Solutions Quiz 2 ECRMMAT, 28-10-2011 (open book)

Problem 1. [70 pts] a. Consider the following optimal control problem. For fixed end time $T > 0$ minimize $\int_0^T \frac{1}{2}(u^2(t) + 1)dt$ over all control functions $u : [0, T] \to [-2, 2]$ such that $x(0) = 5$ and $x(T) = 0$. Here $\dot{x} = u$ is the dynamical system. Determine all candidate-optimal control functions, using the minimum principle.

b. Next, consider the same optimal control problem, but *now suppose that the end time can also vary*: minimize $\int_0^T \frac{1}{2}(u^2(t) + 1)dt$ over all $T > 0$ and all control functions $u : [0, T] \to [-2, 2]$ such that $x(0) = 5$ and $x(T) = 0$. As before, $\dot{x} = u$ is the dynamical system. Determine all candidate-optimal control functions, using the minimum principle.

Solution. a. The Hamiltonian is $H(x, u, p(t)) = \frac{1}{2}(u^2 + 1) + p(t)u$. It has $H_x = 0$, so the adjoint equation gives $\dot{p} = 0$, whence $p(t) \equiv c_1$, a constant. For every $t \in [0, T]$ it follows from the minimum principle that $u^*(t)$ minimizes $\phi(u) := \frac{1}{2}u^2 + c_1 u$ over $u \in [-2, 2]$. So setting $\phi'(u)$ equal to zero gives $u^*(t) = -c_1$, provided that $|c_1| \leq 2$; however, for $c_1 > 2$ minimizing would still give $u^*(t) = -2$ whereas $c_1 < -2$ would give $x^*(t) = 2$. Equivalently, we can say that $u^*(t) \equiv -c_1$ holds, with $|c_1| \leq 2$. The associated trajectory is $x^*(t) = -c_1 t + c_2$ and by the initial condition we get $c_2 = x^*(0) = 5$. Then c_1 follows from the end time condition: $0 = x(T) = -c_1 T + 5$ gives $c_1 = 5/T$. Observe that $|c_1| \leq 2$ implies $|T| \geq 5/2$. Conclusion: if $T \geq 5/2$ then the minimum principle yields $u^* \equiv -5/T$ as the candidate-optimal solution.

In contrast,[1] for $T < 5/2$ the entire optimal control problem is meaningless. Namely, from $x(T) - x(0) = -5 = \int_0^T \dot{x}$ it follows that any control function u for this problem – whether optimal or not – must satisfy $\int_0^T u(t)dt = -5$. But by $-2 \leq u(t) \leq 2$ for all t it also follows that $\int_0^T -2 \leq \int_0^T u \leq \int_0^T 2$, i.e., that $-2T \leq -5 \leq 2T$. For $T < 5/2$ this is impossible.

b. The analysis in part a can still be used up to the determination of c_1. Now the extra hypothesis about the variable end time, combined with the stationarity of the problem, implies $H(x^*(t), u^*(t), p(t)) \equiv 0$. This gives $\frac{1}{2}(c_1^2 + 1) - c_1^2 = 0$. This equation yields $c_1^2 = 1$, whence either $c_1 = 1$ or $c_1 = -1$ (note that the above condition $|c_1| \leq 2$ is satisfied in either case).

Case 1: $c_1 = 1$. This gives $u^* \equiv -1$ and $x^*(t) = 5 - t$. So $T^* = 5$ follows by the end time condition.

Case 2: $c_1 = -1$. This gives $u^* \equiv 1$ and $x^*(t) = 5 + t$. But now the end time condition leads to $T^* = -5 < 0$, which is not allowed.

[1] No points were subtracted in case you missed this point.

Conclusion: there is essentially one candidate-optimal solution and it is $u^* \equiv -1$.

Alternative solution: Part b can also be solved as follows. According to part a, the "cost" of using time T is

$$f(T) := \int_0^T \frac{1}{2}((u^*(t))^2 + 1)dt = \frac{1}{2}\int_0^T (\frac{25}{T^2} + 1)dt = \frac{1}{2}(\frac{25}{T} + T),$$

and this is to be minimized over all $T \geq 5/2$. It is easy to see that $f(T)$ is strictly convex, because of $f''(T) = 25/T^3 > 0$. So setting $0 = f'(T) = \frac{1}{2}(-\frac{25}{T^2} + 1)$ gives the optimal value $T^* = 5$.

Problem 2. [30 pts] Consider the discrete-time optimal control problem to minimize $\sum_{k=0}^{N-1} x_k^2$ over all sequences $(u_0, u_1, \ldots, u_{N-1})$ with $|u_k| \leq 1$ for all k. Here the initial state $x_0 > 0$ is given and the dynamical system is $x_{k+1} = x_k + u_k$.

a. Intuitively, it is obvious what the optimal solution $(u_0^*, u_1^*, \ldots, u_{N-1}^*)$ should be; state what you think it should be and state the associated trajectory as well. *Note:* No further calculations or justifications of your guess are needed in this part.

b. Verify that your $(u_0^*, u_1^*, \ldots, u_{N-1}^*)$ in part a meets the necessary conditions for optimality, as established by the discrete-time minimum principle.

c. *For 10 extra points:* Prove that your guess in part a, assuming that the verification in part b turned out to be correct (which made it candidate-optimal), is actually optimal. *Hint:* The theorem of Weierstrass states that any continuous function, when minimized over a closed and bounded subset of $\mathbb{R}^d$, has a global minimum.

Solution. a. Because $\sum_{k=0}^{N-1} x_k^2$ only contains state variables x_k, one should bring the state (which starts at $x_0 > 0$) down in value as quickly as possible and eventually make it equal to zero or as close to zero as possible (this idea is similar to what was discussed in class about the homework Exercise 3.10 in Bertsekas). This leads to the folowing conjecture about the optimal control sequence $(u_0^*, \ldots, u_{N-1}^*)$.

(*i*) if $x_0 > N$ then $u_k^* \equiv -1$ should be optimal; note that the associated trajectory is $x_k^* := x_0 - k$.

(*ii*) if $x_0 > 0$, it must be of the form $x_0 = m + r$, with $m \geq 0$ an integer and $0 \leq r < 1$. Then one should take $u_k^* = -1$ for $k = 0, \ldots, m-1$, $u_m^* = -r$ and $u_k^* = 0$ for $k = m+1, \ldots, N-1$ (if $m = 0$ this just means taking $u_0^* := -r$). Note that the associated trajectory is $x_k^* := x_0 - k$ if $k \leq m$ and $x_k^* := 0$ if $k > m$ (for instance, if $x_0 = 4.3 = m + r$ with $m = 4$ and $r = 0.3$, then $x_1^* = 3.3, ..., x_4^* = 0.3$ and $x_k^* = 0$ for $k \geq 5$, assuming that $N \geq 5$).

b. Each $U_k := [-1, 1]$ is a convex set; this satisfies an essential underlying condition for the discrete-time minimum principle. The Hamiltonian for this problem is $H(x_k, u_k, p_{k+1}) := x_k^2 + p_{k+1}(x_k + u_k)$, so $H_{x_k} = 2x_k + p_{k+1}$ and the adjoint equation gives (*a*) $p_k = \nabla_{x_k} H = 2x_k^* + p_{k+1}$ for $k = 1, \ldots, N$ and with (*b*) $p_N = 0$ by transversality. Because $H(x_k, u_k, p_{k+1})$ is obviously convex (even linear) in u_k, formula (3.44) in Bertsekas can be applied instead of (3.43). This says

$$u_k^* \text{ minimizes } p_{k+1}u_k \text{ over all } u_k \in [-1, 1] \text{ for } k = 0, \ldots, N-1$$

Observe that the latter is *equivalent* to the following: for every $k = 0, \dots, N-1$

$$u_k^* = \begin{cases} -1 & \text{if } p_{k+1} > 0 \\ ? & \text{if } p_{k+1} = 0 \\ 1 & \text{if } p_{k+1} < 0 \end{cases} \tag{1}$$

where "?" means: undetermined. We must verify that conditions (a), (b) and (1) hold for the conjectured $(u_0^*, \dots, u_{N-1}^*)$ in each of the cases (i), (ii) in part a. Let $(p_N, p_{N-1}, \dots, p_1)$ be the solution of (a)-(b). Then, equivalently,

$$p_N = 0 \text{ and } p_k = 2x_k^* + 2x_{k+1}^* + \cdots 2x_{N-1}^* \text{ for } k = N-1, \dots, 1, \tag{2}$$

since $p_{N-1} = 2x_{N-1}^* + \underbrace{p_N}_{=0}$, $p_{N-2} = 2x_{N-2}^* + \underbrace{p_{N-1}}_{=2x_{N-1}^*}$, etc. It remains to verify (1) in each case:

Case (i). If $x_0 > N$ and $u_k^* = -1$, then by part a $x_k^* = x_0 - k > N - N = 0$ for $k = 1, \dots, N$, so (2) implies $p_k > 0$ for all $k \leq N-1$. It follows that (1) holds for the conjectured $u_k^* \equiv -1$.

Case (ii). If $x_0 \geq 0$ is of the form $x_0 = m + r$ with $m \geq 0$ integer and $0 \leq r < 1$, then by part a $x_k^* \geq 0$ for $k = 1, \dots, N$, so (2) implies $p_k \geq 0$ for all $k \leq N-1$. It follows that (1) holds for the conjectured $(u_0^*, \dots, u_{N-1}^*)$.