Extra exercises
Analysis on Manifolds, 2013

Erik van den Ban and Marius Crainic

Exercises Lecture 6

Exercise 6.1.9 (Revised formulation).

Let $U \subset \mathbb{R}^n$ be an open subset, and let $P : C_c^\infty(U) \to C^\infty(U)$ be a linear operator such that for all $\chi, \psi \in C_c^\infty(U)$ the operator $M_\chi \circ P \circ M_\psi$ belongs to $\Psi^d(U)$.

(a) Show that for all $\chi, \psi \in C_c^\infty(U)$ there exist $p \in S^d(U)$ and $K \in C^\infty(U \times U)$ with $\operatorname{supp} p \subset \operatorname{supp} \chi \times \mathbb{R}^n$ and with $\operatorname{supp} K \subset \operatorname{supp} \chi \times \operatorname{supp} \psi$, such that

$$M_\chi \circ P \circ M_\psi = \Psi_p + T_K.$$

(b) Show that $P \in \Psi^d(U)$.

Exercise 3.8.7 (Extension of the original exercise) Let $P_r : H_r(M, E) \to H_{r-k}(M, F)$ be an operator as in Remark 3.8.6 of the lecture notes, and let $Q : H_{r-k}(M, F) \to H_r(M, E)$ be such that both $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing.

(a) Show that the kernel of the operator $P : \Gamma(E) \to \Gamma(F)$ is finite dimensional.

(b) What can you say about the cokernel of the operator $P : \Gamma(E) \to \Gamma(F)$.

(e) Show that the index of the operator $P_r : H_r(M, F) \to H_{r-k}(M, F)$ only depends on the principal symbol of P.

Exercises Lecture 7

In the extra exercises for Lecture 7, the following revised version of Lemma 7.3.8 will be needed.

Lemma 7.3.8 *Let $\{U_j\}$ be an open covering of the manifold M.*

(a) *Let $P, Q \in \Psi^d(M)$ be such that $P_{U_j} - Q_{U_j} \in \Psi^{-\infty}(U_j)$ for all j. Then $P - Q \in \Psi^{-\infty}(M)$.*

(b) *Assume that for each j a pseudo-differential operator $P_j \in \Psi^d(U_j)$ is given. Assume furthermore that $(P_i)_{(U_i \cap U_j)} = (P_j)_{(U_i \cap U_j)}$ for all indices i, j with $U_i \cap U_j \neq \emptyset$. Then there exist a $P \in \Psi^d(M)$ such that $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$ for all j. The operator P is uniquely determined modulo $\Psi^{-\infty}(M)$.*

Exercise 7.3.9 (Revised formulation). Let Ω be smooth manifold and E a vector bundle on Ω. Let $\{\Omega_j\}_{j \in J}$ be an open cover of Ω. Assume that for each pair of indices (i, j) with $\Omega_{ij} := \Omega_i \cap \Omega_j \neq \emptyset$ a smooth section $g_{ij} \in \Gamma(\Omega_{ij}, E)$ is given and that

$$g_{ij} + g_{jk} + g_{ki} = 0 \qquad \text{on} \quad \Omega_{ijk} := \Omega_i \cap \Omega_j \cap \Omega_k$$

for all $i, j, k \in J$ with $\Omega_{ijk} \neq \emptyset$.

There exists a partition of unity $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}$ on Ω which is subordinate to the covering $\{\Omega_j\}$. The latter requirement means that there exists a map $j : \mathcal{A} \to J$ such that $\operatorname{supp} \psi_\alpha \subset \Omega_{j(\alpha)}$ for all $\alpha \in \mathcal{A}$.

(a) Show that $g_j := \sum_\alpha \psi_\alpha g_{jj(\alpha)}$ defines a smooth section in $\Gamma(\Omega_j, E)$.

(b) Show that $g_i - g_j = g_{ij}$ on Ω_{ij}, for all $i, j \in J$.

Exercise 7.3.10 (Revised formulation). Let M be a smooth manifold and let $d \in \mathbb{R}$. For $P \in \Psi^d(M)$ and every open subset $U \subset M$ the operator $P_U : f \mapsto (Pf)|_U, C_c^\infty(U) \to C^\infty(U)$ belongs to $\Psi^d(U)$. Let $U \subset V$ be open subsets of M. Then $P \mapsto P_U$ defines a map $\Psi^d(V) \to \Psi^d(U)$.

(a) Show that the map $P \mapsto P_U$ maps $\Psi^{-\infty}(V)$ to $\Psi^{-\infty}(U)$.

Thus the map $P \mapsto P_U$ induces a restriction map

$$\rho_U^V : \Psi^d(V)/\Psi^{-\infty}(V) \to \Psi^d(U)/\Psi^{-\infty}(U)$$

which is a homomorphism of vector spaces. It is obvious that the restriction maps satisfy the conditions

$$\rho_U^U = \mathrm{I}, \qquad \rho_U^V \circ \rho_V^W = \rho_U^W,$$

for all open subsets $U, V, W \subset M$ with $U \subset V \subset W$. Because of these properties, the assignment $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ together with the system of restriction maps ρ_U^V is called a **presheaf** of vector spaces. The purpose of this exercise is to show that the presheaf $\Psi^d/\Psi^{-\infty}$ is in fact a **sheaf**. This means that for every open covering $\{U_j\}_{j \in J}$ of M the following conditions should be fulfilled.

(1) **Restriction property.** Let $P, Q \in \Psi^d(M)$ and assume that for all $j \in J$ the operator $P_{U_j} - Q_{U_j}$ belongs to $\Psi^{-\infty}(U_j)$. Then $P - Q \in \Psi^{-\infty}(M)$. (This and the next condition can be formulated more naturally in terms of the restriction maps, see the text following this exercise).

(2) **Gluing property.** Let for each $j \in J$ an operator $P_j \in \Psi^d(U_j)$ be given and assume that $(P_i)_{U_{ij}} - (P_j)_{U_{ij}} \in \Psi^{-\infty}(U_{ij})$ for all $i, j \in J$ with $U_{ij} := U_i \cap U_j \neq \emptyset$. Then there exists an operator $P \in \Psi^d(M)$ such that $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$ for all $j \in J$.

The exercise now proceeds as follows.

(b) Show that condition (1) is fulfilled. Hint: the proof is an adaptation of the proof of Lemma 7.3.8 (a). As in that proof, let $\Omega \subset M \times M$ be the union of the open subsets $U_j \times U_j \subset M \times M$, for $j \in J$. Let $K_P, K_Q \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ be the distribution kernels of P and Q. Show that $K_P - K_Q$ is smooth on Ω.

(c) With P_j as in condition (2), show that there exist $T_j \in \Psi^{-\infty}(U_j)$ such that $(P_i + T_i)_{U_{ij}} = (P_j + T_j)_{U_{ij}}$ for all $i, j \in J$.

Hint: put $\Omega_j = U_j \times U_j$. For all $i, j \in J$ with $U_i \cap U_j \neq \emptyset$, let $g_{ij} \in \mathcal{D}'(\Omega_{ij}, \mathbb{C}_M \boxtimes D_M)$ be the distribution kernel of the operator $(P_i)_{U_{ij}} - (P_j)_{U_{ij}}$. Show that the g_{ij} are smooth and apply Exercise 7.3.9 to find g_j. Define T_j in terms of g_j.

(d) Use (c) combined with Lemma 7.3.8 (b) to prove that condition (2) is fulfilled.

Final remark. The above conditions (1) and (2) are readily seen to be equivalent to the following conditions, formulated in terms of the restriction maps ρ_U^V.

(1)' Let $P, Q \in \Psi^d(M)$ (their images in $\Psi^d(M)/\Psi^{-\infty}(M)$ are denoted by $[P], [Q]$). Assume that $\rho^M_{U_j}([P]) = \rho^M_{U_j}([Q])$ for all $j \in J$. Then $[P] = [Q]$.

(2)' Let for each j an operator $P_j \in \Psi^d(U_j)$ be given and assume that

$$\rho^{U_i}_{U_{ij}}([P_i]) = \rho^{U_j}_{U_{ij}}([P_j])$$

for all $i, j \in J$ with $U_{ij} \neq \emptyset$. Then there exists a $P \in \Psi^d(M)$ such that $\rho^M_{U_j}([P]) = [P_j]$ for all j.

Exercise 7.5.6.
We consider the differential operator $P = -\Delta + e^{-\|x\|^2}$, where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

denotes the Laplacian on $\mathbb{R}^n$.

(a) Determine a symbol $p \in S^2(\mathbb{R}^n)$ such that $P = \Psi_p$. Do not forget to show that p belongs to $S^2(\mathbb{R}^n)$.
(b) Show that the operator P is properly supported.
(c) Show that the function $q(x, \xi) := (1 + \|\xi\|^2)^{-1}$ defines an element of $S^{-2}(\mathbb{R}^n)$.

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ and let $\chi' \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp} \chi$. Put $Q = M_{\chi'} \circ \Psi_q \circ M_\chi$.

(d) Show that $Q \in \Psi^{-2}$ and that Q is properly supported.
(e) Show that

$$Q \circ P - M_\chi \in \Psi^{-1}(\mathbb{R}^n).$$

Hint: use the principal symbol.

Exercise 9.3.3
We assume that M is a connected compact manifold of **dimension at least 2**. A differential operator $P : C^\infty(M) \to C^\infty(M)$ is said to be real if Pf is a real valued function whenever $f \in C^\infty(M)$ is real valued. Let $\mathcal{D}$ denote the algebra of **real** differential operators on M, and let $\mathcal{D}_k$ denote the real subspace consisting of $P \in \mathcal{D}$ of degree at most k. We will write $\sigma^k(P)$ for the k-th order principal symbol of $P \in \mathcal{D}_k$ in the sense of differential operators. Thus, $\sigma^k(P)$ is a function on T^*M which restricts to a homogeneous polynomial function of degree k on each cotangent space $T^*_x M$, for $x \in M$. We modify the principal symbol by a factor $1/i^k$ and put

$$\underline{\sigma}^k(P) = i^{-k} \sigma^k(P).$$

(a) Show that for each $P \in \mathcal{D}_k$ the modified principal symbol $\underline{\sigma}^k(P)$ is real-valued.
(b) Let $P \in \mathcal{D}_k$ be elliptic. Show that either $\underline{\sigma}^k(P)(\xi_x) > 0$ for all $x \in M$ and $\xi_x \in T^*_x M \setminus \{0\}$, or $\underline{\sigma}^k(P)(\xi_x) < 0$ for all $x \in M$ and $\xi_x \in T^*_x M \setminus \{0\}$.
(c) Show that $\mathcal{D}_k$ has no elliptic operators if k is odd.

(d) Let $P_0, P_1 \in \mathcal{D}_k$ be elliptic. Show that $\text{index}\,(P_0) = \text{index}\,(P_1)$. Hint: observe that we may as well assume that $\underline{\sigma}^k(P_0)$ and $\underline{\sigma}^k(P_1)$ have the same sign. Now consider a homotopy of operators on the level of suitable Sobolev spaces.

We will denote the common value of the indices of the elliptic operators in $\mathcal{D}_k$ by n_k.

We now assume that M is equipped with a Riemannian metric g. This means that each tangent space T_xM, for $x \in M$, is equipped with a positive definite inner product g_x. Furthermore, $x \mapsto g_x$ depends smoothly on $x \in M$ in the sense that it defines a smooth section of the tensorbundle $\otimes^2 T^*M$. By means of partitions of unity it can be shown that each manifold can be equipped with a Riemannian metric.

Let dV be the associated Riemannian volume density on M, i.e., dV is the section of the density bundle D_M determined by $dV_x(f_1, \dots f_n) = 1$ for each $x \in M$ and every orthonormal basis $f_1, \dots f_n$ of T_xM.

(e) Let $\mathfrak{X}(M)$ denote the space of smooth vector fields on M. Thus, $\mathfrak{X}(M) = \Gamma^\infty(TM)$. Show that the operator $\text{grad} : C^\infty(M) \to \mathfrak{X}(M)$ defined by

$$\langle \text{grad} f(x)\,,\, v_x \rangle = df(x) v_x$$

for $x \in M$, $v_x \in T_xM$ is a differential operator from the trivial bundle $\mathbb{C}_M$ to the complexified tangent bundle $(TM)_\mathbb{C}$. Show that the principal symbol of grad is given by

$$g_x(\sigma^1(\text{grad})(\xi_x)(1_x),\, \cdot\,) = i\xi_x, \qquad (x \in M,\ \xi_x \in T_x^*M).$$

Here 1_x denotes the element $(x, 1)$ of the fiber $\{x\} \times \mathbb{C}$ of the trivial bundle $\mathbb{C}_M = M \times \mathbb{C}$. Hint: use the characterization of Lemma 1.2.2.

(f) Show that there exists a unique first order differential operator $\text{div} : \mathfrak{X}(M) \to C^\infty(M)$ such that

$$\int_M (\text{div}\, v)(x) f(x)\, dV = -\int_M g_x(v(x), \text{grad} f(x))\, dV. \qquad (*)$$

Show that div is a first order differential operator from $(TM)_\mathbb{C}$ to the trivial bundle $\mathbb{C}_M$. Show that the principal symbol of div is given by

$$\sigma^1(\text{div}\,)(\xi_x) = i(\xi_x)_\mathbb{C} : (T_xM)_\mathbb{C} \to \mathbb{C}_M.$$

Hint: apply the characterization of Lemma 1.2.2 with uniformity in the variable x to the integrals of (*).

(g) Determine the principal symbol of the (Riemannian) Laplace operator

$$\Delta := \text{div} \circ \text{grad} : C^\infty(M) \to C^\infty(M).$$

Show that Δ is real elliptic of order 2.

Hint: use the dual inner product g_x^* on T_x^*M. This inner product is defined as follows. Write g_x for the (invertible) linear map $T_xM \to T_x^*M$ given by $g_x(v) = g_x(v,\, \cdot\,)$. Define $g_x^*(v^*, w^*) := g_x(g_x^{-1}(v^*), g_x^{-1}(w^*))$, for $v^*, w^* \in T_x^*M$.

(h) Show that $\langle \Delta f\,,\, f \rangle_{L^2(M)} < 0$ for every non-constant smooth function $f : M \to \mathbb{R}$.

(i) Show that $\dim \ker \Delta = 1$. Show that $\operatorname{index} \Delta = 0$. Hint: use that Δ is the transpose of Δ relative to dV, and show that $\operatorname{im}(\Delta) = \ker(\Delta)^{\perp}$.

(k) Show that $n_{2k} = 0$ for all $k \in \mathbb{N}$. Hint: use a general result on the index of the composition of Fredholm operators.

(x) Extra question for bonus points: discuss what can happen if M is one-dimensional (the circle).