Homework for WISB372/ECRMMAT, week 39

At this stage all examples and solved problems in Chapter 20, except for the ones dealing with the skipped section 20.7 (i.e., problems 20.34, 20.35), are part of the quizz and exam material. This also holds for all examples and solved problems in Chapter 21, with the following exception for ECRMMAT students: examples and solved problems that involve solving *systems* of first order differential equations are only exam material *after* quizz 1. However, sometimes such a task can easily be converted into solving a single second order differential equation (see exercise 1 below) and this is required knowledge also for quizz 1.

Exercise 1 Consider the OC-problem to maximize $J(y) := \int_0^T (x^2(t) - y^2(t))dt$ over all $y : [0, T] \to \mathbb{R}$ such that $x(0) = 0$. As usual, $T > 0$ is a fixed parameter of the model and $x(T)$ is free.

a. First, show that according to 1. and 2. on Dowling's p. 494, the following must hold (among other things): $y(t) = \lambda(t)/2$ and $\dot{\lambda}(t) = -2x(t)$. Then show that this implies the second order differential equation $\ddot{\lambda} = -\lambda$.

b. Use the outcome of part a to show that

i. If $T = \pi/2$, then all functions $y(t) = c\cos(t)$, $c \in \mathbb{R}$, are candidate-optimal.

ii. If $T < \pi/2$, then $y^*(t) = 0$ for all t is the only candidate-optimal control function.

c. If $T > \pi/2$ a special situation obtains: there does not exist an optimal control, because the values of $\int_0^T (x^2(t) - y^2(t))dt$ can be made arbitrarily high. Show this. *Hint:* Consider first $\pi/2 < T < \pi$ and show that $J(y_n) \to +\infty$ as $n \to \infty$ holds for $y_n(t) := n\cos(t)$, $n = 1, 2, \ldots$. Next, for $T \geq \pi$ choose suitable modifications $\tilde{y}_n$ of the y_n.

d. Does the sufficiency theorem apply here? If so, what is your conclusion about the control functions obtained in part b?

Exercise 2 [only for mathematics students] a. Consider the three modifications "Mod 1", "Mod 2a" and "Mod 2b" of the simplest problem in optimal control theory, as discussed in class. Formulate and prove the corresponding versions of the Sufficiency Theorem for these modifications. *Hint:* By introducing additional concavity conditions, your task is to show that the former necessary conditions for "Mod 1", "Mod 2a" and "Mod 2b" are also sufficient.

b. Formulate for the simplest problem in the CoV the analogous version of "Mod 4". Prove, similar to the earlier proofs of necessary conditions for the simplest CoV-problem and its modifications, the *necessary* conditions for optimality in this

"Mod 4" problem.[1] *Hint:* Consider variations $x^* + mh$ in Dowling's proof on pp. 462-463, with $h(t_0) = 0$ and $h(t_1) \ldots$ (your guess), depending on whether $x^*(t_1) = x_{min}$ or $x^*(t_1) > x_{min}$.

Exercise 3 Consider the following "Mod 4" problem: maximize $\int_0^1 (x(t) - y^2(t))dt$ over all control functions $y : [0,1] \to \mathbb{R}$ subject to $x(0) = 2$ and $x(1) \geq x_{min}$. Here the DS is $\dot{x} = y$ and x_{min} is a fixed parameter.

a. Show that the candidate-optimal solution is of the form $y^*(t) = -\frac{1}{2}t + C$.

b. Show that the nonbinding case $x^*(1) > x_{min}$ can only hold if $x_{min} < 9/4$.

c. Show that the binding case $x^*(1) = x_{min}$ can only hold if $x_{min} \geq 9/4$.

d. Determine the candidate-optimal control function for either case mentioned in parts b-c.

e. Does the sufficiency theorem apply here? If so, what is your conclusion about the control functions obtained in part d?

Exercise 4 Reformulate Example 21.2 as a CoV-problem and solve it completely by using the Sufficiency Theorem for CoV-problems. Check your answer for consistency with Example 21.2.

Exercise 5 Consider the following optimal control problem: *minimize* $J(y) := \int_0^T (x(t) + y^2(t))dt$ over all control functions $y(\cdot) : [0,1] \to \mathbb{R}$ such that $x(0) = 0$ and $x(1)$ is free. Here $\dot{x} = y$ is the DS.

a. Determine the optimal solution $y^*(\cdot)$ for this problem by applying the required necessary conditions, followed by an application of the Sufficiency Theorem. Verify that the associated state function $x^*(\cdot)$ satisfies $x^*(1) = -1/4$.

b. Next, add the end time condition $x^*(1) = a$ to the above problem and again determine its optimal solution $y_a^*(\cdot)$ for any value of the parameter $a \in \mathbb{R}$.

c. Define $\phi(a) := J(y_a)$ and show that $\phi(a)$ is a quadratic function in a that takes its minimum at $a_* = -1/4$.

d. Explain the connection between $a_* = -1/4$ and $x^*(1) = -1/4$, as found above.

e. Using the function $\phi(\cdot)$, determine also the optimal solution of the following "Mod 4" problem: minimize the above $J(y)$ over all $y(\cdot)$ such that $x(0) = 0$, $\dot{x} = y$ and $x(1) \geq b$. Do this for every value of the parameter b.

[1] Observe the following pattern so far: 1. CoV problems are simple enough to allow proofs of both their necessary and sufficient conditions, 2. OC problems, which are much harder, still allow fairly easy proofs of their sufficient conditions (thanks to concavity of course).