Prerequisites from differential geometry for Analysis on Manifolds

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1 Manifolds

Notation and preliminaries

Let V, V' be finite dimensional real linear spaces, and let Ω be an open subset of V. We recall that a map $\varphi : \Omega \to V'$ is called differentiable at a point $a \in \Omega$ in the direction $v \in V$ if

$$\partial_v \varphi(a) = \frac{d}{dt}[\varphi(a + tv)]_{t=0}$$

exists.

The map f is called differentiable at $a \in \Omega$ if there exists a linear map $Df(a) : V \to V'$ such that

$$f(a + h) - f(a) = Df(a)h + o(h) \qquad (h \to 0).$$

The linear map $Df(a)$, which is unique when it exists, is called the derivative of f at a. If f is differentiable in a, then $\partial_v f(a)$ exists for every $v \in V$ and we have

$$\partial_v f(a) = Df(a)v.$$

When $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$, then the above formula may be used to express the matrix of $Df(a)$ in terms of the partial derivatives $\partial_j f_i = \partial_{e_j} f_i$ (the Jacobi matrix).

If f is differentiable in (any point of) Ω, then Df is a map from Ω to the space $\mathrm{Hom}(V, V')$ of linear maps $V \to V'$. If this map is differentiable, then f is called twice differentiable. The derivative of Df is denoted by $D^2 f$. It is now clear how to define the notion of a p times differentiable function and its p–th derivative $D^p f$. A function f is called p times continuously differentiable, or briefly C^p, if it is $p$times differentiable and $D^p f$ is continuous. We recall that f is C^p on Ω if all mixed partial derivatives of f order at most p exist and are continuous on Ω. Let $C^p(\Omega, V')$ denote the linear space of C^p–maps $\Omega \to V'$. Then the effect of any sequence of at most p partial derivatives applied to $C^p(\Omega, V')$ is independent of the order of the sequence.

A map $f : \Omega \to V'$ is called smooth (or C^∞) if it is C^p for every $p \geq 0$. We put

$$C^\infty(\Omega, V') = \cap_{p \geq 0} C^p(\Omega, V)$$

for the space of smooth maps $\Omega \to V'$.

Let $e_1, \ldots, e_n$ be a basis of V, and abbreviate $\partial_j = \partial_{e_j}$. Then ∂_j is a linear operator on the space $C^\infty(\Omega, V')$. By the above mentioned result on the order of mixed partial derivatives we have that ∂_i and ∂_j commute ($1 \le i, j \le n$). Hence, as an endomorphism of $C^\infty(\Omega, V')$ every mixed partial derivatives of order at most p is of the form:

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n},$$

with $|\alpha| := \alpha_1 + \cdots + \alpha_n \le p$.

We briefly write $C^p(\Omega)$ for $C^p(\Omega, \mathbb{C})$ ($0 \le p \le \infty$). By a linear partial differential operator with C^∞–coefficients on Ω we mean a linear endomorpism P of $C^\infty(\Omega)$ of the form:

$$P = \sum_\alpha c_\alpha \, \partial^\alpha$$

with finitely many non-trivial functions $c_\alpha \in C^\infty(\Omega)$. The number $k = \max\{|\alpha|;\ c_\alpha \neq 0\}$ is called the order of P.

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Let $\varphi : \Omega \to \Omega'$ be a bijection between open subsets of finite dimensional real linear spaces. Then φ is called a C^p–diffeomorphism ($1 \le p \le \infty$) if φ and φ^{-1} are C^p. Note that by the inverse function theorem this is equivalent to the requirement that $\varphi \in C^p$ and $D\varphi(a)$ is bijective for every $a \in \Omega$.

We shall now develop the theory of C^∞–manifolds (we leave it to reader to keep track of what can be done in a C^p–context, for further reading we suggest the references [La][1], [Wa][2]).

Let X be a Hausdorff topological space. A pair (U, χ), consisting of an open subset $U \subset X$ and a homeomorphism χ from U onto an open subset of $\mathbb{R}^n$ is called an n–dimensional chart of X. If (U', χ') is a second n–dimensional chart of X, such that $U \cap U' \neq \emptyset$, then the map $\chi' \circ \chi^{-1}$ is a homeomorphism from $\chi(U \cap U')$ onto $\chi'(U \cap U')$. This homeomorphism is called the transition map from the chart χ to the chart χ'.

A set $\{(U_\alpha, \chi_\alpha)\,;\ \alpha \in \mathcal{A}\}$ of n–dimensional charts is called a C^∞ (or smooth) n–dimensional atlas of X, if

(a) $\cup_{\alpha \in \mathcal{A}} U_\alpha = X$;

(b) all transition maps $\tau_{\beta\alpha} = \chi_\beta \circ \chi_\alpha^{-1}$ are smooth (i.e. C^∞).

Remark. Note that since $\tau_{\beta\alpha}$ is the inverse of $\tau_{\alpha\beta}$, it actually follows that all transition maps are diffeomorphisms.

An n–dimensional smooth (or C^∞) manifold is a Hausdorff topological space X equipped with a smooth n–dimensional atlas $\{(U_\alpha, \chi_\alpha)\,;\ \alpha \in \mathcal{A}\}$. An n–dimensional chart (U, χ) of the manifold X is called smooth if all the transition maps $\chi_\alpha \circ \chi^{-1}$ are diffeomorphisms. The components $\chi_1, \ldots, \chi_n$ will then be

[1][La]: S. Lang, Differential Manifolds, Addison Wesley, Reading Massachusetts 1972

[2][Wa]: F. Warner, Foundations of differentiable manifolds and Lie groups, Scott, Foresman and Co., Glenview Illinois 1971.

called a system of local coordinates of X. The collection of all smooth charts of X is an atlas by its own right, called the maximal atlas of the smooth manifold X.

Remark. Any open subset of a finite dimensional linear space is a smooth manifold in a natural way, its dimension being the dimension of the linear space. More generally any open subset of an n–dimensional smooth manifold X is a smooth manifold of dimension n in a natural way.

A map $f : X \to Y$ of smooth manifolds (of possibly different dimensions) is called C^p at a point $x \in X$ if there exist smooth charts (U, χ) and (V, ψ) of X and Y respectively, such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \chi^{-1}$ is a C^p map from $\chi(U)$ to $\psi(V)$. (Similarly one may define the concept of a p times differentiable map between smooth manifolds.)

One readily checks that the composition of C^p maps between smooth manifolds is C^p, etc.

The map $f : X \to Y$ is called a (C^∞) diffeomorphism if it is bijective, and if f and its inverse f^{-1} are smooth (i.e C^∞). Note that diffeomorphic manifolds have the same dimension. The present notion generalizes that of a diffeomorphism of open subsets of finite dimensional real linear spaces.

Our next objective is to generalize the notion of derivative of a differentiable smooth map between manifolds. The key to this is the concept of a tangent vector. Since our manifold is not contained in an ambient linear space, it may seem strange that tangent vectors can be defined at all. The basic idea is that it makes sense to say that two curves are tangent at a point. A tangent vector of a manifold is then defined as an equivalence class of tangential curves. More precisely, let X be smooth manifold of dimension n. Then by a differentiable curve in X, we mean a differentiable map $c : I \to X$, where $I \subset \mathbb{R}$ is some open interval containing 0. The point $c(0)$ is called the initial point of c. Let $x \in X$ be a fixed point. Then two differentiable curves c, d with initial point x are said to be tangent at x if there exists a smooth chart (U, χ) containing x, such that

$$\frac{d}{dt}\, \chi \circ c(t)|_{t=0} = \frac{d}{dt}\, \chi \circ d(t)|_{t=0}. \tag{1}$$

Suppose now that (V, ψ) is another smooth chart, and let $\tau = \psi \circ \chi^{-1}$ be the associated transition map. Then by the chain rule we have:

$$\frac{d}{dt}\, \psi \circ c(t)|_{t=0} = D(\tau)(\chi(x))\, \frac{d}{dt}\, \chi \circ c(t)|_{t=0}. \tag{2}$$

From this we see that if (1) holds in one chart containing x, then it holds in any other chart containing x. Let $\mathcal{C}_x$ denote the set of all differentiable curves in X with initial point x. Define the equivalence relation $\sim$ on $\mathcal{C}_x$ by $c \sim d$ if and only if c and d are tangential at x. We define

$$T_x X := \mathcal{C}_x / \sim .$$

The class of an element $c \in \mathcal{C}_x$ is denoted by $c'(0)$. The elements of $T_x X$ are called the tangent vectors of X at x.

Let (U, χ) be a chart containing x. Then for every $c \in \mathcal{C}_x$, the vector $d/dt\,[\chi \circ c](0)$ only depends on the equivalence class $c'(0)$. We denote it by $T_x\chi\ c'(0)$ (this notation will be justified at a later stage).

Lemma 1.1 *The map $T_x\chi : T_xX \to \mathbb{R}^n$ is bijective.*

Proof. The injectivity of $T_x\chi$ is an immediate consequence of the definitions. To establish its surjectivity, let $v \in \mathbb{R}^n$ and fix any differentiable curve $\underline{c}$ in $\chi(U)$, with initial point $\chi(x)$, and with $d/dt[\underline{c}](0) = v$. Let $c = \chi^{-1}\underline{c}$. Then $c \in \mathcal{C}_x$, and by definition we have $T_x\chi\ c'(0) = v$. $\square$

Let now (V, ψ) be another chart containing x. Then by (2) we have that

$$T_x\psi = D[\psi \circ \chi^{-1}]\,(\chi(x)) \circ T_x\chi \qquad \text{on} \qquad T_xX.$$

This implies the following.

Corollary 1.2 *The set T_xX has a unique structure of real linear space such that for every chart (U, χ) containing x the map $T_x\chi : T_xX \to \mathbb{R}^n$ is linear.*

The set T_xX, equipped with the structure of linear space described in the above corollary, is called the tangent space of X at x.

The tangent map

We can now generalize the concept of derivative to manifolds. Let $f : X \to Y$ be a map between smooth manifolds, and suppose that f is differentiable at the point $x \in X$. If $c, d \in \mathcal{C}_x$, then $f \circ c, f \circ d \in \mathcal{C}_{f(x)}$. Let (U, χ), and (V, ψ) be charts of X and Y such that $x \in U$, $f(U) \subset V$. Then the map $F = \psi \circ f \circ \chi^{-1}$ is differentiable. Moreover, if $c'(0) = d'(0)$, then by definition we have

$$\frac{d}{dt}[\chi \circ c](0) = \frac{d}{dt}[\chi \circ d](0).$$

If we apply $DF(\chi(x))$ to this expression we obtain

$$\frac{d}{dt}[\psi \circ f \circ c](0) = \frac{d}{dt}[\psi \circ f \circ d](0),$$

by the chain rule. It follows from this that $f \circ c$ and $f \circ d$ are equivalent elements of $\mathcal{C}_{f(x)}$. This shows that the map $\mathcal{C}_x \to \mathcal{C}_{f(x)}, c \mapsto f \circ c$ induces a map $T_xX \to T_{f(x)}Y$, which we denote by T_xf.

Note that it is immediate from the above discussion that the following diagram commutes:

$$\begin{array}{ccc} T_xX & \xrightarrow{T_xf} & T_{f(x)}Y \\ {\scriptstyle T_x\chi}\downarrow & & \downarrow{\scriptstyle T_{f(x)}\psi} \\ \mathbb{R}^n & \xrightarrow{D(\psi \circ f \circ \chi^{-1})(\chi(x))} & \mathbb{R}^m. \end{array} \qquad (3)$$

Hence it follows from Lemma 1.1 and Cor. 1.2 that the map $T_xf : T_x \to T_{f(x)}$ is linear; it is called the tangent map of f at x.

Theorem 1.3 (The chain rule). *Let $f : X \to Y$ and $g : Y \to Z$ be maps, such that f is differentiable at $x \in X$ and g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x, and*

$$T_x(g \circ f) = T_{f(x)} g \circ T_x f.$$

Proof. This follows from the ordinary chain rule by using the commutative diagram (3) three times, once for $f : X \to Y$ at x, once for $g : Y \to Z$ at $f(x)$, and once for $g \circ f : X \to Z$ at x. $\square$

Let U be an open subset of X. Then if $x \in U$ one readily checks that the tangent map $T_x i$ of the inclusion map $i : U \to X$ is an isomorphism $T_x U \to T_x X$. Via this isomorphism we shall identify $T_x U \simeq T_x X$. In particular, if U is an open subset of $\mathbb{R}^n$, then $T_x U \simeq T_x \mathbb{R}^n$. The latter space is identified with $\mathbb{R}^n$ as follows. For $v \in \mathbb{R}^n$, define the curve $c_v : [0,1] \to \mathbb{R}^n$ by $c_v(t) = x + tv$. Then we identify $\mathbb{R}^n \simeq T_x \mathbb{R}^n$ via the map $v \mapsto c_v'(0)$. We leave it to the reader to check the following. If $f : U \to V$ is a map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ then via the identifications discussed above, the tangent map $T_x f : T_x U \to T_{f(x)} V$ $(x \in U)$ corresponds to the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$. Also, if (U, χ) is a chart of a smooth manifold X containing the point $x \in X$, then the map $T_x \chi : T_x X \to \mathbb{R}^n$ of Lemma 1.1 corresponds to the map $T_x \chi : T_x X \to T_x \mathbb{R}^n$. Finally, observe that when $c \in \mathcal{C}_x$, then the element $c'(0)$, defined as the $\sim$ class of c, equals $T_0 c(1)$.

Remark. In the literature one also finds the notation $df(x)$ and $Df(x)$ for $T_x f$.

Submanifolds

Let X be an n–dimensional smooth manifold. A subset $Y \subset X$ is called a smooth submanifold of dimension k if for every $y \in Y$ there exists a chart (U, χ) containing y, such that $\chi(U \cap Y) = \chi(U) \cap \mathbb{R}^k$. Here we agree to identify $\mathbb{R}^k$ with the subspace $\{x \in \mathbb{R}^n\,;\ x_j = 0 \ \ (j \geq k)\}$.

Suppose that Y is a submanifold of X, and let $i_Y : Y \to X$ denote the inclusion map. Then one readily verifies that for every $y \in Y$ the map $T_y i_Y$ is an injective linear map. Via this map we shall identify $T_y Y$ with a linear subspace of $T_y X$.

Notice that the notion of submanifold as defined above generalizes the notion of smooth submanifold of $\mathbb{R}^n$. Notice also that a subset $Y \subset X$ is a submanifold of X if it looks like a submanifold of $\mathbb{R}^n$ in any set of local coordinates. More precisely, Y is a submanifold if for every smooth chart (U, χ) of X the set $\chi(U \cap Y)$ is a smooth submanifold of $\mathbb{R}^n$ (which may be empty).

Let X, Y be smooth manifolds. A map $f : X \to Y$ is called an immersion at a point $x \in X$ if the tangent map $T_x f : T_x X \to T_{f(x)} Y$ is injective. It is called a submersion at $x \in X$ if the tangent map $T_x f$ is surjective.

One now has the following useful result, which is a consequence of the implicit function theorem for $\mathbb{R}^n$.

Lemma 1.4 (Immersion Lemma). *Let $f : X \to Y$ be a smooth map, and let $x \in X$. Then f is an immersion at x if and only if $\dim X - \dim Y = p \geq 0$ and there exist open neigbourhoods $X \supset U \ni x$ and $Y \supset V \ni f(x)$ and a diffeomorphism φ of V onto a product $U \times \Omega$ with $\Omega \ni 0$ an open subset of $\mathbb{R}^p$, such that the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle I_U}\downarrow & & \downarrow{\scriptstyle \varphi} \\ U & \xrightarrow{i_1} & U \times \Omega \end{array}$$

Here i_1 denotes the inclusion $x \mapsto (x, 0)$.

Lemma 1.5 (Submersion Lemma). *Let $f : X \to Y$ be a smooth map between smooth manifolds. Then f is submersive at $x \in X$ if and only if* $\dim X - \dim Y = p \geq 0$, *and there exist open neigbourhoods $X \supset U \ni x$ and $Y \supset V \ni f(x)$ and a diffeomorphism $\varphi : U \to V \times \Omega$ with Ω an open subset of $\mathbb{R}^p$, such that the following diagram commutes*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle I_V} \\ V \times \Omega & \xrightarrow{\mathrm{pr}_1} & V \end{array}$$

Here pr_1 *denotes the projection onto the first component.*

In particular it follows from the above lemmas that an immersion is locally injective, and that a submersion always has an open image. From the definitions given before combined with the two lemmas above we now obtain:

Theorem 1.6 *Let X be a smooth manifold, and let $Y \subset X$ be a subset. Then the following conditions are equivalent.*

(a) *Y is a smooth submanifold;*

(b) *Y is locally closed in X, and for every $y \in Y$ there exists an open neighbourhood $U \ni y$ such that $U \cap Y$ is the image of an injective proper immersion;*

(c) *for every $y \in Y$ there exists an open neigbourhood $X \supset U \ni y$ and a submersion φ of U onto a smooth manifold Z such that $Y \cap U = \varphi^{-1}z$ for some $z \in Z$.*

Here we recall that a continuous map between locally compact Hausdorff topological spaces is proper if the preimage of every compact set is compact.

2 Vector fields

If X is a smooth manifold, we write TX for the disjoint union of the tangent spaces T_xX, $x \in X$. The set TX is called the tangent bundle of X.[3] We define the map $\pi : TX \to X$ by the requirement that $\pi(T_xX) \subset \{x\}$ for every $x \in X$.

[3] usually one reserves this name for TX equipped with a canonical structure of vector bundle

A map $v : X \to TX$ with $v(x) \in T_xX$ for every $x \in X$ is called a vector field on X. The set of vector fields on X is denoted by $\Gamma(TX)$. By defining addition and scalar multiplication of vector fields pointwise, we turn $\Gamma(TX)$ into a linear space.

Recall that if $U \subset \mathbb{R}^n$ is open, and $x \in U$, then we have a natural identification $\nu_x : T_xU \xrightarrow{\simeq} \mathbb{R}^n$. We identify TU with $U \times \mathbb{R}^n$ via the map ν given by $\nu(\xi) = (x, \nu_x(\xi))$, for $x \in U$, $\xi \in T_xU$. Via this identification a vector field on U may be viewed as a map $v : U \to U \times \mathbb{R}^n$, with $v(x) \in \{x\} \times \mathbb{R}^n$ for all $x \in U$. Thus $v(x) = (x, f(x))$ for a uniquely defined function $U \to \mathbb{R}^n$. In this way we may identify $\Gamma(TU)$ with the linear space of maps $U \to \mathbb{R}^n$. We now agree to call a vector field $v \in \Gamma(U)$ of class C^p, if it is C^p as a map $U \to \mathbb{R}^n$.

If $f : X \to Y$ is a differentiable map then we define the map $Tf : TX \to TY$ by $Tf = T_xf$ on T_xX.

A smooth map $f : X \to Y$ is a diffeomorphism if and only if $Tf : TX \to TY$ is bijective. Moreover, if this is the case then we have an induced bijective map $f_* : \Gamma(TX) \to \Gamma(TY)$, defined by the formula:

$$[f_*v](f(x)) = T_xfv(x).$$

After these preliminaries we can introduce the notion of a C^p vector field. On open subsets of $\mathbb{R}^n$ this has been done already. If X is a smooth manifold, then a vector field $v \in \Gamma(TX)$ is said to be C^p if for every $x \in X$ there exists a chart (U, χ) containing x so that $\chi_*(v|U)$ is smooth. By what we said above the latter assertion can also be rephrased as: the map

$$\chi_*(v|_U) \circ \chi : x \mapsto T_x\chi\, [v(x)] \tag{4}$$

is C^p. By the chain rule it follows that if v is a C^p vector field on X, then for *any* smooth chart (U, χ) the map (4) is C^p. The set $\Gamma^p(TX)$ of C^p vector fields on X is obviously a linear subspace of $\Gamma(TX)$.

From now on we assume that $1 \leq p \leq \infty$, that X is a smooth manifold, and that $v \in \Gamma^p(X)$. Let $x \in X$. Then by an integral curve for v with initial point x we mean a differentiable map $c : I \to X$, with I an open interval containing 0, such that

$$\begin{aligned} c(0) &= x \\ \dot{c}(t) &= v(c(t)) \qquad (t \in I). \end{aligned}$$

Here we have written $\dot{c}(t)$ for $\frac{d}{dt}c(t) = T_tc \cdot 1$. We now come to a nice reformulation of the existence and uniqueness theorem for systems of first order ordinary differential equations (use local coordinates to see this).

Theorem 2.1 *Let $v \in \Gamma^p(TX)$, $x \in X$. Then there exists an open interval $I \ni 0$ such that:*

(a) *there exists an integral curve $c : I \to X$ for v with initial point x;*

(b) *if $d : J \to X$ is a second integral curve for v with initial point x, then $d = c$ on $I \cap J$.*

Lemma 2.2 *Let $c : I \to X$ be an integral curve for v with initial point x. Fix $t_1 \in I_x$, let $I_1 = I - t_1$ be the translated interval, and let $c_1 : I_1 \to X$ be defined by $c_1(t) = c(t + t_1)$. Then c_1 is an integral curve for v with initial point x_1.*

Proof. By an easy application of the chain rule it follows that

$$\dot{c}_1(t) = \dot{c}(t + t_1) = v(c(t + t_1)) = v(c_1(t)).$$

Moreover, $c_1(0) = x_1$ by definition. $\square$

Corollary 2.3 *Let $c, d : I \to X$ be integral curves for v with initial point x. Then $c = d$.*

Proof. Let J be the set of $t \in I$ for which $c(t) = d(t)$. Then J is a closed subset of I by continuity of c and d. On the other hand, if $t_1 \in J$, then $c(t+t_1) = d(t+t_1)$ for t in a neighbourhood of 0, in view of Lemma 2.2 and Theorem 2.1. This implies that J is open in I as well. Hence J is an open and closed subset of I containing 0, and we see that $J = I$. $\square$

From this corollary it follows that there exists a maximal open interval $I_x \ni 0$ for which there exists an integral curve $c : I_x \to X$ for v with initial point x. Indeed I_x is the union of all the intervals which are domain for an integral curve with initial point x.

The associated unique integral curve $I_x \to X$ is called the maximal integral curve with initial point x.

Exercise 2.4 Let v be a C^1 vector field on a compact manifold X, and let $x \in X$. Show that $I_x = \mathbb{R}$. Hint: assume that I_x is bounded from above, and let s be its sup. Let $\alpha : I_x \to X$ be the maximal integral curve. Show that there exists a sequence $s_n \in I_x$ with $s_n \to s$ so that $\alpha(s_n) \to x_1$. Now apply the existence and uniqueness theorem to v and the starting point x_1.

The following results will be of crucial importance in the theory of Lie groups.

Corollary 2.5 *Let v be a C^p vector field on a smooth manifold X. Let $x \in X$, and let $\alpha : I_x \to X$ be the associated maximal integral curve. Let $t_1 \in I_x$, and let α_1 be the maximal integral curve with initial point $x_1 = \alpha(t_1)$. Then I_{x_1} equals the translated interval $I_x - t_1$. Moreover, for $t \in I_x$ we have:*

$$\alpha_1(t - t_1) = \alpha(t).$$

Proof. It follows from Lemma 2.2 that $c : I_x - t_1 \to X,\ t \mapsto \alpha(t_1 + t)$ defines an integral curve with initial point x_1. Hence $I_x - t_1 \subset I_{x_1}$. Moreover, $c = \alpha_1$ on $I_x - t_1$. In particular it follows that $-t_1 \in I_{x_1}$. Applying the same argument to α_1 and $x = \alpha_1(-t_1)$ we see that $I_{x_1} + t_1 \subset I_x$. Hence $I_{x_1} = I_x - t_1$. The desired equality now follows from $c = \alpha_1$ on $I_x - t_1$. $\square$

The following result, which is stated without proof, expresses that the integral curve depends smoothly on the initial value. Let Ω be the union of the subsets $I_x \times \{x\}$ of $\mathbb{R} \times X$. For $x \in X$, let $\alpha_x : I_x \to X$ be the maximal integral curve of v with initial point x. Then we define the flow of the vector field v to be the map $\Phi : \Omega \to X$ given by $\Phi(t, x) = \Phi_t(x) = \alpha_x(t)$.

Theorem 2.6 *Let v be a C^p vectorfield on X. Then Ω is an open subset of $\mathbb{R} \times X$, and the flow $\Phi : \Omega \to X$ is a C^p map.*