Corrections to section 2.2 of Feichtinger-Hartl

E.J. Balder

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This is an addendum to section 2.2 of *Optimale Kontrolle ökonomischer Prozesse* by G. Feichtinger and R.F. Hartl.

A correct (i.e., improved) approach to the Pontryagin maximum principle, based on the heuristic assumption of second order differentiability for the optimal value function, goes by the following steps:

1. Define $V(x,t)$ and assume its differentiability as on p. 25.
2. Now use the following result:

Lemma 0.1 *i. For all* $(x,u,t) \in \mathbb{R}^n \times \Omega \times \mathbb{R}_+$

$$0 \geq F(x,u,t) + V_x(x,t)f(x,u,t) - rV(x,t) + V_t(x,t).$$

ii. For the optimal pair (u^*, x^*)

$$0 = F(x^*(t), u^*(t), t) + V_x(x^*(t), t)f(x^*(t), u^*(t), t) - rV(x^*(t), t) + V_t(x^*(t), t)$$

for all $t \in \mathbb{R}_+$.

Proof. i. Fix x and t; fix also $\tilde{u} \in \Omega$ and let $h > 0$ be arbitrary. In the definition of $V(x,t)$ in line 6 of p. 25, the right hand side certainly does not increase if we specialize the supremum to the class of all control functions that are constant and equal to $\tilde{u}$ on the interval $[t, t+h]$ (call such a function an "initially $\tilde{u}$" control function). This gives

$$V(x,t) \geq \int_t^{t+h} e^{-r(s-t-h)} F(\bar{x}(s), \tilde{u}, s) ds + e^{-rh}\alpha.$$

Here $\bar{x} : [t, t+h] \to \mathbb{R}^n$ is the solution of $\dot{x}(s) = f(x(s), \tilde{u}, s)$ with the "initial" condition $\bar{x}(t) = x$. Also

$$\alpha := \sup_v [\int_{t+h}^T e^{-r(s-t-h)} F(y(s), v(s), s) ds + e^{-r(T-t-h)} S(y(T), T)],$$

where the supremum is taken over all control functions $v : [t+h, T] \to \Omega$, with y the corresponding trajectory with "initial" condition $y(t+h) = \bar{x}(t+h)$ [note that every "initially $\tilde{u}$" control function leads to a trajectory y with $y(t+h) = \bar{x}(t+h)$]. From this we conclude that α equals $V(\bar{x}(t+h), t+h)$. As the correct analogue of formula (2.25) on p. 26, the above inequality gives

$$V(x,t) \geq hF(x, \tilde{u}, t) + e^{-rh} V(\bar{x}(t+h), t+h) + o(h),$$

and Taylor expansion (exactly as on p. 26) then gives the desired inequality.

ii. As in the book, observe that by optimality of u^*, for any time point t continuing with u^* from t onwards is also optimal for the remaining time interval $[t, T]$, provided that the "initial" point x at time t satisfies $x = x^*(t)$. Hence,

$$V(x^*(t), t) = \int_t^T e^{-r(s-t)} F(x^*(s), u^*(s), s) ds + e^{-r(T-t)} S(x^*(T), T).$$

Let $h > 0$ be arbitrary. The integral term over $[t, T]$ can be decomposed as the sum of the corresponding integrals over $[t, t+h]$ and $[t+h, T]$. By definition of $V(x^*(t+h), t+h)$ and the "persistent optimality" of u^*, noted above, this gives

$$V(x^*(t), t) = \int_t^{t+h} e^{-r(s-t)} F(x^*(s), u^*(s), s) ds + e^{-rh} V(x^*(t+h), t+h).$$

So now the same Taylor expansion argument as in i can be used, but this time it leads to an identity. Q.E.D.

3. Note that the above lemma implies that (2.31) on p. 26 is valid, *but only for* $x = x^*(t)$ (which is all we need for the maximum principle anyway). Thus, the maximum principle follows as on p. 26.

4. Note that the above lemma – not the argument given in the book – also implies the correctness of the statement involving (2.33) on p. 27. Hence, the remaining derivation of the adjoint equation proceeds as on p. 27.

5. The transversality statement follows by (2.28), as described correctly in the book.