Solutions First Quizz M & I, 31-3-11

Problem 1 [35 pt]. Let $\mathcal{C}$ be a collection of subsets of the set X and, as usual, let $\sigma(\mathcal{C})$ be the σ-algebra on X which is generated by $\mathcal{C}$. Demonstrate that for each set $A \in \sigma(\mathcal{C})$ there exists a countable[1] subcollection $\mathcal{C}_0 \subset \mathcal{C}$, such that $A \in \sigma(\mathcal{C}_0)$.

Hint: Denote by $\mathbb{D}$ the collection of all countable collections $\mathcal{D} \subset \mathcal{C}$. Consider $\mathcal{U} := \cup_{\mathcal{D}\in\mathbb{D}}\sigma(\mathcal{D})$ and show that $\mathcal{U}$ is a σ-algebra.

SOLUTION. *Step 1: proof of the hint.* First, take any $\mathcal{D} \in \mathbb{D}$ (for instance, it could even be the empty collection). Then $\emptyset \in \sigma(\mathcal{D}) \subset \mathcal{U}$ by the first property of a σ-algebra. Second, if $A \in \mathcal{U}$, then there exists $\mathcal{D} \in \mathbb{D}$ with $A \in \sigma(\mathcal{D})$ and then also $A^c \in \sigma(\mathcal{D})$ by the second property of a σ-algebra. Third, if $\{A_j\}_j$ is an arbitrary countable collection of sets in $\mathcal{U}$, then for every j there exists $\mathcal{D}_j \in \mathbb{D}$ such that $A_j \in \sigma(\mathcal{D}_j)$. Form $\bar{\mathcal{D}} := \cup_j \mathcal{D}_j$. Then $\bar{\mathcal{D}}$ is evidently countable, so $\bar{\mathcal{D}} \in \mathbb{D}$. Now $\{A_j\}_j$ belongs to $\sigma(\bar{\mathcal{D}})$, which implies $\cup_j A_j \in \sigma(\bar{\mathcal{D}}) \subset \mathcal{U}$ by the third property of a σ-algebra. This proves the hint.

Step 2: $\mathcal{U} = \sigma(\mathcal{C})$. First, we claim $\mathcal{C} \subset \mathcal{U}$. Let $C \in \mathcal{C}$ be arbitrary. Then $\tilde{\mathcal{D}} := \{C\}$ belongs to $\mathbb{D}$ and hence $C \in \sigma(\tilde{\mathcal{D}}) \subset \mathcal{U}$, which proves the claim. By definition of $\mathcal{U}$, we also have $\mathcal{U} \subset \sigma(\mathcal{C})$, so it follows that $\mathcal{C} \subset \mathcal{U} \subset \sigma(\mathcal{C})$. But then $\sigma(\mathcal{C}) \subset \sigma(\mathcal{U}) \overset{\textit{step } 1}{=} \mathcal{U} \subset \sigma(\mathcal{C})$, which implies $\mathcal{U} = \sigma(\mathcal{C})$.

Step 3: proof of the desired result. Give any $A \in \sigma(\mathcal{C})$. Then by step 2 $A \in \cup_{\mathcal{D}\in\mathbb{D}}\sigma(\mathcal{D})$. So there exists $\mathcal{C}_0 \in \mathbb{D}$ such that $A \in \sigma(\mathcal{C}_0)$. By definition of $\mathbb{D}$, this finishes the proof.

Problem 2 [40 pt]. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $\nu : \mathcal{A} \to [0, +\infty]$ by $\nu(A) := \sup\{\mu(B) : B \in \mathcal{A}, B \subset A, \mu(B) < +\infty\}$.

a. Prove that $\nu(A) \leq \mu(A)$ holds for every $A \in \mathcal{A}$.
b. Prove that ν is a measure on $(X, \mathcal{A})$.
c. Prove that if μ is σ-finite, then $\nu = \mu$.
d. Does the converse implication in part c also hold? If yes, then give a proof. If no, then give a counterexample.
e. Determine ν for the following special measure μ: $\mu(A) := +\infty$ if $A \neq \emptyset$ and $\mu(\emptyset) = 0$.

SOLUTION. Notation: let $\mathcal{A}_f := \{A \in \mathcal{A} : \mu(A) < +\infty\}$.

a. If $A \in \mathcal{A}$, then $\mu(B) \leq \mu(A)$ for every $B \in \mathcal{A}_f$ that is contained in A. Hence, $\mu(A)$ is an upper bound for the supremum expression $\nu(A)$.

b. We make two preliminary observations:
(i) ν is monotone: if $A \subset A'$ then $\nu(A) \leq \nu(A')$
(ii) $\nu(A) = \mu(A)$ whenever $A \in \mathcal{A}_f$.

[1] Note: as usual "countable" means "at most countable" (i.e., finite sets are also considered to be countable).

Here (i) is obvious, because the set over which the supremum in the definition of ν is taken, is at least as large for A' as for A. To see (ii), note that $\nu(A) \leq \mu(A)$ follows from part a and the converse inequality follows by the supremal definition of $\nu(A)$, since $A \in \mathcal{A}_f$ and $A \subset A$.

To prove part b, note first that $\nu(\emptyset) = 0$ follows trivally from (ii) above. To prove σ-additivity of ν, let $\{A_j\}$ be an arbitrary countable, mutually disjoint collection in $\mathcal{A}$. Denote $A := \cup_j A_j$. Let $B \in \mathcal{A}_f$ be arbitrary, with $B \subset A$. Note that $B = \cup_j B_j$, where $B_j := B \cap A_j \in \mathcal{A}_f$. Since the B_j's are obviously disjoint and have $B_j \subset A_j$, we have

$$\mu(B) = \sum_j \mu(B_j) \overset{(ii)}{=} \sum_j \nu(B_j) \overset{(i)}{\leq} \sum_j \nu(A_j),$$

Taking the supremum over all such $B \in \mathcal{A}_f$ with $B \subset A$ then gives $\nu(A) \leq \sum_j \nu(A_j)$. Conversely, fix $N \in \mathbb{N}$. For each $1 \leq j \leq N$ let $B_j \in \mathcal{A}_f$ be arbitrary, with $B_j \subset A_j$. Then also the B_j's are disjoint and $\cup_{j=1}^N B_j \in \mathcal{A}_f$, so

$$\sum_{j=1}^N \mu(B_j) = \mu(\cup_{j=1}^N B_j) \overset{(ii)}{=} \nu(\cup_{j=1}^N B_j) \overset{(i)}{\leq} \nu(A).$$

Taking first the supremum over all B_1 on the left gives $\nu(A_1) + \sum_{j=2} \mu(B_j) \leq \nu(A)$. Clearly, this procedure can succesively be extended to all other $B_2, \ldots, B_N$ to yield $\sum_{j=1}^N \nu(A_j) \leq \nu(A)$.

c. If μ is σ-finite, then there exists a countable (i.e., at most countable) collection $\{E_j\} \subset \mathcal{A}_f$ such that $\cup_j E_j = X$ and without loss of generality we can suppose that such E_j's are disjoint. Give an arbitrary $A \in \mathcal{A}$; then $\nu(A) = \sum_j \nu(A \cap E_j)$ and $\mu(A) = \sum_j \mu(A \cap E_j)$. By (ii) above it follows from $A \cap E_j \in \mathcal{A}_f$ that $\nu(A \cap E_j) = \mu(A \cap E_j)$ for every j. So $nu(A) = \mu(A)$.

d. The converse to part c does not hold. Consider $X := \mathbb{R}$ equipped with the counting measure. Then $\nu = \mu$. Indeed, for $A \in \mathcal{A}$ it follows by (ii) above that $\nu(A) = \mu(A)$ if $A \in \mathcal{A}_f$ and if $\mu(A) = \infty$, then A has infinitely many elements, so $\nu(A) = \mu(A)$ follows by considering arbitrarily large but finite subsets of A. If the converse to part c were true, there would exist finite subsets E_j, monotonically increasing to X. This would imply that $X = \cup_j E_j$ has at most countably many elements, which is not true.

e. The case $X = \emptyset$ is trivial and leads to $\mu(\emptyset) = \nu(\emptyset) = 0$. If $X \neq \emptyset$, then $\mathcal{A}_f = \{\emptyset\}$ gives $\nu(A) = \mu(\emptyset) = 0$ for every $A \in \mathcal{A}$. So in both cases ν is the null measure on $(X, \mathcal{A})$.

Problem 3 [25 pt]. a. Prove that in $\mathbb{R}^2$ the line $L := \{(x_1, x_2) : x_2 = 0\}$ has $\lambda^2(L) = 0$ (i.e., has zero two-dimensional Lebesgue measure).

b. Prove that every line in $\mathbb{R}^2$ has zero two-dimensional Lebesgue measure.

Solution. a. For $n \in \mathbb{Z}$ let $L_n := \{(x_1, x_2) : x_1 \in [n, n+1), x_2 = 0\} = [n, n+1) \times \{0\}$. Then $\lambda^2(L_n) = 1 * 0 = 0$. Because $L = \cup_{n \in \mathbb{Z}} L_n$, where the union is disjoint, σ-additivity of λ^2 gives $\lambda^2(L) = 0$.

b. If the line is parallel to L, then the result follows from part a by the invariance of the Lebesgue measure with respect to translations. Otherwise, the line intersects

L at a unique point. Rotation around this point shows that the new line is a rotation of L, so the result follows from part a by the invariance of the Lebesgue measure with respect to rotations. Alternatively, one could work with a generic line equation (e.g., $x_2 = ax_1 + b$, etc.) and imitate the proof of part a.