Solutions Final Exam M & I, 24-6-10

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Problem 1 [16 pt]. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $T : X \to X$ be an $\mathcal{A}/\mathcal{A}$-measurable mapping. Then T is said to *preserve* the measure μ if $\mu(T^{-1}(A)) = \mu(A)$ for every $A \in \mathcal{A}$.

a. Denote by $T^n : X \to X$ the n-fold composition of T with itself (i.e., $T^1 := T$, $T^2 := T \circ T$, $T^3 := T \circ T \circ T$, etc.). Prove by means of induction that T^n preserves the measure μ for every $n \in \mathbb{N}$.

b. For fixed $B \in \mathcal{A}$ let $C := \{x \in B : T^n(x) \notin B \text{ for all } n \in \mathbb{N}\}$. Prove that C belongs to $\mathcal{A}$.

c. For $m \in \mathbb{N}$ define $C_m := (T^m)^{-1}(C)$. Prove that the sets C_m are mutually disjoint.

d. Prove that $\mu(C) = 0$.

e. Provide a concrete counterexample to show that the result in d does not continue to hold if $\mu(X) = \infty$.

Solution. a. Let (H_n) : T^n is measure preserving. Then $(H_n) \Rightarrow (H_{n+1})$ by $\mu((T^{n+1})^{-1}(A)) = \mu(T^{-1}((T^n)^{-1}(A))) = \mu((T^n)^{-1}(A)) \stackrel{(H_n)}{=} \mu(A)$.

b. Clearly, $C = B \cap (\cap_n D_n)$, with $D_n := (T^n)^{-1}(X \backslash B)$. Every D_n belongs to $\mathcal{A}$, because T^n, the composition of measurable mappings, is $\mathcal{A}/\mathcal{A}$-measurable. Hence, $C \in \mathcal{A}$.

c. Consider $k \neq m$ and suppose $k > m$ without loss of generality. Then $x \in C_k \cap C_m$ would imply $T^m(x) \in C$, whence $T^{m+n}(x) \notin B$ for all $n \in \mathbb{N}$. Hence, $n = k - m$ gives $T^k(x) \notin B$, which contradicts $x \in C_k$, because the latter implies $T^k(x) \in C \subset B$.

d. We have $\mu(C_m) = \mu(C)$ by part a. So $\mu(C) > 0$ would imply $\mu(\cup_m C_m) = \sum_m \mu(C_m) = \infty$ by part c. By $\mu(X) < \infty$ this is impossible. Conclusion: $\mu(C) = 0$.

e. Take $X := \mathbb{R}_+$, equipped with the Lebesgue measure μ, and take $B := [0, 1[$ and $T(x) := x + 1$. Then the above definition of C gives $C = B$ and $\mu(C) = 1$.

Problem 2 [16 pt]. Let $(X_i, \mathcal{A}_i, \mu_i)$. $i = 1, 2, 3$ be three finite measure spaces. By complete analogy to the case of two measure spaces in the book, one can introduce the following objects (you need not prove this!):

(*i*) $\mathcal{A} := \sigma(\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3)$; this is called the *product* σ-algebra on $X := X_1 \times X_2 \times X_3$.

(*ii*) The unique extension $\rho : \mathcal{A} \to [0, \infty]$ which extends $\rho : \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \to [0, \infty]$, given by $\rho(A \times B \times C) := \mu_1(A)\mu_2(B)\mu_3(C)$, to a σ-finite measure on $(X, \mathcal{A})$; this is called the *product measure* of μ_1, μ_2 and μ_3.

a. Prove that $\mathcal{A} = (\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes \mathcal{A}_3$.

b. Prove that $\rho = (\mu_1 \times \mu_2) \times \mu_3$. *Important:* Every major result from the course that you wish to invoke to prove parts a and b *must be written out completely* in your solution.

Solution. a. We invoke Lemma 13.3[1] with $\mathcal{F} = \mathcal{A}_1 \times \mathcal{A}_2$ and $\mathcal{G} = \mathcal{A}_3$ (here the exhaustive sequences are trivial: use $F_j \equiv X_1 \times X_2$ and $G_i \equiv X_3$). This yields the desired identity by $\mathcal{A} := \sigma(\mathcal{F} \times \mathcal{A}_3) \stackrel{L.\ 13.3}{=} \sigma(\mathcal{F}) \otimes \mathcal{A}_3$, with $\sigma(\mathcal{F}) =: \mathcal{A}_1 \otimes \mathcal{A}_2$. Of course, an independent proof of the nontrivial inclusion $\supset$ in (*i*) (which rather resembles the proof of Lemma 13.3) can also be given.

b. Let $\pi := (\mu_1 \times \mu_2) \times \mu_3$. Then, by the definition of the product of *two* measures applied twice, we have $\pi(A \times B \times C) = \mu_1(A)\mu_2(B)\mu_3(C) = \rho(A \times B \times C)$ for every $A \times B \times C$

[1] Lemma 13.3: if $\mathcal{B} = \sigma(\mathcal{F})$, $\mathcal{C} = \sigma(\mathcal{G})$ then $\sigma(\mathcal{F} \times \mathcal{G}) = \mathcal{B} \otimes \mathcal{C}$, provided that $\mathcal{F}$ and $\mathcal{G}$ contain exhaustive sequences (F_j) and (G_i).

in the class $\mathcal{H}$ of all measurable rectangles. We now apply the uniqueness Theorem 5.7.[2] This is allowed because the measures μ_i are finite (just take $H_j \equiv X$).

Problem 3 [18 pt]. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $f : X \to \mathbb{R}_+$ be a nonnegative μ-integrable function with the following property: there exists a constant $c \in \mathbb{R}$ such that $\int_X f^n \, d\mu = c$ for every $n \in \mathbb{N}$. Prove that there exists $A \in \mathcal{A}$ such that $f(x) = 1_A(x)$ for almost every x in X.

Solution. *Step 1:* $0 \le f \le 1$ *a.e.* Let $B := \{f > 1\}$; then $\infty > c \ge \int_B f^n$ and on B we have $f^n(x) = (f(x))^n \uparrow \infty$, so $\mu(B) = 0$ by the monotone convergence theorem.

Step 2: $f \in \{0, 1\}$ *a.e.* For $C := X \backslash B$ step 1 implies $\int_C f^2 = c = \int_C f$, so $\int_C (f - f^2) = 0$, where $f - f^2 \ge 0$. Hence, $f = f^2$ a.e. on C, i.e., $f \in \{0, 1\}$ a.e. on C.

Step 3. By steps 1-2 we have $f \in \{0, 1\}$ a.e. on X. Let $A := \{f = 1\}$ and we are done.

Alternative step 2: $c = \mu(\{f = 1\})$. By $f^n \downarrow 0$ on $\{f = 1\}$ we get $c \stackrel{*}{=} \int_{\{f=1\}} 1 + \int_{\{0<f<1\}} f^n \to \mu(\{f = 1\})$ (MCT), so $c = \mu(\{f = 1\})$, but then $\stackrel{*}{=}$ with $n := 1$ becomes $0 = \int_{\{0<f<1\}} f$, causing $\mu(\{0 < f < 1\}) = 0$. Now go to step 3.

Problem 4 [16 pt].[3] Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(u_j)_j$ be a sequence of $\mathcal{A}$-measurable functions $u_j : X \to \mathbb{R}$. Let $u : X \to \mathbb{R}$ also be $\mathcal{A}$-measurable. Prove the following equivalence: the sequence $(u_j)_j$ converges to u in measure if and only if $\int_X \frac{|u_j - u|}{1+|u_j-u|} d\mu \to 0$ for $j \to \infty$.

Solution. Write $v_j := u_j - u$ and note: $\xi \mapsto \xi/(1 + \xi)$ is strictly increasing on $\mathbb{R}_+$.

$\Rightarrow$: Give $\eta > 0$; then for any $\epsilon > 0$ we have $\mu(\{|v_j| > \epsilon) < \eta/2$ for j large enough, so $\int_{\{|v_j|>\epsilon\}} |v_j|/(1+|v_j|) + \int_{\{|v_j|\le\epsilon\}} |v_j|/(1+|v_j|) \le \eta/2 + \epsilon/(1+\epsilon)\mu(X) < \eta$ for j large enough; namely, choose $\epsilon < \eta/(2\mu(X))$.

$\Leftarrow$: Give $\epsilon > 0$. By Markov's inequality $\epsilon\mu(|v_j| > \epsilon)/(1 + \epsilon) \le \int_X |v_j|/(1 + |v_j|) \to 0$. This implies $\mu(|v_j| > \epsilon) \to 0$.

Problem 5 [18 pt]. Let $(X, \mathcal{A}, \mu)$ be a measure space and let λ be another measure on $(X, \mathcal{A})$, with $\lambda(X) < \infty$. Recall that λ is defined to be *absolutely continuous* with respect to μ if $\mu(A) = 0 \Rightarrow \lambda(A) = 0$ for every $A \in \mathcal{A}$. Prove that λ is absolutely continuous with respect to μ if and only if $\lim_n \lambda(A_n) = 0$ holds for every sequence $(A_n)_n$ in $\mathcal{A}$ with $\lim_n \mu(A_n) = 0$. *Hint.* Use contradiction and apply the Borel-Cantelli lemma (exercise 6.9, week 9): $\sum_k \nu(B_k) < \infty$ implies $\nu(\cap_{p=1}^\infty \cup_{k\ge p} B_k) = 0$; this holds for any measure ν on $(X, \mathcal{A})$.

Solution. $\Leftarrow$: Let $\mu(A) = 0$ and take $A_n \equiv A$. Then $\lambda(A) = 0$ follows.

$\Rightarrow$: If there is $(A_n)_n$ with $\lim_n \mu(A_n) = 0$ but $\lambda(A_n) \not\to 0$, then there is a subsequence (n_j) and $\epsilon > 0$ such that $\lambda(A_{n_j}) \ge \epsilon$ for all j. Now pick from (n_j) a further subsequence (m_k) as follows: let m_1 be the first index n_j with $\mu(A_{n_j}) < 2^{-1}$, let m_2 be the first index $n_j > m_1$ with $\mu(A_{n_j}) < 2^{-2}$, etc., etc. Then still $\lambda(A_{m_k}) \ge \epsilon$ for all k and now also $\sum_k \mu(A_{m_k}) < \infty$, causing $\mu(A_*) = 0$ for $A_* := \cap_{p=1}^\infty C_p$ with $C_p := \cup_{k\ge p} A_{m_k}$ (by Borel-Cantelli, as suggested). However, now $C_p \downarrow A_*$ implies $\lambda(C_p) \downarrow \lambda(A_*)$, for λ is a finite measure. Also, $\lambda(C_p) \ge \epsilon$ is evident, so $\lambda(A_*) \ge \epsilon > 0$, which contradicts $\mu(A_*) = 0$ above.

Problem 6 [16 pt]. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(u_j)_j$ be a sequence of $\mathcal{A}$-measurable functions $u_j : X \to \mathbb{R}$. Let $u : X \to \mathbb{R}$ also be $\mathcal{A}$-measurable. Suppose that $(u_j)_j$ converges almost everywhere to u. Suppose also that the sequence $(u_j^-)_j$ of negative parts $u_j^- := \max(0, -u_j)$ is uniformly integrable. Then prove that the following extension of Fatou's lemma holds: $\liminf_{j\to\infty} \int_X u_j \, d\mu \ge \int_X u \, d\mu$.

[2] Theorem 5.7: if two measures π and ρ coincide on a class $\mathcal{H} \subset \mathcal{A}$, closed for finite intersections and generating $\mathcal{A}$, and if a monotone sequence $(H_j)_j$ exists with $\pi(H_j) = \rho(H_j) < \infty$ for all j and $H_j \uparrow X$, then π and ρ coincide on $\mathcal{A}$.

[3] This was course Exercise 16.8.

Hint 1: Time can be saved by employing Vitali's theorem. If you use it, then make sure that what you want to use from it is *written out completely* in your solution. *Hint 2:* In general, if $\alpha := \liminf_j \alpha_j$ in $[-\infty, +\infty]$, then a subsequence of $(\alpha_j)_j$ converges to α.

Solution method 1: use Vitali. By Vitali's theorem,[4] applied to the sequence $(u_j^-)_j$, we have $\int u_j^- \to \int u^- \in \mathbb{R}_+$ (here $(u_j^-)_j$ converges a.e., whence also in measure, to u^-). So $\liminf_j \int u_j \overset{*}{=} \liminf_j \int u_j^+ - \int u^-$ and now Fatou's lemma (p. 73) gives $\liminf_j \int_X u_j^+ \geq \int u^+$, because $u_j^+ \geq 0$ and $u_j^+ \to u^+$ a.e. Together, this gives $\liminf_j \int u_j \geq \int u^+ - \int u^-$. Note 1: hint 2 is not really needed (but handy for those not aware of identities like $\overset{*}{=}$). Note 2: although $\int u^- < \infty$, it could happen that $\int u^+ = \infty$, but then $\int u := \int u^+ - \int u^-$ is still a meaningful value in $(-\infty, +\infty]$, just as in the Fatou lemma on p. 73.

Solution method 2: use Fatou and UI. By the UI hypothesis, for every fixed $\epsilon > 0$ there exists an integrable $w_\epsilon : X \to \mathbb{R}_+$ such that $\sup_j \int_{\{u_j < -w_\epsilon\}} u_j^- < \epsilon$. In succession the above, $u_j \geq -u_j^-$ and the definition of $w_j := \max(u_j, -w_\epsilon)$ give

$$\int u_j = \int_{\{u_j < -w_\epsilon\}} u_j + \int_{\{u_j \geq -w_\epsilon\}} u_j \geq -\epsilon + \int_{\{u_j \geq -w_\epsilon\}} w_j = -\epsilon + \int_X w_j + \int_{\{u_j \geq -w_\epsilon\}} w_\epsilon.$$

By $w_\epsilon \geq 0$ this gives $\int u_j \geq -\epsilon + \int w_j$. Now Fatou's lemma can be applied to $(w_j)_j$, because of $w_j \geq -w_\epsilon$ with $w_\epsilon \in \mathcal{L}^1(\mu)$ (this uses exactly the same elementary reasoning as the reverse Fatou lemma in course Exercise 9.8), so $\liminf_j \int w_j \geq \int \max(u, -w_\epsilon)$ because $w_j \to \max(u, -w_\epsilon)$ a.e. Combined with the above, this yields $\liminf_j \int u_j \geq -\epsilon + \int \max(u, -w_\epsilon) \geq -\epsilon + \int u$, using $\max(u, -w_\epsilon) \geq u$. The proof is finished by letting $\epsilon \downarrow 0$.

[4] From Vitali's Theorem 16.6 (for $p = 1$): if (v_j) converges in measure to v and if $(|v_j|)$ is uniformly integrable, then $\int |v_j - v| \to 0$ and a fortiori $\int v_j \to \int v$.