also set $f_i(t) := \text{bar}\, \delta(t)$ on the complement $\bar{T}$ of $\hat{T}$, $i = 1, \ldots, l+1$, then it follows that $f_1, \ldots, f_{l+1}$ belong to $\mathcal{L}^1_X$. Let $\hat{f}$ correspond to $f_0 := \text{bar}\, \delta$ as in the above (1)–(4) from Cornet and Topuzu (2005, Assumption EC). Then setting $f := \hat{f}$ fulfills the conditions of Proposition 4.1. Namely, (1) follows from (i) by our choice $f_0 := \text{bar}\,\delta$, (2) implies (ii) by $\{f_i(t)\}_i \subset \text{supp}\,\delta(t)$, (1) and (3) imply (iii), and (iv) follows by (4) and $e_t(p, \delta) := d_t(p, \text{bar}\,\delta) = d_t(p, f_0)$.

Next, we show that the weak continuity conditions for d_t, $t \in T$, in Cornet and Topuzu (2005, Assumptions E,EB) cause Assumption 3.4$'$ to be fulfilled for the mixed externality of (5.1). Fix $t \in T$, $m \in \mathbb{N}$ and let $\{\delta_k\}_k$ converge narrowly to δ_0 in $\mathcal{R}_X(m)$. Because the multifunction $t \mapsto X_t^m$ is integrably bounded, it follows that all δ_k satisfy (2.6) with $\psi := m\tilde{\omega}$. Hence, by Section 2(x) we have that $\{\text{bar}\,\delta_k\}_k$ converges to $\text{bar}\,\delta_0$ in the weak topology $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$. So by (5.1) the weak continuity required in Cornet and Topuzu (2005, Assumption E(iii)) implies that our continuity Assumption 3.4$'$(ii) is fulfilled. Also, Cornet and Topuzu (2005, Assumption E(i)) is identical to Assumption 3.4$'$(i). Moreover, Assumption 3.4$'$(iii) is fulfilled by Cornet and Topuzu (2005, Assumption EB) and the fact that the space Y in Cornet and Topuzu (2005) is supposed to be finite-dimensional. The remaining conditions of Theorem 3.2 are seen to hold in Cornet and Topuzu (2005, Theorems 1 and 2) by means of Proposition 3.1(Assumption 3.3$'$ is fulfilled by $\check{f} := 0$, because $\int_T \omega \in \mathbb{R}^l_{++}$ in Cornet and Topuzu, 2005). Thus, Theorem 4.2 implies the main results of Cornet and Topuzu (2005). To conclude this subsection, we illustrate the limitations of Cornet and Topuzu (2005, Assumption EC):

Example 5.1. Take $l = 1$; hence $P = \{1\}$. Take $T := [0, 1]$, equipped with the Lebesgue measure and σ-algebra, and set $\hat{T} := T$, $X_t := \mathbb{R}_+$. For $Y := \mathbb{R}$ consider $d_t(1, f) := \int_T (f(t))^2\, \mathrm{d}t$. Then Cornet and Topuzu (2005, Assumption EC), reproduced in (1)–(4) above, does not hold. To see this, take for instance $f_0 = 1$, $f_1 = 0$ and $f_2 = 2$ (all constant). Then $f_0(t) \in \text{co}\{f_1(t), f_2(t)\}$ a.e. on $\hat{T}$, but no $\hat{f} \in \mathcal{L}^1_X$ exists with $\hat{f}(t) \in \{0, 2\}$ a.e., $\int_T \hat{f} = \int_T f_0$ and $\int_T (\hat{f})^2 = \int_T (f_0)^2$, which would fulfill (2), (3) and (4), respectively. Indeed, the existence of such a function would lead to $0 = \int_T ((\hat{f})^2 - 2\hat{f}) = \int_T ((f_0)^2 - 2f_0) = -1$, which is absurd.

5.3. Hybrid price-consumption externalities

The limitations demonstrated in Example 5.1 can be overcome by working with a somewhat richer class of externality mappings. This allows for a separate treatment of the non-atomic part T^{na} (or its subset $\hat{T}$), upon which one can adopt the externality also used in Balder (2003). This could be done in the context of the previous subsection, but that would still leave us with conditions of the kind used in Proposition 4.1 or Cornet and Topuzu (2005, Assumption EC), which are non-explicit in terms of the original components of the economy $\mathcal{E}$. Therefore, in this subsection we adopt a more concrete model for the price and consumption externalities. It is inspired by Balder (2002) and Balder et al. (2004) and coincides with the model used in Balder (2003) in the situation studied there, which has $\bar{T} = \emptyset$. Let $r, s \in \mathbb{N}$. We define $Y := P \times \mathbb{R}^r \times \mathbb{R}^s$. Let $\{g_1, \ldots, g_r\}$ be a finite collection of $\hat{\mathcal{D}} \otimes \mathcal{B}(P)$-measurable functions $g_j : \hat{D} \times P \to \mathbb{R}$, each of which is integrably bounded and such that $g_j(t, \cdot, \cdot)$ is continuous on $X_t \times P$ for every $t \in \hat{T}$. Also, let $\{\ell_1, \ldots, \ell_s\}$ be a finite collection of $\mathcal{T} \otimes \bar{\mathcal{D}} \otimes \mathcal{B}(P)$-measurable functions $\ell_j : T \times \bar{D} \times P \to \mathbb{R}$, such that for every j the function $\ell_j(t, \tau, \cdot, \cdot)$ is continuous on $X_\tau \times P$ for every $t \in T$, $\tau \in \bar{T}$ and such that for every $t \in T$:

$$\ell_j(t, \tau, \cdot, p) \text{ is affine on } X_\tau \quad \text{for every } \tau \in \bar{T} \text{ and } p \in P, \tag{5.2}$$

and such that for every $t \in T$ there exist $k_{j,t} \in \mathbb{R}_+$ and $\phi_{j,t} \in \mathcal{L}^1(T; \mathbb{R})$ with

$$|\ell_j(t, \tau, x, p)| \le \phi_{j,t}(\tau) + k_{j,t}|x| \quad \text{for every } \tau \in T,\ x \in X_\tau \text{ and } p \in P.$$

For $t \in T$, define $d_t : P \times \mathcal{L}^1_X \to Y$ as follows:

$$d_t(p, f) := (p, \hat{d}(p, f), \bar{d}_t(p, f)), \tag{5.3}$$

where

$$\hat{d}(p, f) := \left(\int_{\hat{T}} g_1(t, f(t), p)\mu(\mathrm{d}t), \ldots, \int_{\hat{T}} g_r(t, f(t), p)\mu(\mathrm{d}t) \right),$$

and

$$\bar{d}_t(p, f) := \left(\int_{\bar{T}} \ell_1(t, \tau, f(\tau), p)\mu(\mathrm{d}\tau), \ldots, \int_{\bar{T}} \ell_s(t, \tau, f(\tau), p)\mu(\mathrm{d}\tau) \right).$$

Thus, information about the consumption profile f on the part $\hat{T} \subset T^{\text{na}}$, i.e., the restriction of the function f to $\hat{T}$, is only measured via the finite-dimensional "statistic" $\hat{d}(p, f)$, which is common to all $t \in \hat{T}$. On the complementary part

$\bar{T}$ each externality mapping is allowed to depend on t, but the price paid for this generality is that the corresponding integrands ℓ_j must be affine in the consumption variable. Corresponding to (5.3), we define $e_t : P \times \mathcal{R}^1_X \to Y, t \in T$, as follows. We set $e_t(p, \delta) := (p, \hat{e}(p, \delta), \bar{e}_t(p, \delta))$, where

$$\hat{e}(p, \delta) := \left(\int_{\hat{T}} \left[\int_{X_t} g_1(t, x, p)\delta(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t), \ldots, \int_{\hat{T}} \left[\int_{X_t} g_r(t, x, p)\delta(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t) \right).$$

and

$$\bar{e}_t(p, \delta) : = \left(\int_{\bar{T}} \left[\int_{X_\tau} \ell_1(t, \tau, x, p)\delta(\tau)(\mathrm{d}x) \right] \mu(\mathrm{d}\tau), \ldots, \int_{\bar{T}} \left[\int_{X_\tau} \ell_s(t, \tau, x, p)\delta(\tau)(\mathrm{d}x) \right] \mu(\mathrm{d}\tau) \right).$$

We claim that, because of these choices, all conditions of Proposition 4.1 are fulfilled, including the crucial condition (iv). To prove this, we imitate the proof of Proposition 4.1(b) as follows. Fix (p, δ) in $\boldsymbol{CE}_k(\mathcal{RE})$. As in that proof, we define $g_i(t, x) := x^i, i = 1, \ldots, l$, but now we define r additional functions: we set $g_{l+j}(t, x) := g_j(t, x, p)$ if $x \in X_t$ and $g_{l+j}(t, x) := 0$ if $x \in \mathbb{R}^l \setminus X_t$. Then Lyapunov's theorem for Young measures, applied to the collection $\{g_1, \ldots, g_{l+r}\}$, yields a function $f \in \mathcal{L}^1_X(\hat{T})$ that has $f(t) \in \operatorname{supp} \delta(t)$ a.e. in $\hat{T}$, $\int_{\hat{T}} f = \int_{\hat{T}} \operatorname{bar} \delta$ and

$$\int_{\hat{T}} g_j(t, f(t), p)\mu(\mathrm{d}t) = \int_{\hat{T}} \left[\int_{X_t} g_j(t, x, p)\delta(t)(\mathrm{d}x) \right] \mu(\mathrm{d}t), \ j = 1, \ldots, r.$$

Next to these r identities, we have

$$\int_{\bar{T}} \ell_j(t, \tau, f(\tau), p)\mu(\mathrm{d}\tau) = \int_{\bar{T}} \left[\int_{X_\tau} \ell_j(t, \tau, x, p)\delta(\tau)(\mathrm{d}x) \right] \mu(\mathrm{d}\tau)$$

for every $t \in T$, $j = 1, \ldots, s$.

These follow from the affinity in (5.2) and the definition of barycenter by setting $f := \operatorname{bar} \delta$ on $\bar{T}$ (the same choice for f on $\bar{T}$ was made in the proof of Proposition 4.1). This proves that all four conditions of Proposition 4.1 are fulfilled. We thus obtain the following corollary of Theorem 4.2:

Corollary 5.1. *Under* Assumptions 3.1, 3.2′–3.6′ and 4.2 *there exists an equilibrium pair in* $\boldsymbol{CE}_2(\mathcal{E})$ *for the model with ordinary externality mappings* d_t, $t \in T$, *as given by* (5.3).

Proof. Above, we already verified that the four conditions of Proposition 4.1 are fulfilled. Now Assumption 3.4′ is valid here by Example 3.2(in particular, the last line of that example applies here). So application of Theorem 4.2 gives the result. □

In Example 5.1 it was shown that the externality $d_t(f) := \int_T (f)^2$ is not covered by the results in Cornet and Topuzu (2005). In contrast, Corollary 5.1 can be applied to it, because it is of the type (5.3): choose $\hat{T} = T$ and set $g_1(t, x) := (x)^2$.

Of course, next to Theorem 4.2, one can also apply Theorem 4.1 to (5.3), with the same associated mixed externalities e_t, $t \in T$ as above. For these to satisfy the continuity Assumption 3.4(ii), which is stronger than Assumption 3.4′(ii), we must place much heavier conditions on the components $\bar{e}_t$, $t \in T$, introduced above. Namely, we must now require in addition that the multifunctions $t \mapsto X_t$ are integrably bounded on $\bar{T}$, for only in this way we can meet (2.6). Once this is done, we are in a setup with compact feasible consumption sets on $\bar{T}$, where much more general results can be obtained using the feeble topology of Balder (1999b) and Balder (2002). But if we stick to the present model, then the remaining details are quite similar to the ones discussed for Corollary 5.1. The following result can be obtained:

Corollary 5.2. *For the model with ordinary externality mappings* d_t, $t \in T$, *as given by* (5.3), *the following holds. Under* Assumptions 3.1–3.6 and 4.2 *there exists an equilibrium pair in* $\boldsymbol{CE}_1(\mathcal{E})$, *provided that there exists an integrable function* $\psi : T \to \mathbb{R}$ *such that*

$$\sup_{x \in X_t} |x| \leq \psi(t) \quad \text{for every } t \in \bar{T}.$$

As a further specialization, which is quite similar to Balder (2003, Corollary 3.1), this existence result can also be applied to games with a measure space of players, as introduced by Schmeidler (1973). Namely, the budget constraint can be made to hold vacuously by choosing the endowment function ω sufficiently large: taking $\omega^i(t) := \psi(t), i = 1, \ldots, l$, one gets $B_t(p) = X_t$ for every $t \in \bar{T}$ and $p \in P$. Theorem 1 of Schmeidler (1973) thus follows (note: in a similar manner Theorem 3.1 generalizes the other main existence result in Schmeidler (1973, Theorem 2)—see Remark 3.7 and Balder (2003)). But here again it should be observed that when the sets X_t are compact, much better results are available (Balder, 2002; Khan, 1984).

6. Proofs of Theorems 3.1 and 3.2

6.1. *Proof of Theorem 3.1*

In this subsection we assume from now on, unless the contrary is explicitly stated, that all assumptions of Theorem 3.1 are fulfilled. We start by introducing $P^m := \{p \in P : \min_i p^i \geq 1/m\}$ for each $m \in \mathbb{N}$. Let us observe that for every $t \in T$ and $p \in P^m$ we have that $px \leq p\omega(t)$ implies $\sum_i x^i \leq m\tilde{\omega}(t)$. Hence the important inclusion $B_t(p) \subset X_t^m$ holds for every $t \in T$ and $p \in P^m$.

Lemma 6.1. *$P^m \times \mathcal{R}_X(m)$ is non-empty, convex and compact for every $m \in \mathbb{N}$.*

Proof. Define $h^m(t, x) := 0$ if $x \in X_t^m$ and $h^m(t, x) := +\infty$ if $x \notin X_t^m$. Then compactness of the sets X_t^m, which follows by virtue of Assumption 3.2 from the definition of those sets, together with Assumption 3.1, implies that h^m belongs to the class $\mathcal{H}^{\text{bb}}(t; \mathbb{R}^l)$ of inf-compact normal integrands in the sense of Section 2(iii). Now $\mathcal{R}_X(m)$ is easily seen to be the set of all $\delta \in \mathcal{R}(T; \mathbb{R}^l)$ such that $I_{h^m}(\delta) \leq 0$. Therefore, $\mathcal{R}_X(m)$ is compact for the narrow topology by Section 2(viii). As observed before, $\mathcal{R}_X(m)$ is non-empty by Assumption 3.3 and it is trivially convex. □

For $p \in P$ and $\delta \in \mathcal{R}_X^1$ let

$$F_1(p, \delta) := \{\eta \in \mathcal{R}_X^1 : \operatorname{supp} \eta(t) \subset M_t(p, e_t(p, \delta)) \quad \text{for a.e. } t \text{ in } T\}$$

and for $m \in \mathbb{N}$ and $\delta \in \mathcal{R}^m$ let

$$F_2^m(\delta) := \operatorname{argmax}_{q \in P^m} q \left[\int_T (\operatorname{bar} \delta - \omega) \, d\mu \right].$$

Lemma 6.2.

(a) *The graph*

$$G_1 := \{(p, \delta, \eta) \in P \times \mathcal{R}_X^1 \times \mathcal{R}_X^1 : \eta \in F_1(p, \delta)\}$$

of the multifunction F_1 is closed.

(b) *For every $m \in \mathbb{N}$ the graph*

$$G_1^m := G_1 \cap (P^m \times \mathcal{R}_X(m) \times \mathcal{R}_X(m))$$

of the restriction of F_1 to $P^m \times \mathcal{R}_X(m)$ is closed.

(c) *$F_1(p, \delta)$ is non-empty and convex for every $m \in \mathbb{N}$, $p \in P^m$ and $\delta \in \mathcal{R}_X(m)$.*

Proof.

(a) We may prove the closedness of G_1 by an argument that only involves sequences (recall that $\mathcal{R}(T;\mathbb{R}^l)$ is semi-metrizable). Let $\{(p_k, \delta_k, \eta_k)\}$ converge to (p_0, δ_0, η_0) and suppose that $(p_k, \delta_k, \eta_k) \in G_1$ for all $k \in \mathbb{N}$. Then by Section 2(vii) we have

$$\mathrm{supp}\,\eta_0(t) \subset \mathrm{Ls}_k\,\mathrm{supp}\,\eta_k(t) \subset \mathrm{Ls}_k M_t(p_k, e_t(p_k, \delta_k)) \quad \text{for a.e. } t.$$

In other words, we have

$$\mathrm{supp}\,\eta_0(t) \subset \mathrm{Ls}_k\, M_t(p_k, e_t(p_k, \delta_k)) \quad \text{for a.e. } t.$$

By Assumption 3.5(ii) this gives $\mathrm{supp}\,\eta_0(t) \subset M_t(p_0, e_t(p_0, \delta_0))$ for a.e. t, because of $e_t(p_k, \delta_k) \to e_t(p_0, \delta_0)$, which holds by Assumption 3.4(ii). This proves that (p_0, δ_0, η_0) belongs to G_1, so G_1 is closed.

(b) The closedness of the subsets G_1^m of G_1 follows trivially from (a), because of Lemma 6.1.

(c) Fix $m \in \mathbb{N}$, $p \in P^m$ and $\delta \in \mathcal{R}_X(m)$. The set $\{(t, x) \in T \times \mathbb{R}^l : x \in M_t(p, e_t(p, \delta))\}$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable because of Assumptions 3.5(i) and 3.4(i). So by $p \in P^m \subset \mathbb{R}^l_{++}$ and Assumption 3.6 it follows from the von Neumann–Aumann measurable selection theorem (Castaing and Valadier, 1977, III) that there exists a function $f : T \to \mathbb{R}^l$, measurable with respect to $\mathcal{T}$, such that $f(t) \in M_t(p, e_t(p, \delta))$ for every $t \in T$. Then the Dirac transition probability ϵ_f belongs to $\mathcal{R}^1_X$ and $\mathrm{supp}\,\epsilon_f(t) = \{f(t)\} \subset M_t(p, e_t(p, \delta))$. This shows the non-emptiness of $F_1(p, \delta)$, and convexity of this set holds trivially. □

Remark 6.1. In connection with the proof of Theorem 3.2, which is to follow, we observe that closedness of G_1^m could also have been proven directly, i.e., without first proving closedness of G_1. To do this, one simply imitates the argument used in part (a) and uses also Lemma 6.1. This still requires the use of Assumption 3.5 and 3.4$'$, but not the full strength of Assumption 3.4.

Lemma 6.3. *For every $m \in \mathbb{N}$ the following hold*:

(a) *The graph*

$$G_2^m := \{(\delta, q) \in \mathcal{R}_X(m) \times P^m : q \in F_2^m(\delta)\}$$

of the multifunction F_2^m is closed.

(b) *$F_2^m(\delta)$ is non-empty and convex for every $\delta \in \mathcal{R}_X(m)$.*

Proof.

(a) Let $\{(\delta_k, q_k)\}$ converge to (δ_0, q_0) and suppose that $(\delta_k, q_k) \in G_2^m$ for all $k \in \mathbb{N}$. By definition of $\mathcal{R}_X(m)$, (2.6) holds with $\psi := m\tilde{\omega}$. Therefore, Section 2(x) implies that $\delta \mapsto \int_T (\mathrm{bar}\,\delta - \omega)^j \,\mathrm{d}\mu$ is continuous on $\mathcal{R}_X(m)$. By Berge's theorem of the maximum, this implies $q_0 \in \mathrm{argmax}_{q \in P^m}\, q[\int_T (\mathrm{bar}\,\delta_0 - \omega)\,\mathrm{d}\mu]$. It proves that G_2^m is closed.

(b) Non-emptiness of $F_2^m(\delta)$ follows by the Weierstrass theorem from compactness of P^m and continuity of the function $q \mapsto q \int_T (\mathrm{bar}\,\delta - \omega)\,\mathrm{d}\mu$. The convexity of $F_2^m(\delta)$ follows simply from linearity of the same function and convexity of P^m. □

Lemma 6.4. *For every $m \in \mathbb{N}$ there exist $\hat{p}_m \in P^m$ and $\hat{\delta}_m \in \mathcal{R}_X(m)$ such that*

$$\mathrm{supp}\,\hat{\delta}_m(t) \subset M_t(\hat{p}_m, e_t(\hat{p}_m, \hat{\delta}_m)) \quad \textit{for a.e. } t \textit{ in } T$$

and such that for $m \geq d+1$

$$\max_{1\leq i\leq d} \int_T (\mathrm{bar}\,\hat{\delta}_m - \omega)^i \,\mathrm{d}\mu \leq \frac{1}{m-d}\int_T \tilde{\omega}\,\mathrm{d}\mu.$$

Proof. Fix $m \geq d+1$. The inclusion $B_t(p) \subset X_t^m$, observed earlier, implies that $F_1(p,\delta) \subset \mathcal{R}_X(m)$ if $p \in P^m$. Thus, the multifunction $F^m : (p,\delta) \mapsto F_1(p,\delta) \times F_2^m(\delta)$ maps $P^m \times \mathcal{R}_X(m)$ into itself. By Lemma 6.1 the set $P^m \times \mathcal{R}_X(m)$ is non-empty convex and compact and by Lemmas 6.2(b),(c) and 6.3 it follows that F^m has a closed graph with non-empty, convex and closed values. So we can apply the non-Hausdorff transcription of Kakutani's theorem, as given in Balder (1999b), which gives the existence of $(\hat{p}_m, \hat{\delta}_m) \in P^m \times \mathcal{R}_X(m)$ such that $(\hat{p}_m, \hat{\delta}_m) \in F^m(\hat{p}_m, \hat{\delta}_m)$. Alternatively, one can introduce μ-a.e. equivalence classes in $\mathcal{R}(T;\mathbb{R}^l)$ and use the standard Kakutani theorem; a second alternative to work with the standard Kakutani theorem (i.e., in a Hausdorff space setting) is to form product measures $\mu \otimes \delta$ and to use the ws-topology on $\mathrm{Prob}(T \times \mathbb{R}^l)$—see Section 2($xi$)). Now the first property stated in the lemma follows immediately. To prove the second property, we first combine the elementary identity:

$$\sup_{q\in P^m} qa = \frac{1}{m}\sum_i a^i + \left(1-\frac{d}{m}\right)\max_i a^i \quad \text{for every } a \in \mathbb{R}^l$$

with the fact that for $a_m := \int_T (\mathrm{bar}\,\hat{\delta}_m - \omega)\,\mathrm{d}\mu$

$$\sup_{q\in P^m} qa_m = \hat{p}_m a_m = \int_T \left[\int_{\mathbb{R}^l} (\hat{p}_m x - \hat{p}_m\omega(t))\hat{\delta}_m(t)(\mathrm{d}x)\right]\mu(\mathrm{d}t) \leq 0,$$

where we use that $\mathrm{supp}\,\hat{\delta}_m(t) \subset M_t(\hat{p}_m, e_t(p_m, \hat{\delta}_m)) \subset B_t(\hat{p}_m)$ a.e. to get the inequality on the right. This results in $\int_T \tilde{\omega}\,\mathrm{d}\mu \geq -\sum_i a_m^i \geq (m-d)\max_i a_m^i$, where we use $\mathrm{bar}\,\hat{\delta}_m(t) \in \mathbb{R}_+^l$. □

Lemma 6.5. *The sequence* $\{(\hat{p}_m, \hat{\delta}_m)\}_m$ *in* $P \times \mathcal{R}_X^1$ *contains a subsequence* $\{(\hat{p}_{m_k}, \hat{\delta}_{m_k})\}_k$ *that converges to a certain* $(p_*, \delta_*) \in P \times \mathcal{R}_X^1$.

Proof. The unit simplex P is compact. Also, Lemma 6.4 implies:

$$\sigma := \sup_m \int_T \left[\int_{\mathbb{R}^l} \sum_i x^i \hat{\delta}_m(t)(\mathrm{d}x)\right]\mu(\mathrm{d}t) < +\infty. \tag{6.1}$$

So $\sup_m I_h(\hat{\delta}_m) < +\infty$, with $h(t,x) := \sum_i x^i$ if $x \in X_t$ and $h(t,x) := +\infty$ if $x \notin X_t$. This defines an inf-compact normal integrand $h \in \mathcal{H}^{\mathrm{bb}}(T;\mathbb{R}^l)$ (use Assumption 3.1). Then $\{\hat{\delta}_m\}_m$ is relatively sequentially compact by Prohorov's theorem for transition probabilities in Section 2(viii). So the existence of a subsequence $\{(\hat{p}_{m_k}, \hat{\delta}_{m_k})\}_k$ that converges to a certain $(p_*, \delta_*) \in P \times \mathcal{R}(T;\mathbb{R}^l)$ follows. Now by $h \in \mathcal{H}^{\mathrm{bb}}(T;\mathbb{R}^l) \subset \mathcal{G}^{\mathrm{bb}}(T;\mathbb{R}^l)$ the mapping I_h is lower semi-continuous, in view of Section 2(v). So we have $I_h(\delta_*) \leq \sigma < +\infty$, which implies $\delta_* \in \mathcal{R}^1(T;\mathbb{R}^l)$. Also, because $h = +\infty$ on the complement of the set it D, the previous inequality implies $\delta_* \in \mathcal{R}_X$. We conclude that δ_* belongs to $\mathcal{R}_X^1 = \mathcal{R}_X \cap \mathcal{R}^1(T;\mathbb{R}^l)$. □

Lemma 6.6. *The pair* (p_*, δ_*) *in* Lemma 6.5 *belongs to* $\boldsymbol{CE}_1(\mathcal{RE})$.

Proof. Let $\{(\hat{p}_{m_k}, \hat{\delta}_{m_k})\}_k$ be as in Lemma 6.5. By Lemma 6.4 we have $(\hat{p}_{m_k}, \hat{\delta}_{m_k}, \hat{\delta}_{m_k}) \in G_1$ for every k. So by Lemmas 6.5 and 6.2(a) we obtain $(p_*, \delta_*, \delta_*) \in G_1$, i.e., $\mathrm{supp}\,\delta_*(t) \subset M_t(p_*, e_t(p_*, \delta_*))$ a.e.

Next, fix $i \in \{1,\ldots,d\}$. Obviously, $g_i(t,x) := \max(x^i, 0)$ defines a normal integrand in $\mathcal{G}^{\mathrm{bb}}(T;\mathbb{R}^l)$. Evidently

$$I_{g_i}(\hat{\delta}_{m_k}) = \int_T (\mathrm{bar}\,\hat{\delta}_{m_k})^i \,\mathrm{d}\mu = a_{m_k}^i + \int_T \omega^i \,\mathrm{d}\mu,$$

where $a_m := \int_T (\mathrm{bar}\,\hat{\delta}_m - \omega)\,\mathrm{d}\mu$, as in the proof of Lemma 6.4. So by Section 2(v)

$$I_{g_i}(\delta_*) \leq \liminf_k I_{g_i}(\hat{\delta}_{m_k}) \leq \limsup_k a_{m_k}^i + \int_T \omega^i \,\mathrm{d}\mu.$$

Here $\limsup_k a^i_{m_k} \leq 0$, as a consequence of the inequality in Lemma 6.4. We conclude that $\int_T (\text{bar}\,\delta_*)^i \, d\mu \leq \int_T \omega^i \, d\mu$ for every i. □

Clearly, with the last lemma our proof of Theorem 3.1 has come to an end. We observe that truncations are also used in the usual literature on existence of competitive equilibria in continuum economies. However, what makes the above proof stand out – and at the same time this explains the novelty of the strand of results starting with Theorem 3.1– is the final limit argument involving the narrow – i.e., very weak – limit δ_* in the proof of Lemma 6.6, where integrable boundedness, as enjoyed by the truncations $t \mapsto X^m_t$, is completely gone and one can only use tightness as in (6.1), in connection with Prohorov's theorem for transition probabilities. To handle consumption externalities, Fatou's lemma in several dimensions, which is the standard sort of tool in these limit arguments, falls short. Interestingly, that lemma is a direct consequence of (v) to (ix) in Section 2(see Balder, 2000a, Theorem 5.9, Balder, 2000b, Theorem 5.5 and their proofs), so in our proof that lemma is replaced by a considerably more powerful collection of results.

6.2. Proof of Theorem 3.2

From now on we suppose in this subsection that all assumptions used in Theorem 3.2 are valid. The difference between the assumptions in Theorems 3.1 and 3.2 concerns, on the one hand, the replacement of Assumptions 3.4 and 3.5 by the weaker Assumptions 3.4$'$, 3.5$'$ and on the other hand the replacement of Assumptions 3.2, 3.3 and 3.6 by the stronger Assumptions 3.2$'$, 3.3$'$ and 3.6$'$. To compensate for the replacement of Assumption 3.5 by Assumption 3.5$'$, it is standard to introduce *quasi-demand*. In the present context this is in the form of quasi-demand multifunctions $\bar{M}_t : P \times Y \to 2^{X_t}$, $t \in T$, defined by

$$\bar{M}_t(p, y) := \begin{cases} M_t(p, y) & \text{if } B_t(p) = \text{cl } B^0_t(p) \\ B_t(p) & \text{if } B_t(p) \neq \text{cl } B^0_t(p) \end{cases}$$

Let us stablish some basic facts about these multifunctions:

Lemma 6.7. $\bar{M}_t(p, y) = M_t(p, y)$ *for every* $t \in T$, $p \in P \cap \mathbb{R}^l_{++}$ *and* $y \in Y$.

Proof.

Case 1 $t \in \omega^{-1}(0)$. In this case the strict positivity of p implies $B_t(p) = \{0\} = \{\omega(t)\}$. So by Assumption 3.6$'$(i) we have $\emptyset \neq M_t(p, y) \subset \bar{M}_t(p, y) \subset \{0\}$, which implies $M_t(p, y) = \bar{M}_t(p, y) = \{0\}$.

Case 2 $t \notin \omega^{-1}(0)$. In this case X_t is convex by Assumption 3.2$'$ and the strict positivity of p, combined with Assumption 3.3', implies $p\check{f}(t) < p\omega(t)$. Hence, Remark 3.1 gives $B_t(p) = \text{cl}B^0_t(p)$, so $\bar{M}_t(p, y) = M_t(p, y)$ holds by definition of $\bar{M}_t$. □

Lemma 6.8. *For every* $p \in P$ *and* $y \in Y$ *the following hold*:

(i) $\{(t, x) \in D : x \in \bar{M}_t(p, y)\}$ is *$\mathcal{D}$-measurable*,

(ii) $\text{Ls}_k\, \bar{M}_t(p_k, y_k) \subset \bar{M}_t(p, y)$ *for every* $t \in T$ *and every sequence* $\{(p_k, y_k)\}_k \subset P \times Y$ *converging to* (p, y).

Proof.

(i) Fix $p \in P$ and $y \in Y$. Because of Assumption 3.1 and measurability of $(t, x) \mapsto p(x - \omega(t))$, the graph of $t \mapsto B_t(p)$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable. Therefore, given Assumption 3.5$'$(i), it is enough to to prove that that $A := \{t \in T : B_t(p) = \text{cl}B^0_t(p)\}$ belongs to $\mathcal{T}$. Let $V \subset \mathbb{R}^l$ be open and define $T_V := \{t \in T : \text{cl}B^0_t(p) \cap V \neq \emptyset\}$. Then also $T_V = \{t \in T : B^0_t(p) \cap V \neq \emptyset\}$. By Castaing and Valadier (1977, III.30), Assumption 3.2 implies existence of a sequence $\{f_k\}_k$ of measurable functions $f_k : T \to \mathbb{R}^l$ such that $X_t = \text{cl}\{f_k(t)\}_k$ for every $t \in T$. Then openness of V implies $T_V = \cup_k\{t \in T : f_k(t) \in V, pf_k(t) < p\omega(t)\}$; hence, T_V is measurable. By Castaing and Valadier (1977, III.30) we can now conclude that the graph of $t \mapsto B^0_t(p)$ is $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable, as is then the graph G of

$t \mapsto B_t(p) \setminus \mathrm{cl} B_t^0(p)$. Because $T \setminus A$ is the projection of G on T, the measurability of A follows from Castaing and Valadier (1977, III.23).

(ii) Case 1: $B_t(p) = \mathrm{cl} B_t^0(p)$. In this case $t \notin \omega^{-1}(0)$ and there certainly exists $\tilde{x} \in B_t^0(p)$, i.e., $p\tilde{x} < p\omega(t)$. This implies $p_k\tilde{x} < p_k\omega(t)$ for all k large enough. By Assumption 3.3$'$ the set X_t is convex, so it follows from Remark 3.1 that $B_t(p_k) = \mathrm{cl} B_t^0(p_k)$, whence also $\bar{M}_t(p_k, y_k) = M_t(p_k, y_k)$, for all k large enough. Hence, the desired inclusion holds by Assumption 3.5$'$.

Case 2: $B_t(p) \neq \mathrm{cl} B_t^0(p)$. In this case $\bar{M}_t(p, y) = B_t(p)$. Now the desired inclusion follows from $\bar{M}_t(p_k, y_k) \subset B_t(p_k)$ for all k and the elementary inclusion $\mathrm{Ls}_k B_t(p_k) \subset B_t(p)$. □

Next, we extract the following result from the previous subsection:

Proposition 6.1. *There exists a sequence $\{(\hat{p}_k, \hat{\delta}_k)\}_k$ in $P \times \mathcal{R}_X^1$, converging to some $(p_*, \delta_*) \in P \times \mathcal{R}_X^1$, such that for every $k \in \mathbb{N}$:*

$$\mathrm{supp}\, \hat{\delta}_k(t) \subset \bar{M}_t(\hat{p}_k, e_t(\hat{p}_k, \hat{\delta}_k)) \quad \textit{for a.e. } t \textit{ in } T$$

and $\int_T \mathrm{bar}\, \delta_*\, \mathrm{d}\mu \leq \int_T \omega\, \mathrm{d}\mu$.

Proof. We turn to the results proven in Section 6.1. First of all, we can mimic the continued validity of Assumption 3.5(instead of the weaker Assumption 3.5$'$) by systematically replacing M_t by $\bar{M}_t$ everywhere. This is due to Lemma 6.8. Next to Assumption 3.5$'$, the only other assumption in Theorem 3.2 that is weaker than its counterpart in Theorem 3.1, is Assumption 3.4$'$. Its part (i) still coincides with Assumption 3.4(i), but its part (ii) only coincides with Assumption 3.4(ii) insofar as restrictions of the mappings e_t to $P^m \times \mathcal{R}_X(m)$, with $m \in \mathbb{N}$ fixed, are concerned. Therefore, with thanks to Remark 6.1(which already allows for Assumption 3.4$'$ explicitly), the results in Lemmas 6.1, 6.2(b), 6.3, 6.4 continue to hold. It can also be seen easily from its proof that Lemma 6.5 continues to hold. Now take $(\hat{p}_k, \hat{\delta}_k) := (\hat{p}_{m_k}, \hat{\delta}_{m_k})$, where $\{(\hat{p}_{m_k}, \hat{\delta}_{m_k})\}_k$ is as obtained in Lemma 6.5. Then observe, by Lemma 6.4 and what was said above, that the stated inclusion holds. Since the elementary reasoning in the second part of the proof of Lemma 6.6 continues to hold here as well, the stated inequality follows from Lemma 6.4. □

Following this, we refine the result in Proposition 6.1 by exploiting those assumptions in Theorem 3.2 that are stronger than their counterparts in Theorem 3.1:

Lemma 6.9. *In Proposition 6.1 the following actually hold*:

(i) $p_* \in \mathbb{R}^l_{++}$,
(ii) $\mathrm{supp}\, \delta_*(t) \subset M_t(p_*, e_t(p_*, \delta_*))$ *for a.e. t in T*,
(iii) $\delta_* \in \cup_m \mathcal{R}_X(m)$,
(iv) $\int_T \mathrm{bar}\, \delta_*\, \mathrm{d}\mu = \int_T \omega\, \mathrm{d}\mu$.

Proof.

(i) By Assumption 3.3$'$ the set $A := \{t \in T : p_*\omega(t) > p_*\check{f}(t)\}$ has positive measure. Note that $A \subset T \setminus \omega^{-1}(0)$, so by Assumption 3.2$'$ the set X_t is convex for every $t \in A$. Hence, Remark 3.1 gives $B_t(p_*) = \mathrm{cl} B_t^0(p_*)$ for all $t \in A$, and then it follows also that $\bar{M}_t(p_*, e_t(p_*, \delta_*)) = M_t(p_*, e_t(p_*, \delta_*))$ for all $t \in A$. By the inclusion in Proposition 6.1 and by applying Section 2(*vii*) to $\{\hat{\delta}_k\}_k$, which we know to converge to δ_*, it follows from Lemma 6.8(ii) that

$$\mathrm{supp}\, \delta_*(t) \subset \mathrm{Ls}_k\, \mathrm{supp}\, \hat{\delta}_k(t) \subset \bar{M}_t(p_*, e_t(p_*, \delta_*)) \quad \text{for a.e. } t \text{ in } T. \tag{6.2}$$

So by the above we find $\mathrm{supp}\, \delta_*(t) \subset M_t(p_*, e_t(p_*, \delta_*))$ for a.e. t in A. Therefore, Assumption 3.6$'$(ii) implies $p_* \in \mathbb{R}^l_{++}$.

(ii) By $p_* \in \mathbb{R}^l_{++}$, just proven, the above inclusion (6.2) turns into $\mathrm{supp}\, \delta_*(t) \subset M_t(p_*, e_t(p_*, \delta_*))$ for a.e. t in T, because now $\bar{M}_t(p_*, e_t(p_*, \delta_*)) = M_t(p_*, e_t(p_*, \delta_*))$ holds for all $t \in T$, thanks to Lemma 6.7.

(iii) By $\hat{p}_k \to p_*$ and (i) above, there exists $\hat{m} \in \mathbb{N}$ such that $\hat{p}_k \in P^{\hat{m}}$ for all k sufficiently large. Therefore, it follows from $\bar{M}_t(\hat{p}_k, e_t(\hat{p}_k, \hat{\delta}_k)) \subset B_t(\hat{p}_k)$ that $\hat{\delta}_k \in \mathcal{R}_X(\hat{m})$ for all k sufficiently large (see the observation made in the beginning of Section 6.1). By closedness of $\mathcal{R}_X(\hat{m})$, this implies $\delta_* \in \mathcal{R}_X(\hat{m})$.

(iv) Assumption 3.6$'$(i) and the above part (ii) give $\text{supp}\,\delta_*(t) \subset S_t(p_*)$ for a.e. t in T. This implies $p_* \int_T \text{bar}\,\delta_* = p_* \int_T \omega$. By strict positivity of p_* (see (i)), the inequality already obtained in Proposition 6.1 then turns into the desired market clearing identity. $\square$

Clearly, Lemma 6.9 implies that the pair (p_*, δ_*) in Proposition 6.1 belongs to $\boldsymbol{CE}_2(\mathcal{RE})$. Thus, Theorem 3.2 has been proven.

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