Exact and Useful Optimization Methods for Microeconomics

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Summary

Presentation of exact methods for determination of all global optimal solutions of the *utility maximization problem* (UMP) in microeconomics:

$$\text{maximize } u(x_1, x_2, \ldots, x_\ell) \text{ over all bundles } (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell_+$$

subject to the *budget constraint* $p_1x_1 + p_2x_2 + \cdots p_\ell x_\ell \leqslant y$

$p_1, p_2, \ldots, p_\ell > 0$: *prices* of goods $1, 2, \ldots, \ell$ and $y > 0$: *income*

Similar methods also exist for UMP with $\mathbb{R}^\ell_+$ replaced by $\mathbb{R}^\ell_{++}$, but not discussed here

Will present two solution methods:

1. Solution method 1 via parts (a)-(b) of Main Theorem: no (quasi-)concavity conditions

2. Solution method 2 via parts (c)-(d) of Main Theorem: need (quasi-)concavity conditions, including a new *stringent quasiconcavity* condition, custom-made for microeconomics

Presentation stresses how imposing precise microeconomic specifications can inspire the mathematics to be used

Origins lie in teaching solution method 1 and simplified version of solution method 2. Similar approach is also possible for expenditure minimization problem

Survey of this presentation

Criteria for operational usefulness of UMP solution methods: four test cases

A remarkable modelling error in the advanced literature

Main Theorem, first part, and UMP-solution method 1

Unusual application of method 1 to a test case

Preparations for Main theorem, second part

Main Theorem, second part, and UMP-solution method 2

Application of method 2 to a test case

Criteria for operational usefulness of solution methods: test cases

Method(s) must work to determine all global optimal UMP-solutions for at least the following test cases on $\mathbb{R}^\ell_+$:

(i) Cobb-Douglas utility functions, i.e., u's of the type

$$u(x_1, x_2, \ldots, x_\ell) \stackrel{def}{=} A\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$$

with $A, \alpha_1, \alpha_2, \ldots, \alpha_\ell > 0$

(ii) standard CES utility functions, i.e., u's of the type

$$u(x_1, x_2, \ldots, x_\ell) \stackrel{def}{=} [a_1 x_1^\rho + a_2 x_2^\rho + \cdots + a_\ell x_\ell^\rho]^{1/\rho}$$

with $a_1, a_2, \ldots, a_\ell > 0$ and $0 < \rho < 1$

(*iii*) u's of the type

$$u(x_1, x_2) = x_1^2(x_2 + 1),$$

whose optima can be interior solutions (i.e. in $\mathbb{R}^\ell_{++}$) or boundary/corner solutions (i.e., in $\mathbb{R}^\ell_+ \backslash \mathbb{R}^\ell_{++}$), depending on prices and income

(*iv*) Leontiev utility functions, i.e., u's of the type

$$u(x_1, x_2, \ldots, x_\ell) \stackrel{def}{=} \min(b_1 x_1, b_2 x_2, \cdots, b_\ell x_\ell),$$

with $b_1, b_2, \ldots, b_\ell > 0$.

Note: Leontiev utility functions are non-differentiable ...

Common features of the test cases

Basic features of utility functions u on $\mathbb{R}^\ell_+$ in all four test cases:

- u is continuous on $\mathbb{R}^\ell_+$
- u is strictly increasing on $\mathbb{R}^\ell_+$, i.e., $x'_1 > x_1, x'_2 > x_2, \ldots, x'_\ell > x_\ell$ implies $u(x'_1, x'_2, \ldots, x'_\ell) > u(x_1, x_2, \ldots, x_\ell)$
- u is differentiable on an open set $\Omega \subset \mathbb{R}^\ell_{++}$

$\Omega \stackrel{def}{=} \mathbb{R}^\ell_{++}$ works in cases (i) to (iii), but in case (iv) take

$$\Omega \stackrel{def}{=} \{(x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell_{++} : b_i x_i \neq b_j x_j \text{ for all } i \neq j\}$$

From now on: above three basic conditions for u are in force. Note: these conditions are *unconventional* from classical optimization viewpoint, so use of standard literature might be restrictive ...

... and turns out to be so!

A remarkable modelling error in the advanced literature

Observe: to take a differentiability set Ω for u with $\Omega \supset \mathbb{R}^\ell_+$, as encountered frequently, is *not* a common characteristic!

Reason 1: Differentiability need not hold on $\mathbb{R}^\ell_+ \setminus \mathbb{R}^\ell_{++}$. For instance, think of $u(x_1, x_2) = \sqrt{x_1 x_2}$. This holds for cases (i) (Cobb-Douglas) and (ii) (CES), i.e., the most popular applications

Reason 2: Defining u outside $\mathbb{R}^\ell_+$ may be problematic or impossible: again cf. cases (i) (Cobb-Douglas) and (ii) (CES). Supported by intuition: defining u outside $\mathbb{R}^\ell_+$ is economically unnatural.

Reasons 1-2 explain a remarkable error in the literature (including Mas-Colell-Whinston-Green, Simon-Blume, Luenberger ...)

Intermezzo: the subtle test case (*iii*)

Illustrations:

Case (i) for $\ell = 2$: only *interior* UMP-solutions $\mathbf{x}^* \stackrel{def}{=} (x_1^*, x_2^*)$:

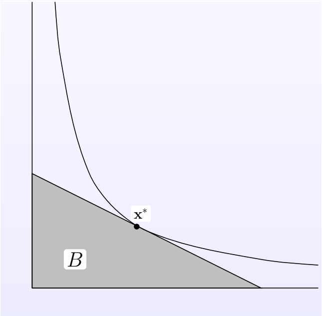

Figure: INTERIOR SOLUTION OF UMP: $\mathbf{x}^* \in \mathbb{R}^{\ell}_{++}$

Here $B \stackrel{def}{=} \{(x_1, x_2) \in \mathbb{R}^2_+ : p_1x_1 + p_2x_2 \leqslant y\}$ is the *budget set*

In contrast, case (*iii*) can have interior UMP-solutions (x_1^*, x_2^*) if $\frac{p_1}{p_2}$ is sufficiently large:

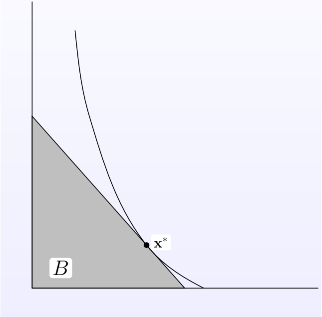

Figure: INTERIOR SOLUTION OF UMP IN CASE (*iii*)

But ...

... case (*iii*) can also have corner UMP-solutions $\mathbf{x}^*$ if $\frac{p_1}{p_2}$ is sufficiently small, i.e., when good 2 is so expensive that consumer wants none of it:

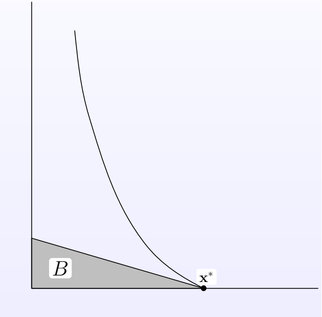

Figure: CORNER SOLUTION OF UMP IN CASE (iii): $\mathbf{x}^* = (\frac{y}{p_1}, 0)$

Main Theorem, parts (a)-(b)

Use *vector notation*: e.g., $\mathbf{x} \stackrel{def}{=} (x_1, x_2, \ldots, x_\ell)$, $\mathbf{p} \stackrel{def}{=} (p_1, p_2, \ldots, p_\ell)$, etc.

Main Theorem (a) The UMP has an optimal solution and every optimal solution $\mathbf{x}^*$ is such that

$$p_1 x_1^* + p_2 x_2^* + \cdots + p_\ell x_\ell^* = y \ \textit{(budget-balancedness)}$$

(b) If the UMP has a solution $\mathbf{x}^*$ in Ω, then there exists $\lambda \geqslant 0$ such that

$$\frac{\frac{\partial u}{\partial x_1}(\mathbf{x}^*)}{p_1} = \frac{\frac{\partial u}{\partial x_2}(\mathbf{x}^*)}{p_2} = \cdots \frac{\frac{\partial u}{\partial x_\ell}(\mathbf{x}^*)}{p_\ell} = \lambda$$

This is *Gossen's second law* (1854): the marginal cost of acquisition is constant across goods

Remark: λ in part (b) acts as a *Lagrange/Kuhn-Tucker multiplier* (but (b) also has an elementary direct proof)

Solution method 1 – based on (a)-(b) in Main Theorem

STEP 0: Verify that u is continuous and strictly increasing on $\mathbb{R}^\ell_+$

STEP 1: Determine the set C of all budget-balanced bundles $\mathbf{x} \in \mathbb{R}^\ell_+$ for which either

(a) $\mathbf{x}$ is in Ω and satisfies Gossen's second law

or

(b) $\mathbf{x} \in \mathbb{R}^\ell_+ \setminus \Omega$

STEP 2: Determine $\mu \stackrel{def}{=} \max_{\mathbf{x} \in C} u(\mathbf{x})$; then $\{\mathbf{x} \in C : u(\mathbf{x}) = \mu\}$ is the set of all globally optimal solutions

Name: C is called the set of all optimality *candidates*; in above method it is certainly nonempty

Observe: method 1 is only practical if $\mathbb{R}^\ell_+ \setminus \Omega$ is "small", but even then step 2 can be hard; this is particularly true for case (*iii*). It explains forthcoming method 2

Example: application of method 1 to test case (*iv*)

Example: Leontiev case (*iv*) for $\ell = 2$, i.e., $u(x_1, x_2) = \min(b_1 x_1, b_2 x_2)$, with $b_1, b_2 > 0$

Recall choice of $\Omega \stackrel{def}{=} \{(x_1, x_2) \in \mathbb{R}^2_{++} : x_2 \neq \frac{b_1}{b_2} x_1\}$

STEP 0 is OK: u is evidently continuous and strictly increasing on $\mathbb{R}^\ell_+$

STEP 1(a): Gossen's law states $\frac{b_1}{p_1} = 0$ if $x_2 > \frac{b_1}{b_2} x_1$ and $\frac{b_2}{p_2} = 0$ if $x_2 < \frac{b_1}{b_2} x_1$: both are impossible, so step 1(a) gives no candidates

STEP 1(b) gives $(\frac{y}{p_1}, 0)$, $(0, \frac{y}{p_2})$ and all budget balanced (x_1, x_2) with $x_2 = \frac{b_1}{b_2} x_1$, i.e., the single bundle $\bar{\mathbf{x}} \stackrel{def}{=} (\frac{b_2 y}{b_2 p_1 + b_1 p_2}, \frac{b_1 y}{b_2 p_1 + b_1 p_2})$

Summary: step 1 yields $C = \{(\frac{y}{p_1}, 0), (0, \frac{y}{p_2}), \bar{\mathbf{x}}\}$

STEP 2: Obviously $\mu = u(\bar{\mathbf{x}}) > 0 = u(\frac{y}{p_1}, 0) = u(0, \frac{y}{p_2})$, so $\bar{\mathbf{x}}$ is the unique UMP-solution, i.e., the *Marshallian demand bundle*

Preparations for parts (*c*)-(*d*) of Main Theorem

Definition: A set $D \subset \mathbb{R}^\ell_+$ is *convex* if for every $\mathbf{x}, \mathbf{x}' \in D$

$$t\mathbf{x} + (1-t)\mathbf{x}' \in D \text{ for every } 0 < t < 1$$

Definition: 1. u is *quasiconcave* on a convex set $D \subset \mathbb{R}^\ell_+$ if for every $\mathbf{x}, \mathbf{x}' \in D$

$$u(t\mathbf{x} + (1-t)\mathbf{x}') \geqslant \min(u(\mathbf{x}), u(\mathbf{x}')) \text{ for every } 0 < t < 1$$

2. u is *strictly quasiconcave* on a convex set $D \subset \mathbb{R}^\ell_+$ if for every $\mathbf{x}, \mathbf{x}' \in D$ with $\mathbf{x} \neq \mathbf{x}'$

$$u(t\mathbf{x} + (1-t)\mathbf{x}') > \min(u(\mathbf{x}), u(\mathbf{x}')) \text{ for every } 0 < t < 1$$

Distinction: indifference curves of strictly quasiconcave u's cannot have straight line segments

Graphical illustration of (strict) quasiconcavity:

The function u on $\mathbb{R}^\ell_+$ is [strictly] quasiconcave if and only if for every $\alpha \in \mathbb{R}$

$$\{u \geqslant \alpha\} \stackrel{def}{=} \{\mathbf{x} \in \mathbb{R}^\ell_+ : u(\mathbf{x}) \geqslant \alpha\} \text{ is a [strictly] convex set}$$

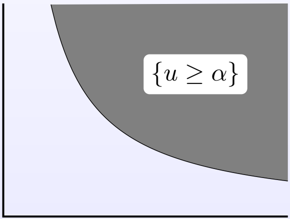

Figure: CONVEXITY OF $\{u \geqslant \alpha\}$

Question: which u's in cases (i) to (iv) are quasiconcave or strictly quasiconcave on $\mathbb{R}^\ell_+$?

Case (i) (C-D): quasiconcave on $\mathbb{R}^\ell_+$, but *only* strictly quasiconcave on $\mathbb{R}^\ell_{++}$

Case (ii) (CES, $0 < \rho < 1$): strictly quasiconcave on $\mathbb{R}^\ell_+$

Case (iii) ("subtle case"): quasiconcave on $\mathbb{R}^\ell_+$, but *only* strictly quasiconcave on $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$

Case (iv) (Leontiev): quasiconcave on $\mathbb{R}^\ell_+$, but not strictly quasiconcave

Conclusion: quasiconcavity is common to all test cases, but strict quasiconcavity, more desirable because it leads to uniqueness of UMP-solutions, is *not*

A new property: *stringent* quasiconcavity

Follow-up question: is there a useful property “between” quasiconcavity and strict quasiconcavity that is shared by all test cases ?

Answer: yes, there is, but with exclusion of case (*iv*), which is “too linear” (but recall: solution method 1 can handle it)

Definition: u is *stringently quasiconcave* on $\mathbb{R}^\ell_+$ if it is quasiconcave on $\mathbb{R}^\ell_+$ and if it has the following property: for every $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^\ell_+$ with $\mathbf{x} \neq \mathbf{x}'$ and $u(\mathbf{x}) = u(\mathbf{x}') > u(0)$

$$u(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}') > u(\mathbf{x}) = u(\mathbf{x}')$$

Observe how stringent quasiconcavity exploits that in microeconomics u is strictly increasing!

Such monotonicity is unparalleled in ordinary nonlinear programming, operations research, etc.

Employing stringent quasiconcavity

Of course, for any u

strictly quasiconcave $\Rightarrow$ stringently quasiconcave $\Rightarrow$ quasiconcave

Sufficient conditions for stringent quasiconcavity of u:

(1) $C_0 \stackrel{def}{=} \{x \in \mathbb{R}^\ell_+ : u(\mathbf{x}) > u(\mathbf{0})\}$ is convex

(2) there is a strictly increasing $h : (u(0), +\infty) \to \mathbb{R}$ such that $h(u(\mathbf{x}))$ is strictly quasiconcave on C_0

Can show: u's in test cases (i) to (iii) are stringently quasiconcave

Example: case (iii)**:** $u(x_1, x_2) = x_1^2(x_2 + 1)$, so $u(0,0) = 0$ and $C_0 = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 > 0\}$; now pick $h(t) \stackrel{def}{=} \log(t)$ on $(0, +\infty)$

Then $h(u(x_1, x_2)) = 2\log(x_1) + \log(x_2 + 1)$ is strictly concave, whence strictly quasiconcave on C_0. So u is stringently quasiconcave in case (iii)

Adding to the Main Theorem

First recall parts (a)-(b) of Main Theorem:

(a) The UMP has an optimal solution and every optimal solution $\mathbf{x}^*$ is budget balanced.

(b) If $\mathbf{x}^* \in \Omega$ is an optimal UMP-solution, then $\mathbf{x}^*$ satisfies Gossen's law: there exists $\lambda \geqslant 0$ such that all $\frac{\frac{\partial u}{\partial x_i}(\mathbf{x}^*)}{p_i} = \lambda$ for all $i = 1, 2, \ldots, n$.

Now add a well-known part (c) and a new part (d):
(c) Suppose u is quasiconcave. If $\mathbf{x}^* \in \Omega$ is budget-balanced and satisfies Gossen's law for $\lambda > 0$, then $\mathbf{x}^*$ is an optimal UMP-solution

(d) Suppose u is stringently quasiconcave. If $\mathbf{x}^* \in \Omega$ is budget-balanced and satisfies Gossen's law for $\lambda > 0$, then $\mathbf{x}^*$ is the unique optimal UMP-solution

Only part (d) is new; under its conditions one finds *all* optimal UMP-solutions, i.e., a single one

Solution method 2

Use parts (c)-(d) to build solution method 2:

STEP 0. Verify that u is continuous, strictly increasing and stringently quasiconcave on $\mathbb{R}^{\ell}_{+}$

STEP 1: Determine if there is a budget-balanced bundle $\mathbf{x} \in \mathbb{R}^{\ell}_{+}$ for which

$\mathbf{x}$ is in Ω and satisfies Gossen's second law

If such $\mathbf{x}$ exists, then STOP: there can only be one and it is the unique optimal UMP-solution

If not, then CONTINUE:
STEP 2: Determine the set C of all budget-balanced bundles in $\mathbb{R}^{\ell}_{+} \setminus \Omega$

STEP 3: Determine $\mu \overset{def}{=} \max_{\mathbf{x} \in C} u(\mathbf{x})$. Then $\{\mathbf{x} \in C : u(\mathbf{x}) = \mu\}$, is the set of all globally optimal solutions

Testing solution method 2

Solution method 2 can deal with all three cases (*i*)-(*iii*)

Example: case (*iii*). Here $u(x_1, x_2) = x_1^2(x_2 + 1)$, $\Omega = \mathbb{R}^\ell_{++}$

STEP 0: OK, we already saw that u is stringently quasiconcave

STEP 1: Gossen's law gives $2(x_2 + 1) = \frac{p_1}{p_2} x_1$; together with budget balancedness get one solution, i.e.

$$(\bar{x}_1, \bar{x}_2) = \left(2\,\frac{y + p_2}{3p_1}, \frac{y - 2p_2}{3p_2}\right)$$

Case 1: $y > 2p_2$. Then $(\bar{x}_1, \bar{x}_2) \in \Omega$ and $\lambda > 0$. So step 1 says STOP: $(\bar{x}_1, \bar{x}_2)$ is the unique optimal solution in case 1

Case 2: $y \leqslant 2p_2$. Then step 1 says CONTINUE with C, formed by the two corner points. By $u(\frac{y}{p_1}, 0) > 0 = u(0, \frac{y}{p_2})$ this implies that $(\frac{y}{p_1}, 0)$ is the unique optimal solution in case 2

Reference: E.J. Balder, ”Exact and useful optimization methods for microeconomics” in: *New Insights into the Theory of Giffen Goods* (W. Heijman and P. Mouche, eds.), Lecture Notes in Economics and Mathematical Systems 655, Springer, 2012, pp. 21-38

Related literature: Existence and Optimality of Competitive Equilibria, Aliprantis, Brown and Burkinshaw, Springer, 1989

The above article provides a comparison with the much more restrictive contribution by Aliprantis et al. and its Appendix gives a detailed critique of the treatment of optimization in the advanced microeconomics literature, as regards exactness, usefulness and completeness

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