Homework 1*

To be turned in next Monday!

Exercise 1-1 Has now been changed into: Exercise 1.1.10 (Stroock).

Exercise 1-2 Prove: on p. 4

$$\mathcal{U}(f;\mathcal{C}) = \sup_{\xi\in\Xi(\mathcal{C})} \mathcal{R}(f;\mathcal{C},\xi)$$

and

$$\mathcal{L}(f;\mathcal{C}) = \inf_{\xi\in\Xi(\mathcal{C})} \mathcal{R}(f;\mathcal{C},\xi).$$

Exercise 1-3 (*i*) For every collection $\mathcal{C}$ of rectangles in $\mathbb{R}^N$ let $\alpha(\mathcal{C})$ be some unique real number associated to $\mathcal{C}$. Recall:

$$\underline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C}) := \lim_{\delta\to 0} \inf_{\mathcal{C}} \{\alpha(\mathcal{C}) : \|\mathcal{C}\| \leq \delta\},$$

$$\overline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C}) := \lim_{\delta\to 0} \sup_{\mathcal{C}} \{\alpha(\mathcal{C}) : \|\mathcal{C}\| \leq \delta\}.$$

Prove that in both definitions the limit on the right always exists, provided that $\underline{\lim}_{\|\mathcal{C}\|\to 0}\alpha(\mathcal{C})$ is allowed to be $-\infty$ and $\overline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C})$ is allowed to be $+\infty$. *Hint:* From your Analysis course you know that every monotone bounded *sequence* in $\mathbb{R}$ has a limit. Adapt this result to unbounded sequences and then make a connection with the obvious fact that both $\beta(\delta) := \inf_{\mathcal{C}}\{\alpha(\mathcal{C}) : \|\mathcal{C}\| \leq \delta\}$ and $\gamma(\delta) := \sup_{\mathcal{C}}\{\alpha(\mathcal{C}) : \|\mathcal{C}\| \leq \delta\}$ are monotone in the (uncountable!) parameter δ.

(*ii*) Prove the following: always $\overline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C}) \geq \underline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C})$. Moreover, we have $\overline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C}) = \underline{\lim}_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C})$ if and only if $\lim_{\|\mathcal{C}\|\to 0}\, \alpha(\mathcal{C})$ exists (in $[-\infty,+\infty]$!).

(*iii*) Prove the following claim made in the middle of p. 4: for every bounded f we have $\underline{\lim}_{\|\mathcal{C}\|\to 0}\, \mathcal{L}(f;\mathcal{C}) \geq \overline{\lim}_{\|\mathcal{C}\|\to 0}\, \mathcal{U}(f;\mathcal{C})$ if and only if f is Riemann-integrable.

(*iv*) Explain why in (*iii*) the function f has to be *bounded*.

Exercise 1-4 = Exercise 1.1.9

*Re: Measure and Integration, 7 January, 2002