Homework for WISB372/ECRMMAT, week 37

You should *actively* practice with Examples 3,4, the example in section 20.8 and the solved problems 20.2 to 20.24 (except for their sufficiency part, which you will be able to do starting next week).

Comment. Repeatedly, Dowling is in the habit of waving hands when it comes to determining integration constants by means of the boundary conditions. When the optimal candidate function $x(\cdot)$ is complicated, this is acceptable, but it is less so when $x(\cdot)$ is simple.

Example: In solved problem 20.8 one finds $x(t) = k_1 t^4 + k_2$ and k_1, k_2 are left unspecified with the implicit understanding that they follow from $x_0 = x(t_0) = k_1 t_0^4 + k_2$ and $x_1 = x(t_1) = k_1 t_1^4 + k_2$. But this gives immediately $k_1 = (x_1 - x_0)/(t_1^4 - t_0^4)$ and $k_2 = (x_0 t_1^4 - x_1 t_0^4)/(t_1^4 - t_0^4)$, causing

$$x(t) = \frac{x_1 - x_0}{t_1^4 - t_0^4} t^4 + \frac{x_0 t_1^4 - x_1 t_0^4}{t_1^4 - t_0^4}.$$

Here the explicit solution is preferred because it gives more insight into $x(\cdot)$; for instance, this gives insights into the signs of the constants k_1 and k_2.

Exercise 1 [optimal consumption] Let $x(t) \geq 0$ be the amount of money in the bank. Let r be the interest rate and let $\rho > 0$ be a discount factor. The amount of money changes according to $\dot{x}(t) = rx(t) - c(t)$ over the lifetime interval $[0, T]$, where $c(t)$ can be seen as the rate of consumption at time t. The optimal consumption problem is to chose a consumption function $c(\cdot)$ in such a way that the consumer/saver's aggregate utility $\int_0^T e^{-\rho t} \log(c(t)) dt$ over his/her lifetime $[0, T]$ is maximized, subject to the initial condition $x(0) = x_0 > 0$ (initial wealth), the terminal condition $x(T) = x_T \geq 0$ ("inheritance") and the no-borrowing assumption $x(t) \geq 0$ for all $t \in [0, T]$.

a. Assuming that any candidate optimal solution satisfies $x(t) > 0$ for all t ("interiority assumption"), reformulate this problem as the following CoV problem: maximize $\int_0^T e^{-\rho t} \log(rx(t) - \dot{x}(t)) dt$ over all $x \in \mathcal{C}^1$, subject to $x(0) = x_0$ and $x(T) = x_T$.

b. Show that the associated Euler equation is

$$\frac{r\dot{x}(t) - \ddot{x}(t)}{rx(t) - \dot{x}(t)} = r - \rho$$

and that its solution is given by $x^*(t) = c_1 e^{(r-\rho)t} + c_2 e^{rt}$, with c_1, c_2 such that the initial and end time conditions hold, giving

$$c_1 := \frac{x_0 - x_T e^{-rT}}{1 - e^{-\rho T}} \text{ and } c_2 := \frac{x_T e^{-rT} - x_0 e^{-\rho T}}{1 - e^{-\rho T}}.$$

c. Substituting back, show that the candidate optimal consumption plan $c^*(t) := rx^*(t) - \dot{x}^*(t)$ is given by

$$c^*(t) = \rho \frac{x_0 - x_T e^{-rT}}{1 - e^{-\rho T}} e^{(r-\rho)t}.$$

d. Inspect conditions under which $x^*(\cdot)$ satisfies the interiority assumption $x^*(t) > 0$ for all $t \in [0, T]$. First, observe that the interiority assumption holds if and only if $c_1 e^{-\rho t} + c_2 > 0$ for all t.

d-1. Show that $c_1 \geq 0$ can be supposed without loss of economic generality, by the equivalence $c_1 < 0 \Leftrightarrow x_T > x_0 e^{rT}$. *Hint:* Show that any solution of $\dot{x} = rx - c \leq rx$, with $x(0) = x_0$, must satisfy $x(t) \leq x_0 e^{rt}$ for all t. What is the economic explanation for the impossibility of $x_T > x_0 e^{rT}$?

d-2. Conclude in d-1 that $x^*(\cdot)$ satisfies the interiority assumption if and only if $c_1 e^{-\rho T} + c_2 > 0$.

d-3. Show that a sufficient (but not necessary) condition for the interiority of $x^*(\cdot)$ is that $x_0 e^{rT} \geq x_T > x_0 e^{(r-\rho)T}$.

See also the related worked out problems 20.21 to 20.24 in Dowling, where the non-negativity of the consumption rates seems to be taken for granted ...

Exercise 2 [mathematics students only] Consider the *modification* no. 2 of the simplest problem in the CoV: optimize

$$e(x(t_0), x(t_1)) + \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t)) dt$$

over all $x \in \mathcal{C}^1[t_0, t_1]$ (no initial or end time conditions). Here the function $e(\xi_0, \xi_1)$ is supposed to have continuous partial derivatives $e_{\xi_0}(\xi_0, \xi_1)$ and $e_{\xi_1}(\xi_0, \xi_1)$. Show, by adapting the "proof" given in Dowling, as improved in class, that the following necessary condition for optimality of $x^* \in \mathcal{C}^1[t_0, t_1]$ holds: (i) the Euler equation $\forall_{t \in [t_0, t_1]} F_\xi(t, x^*(t), \dot{x}^*(t)) = \frac{d}{dt}[F_\eta(t, x^*(t), \dot{x}^*(t))]$ and (ii) $F_\eta(t_0, x^*(t_0), \dot{x}^*(t_0)) = e_{\xi_0}(x^*(t_0), x^*(t_1))$ and $F_\eta(t_1, x^*(t_1), \dot{x}^*(t_1)) = -e_{\xi_1}(x^*(t_0), x^*(t_1))$.

Exercise 3 Apply the necessary conditions stated in the previous exercise to the following problem: minimize $\int_0^2 [\dot{x}^2(t) + 3tx(t)] dt + x^2(0) + x^2(2)$ over all $x \in \mathcal{C}^1[0, 2]$. *Hint:* use $e(\xi_0, \xi_1) := \xi_0^2 + \xi_1^2$. *Your test for correctness:* The correct candidate-optimal solution $x^*(\cdot)$ of this problem satisfies $x^*(0) = -5/4$.

Exercise 4 a. Apply the necessary conditions stated in class for the *modification* no. 1 of the simplest problem in the CoV to the following problem: minimize $\int_0^T [x(t) + \dot{x}^2(t)]dt$ over all $x \in \mathcal{C}^1[0,T]$ such that $x(0) = 0$.

b. Let x^* be the candidate solution that you determined in part a. Verify that $x^*(T)$ is equal to $-\frac{1}{4}T^2$. Next, consider the following simplest problem in the CoV: minimize $\int_0^4 [3x(t) + 3\dot{x}^2(t)]dt$ over all $x \in \mathcal{C}^1[0,4]$ such that $x(0) = 0$ and $x(4) = -4$. Determine the candidate-optimal solution for this problem and explain completely why this also follows from the outcome obtained in part a.