Solutions Second Quizz M & I, 19-5-11

Problem 1 [35 pt]. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $u : X \to \bar{\mathbb{R}}$ be $\mathcal{A}$-measurable and such that $\int_X |u| d\mu = 0$. Prove that $|u| = 0$ almost everywhere.

SOLUTION. Let $A_j := \{|u| \geq 1/j\}$, $j \in \mathbb{N}$. Then $A := \{|u| > 0\}$ is the monotone limit of the increasing sequence $\{A_j\}_j$, which implies $\mu(A) = \lim_j \uparrow \mu(A_j)$. Also, by Markov's inequality $\frac{1}{j}\mu(A_j) \leq \int_X |u| d\mu = 0$ for every j, which implies $\mu(A_j) = 0$. So it follows that $\mu(A) = 0$.

Problem 2 [35 pt]. a. Let $\{\alpha_j\}_j$ be a monotone sequence of nonnegative real numbers such that $\alpha_j \downarrow 0$. Then prove, using a Cauchy sequence argument, that the series $\sum_{j=0}^{\infty}(-1)^j \alpha_j$ converges (i.e., the corresponding sequence of partial sums $s_n := \sum_{j=0}^{n}(-1)^j \alpha_j$ has a limit in $\mathbb{R}$). *Hint:* Start by proving that $s_{2m} - s_{2n} \geq \alpha_{2m} - \alpha_{2n+1}$ for $m > n$.

b. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{u_j\}_j$ be a sequence of nonnegative functions in $\mathcal{L}^1_{\mathbb{R}}$ such that $u_j(x) \downarrow 0$ for every $x \in X$. Prove the following statement: $f(x) := \sum_{j=0}^{\infty}(-1)^j u_j(x)$ defines a function which is integrable and for which $\int_X f\, d\mu = \sum_{j=0}^{\infty}(-1)^j \int_X u_j\, d\mu$.

PROOF. a. *Step 1: $\{s_{2j}\}_j$ is a Cauchy sequence.* For $m > n$ the inequality in the hint holds by

$$s_{2m} - s_{2n} = -\alpha_{2n+1} + \underbrace{\alpha_{2n+2} - \alpha_{2n+3}}_{\geq 0} + \cdots + \underbrace{\alpha_{2m-2} - \alpha_{2m-1}}_{\geq 0} + \alpha_{2m} \geq -\alpha_{2n+1} + \alpha_{2m}$$

and an opposite bound is provided directly by

$$s_{2m} - s_{2n} = \underbrace{-\alpha_{2n+1} + \alpha_{2n+2}}_{\leq 0}\underbrace{-\alpha_{2n+3} + \alpha_{2n+4}}_{\leq 0} - \cdots + \underbrace{-\alpha_{2m-1} + \alpha_{2m}}_{\leq 0} \leq 0.$$

Hence, $|s_{2m} - s_{2n}| = s_{2n} - s_{2m} \leq \alpha_{2n+1} - \alpha_{2m} \leq \alpha_{2n+1}$ for $m > n$. It thus follows that $\{s_{2j}\}_j$ is Cauchy (given any $\epsilon > 0$, choose N so large that $\alpha_{2n+1} < \epsilon$ for all $n \geq N$; then $m > n \geq N$ implies $|s_{2m} - s_{2n}| < \epsilon$).

Step 2: $\{s_{2j+1}\}_j$ is a Cauchy sequence. Imitating step 1, we find for $m > n$

$$s_{2m+1} - s_{2n+1} = \underbrace{\alpha_{2n+2} - \alpha_{2n+3}}_{\geq 0} + \ldots + \underbrace{\alpha_{2m} - \alpha_{2m+1}}_{\geq 0} \geq 0$$

and

$$s_{2m+1} - s_{2n+1} = \alpha_{2n+2}\underbrace{-\alpha_{2n+3} + \alpha_{2n+4}}_{\leq 0} \cdots \underbrace{-\alpha_{2m-1} + \alpha_{2m}}_{\leq 0} - \alpha_{2m+1} \leq \alpha_{2n+2} - \alpha_{2m+1}.$$

Hence, $|s_{2m+1} - s_{2n+1}| = s_{2m+1} - s_{2n+1} \leq \alpha_{2n+2}$. Just as in step 1, this inequality implies that $\{s_{2j+1}\}_j$ is Cauchy.

Step 3: Conclusion. By steps 1-2 there exists z_1 and z_2 in $\mathbb{R}$ such that $s_{2j} \to z_1$ and $s_{2j-1} \to z_2$. By $z_2 \leftarrow s_{2j+1} = s_{2j} + \alpha_{2j+1} \to z_1 + 0$ it follows that $z_1 = z_2$ and therefore the sequence $\{s_n\}_n$ as a whole converges.[1]

b.By part a, applied pointwise, the function f is well-defined and by another application of part a

$$\text{the series } \sum_{j=0}^{\infty}(-1)^j \int_X u_j \, d\mu \text{ is convergent,} \tag{1}$$

because of $\lim_j \downarrow \int_X u_j = 0$, which is true by the monotone convergence theorem (note that $u_j \leq u_1 \in \mathcal{L}^1(\mu)$ for all j). We can now either use the LDCT or the MCT to finish the proof.

Use of the LDCT. We seek to apply the LDCT to the partial sums $f_n(x) := \sum_{j=0}^{n}(-1)^j u_j(x)$, which clearly belong to $\mathcal{L}^1(\mu)$ and converge to $f(x)$ pointwise for every $x \in X$ by part a. For the even indices we have

$$f_{2n}(x) = \underbrace{u_0(x) - u_1(x)}_{\geq 0} + \cdots + \underbrace{u_{2n-2}(x) - u_{2n-1}(x)}_{\geq 0} + \underbrace{u_{2n}(x)}_{\geq 0} \geq 0$$

and

$$f_{2n}(x) = u_0(x) \underbrace{-u_1(x) + u_2(x)}_{\leq 0} - \cdots \underbrace{-u_{2n-1}(x) + u_{2n}(x)}_{\leq 0} \leq u_0(x).$$

Also, for the odd indices

$$f_{2n+1}(x) = \underbrace{u_0(x) - u_1(x)}_{\geq 0} + \cdots + \underbrace{u_{2n}(x) - u_{2n+1}(x)}_{\geq 0} \geq 0$$

and

$$f_{2n+1}(x) = u_0(x) \underbrace{-u_1(x) + u_2(x)}_{\leq 0} - \cdots \underbrace{-u_{2n-1}(x) + u_{2n}(x)}_{\leq 0} \underbrace{-u_{2n+1}(x)}_{\leq 0} \leq u_0(x)$$

hold. This shows that $\sup_n |f_n| \leq w := u_0 \in \mathcal{L}^1(\mu)$ (alternatively, a similarly useful bound could also be deduced from the inequalities already derived in part a). Hence, it follows that *(i)* f, the pointwise limit of the f_n, is also dominated by w (whence $f \in \mathcal{L}^1(\mu)$) and *(ii)* the LDCT can be applied, giving

$$\sum_{j=0}^{\infty}(-1)^j \int_X u_j \overset{(1)}{=} \lim_n \sum_{j=0}^{n}(-1)^j \int_X u_j = \lim_n \int_X f_n \overset{LDCT}{=} \int_X \lim_n f_n = \int_X f, \tag{2}$$

as had to be proven.

[1] Note from steps 1-2 that $\{s_{2j}\}_j$ decreases monotonically and that $\{s_{2j+1}\}_j$ increases monotonically. That leads to an alternative convergence proof, but not the one asked for in problem 1a – see also the alternative solution of part b below.

Alternative: use of the MCT. As an alternative to the above use of the LDCT, one can observe from the above and part a that (i) $\{f_{2j}\}_j$ decreases monotonically to f with $0 \le f_{2j} \le f_2 \in \mathcal{L}^1(\mu)$, implying $\lim_j \downarrow \int_X f_{2j} = \int_X f$ by the MCT, and (ii) $\{f_{2j+1}\}_j$ increases monotonically to f, implying $\lim_j \uparrow \int f_{2j+1} = \int_X f$ by the MCT. Combined, this gives $\lim_n \int_X f_n = \int_X f$ and the rest is as in (2).

Problem 3 [30 pt]. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let $K : X \times \mathcal{B} \to [0, +\infty]$ be a function which is such that (i) for every $x \in X$ $B \mapsto K_x(B) := K(x, B)$ is a measure on $(Y, \mathcal{B})$ and (ii) for every $B \in \mathcal{B}$ $x \mapsto K_x(B) := K(x, B)$ is a nonnegative $\mathcal{A}$-measurable function on X. Let μ be a measure on $(X, \mathcal{A})$ and let $f : Y \to [0, +\infty]$ be $\mathcal{B}$-measurable.

a. Prove that $\nu : B \mapsto \int_X K(x, B)\mu(dx)$ is a measure on $(Y, \mathcal{B})$.

b. Prove that $g : x \mapsto \int_Y f(y)K_x(dy)$ is an $\mathcal{A}$-measurable function.

c. Prove that $\int_Y f\, d\nu = \int_X g\, d\mu$.

Solution. a. Let $\{B_j\}$ be at most countable and mutually disjoint and denote $B := \cup_j B_j$. Then $\sum_j K_x(B_j) = K_x(B)$ holds by (i) for every $x \in X$. Hence, Beppo Levi's theorem (or the MCT) gives

$$\sum_j \nu(B_j) = \sum_j \int_X K(x, B_j)\mu(dx) \overset{BLT}{=} \int_X \sum_j K(x, B_j)\mu(dx) = \int_X K(x, B)\mu(dx) = \nu(B),$$

so we can conclude that ν is σ-additive. Finally, $\nu(\emptyset) = \int_X 0 = 0$ is totally obvious.

b-c. We follow the well-known three-step procedure to prove the statements in b and c:

Step 1: both statements are true if f is a characteristic function. Let $f = 1_B$. Then $g(x) = K(x, B)$ is measurable by (i) and we have both $\int_Y f\, d\nu = \int_Y 1_B\, d\nu = \nu(B)$ and $\int_X g\, d\mu = \int_X K(x, B)\mu(dx) =: \nu(B)$.

Step 2: the statement is true if f is a step function. Let $f = \sum_{i=1}^N y_i 1_{B_i}$. Then $g(x) = \sum_i y_i K(x, B_i)$ is measurable by (i) and by elementary measurability properties. By step 1 we then also have $\int_Y f\, d\nu = \sum_i y_i \nu(B_i)$ and $\int_X g\, d\mu = \sum_i y_i \int_X K(x, B_i)\mu(dx) = \sum_i y_i \nu(B_i)$.

Step 3: the statement is true if f is a nonnegative measurable function. We know that f is the pointwise monotone limit of a sequence $\{f_k\}$ of step functions. Then the BLT/MCT implies $g(x) = \lim_k \uparrow g_k(x)$ for every $x \in X$, with $g_k(x) := \int_Y f_k(y)K_x(dy)$. So g is measurable, because of step 2. Further, another application of the BLT/MCT (twice) gives

$$\int_Y f\, d\nu \overset{BLT}{=} \lim_k \uparrow \int_Y f_k\, d\nu \overset{step\ 2}{=} \lim_k \uparrow \int_X g_k\, d\mu \overset{BLT}{=} \int_X g\, d\mu.$$