Harmonic Analysis

Lecture Notes

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Contents

1 Prerequisites on locally convex spaces

In these lecture notes, G will be a Lie group, and dx a choice of left Haar measure on G.

The group G is said to be unimodular if $|\det \mathrm{Ad}(x)| = 1$ for all $x \in G$. If G is unimodular, then dx is a right Haar measure as well.

If G is compact, then it is unimodular, so that dx is bi-invariant. Moreover, by adapting the normalization, we may determine dx uniquely by the requirement that $\int_G dx = 1$. This particular choice of measure is called normalized Haar measure on G.

In harmonic analysis on G, the theory of representations plays an important role. Such representations will always be defined in terms of a (complex) topological linear space. By this we mean a linear space V equipped with a topology such that

(a) the addition map $(v_1, v_2) \mapsto v_1 + v_2$ is continuous $V \times V \to V$;

(b) the scalar multiplication map $(\lambda, v) \mapsto \lambda v$ is continuous $\mathbb{C} \times V \to V$.

It is easy to see that a topological linear space V is Hausdorff if and only if $\{0\}$ is closed.

Let V be a topological linear Hausdorff space. By a Cauchy sequence in V we mean a sequence $(v_j)_{j \geq 0}$ in V such that for every neighborhood $\mathscr{O}$ of 0 in V there exists an index $n \geq 0$ such that $v_i - v_j \in \mathscr{O}$ for all $i, j > n$. The space V is said to be sequentially complete if every Cauchy sequence converges.

If the topology on V does not satisfy the first countability axiom (i.e. 0 does not have a countable fundamental system of neighborhoods) then this notion of sequential completeness is not strong enough. The suitable notion requires the concept of a *net*, which generalizes the notion of a sequence.

Recall that a *directed set* is a partially ordered set $(I, \leq)$ such that for all $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$. A *net* in V is a map $i \to v_i, I \to V$, with I a directed set. The net $(v_i)_{i \in I}$ is said to converge with limit $v \in V$ if for every open neighborhood $\mathscr{O}$ of v in V there exists an index $n \in I$ such that $i \geq n \Rightarrow v_i \in \mathscr{O}$. The net is said to be a Cauchy net if for every neighborhood $\mathscr{O}$ of the origin there exists $n \in I$ such that $i, j \geq n \Rightarrow v_i - v_j \in \mathscr{O}$. The topological linear space V is said to be complete ini V if every Cauchy net in V converges (has a limit).

The notion of a complete topological linear Hausdorff space V is still too general to develop a sufficiently rich theory, which allows for instance integration of functions in the space $C_c(G, V)$ of compactly supported continuous functions $G \to V$. On the other hand, for representation theory we definitely want to include the spaces $C(G)$ and $C_c(G)$ of continuous functions and compactly supported continuous functions $G \to \mathbb{C}$. Because of this, the class of Banach spaces is not large enough.

The appropriate subtype of topological linear space we need for representation theory is that of a locally convex space. By this we mean a topological linear space V (always assumed complex and Hausdorff) whose topology has certain convexity properties. This can be stated in terms of the existence of certain systems of seminorms.

To be precise, let V be a complex linear space. By a fundamental system of seminorms on V we mean a set $\mathscr{P}$ of seminorms on V such that

(a) for every pair $p_1, p_2 \in \mathscr{P}$ there exists a $q \in \mathscr{P}$ such that $p_1, p_2 \leq q$;

(b) if $v \in V$ and $p(v) = 0$ for all $p \in \mathscr{P}$, then $v = 0$.

For every seminorm p on V, every point $a \in V$ and every constant $r > 0$ we define the p-ball of center a and radius r by

$$B_p(a;r) = \{v \in V \mid p(v - v_0) < r\}.$$

Condition (a) guarantees that the collection of all these balls, for $p \in \mathscr{P}$, $a \in V$ and $r > 0$, together with the empty set, is a basis for a topology on V, turning V into a topological linear space. Condition (b) guarantees that the topology is Hausdorff.

By a locally convex space we mean a topological linear space whose topology can be described as above by a fundamental system $\mathscr{P}$ of seminorms. Obviously, this system is not uniquely determined. However, all seminorms in the system are continuous for the topology. More generally, the following lemma is immediate from the definitions.

Lemma 1.1. *Let V be a locally convex space, and $\mathscr{P}$ a fundamental system of seminorms determining the topology of V. Let q be a seminorm on V. Then the following conditions are equivalent.*

(a) *q is continuous on V;*

(b) *there exist $p \in \mathscr{P}$ and $C > 0$ such that $q \leq Cp$.*

Clearly, the system $\mathscr{N}_V$ of all continuous seminorms on V is a fundamental system which determines the topology. In fact, it is the maximal fundamental system for V.

Let $(v_i)_{i \in I}$ be a net in the locally convex space V. Convergence of the net now means that there exists a $v \in V$ such that for every continuous seminorm p on V and every $\varepsilon > 0$ there exists an index $n \in I$ such that $i \geq n \Rightarrow p(v_i - v) < \varepsilon$. The net is a Cauchy net if and only if for every continuous seminorm p on V and every $\varepsilon > 0$ there exists an index $n \in I$ such that $i, j \geq n \Rightarrow p(v_i - v_j) < \varepsilon$.

Before proceeding we mention the following characterization of continuity of a linear map. The proof is easy and left to the reader.

Lemma 1.2. *Let $T : V \to W$ be a linear map between locally convex spaces. Then the following conditions are equivalent.*

(a) *the map T is continuous;*

(b) *the map T is continuous at 0;*

(c) *for every continuous seminorm q on W there exists a continuous seminorm p on V such that $q(Tv) \leq p(v)$ for all $v \in V$.*

We will sometimes briefly write $T \circ q \leq p$ for the estimate in (c).

We are now finally prepared for the definition of the notion of continuous representation of G. If V is a locally convex (Hausdorff) space (over the base field $\mathbb{C}$) we denote by $\mathrm{End}(V)$ the space of continuous linear endomorphisms of V, and by $\mathrm{GL}(V)$ the subset of $\mathrm{End}(V)$ consisting of bijective $A \in \mathrm{End}(V)$ for which $A^{-1} \in \mathrm{End}(V)$.

Definition 1.3. By a *continuous representation* (π,V) of G in a complete locally convex space V we mean a group homomorphism $\pi : G \to \mathrm{GL}(V)$ for which the action map $(x,v) \mapsto \pi(x)v$, $G \times V \to V$ is continuous.

Remark 1.4. If V is finite dimensional, then the above continuity implies that $\pi : G \to \mathrm{GL}(V)$ is a continuous group homomorphism. As G and $\mathrm{GL}(V)$ are Lie groups, this implies that π is smooth, i.e. C^∞ (it is even analytic).

Let (π,V) be a continuous representation of G is a locally convex space V. By a closed invariant subspace of this representation we mean a closed subspace $W \subset V$ which is invariant for $\pi(x)$, for all $x \in G$.

Definition 1.5. A continuous representation (π,V) is said to be irreducible if 0 and V are the only closed invariant subspaces of V.

We recall that a Fréchet space is a complete locally convex space V whose topology can be generated by a countable fundamental system of continuous seminorms. Equivalently, this means that there exists a countable set $\mathscr{N}$ of continuous seminorms on V such that for each continuous seminorm q on V there exists a $p \in \mathscr{N}$ such that $q \le Cp$ for some $C > 0$. Note that a Banach space is Fréchet; hence, so is a Hilbert space.

A Fréchet space satisfies the following principle of *uniform boundedness*. Let W be a locally convex space and $\mathscr{T} := \{T_i \mid i \in I\}$ a family of continuous linear maps $V \to W$. If $\mathscr{T}$ is *pointwise bounded*, i.e., $\{T_i(v) \mid i \in I\}$ is bounded in W for every $v \in V$, then $\mathscr{T}$ is *equicontinuous*. The latter means that for every continuous seminorm q on V there exists a continuous seminorm p such that $q \circ T_i \le p$ for all $i \in I$.

The principle of uniform boundedness, which more generally is valid for the larger class of so-called *barrelled spaces*, implies a useful criterion for determining whether a group homomorphism $\pi : G \to \mathrm{GL}(V)$ defines a continuous action.

The strong topology on $\mathrm{End}(V)$ is defined to be the locally convex topology determined by the seminorms

$$A \mapsto \max_{v \in F} p(Av),$$

with $F \subset V$ a finite subset, and p a continuous seminorm on V.

The restriction of the strong topology to $\mathrm{GL}(V)$ is said to be the strong topology on $\mathrm{GL}(V)$.

Proposition 1.6. *Let V be a Fréchet (or more generally a barrelled) space and let $\pi : G \to \mathrm{GL}(V)$ be a group homomorphism. Then the following assertions are equivalent.*

(a) *For each $v \in V$ we have $\lim_{x \to e} \pi(x)v = v$.*

(b) *The map $\pi : G \to \mathrm{GL}(V)$ is continuous for the strong topology on $\mathrm{GL}(V)$.*

(c) *(π,V) is a continuous representation of G.*

Proof. It is clear that (c) implies (a). We will establish the implications '(a) $\Rightarrow$ (b) $\Rightarrow$ (c)'. For the first implication, assume that (a) holds. Then by homogeneity, for (b) to be valid it suffices to show that π is strongly continuous at e. Let ω be an open neighborhood of $I = \pi(e)$ in $\mathrm{GL}(V)$

for the strong topology. Then there exists a finite set $F \subset V$, a continuous seminorm p on V and a constant $\varepsilon > 0$ so that the set $\Omega = \{T \in \mathrm{GL}(V) \mid \forall v \in F : p(Tv - v) < \varepsilon\}$ is contained in ω. Condition (a) implies that for each $v \in F$ there exists an open neighborhood U_v of e in G such that $p(\pi(x)v - v) < \varepsilon$ for all $x \in U_v$. Let $U = \cap_{v \in F} U_v$. Then U is an open neighborhood of e in G and $\pi(U) \subset \Omega \subset \omega$. This implies (b).

We turn to the second implication. Assume that (b) is valid. Let $\mathscr{K}$ be a compact neighborhood of e in G. Then by (b) the family $\{\pi(k) | k \in \mathscr{K}\}$ is pointwise bounded. By the principle of uniform boundedness, the family is equicontinuous.

Fix $v_0 \in V$. Then by homogeneity it suffices to prove the continuity of the action map $G \times V \to V$ in (e, v_0). Let ω be an open neighborhood of v_0 in V. We will show that there exist open neighborhoods U of e in G and $\mathscr{O}$ of v_0 in V such that $\pi(x)v in \omega$ for all $(x, v) \in U \times \mathscr{O}$. Without loss of generality we may assume that $\omega = \{v \in V \mid p(v - v_0) < \varepsilon\}$, for p a continuous seminorm on V and $\varepsilon > 0$. We now note that

$$p(\pi(x)v - v_0) \leq p(\pi(x)(v - v_0)) + p(\pi(x)v_0 - v_0).$$

By equicontinuity, there exists a continous seminorm q on V such that $p \circ \pi(x) \leq q$ for all $x \in \mathscr{K}$. This leads to the estimate

$$p(\pi(x)v - v_0) \leq q(v - v_0)) + p(\pi(x)v_0 - v_0), \quad (x \in \mathscr{K}).$$

There exists an open neighborhood U of e in G contained in $\mathscr{K}$ such that $p(\pi(x)v_0 - v_0) < \varepsilon/2$ for all $x \in U$. Furthermore, $\mathscr{O} = B_q(v_0; \varepsilon/2)$ is an open neighborhood of v_0 in V. For $x \in U$ and $v \in V$ we have

$$p(\pi(x)v - v_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, the action map π maps $U \times \mathscr{O}$ into ω. $\square$

Of particular interest will be the class of unitary representations.

Definition 1.7. By a unitary representation of G we mean a continuous representation $(\pi, \mathscr{H})$ of G in a (complex) Hilbert space $\mathscr{H}$ such that $\pi(x) \in \mathrm{U}(\mathscr{H})$ for all $x \in G$.

The following criterion is useful for determining whether a representation is unitary.

Lemma 1.8. *Let $\mathscr{H}$ be a Hilbert space and $\pi : G \to \mathrm{U}(\mathscr{H})$ a group homomorphism. Let $\mathscr{D}$ be a dense subset of $\mathscr{H}$. Then the following conditions are equivalent.*

(a) *For each $v \in \mathscr{D}$ we have $\lim_{x \to e} \pi(x)v = v$.*

(b) *The representation π is continuous.*

Proof. It is clear that (b) implies (a). For the conversed implication, assume (a). Let $v_0 \in \mathscr{H}$ and $\varepsilon > 0$. Then there exists a $v_1 \in \mathscr{D}$ such that $\|v_1 - v_0\| < \varepsilon/3$. Furthermore, there exists an open neighborhood U of e in G such that $\|\pi(x)v_1 - v_1\| < \varepsilon/3$ for $x \in U$. Let $x \in U$, then by unitarity of $\pi(x)$ we find that

$$\begin{aligned} \|\pi(x)v_0 - v_0\| &\leq \|\pi(x)(v_0 - v_1)\| + \|\pi(x)v_1 - v_1\| + \|v_1 - v_0\| \\ &\leq \|\pi(x)v_1 - v_1\| + 2\|v_1 - v_0\| < \varepsilon. \end{aligned}$$

It follows that π satisfies the hypothesis and condition (a) of Proposition 1.6, hence is continuous. $\square$

The above result has a nice application to the left regular representation L of the group G in the Hilbert space $L^2(G) = L^2(G,dx)$ of square integrable functions $G \to \mathbb{C}$. The inner product on $L^2(G)$ is given by

$$\langle f \,,\, g\rangle = \int_G f(x)\overline{g(x)}\,dx, \qquad (f,g \in L^2(G)).$$

The left regular representation is given by the formula

$$L_x f(y) := f(x^{-1}y), \qquad (f \in L^2(G), x,y \in G).$$

By left invariance of the measure dx, one readily verifies that L_x is unitary for every $x \in G$.

Lemma 1.9. *Let $f \in C_c(G)$. Then the following assertions are valid.*

(a) $\lim_{x\to e} L_x f = f$ *relative to the sup-norm on G.*

(b) $\lim_{x\to e} L_x f = f$ *in the Hilbert space $L^2(G)$.*

Proof. The support $S := \mathrm{supp} f$ is compact. Select a compact subset $\mathscr{K}$ of G whose interior contains S. Then by compactness there exists an open neighborhood U_0 of e in G such that $U_0^{-1}S \subset \mathscr{K}$.

Let $\varepsilon > 0$. By compactness, the function f is uniformly continuous on $\mathscr{K}$ in the sense that there exists an open neighborhood U_ε of e in G such that for $a,b \in \mathscr{K}$ we have

$$ab^{-1} \in U_\varepsilon \;\Rightarrow\; |f(a) - f(b)| < \varepsilon.$$

Let now $x \in U := U_0 \cap U_\varepsilon^{-1}$ and $y \in S$. Then it follows that $x^{-1}y$ and y belong to $\mathscr{K}$, whereas $x^{-1}yy^{-1} = x^{-1} \in U_\varepsilon$, so that

$$|f(x^{-1}y) - f(y)| < \varepsilon.$$

This implies that $\sup_G |L_x f - f| = \sup_S |L_x f - f| < \varepsilon$ for all $x \in U$. Assertion (a) follows. Assertion (b) now follows from the estimate

$$\|L_x^f - f\|_{L^2}^2 = \int_G |L_x f(y) - f(y)|^2\,dy \leq \sup_G |L_x f - f| \int_S dy.$$

$\square$

Corollary 1.10. *The left regular representation $(L, L^2(G))$ is continuous (and unitary).*

Proof. The space $C_c(G)$ is dense in the Hilbert space $L^2(G)$. Now apply Lemma 1.8. $\square$

2 Integration and approximation

Let V be a complete locally convex space. Then it is possible to give a natural extension of the notion of Riemann integral to the space of continuous compactly supported functions $f : \mathbb{R}^n \to V$. More precisely, we have the following. We denote by V' the topological linear dual of V, i.e., the space of continous linear functionals $V \to \mathbb{C}$. Let $A \subset \mathbb{R}^n$ be compact, then by $C_A(\mathbb{R}^n, V)$ we indicate the Banach space of functions $f \in C(\mathbb{R}^n, V)$ which vanish outside A, equipped with the supnorm $\| \cdot \|_\infty$.

Lemma 2.1. *Let $A \subset \mathbb{R}^n$ be compact. There exists a unique continuous linear map $I_A : C_A(\mathbb{R}^n, V) \to V$ such that for every $\xi \in V'$ we have*

$$\xi \circ I_A(f) = \int_{\mathbb{R}^n} \xi(f(x))\, dx, \qquad (f \in C_A(\mathbb{R}^n, V)).$$

Proof. By the Hahn-Banach theorem for locally convex spaces, if $v \in V$ and $\xi(v) = 0$ for all $\xi \in V'$ then $v = 0$. This implies that I is uniquely determined by the above conditions.

We will now indicate how to prove existence. By a block in $\mathbb{R}^n$ we mean a product of segments of the form $X_j[a_j, b_j]$. Without loss of generality, we may assume that A is such a block. Let $\mathscr{P}$ denote the set of partitions of A in subblocks. The set of partitions carries the partial ordering $\preceq$ given by $P_1 \preceq P_2$ if P_2 is a refinement of P_1.

For each partition $P \in \mathscr{P}$ and every $B \in P$, we fix a choice of a point $\xi_B \in B$. For $f \in C_A(\mathbb{R}^n, V)$ and $P \in \mathscr{P}$, we define the Riemann sum

$$S(f, P) := \sum_{B \in P} f(\xi_B) \mathrm{vol}(B).$$

Then by using uniform continuity of $q \circ f$ with respect to every continuous seminorm q of V, one sees that $(S(f, P))_{P \in \mathscr{P}}$ is a Cauchy filter in V, hence convergent. The limit is denoted by $I(f)$. It is now easily verified that I is continuous linear and satisfies the requirement. $\square$

By uniqueness, it follows that I_A and I_B coincide on $C_{A \cap B}(\mathbb{R}^n, V)$. It follows that there exists a unique linear map $I : C_c(\mathbb{R}^n, V)$ which restricts to I_A on $C_A(\mathbb{R}^n, V)$ for every block $A \subset \mathbb{R}^n$. For obvious reasons we will call $I(f)$ the integral of $f \in C_c(\mathbb{R}^n, V)$ and agree to write

$$\int_{\mathbb{R}^n} f(x)\, dx = I(f).$$

By using partitions of unity, one may extend the above result to compactly supported continuous functions on manifolds, in case a positive density is given.

Lemma 2.2. *Let M be a manifold, and dm a positive density on M. Let V be a complete locally convex space. Then there exists a unique linear map $I : C_c(M, V) \to V$ such that*

$$\xi \circ I(f) = \int_M \xi(f(x))\, dx, \qquad (f \in C_c(M, V), \xi \in V').$$

The map I is continuous linear in the sense that it restricts to a continuous linear map $C_A(M, V) \to V$ for every compact set $A \subset M$.

In the setting of this lemma, we write $I(f) = \int_M f(m)\, dm$ and call this the integral of f over M. For future application, we also need the following lemma.

Lemma 2.3. *Let M be a manifold, and dm a positive density on M. Let p be a continuous seminorm on V and let A be a continuous linear operator from V to a second complete locally convex space W. Then the following hold, for all $f \in C_c(M,V)$:*

(a) $p(\int_M f(m)\, dm) \leq \int_M p(f(m))\, dm$;

(b) $A(\int_M f(m)\, dm) = \int_M A(f(m))\, dm$.

Proof. This follows straightforwardly from the constructions indicated above. □

The above results are in particular valid for a Lie group G equipped with a left (or right) Haar measure dx (which is a positive density). This yields an important tool for representation theory.

Let (π, V) be a continuous representation of G in a locally convex space. Given $f \in C_c(G)$ we define the operator $\pi(f) : V \to V$ by

$$\pi(f)v = \int_G f(x)\pi(x)v\, dx;$$

note that the integrand belongs to $C_c(G,V)$. It is readily seen that $\pi(f)$ is linear.

Lemma 2.4. *Let V be Fréchet (or, more generally, barrelled). Then $\pi(f)$ is continuous linear for every $f \in C_c(G)$.*

Proof. Let q be a continuous seminorm on V. Then by the principle of uniform boundedness, there exists a continuous seminorm p on V such that $q \circ \pi(x) \leq p$ for all $x \in \mathrm{supp} f$. It follows that

$$q(\pi(f)v) \leq \int_G |f(x)| q(\pi(x)v)\, dx \leq \int_G |f(x)| dx p(v),$$

showing that $\pi(f)$ is continuous. □

The structure of the group is reflected by the relation of $\pi : C_c(G) \to \mathrm{End}(V)$ with convolution.

The convolution product of two functions $f, g \in C_c(G)$ is defined by $f * g = L(f)g$, with L the left regular representation. Thus,

$$f * g(x) = \int_G f(y) g(y^{-1}x)\, dy.$$

Lemma 2.5. *Let (π, V) be a representation of G in a complete locally convex space. Then*

$$\pi(f * g) = \pi(f) \circ \pi(g), \qquad (f, g \in C_c(G)).$$

Proof. From the definitions it follows that, for $v \in V$,

$$\begin{aligned}\pi(f*g)v &= \int_G f*g(x)\pi(x)v\,dx \\ &= \int_G\int_G f(y)g(y^{-1}x)\pi(x)v\,dy\,dx \\ &= \int_G\int_G f(y)g(y^{-1}x)\pi(x)v\,dx\,dy\end{aligned}$$

where the interchange of integration is an application of the Fubini theorem. By left invariance of dx we now obtain

$$\begin{aligned}\pi(f*g)(x)v &= \int_G\int_G f(y)g(x)\pi(yx)v\,dx\,dy \\ &= \int_G f(y)\pi(y)[\int_G g(x)\pi(x)v\,dx]dy \\ &= \pi(f)\circ\pi(g)v.\end{aligned}$$

□

The above technique can be used in convolution type approximation arguments as follows.

Definition 2.6. An approximation of the identity on G is a sequence of functions $(\psi_j)_j \geq 0$ in $C_c(G)$ such that the following conditions are fulfilled.

(a) for every j we have $\psi_j \geq 0$ and $\int_G \psi_j(x)\,dx = 1$.

(b) for every neighborhood U of e in G there exists an index n such that $j \geq n \Rightarrow \operatorname{supp}\psi_j \subset U$.

The existence of an approximation of the identity is obvious. In fact, it exists under the additional requirement that each of the functions ψ_j is smooth.

Lemma 2.7. *Let $(\psi_j)_{j\geq 0}$ be an approximation of the identity on G, and let (π, V) be a continuous representation of G in a complete locally compact space. Then $\pi(\psi_j) \to I$ $(j \to \infty)$, with respect to the strong topology on* $\mathrm{End}(V)$, *i.e., pointwise.*

Proof. We note that

$$\pi(\psi_j)v - v = \int_G \psi_j(x)\pi(x)v\,dx - \int_G \psi_j(x)v\,dx = \int_G \psi_j(x)[\pi(x)v - v]\,dx.$$

Let q be a continuous seminorm on V. Then we obtain the estimate

$$q(\pi(\psi_j)v - v) \leq \int_G \psi_j(x)q(\pi(x)v - v)\,dx.$$

Let $\varepsilon > 0$. There exists a neighborhood U of e in G such that

$$q(\pi(x)v - v) < \varepsilon \qquad (x \in U).$$

There exists n as in (b) of Definition 2.6. For $j \geq n$ we now have

$$q(\pi(\psi_j)v - v) \leq \int_G \psi_j(x)\varepsilon\,dx = \varepsilon.$$

It follows that $\lim_{j\to\infty} q(\pi(\psi_j)v - v) = 0$. □

Corollary 2.8. *Let (π, V) be a continuous finite dimensional representation of G. Then there exists a function $f \in C_c(G)$ such that $\pi(f) = I_V$.*

Proof. Let (ψ_j) be an approximation of the identity. Then, by finite dimensionality of V, it follows that $\pi(\psi_j) \to I$ in the finite dimensional space $\mathrm{End}(V)$. This implies that I belongs to the closure of the linear subspace $\pi(C_c(G))$ in $\mathrm{End}(V)$. By finite dimensionality, this subspace is closed. Hence $I \in C_c(G)$. □

For later use, we also need the following result.

Lemma 2.9. *Let $f \in C_c(G)$ and $g \in L^2(G)$. Then $f * g \in C(G)$.*

Proof. By right homogeneity it suffices to prove the continuity at e. Let $\mathscr{K}$ be a compact subset of G whose interior contains $\mathrm{supp} f$. Then for x sufficiently close to e we have $\mathrm{supp} L_x f \subset \mathscr{K}$. We now note that

$$f * g(x) = \int_G f(y) g(y^{-1}x)\, dy = \int_G f(xy) g^\vee(y)\, dy,$$

where $g^\vee(y) := g(y^{-1})$. For $x \in G$ sufficiently close to e we find

$$\begin{aligned} |f * g(x) - f * g(e)| &\leq \int_G |f(xy) - f(y)||g^\vee(y)|\, dy \\ &\leq \|L_{x^{-1}} f - f\|_\infty \int_G 1_{\mathscr{K}}(y) |g^\vee(y)|\, dy \\ &\leq C \|L_{x^{-1}} f - f\|_\infty \end{aligned}$$

with $C = \|1_{\mathscr{K}}\|_2 \|1_{\mathscr{K}} g^\vee\|_2$, by the Cauchy-Schwartz inequality. By uniform continuity, it follows that

$$\lim_{x \to e} \|L_x f - f\|_\infty = 0.$$

We conclude that $f * g(x) \to f * g(e)$ for $x \to e$. This establishes the result. □

3 K-finite vectors

We assume that K is a compact Lie group, and that dk is a choice of left Haar measure on K. By compactness we may fix the measure uniquely by requiring that it is normalized, i.e., $\int_K dk = 1$. As K is compact, hence unimodular, the measure dk is also right invariant. Henceforth, the present choice of measure will be called the normalized Haar measure on K.

We denote by $\widehat{K}$ the set of (equivalence classes of) irreducible continuous finite dimensional representations of K.

Let $\delta \in \widehat{K}$, with representation space V_δ. If m is a positive integer, then by $m\delta$ we denote the direct sum of m copies of δ. A nice way to realize the representation space is by

$$V_{m\delta} = V_\delta \otimes \mathbb{C}^m,$$

with the K-action on the first component of the tensor product.

Lemma 3.1. *Let δ_1, δ_2 be inequivalent irreducible representations of K and let m_1, m_2 be positive integers. Then* $\mathrm{Hom}_K(m_1\delta_1, m_2\delta_2) = 0$.

Proof. It is clear that it suffices to show this for $m_1 = 1$. Let $V_1 := V_{\delta_1}$, let $V_2 := V_{\delta_2} \oplus \cdots \oplus V_{\delta_2}$ (m_2 copies) and let $P_j : V_2 \to V_{\delta_2}$ the projection onto the j-th component. Then P_j is K-equivariant. Thus if $T \in \mathrm{Hom}_K(V_1, V_2)$, then $P_j \circ T \in \mathrm{Hom}_K(V_{\delta_1}, V_{\delta_2})$. The latter space is zero by Schur's lemma. As this holds for every T and j the result follows. □

If (π, V) is any finite dimensional continuous representation of K, then V admits a decomposition into a direct sum $V = V_1 \oplus \cdots \oplus V_m$ such that each $\pi_j := \pi|_{V_j}$ is irreducible. We know that the decomposition into irreducibles is not unique. However, the so-called decomposition into isotypical components is unique.

If $v \in V$ we denote by $\mathrm{span}\,(\pi(K)v)$ the linear span of all vectors $\pi(k)v$, for $k \in K$. Clearly, this is an invariant subspace for π. Let $\delta \in \widehat{K}$ and denote by $V[\delta]$ the set of $v \in V$ for which the linear span $\mathrm{span}(\pi(K)v)$ admits a decomposition into irreducibles which are all equivalent to δ. The set $V[\delta]$ is readily seen to be a linear subspace of V which is K-invariant. It is called the isotypical component of π of type δ

Lemma 3.2. *Let $W \subset V[\delta]$ a K-invariant subspace. Then $\pi|_W$ admits a decomposition into a direct sum of copies of δ.*

Proof. If W is trivial, there is nothing to prove. Thus assume that $W \neq 0$ and fix $w \in W \setminus \{0\}$. Then $W_0 := \mathrm{span}(\mathrm{Ad}(K)w) \subset W$ admits a decomposition into a direct sum of copies of δ. As K is compact, and W_0 invariant, there exists an invariant subspace W_1 of W such that $W = W_0 \oplus W_1$. Applying induction on the dimension of W, we may assume that W_1 admits a decomposition of the required sort. Hence, W does. □

Lemma 3.3. *The set $F := \{\delta \in \widehat{K} \mid V[\delta] \neq 0\}$ is finite. Furthermore,*

$$V = \bigoplus_{\delta \in F} V[\delta].$$

Proof. Consider a finite set $S \subset \widehat{K}$. We will first establish the claim that the sum $\sum_{\delta \in S} V[\delta]$ is direct. We proceed by induction on the cardinality of S. For $|S| = 1$ the assertion is obvious. Let now $m > 1$ and assume that the claim has been established for S with $|S| < m$. Let $\delta_1, \ldots, \delta_m$ be distinct elements of $\widehat{K}$. Then the sum $W := V[\delta_1] \oplus \cdots \oplus V[\delta_{m-1}]$ is direct. We will complete the proof by showing that $U := W \cap V[\delta_m]$ is trivial. Arguing by contradiction, assume this is not the case. Then $U \subset V[\delta_m]$, hence by Lemma 3.2 the space U decomposes as a direct sum of copies of δ_m. It follows that there exists a non-trivial K-equivariant embedding $\varphi : V_{\delta_m} \hookrightarrow U$. For each $1 \leq j \leq m-1$ let $P_j : W \to V[\delta_j]$ be the projection along the other components. Then P_j is K-equivariant, hence so is $P_j \circ \varphi$. Furthermore, $V[\delta_j]$ is equivalent to a direct sum of copies of δ_j so by application of Lemma 3.1 it follows that $P_j \circ \varphi = 0$. This shows that $\varphi = 0$, contradiction. This establishes the claim.

From the claim it follows that $V = \oplus_{\delta \in \widehat{K}} V[\delta]$. By finite dimensionality it follows that F is finite. □

Lemma 3.4. *Let (π, V) be a continuous finite dimensional representation of K and $\delta \in \widehat{K}$. Then*

(a) *the natural linear map $\varphi_\delta : V_\delta \otimes \mathrm{Hom}_K(V_\delta, V) \to V$ given by $v \otimes T \mapsto T(v)$ intertwines the representation $\delta \otimes 1$ with π.*

(b) *The map φ_w is a linear isomorphism with image $V[\delta]$.*

(c) *In particular, $\pi|_{V[\delta]}$ decomposes as a direct sum of $\dim_{\mathbb{C}}(\mathrm{Hom}_K(V_\delta, V))$ copies of δ.*

Proof. Assertion (a) is obvious. As $\delta \otimes 1$ is a direct sum of copies of δ, φ_w maps into $V[\delta]$. Let W be the sum of the remaining isotypical components in V. Then $V = V[\delta] \oplus W$ and $\mathrm{Hom}_K(V_\delta, W) = 0$. Thus, the inclusion $V[\delta] \to V$ induces an isomorphism

$$\mathrm{Hom}_K(V_\delta, V) \simeq \mathrm{Hom}_K(V_\delta, V[\delta]).$$

We thus see that it suffices to prove (b) and (c) under the assumption that $V = V[\delta]$. Thus, we may as well assume that $\pi = m\delta$ and $V = V_\delta \otimes \mathbb{C}^m$. By Schur's lemma, $\mathrm{End}_K(V_\delta) = \mathbb{C}I$. It follows that

$$\mathrm{Hom}_K(V_\delta, V_\delta \otimes \mathbb{C}^m) \simeq (\mathbb{C}I) \otimes \mathbb{C}^m,$$

naturally. With this identification, the map φ_w is given by

$$v \otimes (I \otimes z) \mapsto v \otimes z.$$

Clearly, this defines a linear isomorphism from $V_\delta \otimes \mathrm{Hom}_K(V_\delta, V_\delta \otimes \mathbb{C}^m)$ onto $V[\delta] = V_\delta \otimes \mathbb{C}^m$. This establishes (b) and (c). $\square$

We now consider more generally a continuous representation (π, V) of K in a complete locally convex space V. A vector $v \in V$ is said to be K-finite if the linear span of $\pi(K)v$ is finite dimensional. The space of K-finite vectors in V is denoted by V_K. For each $\delta \in \widehat{K}$ we may now define $V[\delta]$ as the space of vectors $v \in V$ such that the linear span of $\pi(K)v$ decomposes as a finite direct sum of copies of the K-module V_δ. Then obviously, $V[\delta] \subset V_K$. As in the finite dimensional case, $V[\delta]$ is called the isotypical component of type δ.

Proposition 3.5. *Let (π, V) be a continuous representation of K in a complete locally convex space V. Then*

(a) *For each $\delta \in \widehat{K}$, the natural map $\varphi_\delta : V_\delta \otimes \mathrm{Hom}_K(V_\delta, V) \to V$ is a linear isomorphism with image $V[\delta]$, which intertwines the K representations $\delta \otimes 1$ and $\pi|_{V[\delta]}$.*

(b) *The space V_K of K-finite vectors decomposes as a direct sum,*

$$V_K = \bigoplus_{\delta \in \widehat{K}} V[\delta];$$

Proof. Each vector of V_K belongs to a finite dimensional K-invariant space to which the isotypical decomposition of Lemma 3.4 applies. The isotypical decompositions of two finite dimensional K-invariant subspaces are compatible on the intersection. From this, (b) follows.

We turn to (a). The asserted intertwining property of the map in (a) is obvious. From the equivariance one sees it maps into $V[\delta]$. As every element of $V[\delta]$ is contained in a finite dimensional K-invariant subspace, it follows from Lemma 3.4 that the map is surjective.

We will complete the proof by showing that the kernel of φ_δ is trivial. Let $\kappa \in \ker(\varphi_\delta)$ be an element in the kernel. Then κ can be written as a finite sum of the form $\sum_j v_j \otimes T_j \in V_\delta \otimes \mathrm{Hom}_K(V_\delta, V)$, with $v_j \in V_\delta$ and $T_j \in \mathrm{Hom}_K(V_\delta, V)$. The vector sum V_0 of the images $T_j(V_\delta)$ is a finite dimensional K-invariant subspace of V. Viewing κ as an element of $V_\delta \otimes \mathrm{Hom}_K(V_\delta, V_0)$, we see that its canonical image in V_0 is zero. In view of Lemma 3.4, it follows that $\kappa = 0$. $\square$

4 The ring of representative functions

In this section we assume that K is a compact Lie group.

Definition 4.1. We define the space $\mathscr{R}(K)$ of *representative functions* to be the space $L^2(K)_K$ of K-finite functions for the left regular representation $(L, L^2(K))$.

Remark 4.2. We leave it to the reader to verify that, equipped with the pointwise addition and multiplication of functions, $\mathscr{R}(K)$ is an algebra over $\mathbb{C}$ with unit. It therefore also called the ring of representative functions.

We write $\mathscr{R}(K)_\delta$ for $L^2(K)[\delta]$, the isotypical component of type δ in $L^2(K)$, with respect to the left regular representation.

It follows by application of Proposition 3.5 that the space $\mathscr{R}(K)$ admits the following direct sum decomposition

$$\mathscr{R}(K) = \bigoplus_{\delta \in \widehat{K}} \mathscr{R}(K)_\delta. \tag{1}$$

Lemma 4.3. *The space $\mathscr{R}(K)$ consists of continuous functions.*

Proof. Let $\delta \in \widehat{K}$. It suffices to show that $\mathscr{R}(K)_\delta \subset C(K)$. By Corollary 2.8 there exists a continous function $f \in C(K)$ such that $\delta(f) = I$. By equivariance of the isomorphism φ_w it now follows that $L(f)$ is the identity on $\mathscr{R}(K)_\delta$. Thus, if g belongs to the latter space, then $g = L(f)g = f * g \in C(K)$, in view of Lemma 2.9. $\square$

Corollary 4.4. *Let $\delta \in \widehat{K}$ and $T \in \mathrm{Hom}_K(V_\delta, L^2(K))$, then* $\mathrm{im}(T) \subset C(K)$.

Proof. Immediate. $\square$

If $k \in K$, then the map $R_k : L^2(K) \to L^2(K)$ commutes with $L_{k'}$, for every $k' \in K$. Thus, R_k is intertwines the left regular representation $(L, L^2(K))$ with itself. It follows that R_k leaves $\mathscr{R}(K)$ and the decomposition (1) invariant. The restriction $R[\delta]$ of R to $\mathscr{R}(k)_\delta$ is a continuous unitary representation of K on $R(K)_\delta$ equipped with the restriction of the L^2-inner product.

We note that $\mathrm{Hom}_K(V_\delta, L^2(K))$ may naturally be viewed as a closed subspace of $V_\delta^* \otimes L^2(K)$, which is invariant for the representation $1 \otimes R$ of K. Accordingly, the restriction of $1 \otimes R$ to this space becomes a continuous representation of K in $\mathrm{Hom}_K(V_\delta, L^2(K))$. This representation is given by $(k, T) \mapsto R_k \circ T$.

We consider the map $\varepsilon : \mathrm{Hom}_K(V_\delta, L^2(K)) \to V_\delta^*$ given by

$$\varepsilon(T) := \mathrm{ev}_e \circ T, \qquad (T \in \mathrm{Hom}_K(V_\delta, L^2(K)),$$

where $ev_e : C(K) \to \mathbb{C}, f \mapsto f(e)$ denotes evaluation at the identity e of the group K.

Lemma 4.5. *Let $\delta \in \widehat{K}$. The map $\varepsilon : \mathrm{Hom}_K(V_\delta, L^2(K)) \to V_\delta^*$ introduced above is a linear isomorphism which is K-intertwining for the restriction of $1 \otimes R$ on the domain and the contragredient $\delta^\vee$ on the codomain.*

Proof. Clearly, ε is a linear map. We will first establish the mentioned intertwining property. Let $T \in \mathrm{Hom}_K(V_\delta, L^2(K))$, $v \in V_\delta$ and $k \in K$. Then

$$\begin{aligned} \varepsilon(R_k \circ T)(v) &= \mathrm{ev}_e \circ R_k \circ T(v) \\ &= T(v)(ek) = L_{k^{-1}} T(v)(e) \\ &= T(\delta(k^{-1})v)(e) = \varepsilon(T) \circ \delta(k)^{-1}(v) \\ &= [\delta^\vee(k)\varepsilon(T)](v). \end{aligned}$$

Therefore, $\varepsilon(R_k \circ T) = \delta^\vee(k)\varepsilon(T)$ and we obtain the mentioned intertwining property.

We will now show that ε is injective. Assume that $\varepsilon(T) = 0$. Let $v \in V_\delta$. Then it follows that $T(v)(k) = T(\delta(k)^{-1}v)(e) = \varepsilon(T) \circ \delta(k)^{-1}(v) = 0$, hence $Tv = 0$. As this holds for every $v \in V_\delta$, we see that $T = 0$. Thus, ε is injective.

To see that ε is surjective, let $\eta \in V_\delta^*$. Define $T_\eta \in \mathrm{Hom}(V_\delta, C(K))$ by $T_\eta(v)(k) = \eta(\delta(k^{-1})v)$. Then it follows that T_η intertwines δ and L, hence belongs to $\mathrm{Hom}_K(V_\delta, C(K))$. It is now immediate that $\varepsilon(T_\eta) = \eta$. The surjectivity follows. $\square$

It follows from Lemma 4.5 that the map φ_δ composed with $I \otimes \varepsilon^{-1}$ yields a linear isomorphism $m_\delta : V_\delta \otimes V_\delta^* \to \mathscr{R}(K)[\delta]$, which intertwines the K-representations $\delta \times 1$ and L. Explictly, we have

$$m_\delta(v \otimes \eta)(k) = \varphi_w(v \otimes \varepsilon^{-1}(\eta))(k) = \eta(\delta(k^{-1})v).$$

Thus, up to inversion, m_δ equals the matrix coefficient map $V_\delta \otimes V_\delta^* \to C(K)$. The space $V_\delta \otimes V_\delta^*$ carries the natural representation $\delta \widehat{\otimes} \delta^\vee$ of $K \times K$ given by

$$\delta \widehat{\otimes} \delta^\vee(k_1, k_2) := \delta(k_1) \otimes \delta^\vee(k_2), \qquad ((k_1, k_2) \in K \times K). \tag{2}$$

On the other hand, the space $L^2(K)$ carries the natural representation $L \times R$ of $K \times K$ given by

$$(L \times R)(k_1, k_2)\varphi = L_{k_1} R_{k_2} \varphi, \qquad (\varphi \in L^2(K),\ (k_1, k_2) \in K \times K).$$

It is readily seen that $\mathscr{R}(K)$ and its decomposition (1) are invariant for the representation $L \times R$.

Lemma 4.6. *The map $m_\delta : V_\delta \otimes V_\delta^* \to \mathscr{R}(K)_\delta$ is a linear isomorphism which intertwines the representations $\delta \otimes \delta^\vee$ and $L \times R$ of $K \times K$.*

Proof. Only the intertwining property remains to be proven. We leave the easy verification to the reader. $\square$

For $v \in V_\delta$ and $\eta \in V_\delta^*$ we define the linear map $L_{v,\eta} : V_\delta \to V_\delta$ by $L_{v,\eta} : u \mapsto \eta(v)(u)$. Then the bilinear map $(v, \eta) \mapsto L_{v,\eta}$ induces a linear isomorphism $L : V_\delta \otimes V_\delta^* \to \mathrm{End}(V_\delta)$. This isomorphism will also be called the canonical isomorphism between these spaces. From now on we will use it to identify

$$V_\delta \otimes V_\delta^\vee \simeq \mathrm{End}(V_\delta). \tag{3}$$

We observe that the contraction map $v \otimes \eta \mapsto \eta(v)$ defines a linear functional on the first of the spaces in (3). It is easily checked that $\eta(v) = \mathrm{tr}(L_{v,\eta})$. Thus, through the isomorphism (3) the contraction map becomes identified with the trace map $A \mapsto \mathrm{tr}(A)$ on the second space in (3). It follows that under the natural identification (3) the linear map m_δ becomes the map $T_\delta : \mathrm{End}(V_\delta) \to \mathscr{R}(K)_\delta$ given by

$$T_\delta(A)(x) = \mathrm{tr}(\delta(x)^{-1}A), \qquad (A \in \mathrm{End}(V_\delta),\, x \in K).$$

Via the natural identification (3) through L, the represention $\delta \widehat{\otimes} \delta^\vee$ becomes a representation on $\mathrm{End}(V_\delta)$, given by

$$\delta \widehat{\otimes} \delta^*(k_1, k_2)(T) = \delta(k_1) \circ T \circ \delta(k_2)^{-1}, \qquad (T \in \mathrm{End}(V_\delta),\, (k_1, k_2) \in K \times K).$$

Corollary 4.7. *The map $T_\delta : \mathrm{End}(V_\delta) \to \mathscr{R}(K)_\delta$ is a linear isomorphism intertwining the $K \times K$ representations $\delta \widehat{\otimes} \delta^\vee$ and $L \times R$.*

Proof. In view of the identification (3) this is an immediate consequence of Lemma 4.6. □

5 The Schur orthogonality relations

We recall that the Haar measure dx of K is invariant under the substitution $x \mapsto x^{-1}$. It follow that for every $f \in L^2(G)$ we may define the function $f^* \in L^2(G)$ by

$$f^*(x) = \overline{f(x^{-1})}.$$

Lemma 5.1. *Let V_δ be equipped with an inner product for which δ is unitary. Then for all $A \in \mathrm{End}(V_\delta)$ we have*

$$T_\delta(A)^* = T_\delta(A^*).$$

Proof. Let $x \in K$. Then

$$T_\delta(A)^*(x) = \overline{\mathrm{tr}(\delta(x)A)} = \mathrm{tr}(A^*\delta(x)^*) = \mathrm{tr}(A^*\delta(x^{-1})) = T_\delta(A^*)(x).$$

□

Corollary 5.2. *The map $f \mapsto f^*$ preserves $\mathscr{R}(K)$ and its decomposition (1) into isotypical components.*

Proof. This is an immediate consequence of Lemma 5.1 and Corollary 4.7. □

Lemma 5.3. *Let $f, g \in C(K)$. Then*

(a) $(f * g)^* = g^* * f^*$;

(b) $\langle f\,,\, g\rangle_{L^2(K)} = (f * g^*)(e)$.

Proof. The proof is straightforward, and left to the reader. $\square$

Corollary 5.4. *Let $\delta_1, \delta_2 \in \widehat{K}$ and $f_j \in \mathscr{R}(G)_{\delta_j}$, for $j = 1, 2$. If $\delta_1 \not\sim \delta_2$ then*

$$f_1 * f_2 = 0 \quad \text{and} \quad f_1 \perp f_2.$$

In particular, the decomposition (1) is orthogonal for the L^2-inner product.

Proof. Since $f_2 \in \mathscr{R}(K)_{\delta_2}$, we have $f_1 * f_2 = L(f_1)f_2 \in \mathscr{R}(K)_{\delta_2}$. Likewise, we have $(f_1 * f_2)^* = f_2^* * f_1^* \in \mathscr{R}(K)_{\delta_1}$, so that

$$f_1 * f_2 \in \mathscr{R}(K)_{\delta_1} \cap \mathscr{R}(K)_{\delta_2} = 0.$$

Since $f_2^* \in \mathscr{R}(K)_{\delta_2}$ it also follows that

$$\langle f_1\,,\, f_2\rangle_{L^2(K)} = (f_1 * f_2^*)(e) = 0.$$

$\square$

The orthogonality of the direct sum (1) is known as part of the Schur orthogonality relations. The full orthogonality relations express the L^2-inner product on the isotypical components in terms of the Hilbert-Schmid inner product on $\mathrm{End}(V_\delta)$. To derive those, we will firt compare the map T_δ with the map $f \mapsto \delta(f)$ from $\mathscr{R}(K)_\delta$ to $\mathrm{End}(V_\delta)$.

Lemma 5.5. *Let $(\pi, \mathscr{H})$ be a unitary representation of the unimodular Lie group G. Then for all $f \in C_c(G)$ we have*

$$\pi(f)^* = \pi(f^*). \tag{4}$$

Proof. Easy and left to the reader. $\square$

In particular, if $\delta \in \widehat{K}$ then V_δ can be equipped with a Hermitian positive definite inner product for which δ is unitary, and then we have $\delta(f)^* = \delta(f^*)$. If V_δ is equipped with such an inner product, then the associated Hilbert-Schmid inner product on $\mathrm{End}(V_\delta)$ is given by

$$\langle A\,,\, B\rangle_{\mathrm{HS}} = \mathrm{tr}(AB^*).$$

Lemma 5.6. *Let V_δ be equipped with an inner product for which δ is unitary, and let $\mathrm{End}(V_\delta)$ be equipped with the associated Hilbert-Schmid inner product. Then the maps $T_\delta : \mathrm{End}(V_\delta) \to \mathscr{R}(G)_\delta$ and $\delta : \mathscr{R}(G)_\delta \to \mathrm{End}(V_\delta)$ are adjoint. In particular, δ is a linear isomorphism.*

Proof. Let $f \in \mathscr{R}(G)_\delta$ and $A \in \mathrm{End}(V_\delta)$. Then

$$\begin{aligned}\langle A\,,\delta(f)\rangle_{\mathrm{HS}} &= \mathrm{tr}(A\delta(f)^*)\\ &= \mathrm{tr}\left(A\int_K [f(x)\pi(x)]^* dx\right)\\ &= \mathrm{tr}\int_K \overline{f(x)}A\pi(x)^{-1}\,dx\\ &= \int_K \overline{f(x)}\,\mathrm{tr}(A\pi(x)^{-1})\,dx\\ &= \langle T_\delta(A)\,,f\rangle_{L^2(K)}.\end{aligned}$$

This completes the proof. □

We will now show that up to a positive scalar, the maps T_δ and δ will turn out to be inverse to each other. A crucial step in the proof is the following.

Lemma 5.7. *Let $f \in \mathscr{R}(K)_\delta$. Then $f * \mathscr{R}(K) \subset \mathscr{R}(K)_\delta$. Furthermore, the convolution operator $C_f : \mathscr{R}(K)_\delta \to \mathscr{R}(K)_\delta$, $g \mapsto f * g$ has trace given by*

$$\mathrm{tr}\, C_f = f(e).$$

Proof. Let $g \in C(K)$. Then by left invariance of the Haar measure, we have

$$f * g(x) = \int_K f(y)g(y^{-1}x)\,dy = \int_K f(xy)g(y^{-1})\,dy = R(g^\vee)f, \qquad (x \in K),$$

where $g^\vee : y \mapsto g(y^{-1})$. Since $\mathscr{R}(K)_\delta$ is invariant under the right regular representation, it follows that $f * g \in \mathscr{R}(K)_\delta$.

Through the substitution $y \mapsto y^{-1}$, which preserves the normalized Haar measure on By bi-invariance of the Haar measure, we may rewrite the above as

$$f * g(x) = \int_K A(x,y)g(y)\,dy, \qquad (x \in K),$$

with $A \in C(K \times K)$ given by $A(x,y) = f(xy^{-1})$. Thus, $f* : L^2(K) \to L^2(K)$ may be viewed as an integral operator with kernel $A \in C(K \times K)$. By compactness of K it is a general principle that such an operator is of trace class, with trace given by the diagonal integral $\int_K A(k,k)\,dk$. In the present context $f*$ has image contained in $\mathscr{R}(G)_\delta$, and we will give an elementary proof of the claim that

$$\mathrm{tr}\,(C_f) = \int_K A(y,y)\,dy. \tag{5}$$

Indeed, let $h_1, \ldots, h_m$ be an orthonormal basis of $\mathscr{R}(K)_\delta$ with respect to the L^2-inner product. The functions h_i are continuous. For each $y \in K$ the function $x \mapsto A(x,y) = f(xy^{-1})$ belongs to $\mathscr{R}(K)_\delta$, hence admits a decomposition of the form

$$A(x,y) = \sum_{i=1}^m A_i(y)h_i(x),$$

with

$$A_i(x) = \int_K A(x,y)\overline{h_i(x)}\, dx.$$

It is clear that $A_i \in C(K)$. We now observe that

$$\begin{aligned}
\int_K A(y,y)dy &= \sum_{i=1}^{m} \int_K A_i(y)h_i(y)\, dy \\
&= \sum_{i=1}^{m} \int_K \int_K A(x,y)\overline{h_i(x)}h_i(y)\, dx dy \\
&= \sum_{i=1}^{m} \int_K \int_K A(x,y)h_i(y)\, dy\, \overline{h_i(x)}\, dx \\
&= \sum_{i=1}^{m} \int_K C_f(h_i)(x)\overline{h_i(x)}\, dx \\
&= \sum_{i=1} \langle C_f(h_i)\, , h_i\rangle
\end{aligned}$$

and the claim (5) follows.

To complete the proof, we observe that

$$\int_K A(x,x)\, dx = \int_K f(xx^{-1})\, dy = \int_K f(e)\, dy = f(e).$$

□

Theorem 5.8.

(a) *The map* $\dim\delta \cdot T_\delta$ *is the inverse to* $\delta : \mathscr{R}_\delta(K) \to \mathrm{End}(V_\delta)$.

(b) *The map* $\sqrt{\dim\delta} \cdot T_\delta : \mathrm{End}(V_\delta) \to \mathscr{R}_\delta(K)$ *is an isometric isomorphism.*

Proof. For $A \in \mathrm{End}(V_\delta)$ let L_A denote the linear map $\mathrm{End}(V_\delta) \to \mathrm{End}(V_\delta)$ given by $X \mapsto AX$. Via the canonical isomorphism () we may identify L_A with the linear endomorphism $A \otimes I_{V_\delta^*}$ of $V_\delta \otimes V_\delta^*$. It follows that

$$\mathrm{tr}(L_A) = \mathrm{tr}(A)tr(I_{V_\delta^*}) = \dim(\delta)\,\mathrm{tr}(A).$$

Let now $f \in \mathscr{R}(K)_\delta$. Then for $g \in \mathscr{R}(K)_\delta$ we have $\delta(f * g) = \delta(f)\delta(g)$. Therefore, $\delta \circ C_f = L_{\delta(f)} \circ \delta$. Thus $\delta : \mathscr{R}(K)_\delta \to \mathrm{End}(V_\delta)$ is a linear isomorphism, intertwining C_f and $L_{\delta(f)}$. It follows that

$$\mathrm{tr}(C_f) = \dim(\delta) \cdot \mathrm{tr}(\delta(f)) = \dim(\delta) \cdot T_\delta(\delta(f))(e),$$

hence

$$f(e) = \dim(\delta) \cdot T_\delta(\delta(f))(e).$$

Applying this result to $L_{x^{-1}}f$ and using that $T_\delta \circ \delta$ is intertwining for L, it follows that

$$f(x) = \dim(\delta) \cdot T_\delta(\delta(f))(x), \qquad (x \in G).$$

This completes the proof of (a).

We turn to (b). The map $\sqrt{\dim(\delta)} \cdot T_\delta$ is both the transpose and inverse of $\sqrt{\dim(\delta)} \cdot \delta$. This implies that the map is unitary, hence an isometry. □

We recall that every finite dimensional continuous representation (π, V) of K has a character $\chi_\pi \in C(K)$ defined by

$$\chi_\pi(x) = \mathrm{tr}(\pi(x)), \qquad (x \in K).$$

If $\delta \in \widehat{K}$ then it follows that $\chi_{\delta^\vee} = T_\delta(I_{V_\delta})$.

Lemma 5.9. *Let $\delta, \delta' \in \widehat{K}$. Then the following orthogonality relations are valid.*

(a) *If $\delta \not\sim \delta'$, then $\chi_\delta \perp \chi_{\delta'}$.*

(b) $\|\chi_\delta\|_{L^2(K)} = 1$.

Proof. Since $\chi_\delta \in \mathscr{R}(G)_{\delta^\vee}$, the orthogonality in (a) follows from the orthogonality of the decomposition (1). For (b) we use that $\sqrt{\dim(\delta)} \cdot T_{\delta^\vee}$ is an isometry. Therefore,

$$\|\chi_\delta\|_{L^2(K)} = \|T_{\delta^\vee}(I_{V_\delta^*})\|_{L^2(K)} = \sqrt{\dim(\delta)}^{-1} \cdot \|I_{V_\delta^*}\|_{\mathrm{HS}} = 1.$$

□

Finally, we note that the irreducible characters behave as follows with respect to convolution.

Lemma 5.10. *Let $\delta, \delta' \in \widehat{K}$. Then*

(a) *If $\delta \not\sim \delta'$, then $\chi_{\delta^\vee} * f = 0$ for all $f \in \mathscr{R}(K)_{\delta'}$.*

(b) $\dim\delta \cdot \chi_{\delta^\vee} * f = f$, *for all $f \in \mathscr{R}(K)_\delta$.*

Proof. Since $\chi_{\delta^\vee} = T_\delta(I_{V_\delta}) \in \mathscr{R}(K)_\delta$, assertion (a) follows from Corollary 5.4. For (b) we note that $\dim(\delta) \cdot \chi_{\delta^\vee} = \dim(\delta) \cdot T_\delta(I_{V_\delta})$, so that

$$\delta(\dim(\delta) \cdot \chi_{\delta^\vee}) = I_{V_\delta}.$$

If $f \in \mathscr{R}(K)_\delta$ then it follows that

$$\delta(\dim(\delta) \cdot \chi_{\delta^\vee} * f) = I_{V_\delta} \circ \delta(f) = \delta(f).$$

The assertion follows from the injectivity of $f \mapsto \delta(f)$ on $\mathscr{R}(K)_\delta$. □

Corollary 5.11. *The convolution operator $\dim(\delta) \cdot \chi_{\delta^\vee} * \cdot : \mathscr{R}(K) \to \mathscr{R}(K)$ equals the projection operator $\mathscr{R}(K) \to \mathscr{R}(K)_\delta$ determined by the decomposition (1).*

6 The Peter-Weyl Theorem

We will now concentrate on the Peter-Weyl theorem. The main step towards it consists of showing that the subspace $\mathscr{R}(K)$ is dense in $L^2(K)$. Equivalently, this means that the orthocomplement $\mathscr{R}(K)^\perp$ in $L^2(K)$ is trivial. A final tool for this is the following result.

Lemma 6.1. *Let $f \in C_c(K)$ be such that $f^* = f$. Then the continuous linear operator $R(f) : L^2(K) \to L^2(K)$ is left K-equivariant, compact and self-adjoint.*

Proof. If $k \in K$ then L_k commutes with R_x for all $x \in K$. This implies that L_k commutes with $R(f)$. Thus, $R(f)$ is left K-equivariant.

Given $g \in L^2(G)$ we have

$$[R(f)g](x) = \int_K f(y)g(xy)\, dy = \int_K f(x^{-1}y)g(y)\, dy, \qquad (x \in K).$$

Thus $R(f)$ is continuous linear integral operator $L^2(K) \to L^2(K)$ with integral kernel $\mathscr{K}(x,y) := f(x^{-1}y)$. It follows that the integral kernel is continuous, hence also $\mathscr{K} \in L^2(K \times K)$. Such an integral operator is compact as a continuous linear operator on $L^2(K)$, see the remark below.

Finally, the self-adjointness follows from the fact that R is a unitary representation of K in $L^2(K)$, so that, by application of Lemma 5.5,

$$R(f)^* = R(f^*) = R(f).$$

$\square$

Remark. Let M be a smooth manifold and dm a positive density on M. Let $\mathscr{K} \in L^2(M \times M, dm \otimes dm)$ and let T be the associated integral operator $L^2(K) \to L^2(K)$ given by

$$Tg(x) = \int_K \mathscr{K}(x,y)g(y)\, dm(y),$$

for $g \in L^2(K)$ and for almost every $x \in M$. Then for $h \in L^2(M)$ we have

$$\langle Tg\,, h\rangle = \int_K \int_K \mathscr{K}(x,y)g(y)\overline{h(x)}\, dm(x)\, dm(y) = \langle \mathscr{K}\,, \bar{h} \otimes g\rangle$$

so that, by the Cauchy-Schwartz inequality,

$$\|\langle Tg\,, h\rangle\| \leq \|\mathscr{K}\|_2 \|g\|_2 \|h\|_2$$

where index 2 indicates that the appropriate L^2-norm has been taken. By the Riesz representation theorem this implies that $\|Tg\|_2 \leq \|cK\|_2\|g\|_2$, so that the operator norm of T is estimated by

$$\|T\|_{\mathrm{op}} \leq \|\mathscr{K}\|_2.$$

Compactness of the operator T may be established as follows. Let $\{\psi_j\}_{j \geq 1}$ be an orthonormal basis of $L^2(K)$. For $\nu \geq 1$ let P_ν denote the orthogonal projection onto the span of the first ν basis vectors $\psi_1, \ldots, \psi_\nu$. Then $T_\nu = P_\nu \circ T$ has rank at most ν, and is the integral operator with integral kernel given by

$$\mathscr{K}_\nu(x,y) := \sum_{j=1}^{m} \int_K \mathscr{K}(z,y)\overline{\psi_j(z)}\, \psi_j(x)\, dz.$$

We have that $\mathscr{K}_\nu \to \mathscr{K}$ in $L^2(K \times K)$. By the above it follows that $\|T - T_\nu\|_{\mathrm{op}} \leq \|\mathscr{K} - \mathscr{K}_\nu\|_2 \to 0$ for $\nu \to \infty$. Thus, T is a limit of finite rank operators with respect to the operator norm. This implies that T is compact.

Proposition 6.2. *The space $\mathscr{R}(K)$ is dense in $L^2(K)$.*

Proof. Since both L and R are unitary representations, the orthocomplement $\mathscr{H} := \mathscr{R}(K)^\perp$ is a closed bi-K-invariant subspace of $L^2(K)$. In particular, it is a Hilbert space of its own right. We will complete the proof by showing that $\mathscr{H}$ is trivial.

Let $f \in C_c(K)$ be as in Lemma 6.1. Then the operator $R(f)$ leaves $\mathscr{H}$ invariant and the restriction $T := R(f)|_{\mathscr{H}}$ is a compact self-adjoint operator on $\mathscr{H}$. Let λ be a non-zero eigenvalue of T. Then the associated eigenspace $\mathscr{H}_\lambda$ is finite dimensional. Since T commutes with the left action of K on $\mathscr{H}$, it follows that $\mathscr{H}_\lambda$ is finite dimensional and K-invariant, hence contained in $\mathscr{R}(K)$. This implies that $\mathscr{H}_\lambda \subset \mathscr{R}(K) \cap \mathscr{H} = 0$. Hence, T has no non-zero eigenvalues.

By the spectral theorem for compact self-adjoint operators, we now conclude that $T = 0$. It follows from this reasoning that $R(f) = 0$ on $\mathscr{H}$ for all $f \in C_c(G)$ satisfying $f = f^*$.

Let $\{\psi_j\}$ be an approximation of the identity in $C_c(G)$. Then it is readily verified that $f_j := \frac{1}{2}(\psi_j + \psi_j^*)$ is an approximation of the identity as well. As $\mathscr{H}$ is right K-invariant, we may apply Lemma 2.7 to conclude that $0 = R(f_j)v \to v$ $(j \to \infty)$ for all $v \in \mathscr{H}$. We thus see that $\mathscr{H} = 0$. $\square$

We now obtain the following result.

Theorem 6.3. (Peter–Weyl) *The space $L^2(K)$ admits the following orthogonal direct sum decomposition of finite dimensional Hilbert spaces*

$$L^2(K) = \widehat{\bigoplus_{\delta \in \widehat{K}}} \mathscr{R}(K)_\delta. \tag{6}$$

This direct sum is bi-K-invariant, i.e., invariant under both the left regular representation L and the right regular representation R. For each $\delta \in \widehat{K}$ the corresponding orthogonal projection $P_\delta : L^2(K) \to \mathscr{R}(K)_\delta$ is bi-K-equivariant, and given by

$$P_\delta(f) = \dim(\delta)\, \chi_{\delta^\vee} * f, \qquad (f \in L^2(K)).$$

Proof. The components $\mathscr{R}(K)_\delta$, for $\delta \in \widehat{K}$, are orthogonal by Corollary 5.4. Their algebraic direct sum equals $\mathscr{R}(K)$, by (1). The latter space is dense in $L^2(K)$, by Proposition 6.2. This implies that (6) is an orthogonal direct sum decomposition of Hilbert spaces. The summands are finite dimensional, and both left and right K-invariant, by Lemma 4.6. The orthogonal projection $P_\delta : L^2(K) \to \mathscr{R}(K)_\delta$ and the convolution operator $\dim(\delta)\chi_{\delta^\vee} *$ are continuous linear endomorphisms of $L^2(K)$. Furthermore, in view of Corollary 5.11, they are equal to each other on the subspace $\mathscr{R}(K)$. As the latter subspace is dense in $L^2(K)$, it follows that the two operators are equal on $L^2(K)$. By bi-K-invariance of the decomposition (6), the operator P_δ is bi-K-invariant. $\square$

Motivated by the theory developed above, we define the direct sum of linear spaces

$$\mathfrak{H}_K := \bigoplus_{\delta \in \widehat{K}} \mathrm{End}(V_\delta)$$

and equip this space with the pre-Hilbert structure given by

$$\langle \varphi\,,\, \psi \rangle := \sum_{\delta \in \widehat{K}} \dim(\delta)\, \langle \varphi_\delta\,,\, \psi_\delta \rangle_{\mathrm{HS}},$$

where the index HS indicates that the Hilbert-Schmid inner product on $\mathrm{End}(V_\delta)$ is taken. Let

$$\mathfrak{H} := \widehat{\bigoplus}_{\delta\in\widehat{K}} \ \mathrm{End}(V_\delta)$$

be the Hilbert completion with respect to this inner product. This Hilbert completion may be equipped with the unitary representation of π of $K\times K$ given by

$$[\pi(x,y)\varphi]_\delta := \delta(x)\circ\varphi_\delta\circ\delta(y)^{-1}.$$

Thus, π is the Hilbert direct sum of the representations $\delta\otimes\delta^\vee$.

For $f\in C(K)$ we define the operator valued Fourier coefficients $\hat{f}(\delta)\in\mathrm{End}(V_\delta)$, for $\delta\in\widehat{K}$, by

$$\hat{f}(\delta) := \delta(f)\in\mathrm{End}(V_\delta).$$

The assignment $\hat{f}:\delta\mapsto\hat{f}(\delta)$ is also called the Fourier transform of f. In terms of this Fourier transform, we have the following decomposition theorem for the unitary representation $(L\times R, L^2(K))$ of $K\times K$.

Theorem 6.4. *For each $f\in C(K)$ the associated sequence $\hat{f}$ of Fourier coefficients belongs to $\mathfrak{H}$. Furthermore,*

(a) *The map $f\mapsto\hat{f}$ extends to an isometric isomorphism $L^2(K)\xrightarrow{\simeq}\mathfrak{H}$, also denoted by $f\mapsto\hat{f}$.*

(b) *The inverse of the Fourier transform equals the map $\mathscr{I}:\mathfrak{H}\to L^2(K)$ given by*

$$\mathscr{I}(A) = \sum_{\delta\in\widehat{K}} \dim(\delta)T_\delta(A_\delta),$$

with the sum converging in $L^2(K)$.

(c) *The extended map $f\mapsto\hat{f}$ is a unitary equivalence of the representations $(L\times R, L^2(K))$ and $(\pi,\mathfrak{H})$ of $K\times K$.*

In particular, Fourier transform induces a unitary equivalence

$$(L\times R, L^2(K)) \xrightarrow{\simeq} \widehat{\bigoplus}_{\delta\in\widehat{K}} \ \delta\otimes\delta^\vee,$$

establishing the decomposition of the unitary representation $L\times R, L^2(K)$ into irreducible unitary representations of $K\times K$.

Proof. Let $\delta\in\widehat{K}$. Then for (a) it suffices to show that $f\mapsto\hat{f}(\delta)$ is an isometric isomorphism from $\mathscr{R}(K)_\delta$ onto $\mathrm{End}(V_\delta)$, equipped with the inner product $\dim(\delta)\langle\cdot\,,\,\cdot\rangle_{\mathrm{HS}}$. Since $\hat{f}(\delta)=\delta(f)$, this follows from Theorem 5.8.

From the same theorem, it follows that the inverse of $f\mapsto\hat{f}$ on $\mathrm{End}(V_\delta)$ is given by $\mathscr{I}$. For a given $A\in\mathfrak{H}$, the sum $\sum_\delta\dim(\delta)\|A_\delta\|^2_{\mathrm{HS}}$ converges, and by the isometric property of $\mathscr{I}$, this in turn implies implies that the sum $\sum_\delta(\dim(\delta)^2\|T_\delta(A_\delta)\|_2$ converges. It follows that the sum for $\mathscr{I}(A)$ converges in $L^2(K)$, and (b) follows.

For (c) it now suffices to establish the intertwining property. Let $f \in \mathscr{R}(K)$, $\delta \in \widehat{K}$, then for $(k_1, k_2) \in K \times K$ we have

$$(\pi(k_1,k_2)\hat{f}\,)_\delta = \delta(k_1) \circ \delta(f) \circ \delta(k_2)^{-1} = \delta(L_{k_1} R_{k_2} f) = (L_{k_1} R_{k_2} f)\hat{\ }(\delta).$$

It follows that $\mathfrak{F} : f \mapsto \hat{f}$ intertwines $(\mathscr{R}(K), L \times R)$ with $(\mathfrak{H}_K, \pi)$. As $\mathfrak{F}$ is an isometry, the representations are unitary and $\mathscr{R}(K)$ is dense in $L^2(K)$, it follows by a standard approximation argument that $\mathscr{F} : L^2(K) \to \mathfrak{H}$ is an intertwining operator. □

In harmonic analysis, the above isometric decomposition of $L \times R$ into irreducible representation for $K \times K$ is also known as the Plancherel decomposition for the compact group K. We note that condition (a) implies that for $f \in L^2(K)$ we have the following Parseval identity:

$$\|f\|_2^2 = \sum_{\delta \in \widehat{K}} \dim(\delta)\, \|\hat{f}(\delta)\|_{\mathrm{HS}}^2.$$

Here we recall that HS refers to the Hilbert-Schmid inner product on $\mathrm{End}(V_\delta)$ induced by any choice of inner product on V_δ which makes δ unitary. In particular, this inner product does not depend on the choice of unitarizing inner product on V_δ, see also one of the exercises.

We now equip $\mathfrak{H}_K$ with the direct sum algebra structure

Proposition 6.5. *The Fourier transform $\mathfrak{F} : f \mapsto \hat{f}$ restricts to an isomorphism of the convolution algebra $(\mathscr{R}(K), *)$ with the algebra $\mathfrak{H}_K$.*

Proof. By equivariance, it follows from the result above, that $f \mapsto \hat{f}$ maps the space $\mathscr{R}(K)$ of left K-finite functions in $L^2(K)$ onto the space of $K \times \{e\}$-finite functions in $\mathfrak{H}$. That image space equals $\mathfrak{H}_K$. Let $\delta \in \widehat{K}$ and let $f, g \in \mathscr{R}(K)$. Then it follows from

$$\mathfrak{F}(f * g))_\delta = \delta(f * g) = \delta(f)\delta(g) = (\mathfrak{F}(f)\mathfrak{F}(g))_\delta.$$

This shows that $\mathfrak{F}(f * g) = \mathfrak{F}(f)\mathfrak{F}(g)$. Hence, $\mathfrak{F}$ is an isomorphism of algebras. □

Let

$$\Delta_K := \{(k,k) \mid k \in K\}$$

be the diagonal subgroup of the group $K \times K$. A function $f \in C(G)$ is said to be conjugation invariant if

$$f(k^{-1}xk) = f(x), \qquad (x, k \in K).$$

Equivalently, this means that f is invariant under $(L \times R)(\Delta_K)$. Functions with this type of invariance are also called class functions, since they may be viewed as functions on the set of conjugation classes. Similarly, we define $\mathscr{R}(K)_{\mathrm{class}}$ and $L^2(K)_{\mathrm{class}}$ to be the subspaces of $(L \times R)(\Delta_K)$-invariantes in $\mathscr{R}(K)$ and $L^2(K)$. Given $\delta \in \widehat{K}$, let I_δ denote the identity element of $\mathrm{End}(V_\delta)$. Note that

$$(V_\delta \otimes V_\delta^*)^{\Delta_K} \simeq \mathrm{End}_K(V_\delta) = \mathbb{C}\mathrm{I}_\delta,$$

by Schur's lemma. We consider the closed subspace

$$\mathfrak{H}_{\mathrm{class}} := \{A \in \mathfrak{H} \mid \forall \delta \in \widehat{K} : \quad A_\delta \in \mathbb{C}\mathrm{I}_\delta\},$$

of $\mathfrak{H}$, which is a Hilbert space of its own right. Then $\mathfrak{H}_{\rm class}$ equals the space of Δ_K-invariants in $\mathfrak{H}$.

Proposition 6.6. *Fourier transform $\mathfrak{F}$ maps $L^2(K)_{\rm class}$ isometrically onto $\mathfrak{H}_{\rm class}$. Furthermore, this map restricts to a linear isomorphism from $\mathscr{R}(K)_{\rm class}$ onto the algebraic direct sum $\oplus_{\delta\in\widehat{K}}\mathbb{C}\mathrm{I}_\delta$.*

Proof. By equivariance of $\mathfrak{F}$ combined with the Plancherel decomposition for $L^2(K)$, it follows that $\mathfrak{F}$ maps $L^2(K)_{\rm class}$ isometrically onto the space of Δ_K-invariants in $\mathfrak{H}$. The latter equals $\mathfrak{H}_{\rm class}$. Restriction to the $K\times\{e\}$-finite elements gives the second assertion, again by equivariance. $\square$

We finally mention the following version of the Parseval identity.

Proposition 6.7. *The characters χ_δ, for $\delta\in\widehat{K}$ form an orthonormal basis of $L^2(K)_{\rm class}$. In particular, if $f\in L^2(K)_{\rm class}$, then*

$$\|f\|_2^2=\sum_{\delta\in\widehat{K}}|\langle f\,,\chi_\delta\rangle|^2.$$

Proof. It follows from the previous result that $\mathscr{R}(K)_{{\rm class},\delta}$ is one dimensional, hence equals the span of $\chi_{\delta^\vee}$. It now follows from the Plancherel decomposition that $L^2(K)_{\rm class}$ decomposes as the orthogonal direct sum of the spaces $\mathscr{R}(K)_{{\rm class},\delta}$. As each character $\chi_{\delta^\vee}$ has L^2-norm 1, it follows that these characters form an orthonormal basis in $L^2(K)_{\rm class}$. The result follows. $\square$

7 Application to K-representations

In this section we assume that K is a compact group, and that (π,V) is a continuous representation of K in a Fréchet space V.

Lemma 7.1. *Let $\delta\in\widehat{K}$. Then the operator*

$$P_\delta:=\dim(\delta)\,\pi(\chi_{\delta^\vee}). \tag{7}$$

is a K-equivariant continuous linear projection with image $V[\delta]$. In particular, $V[\delta]$ is a closed subspace of V.

Proof. Write $\alpha_\delta:=\dim(\delta)\chi_{\delta^\vee}$. Then the convolution operator $\alpha_\delta*(\,\cdot\,)$ restricts to the projection $\mathscr{R}(K)\to\mathscr{R}(K)_\delta$, by Corollary 5.11. In particular, since $\alpha_\delta\in\mathscr{R}(K)_\delta$ it follows that

$$\alpha_\delta*\alpha_\delta=\alpha_\delta.$$

We claim that $P_\delta=\pi(\alpha_\delta)$ satisfies all assertions. The continuity of this operator follows by application of Lemma 2.4. Furthermore, in view of Lemma 2.5 the above equation implies that $P_\delta^2=P_\delta$. It follows that P_δ is a continuous projection operator, hence has closed image. We will finish the proof by showing that $V[\delta]=\mathrm{im}(P_\delta)$. For this, let $\varphi_\delta:V_\delta\otimes\mathrm{Hom}_K(V_\delta,V)\to V[\delta]$ be the canonical isomorphism. Then by equivariance, it follows that

$$\pi(\alpha_\delta)\circ\varphi_\delta=\varphi_\delta\circ[\delta(\alpha_\delta)\otimes 1].$$

Since $\delta(\alpha_\delta) = \mathrm{I}_\delta$, it follows that $\pi(\alpha_\delta) = I$ on $\mathscr{R}[\delta]$ so that $\mathscr{R}[\delta]$ is contained in the image of P_δ. For the converse inclusion, let $v \in V$. Then the map $f \mapsto \pi(f)v$ from $\mathscr{R}(K)$ to V intertwines the K-representations L and π, hence maps $\mathscr{R}(K)_\delta = \mathscr{R}(K)[\delta]$ into $V[\delta]$. In particular, $P_\delta v = \pi(\alpha_\delta)v \in V[\delta]$. □

Using the Peter-Weyl theorem we will be able to show that the space V_K of K-finite vectors is dense in V. The following lemma will be needed in the proof.

Lemma 7.2. *Let K be a compact group, equipped with a choice of Haar measure. Then*

$$C(K) \subset L^2(K) \subset L^1(K).$$

Each of the inclusion maps is continuous, with dense image.

Proof. Without loss of generality we may assume that the Haar measure dx on K is normalized.

We will first prove the claim that for all $f \in C(K)$ we have

$$\|f\|_1 \leq \|f\|_2 \leq \sup_K |f|. \tag{8}$$

Indeed, the first inequality follows from

$$\|f\|_1 = \int_K |f(x)|\, dx = \langle |f|\,,\, 1_K \rangle \leq \|f\|_2 \|1_K\|_2 = \|f\|_2$$

by the Cauchy-Schwartz inequality. For the second inequality, we note that

$$\|f\|_2^2 = \int_K |f(x)|^2\, dx \leq (\sup_K |f|)^2 \int_K dx = (\sup_K |f|)^2.$$

These estimates imply the continuity of the inclusion maps of $C(K)$ into $L^1(K)$ and $L^2(K)$. Furthermore, if $f \in L^2(K)$, then there exists a sequence $(f_j)_{j\geq 1}$ in $C(K)$ such that $\|f_j - f\|_2 \to 0$. Then $f_j \to f$ in measure. The sequence (f_j) is Cauchy for the L^2-norm hence also for the L^1-norm, and we see that there exists a $g \in L^1(K)$ such that $f_j \to g$ in $L^1(K)$. In particular, it follows that $f_j \to g$ in measure, so that $f = g$ almost everywhere. Thus, we see that $f \in L^1(K)$. Furthermore, from the inequalities $\|f_j\|_1 \leq \|f_j\|_2$ we find $\|f\|_1 \leq \|f\|_2$, by taking the limit for $j \to \infty$. It follows that $L^1(K) \subset L^2(K)$, with continuous inclusion map. Finally, since $C(K)$ is dense in both $L^1(K)$ and $L^2(K)$, all density assertions follow. □

Corollary 7.3. *The space $\mathscr{R}(K)$ of representative functions is dense in $L^1(K)$.*

Proof. This follows from combining Proposition 6.2 with the previous lemma. □

Proposition 7.4. *Let (π, V) be a continuous representation of K in a Fréchet space. The space V_K of K-finite vectors is dense in V.*

Proof. Let $v_0 \in V$, p a continuous seminorm on V and $\varepsilon > 0$. We will complete the proof by showing that there exists $v \in V_K$ such that $p(v - v_0) < \varepsilon$.

As $x \mapsto \pi(x)v$ is continuous, and K compact, there exists a constant $C > 0$ such that $p(\pi(x)v_0) < C$ for all $x \in K$. From this estimate it follows that

$$p(\pi(f)v_0) \leq C\,\|f\|_1$$

for all $f \in C(K)$. There exists $g \in C(K)$ such that $p(\pi(f)v_0 - v_0) < \varepsilon/2$, see Lemma 2.7. Furthermore, by Corollary 7.3, there exists a function $f \in \mathscr{R}(G)$ such that $\|f - g\|_1 \leq \varepsilon/2(C+1)$. By equivariance of the map $h \mapsto \pi(h)v_0$ the vector $v := \pi(f)v_0$ is K-finite in V, and by the previous estimates we have

$$p(v - v_0) = p(\pi(f)v_0 - \pi(g)v_0) + p(\pi(g)v_0 - v_0) < C\,\|f - g\|_1 + \varepsilon/2 < \varepsilon.$$

$\square$

Corollary 7.5. *Let (π, V) be a continuous representation of K in a Fréchet space. Let $\delta \in \widehat{K}$. Then there exists a unique K-equivariant continuous linear projection operator $V \to V$ with image $V[\delta]$. This operator is given by (7).*

Proof. We leave this as an exercise to the reader. $\square$

8 Application to compact homogeneous spaces

In this section we assume that K is a compact Lie group, and that H is a closed subgroup. Then the coset space K/H carries a unique structure of smooth manifold, which turns the canonical projection $\pi : K \to K/H$ into a submersion. Let $\pi^* : C(K/H) \to C(K)$ be the map defined by pull-back under π, i.e., $p^*(f) = f \circ \pi$, for $f \in C(K/H)$. Then π^* is an injective linear map with image consisting of the closed subspace $C(K)^{R(H)}$ of functions in $C(K)$ that are invariant for the right regular representation restricted to H. That is, a function $f \in C(K)$ belongs to $\pi^*(C(K/M))$ if and only if

$$R_h f = f, \qquad (h \in H).$$

Accordingly, we shall identify the elements of $C(K/H)$ with the subspace $C(K)^{R(H)} \subset C(K)$ via the linear embedding π^*.

If (π, V) is a continuous finite dimensional representation of a Lie group G, and H a closed subgroup, a vector $v \in V$ is called H-invariant if and only if $\pi(h)v = v$ for all $h \in H$. The space of all such vectors in V is denoted by V^H.

Lemma 8.1. *Let (π, V) be a continuous finite dimensional representation of a Lie group G and let H be a compact subgroup. Let $p : V \to V^H$ be the unique H-equivariant projection operator. Then its transpose p^* is an injection $(V^H)^* \to V^*$ with image $(V^*)^H$. Accordingly, $\eta \mapsto \eta \circ p$ gives a linear isomorphism*

$$(V^H)^* \xrightarrow{\simeq} (V^*)^H. \tag{9}$$

Remark 8.2. In particular it follows that $V^H \neq 0$ if and only if $(V^*)^H \neq 0$.

Proof. The map $p^* : (V^H)^* \to V^*$ is H-equivariant and injective. It follows that p^* maps into $(V^*)^H$. Let $\eta \in V^*$ be H-invariant. We will complete the proof by showing that η belongs to the image of p^*. Let $P : V \to V$ be the unique H-equivariant projection map with image V^H. Let $\iota : V^H \to V$ be the inclusion map. Then $\iota \circ p = P$.

By Corollary 7.5, applied for the compact group H, the map $P : V \to V$ is given by $P = \int_H \pi(h)\, dh$. This implies that

$$P^*(\eta) = \eta \circ P = \int_H \eta \circ \pi(h)\, dh = \eta.$$

Hence,

$$\eta = P^*(\eta) = p^* \iota^*(\eta) \in \mathrm{im}(p^*).$$

$\square$

We define $\mathscr{R}(K/H)$ to be the space of left K-finite functions in $C(K/H)$. Since π^* is left K-equivariant, it then follows that π^* embeds $\mathscr{R}(K/H)$ into $\mathscr{R}(K)$; its image is obviously equal to the space of right H-invariant elements in $\mathscr{R}(K)$. By using the bi-K-equivariance of the Fourier transform

$$\mathfrak{F} : \mathscr{R}(K) \xrightarrow{\simeq} \mathfrak{H}_K$$

of the previous section, we see that

$$\mathfrak{F}(\mathscr{R}(K/H)) = \bigoplus_{\delta \in \widehat{K}_H} V_\delta \otimes (V_\delta^*)^H,$$

Let $C(H\backslash K/H)$ denote the space of bi-H-invariant functions in $C(K)$. Let $\mathscr{R}(H\backslash K/H)$ denote the space bi-H-invariant functions in $\mathscr{R}(K)$. Then by left equivariance of the Fourier transform, it follows that

$$\mathfrak{F}(\mathscr{R}(H\backslash K/H)) = \bigoplus_{\delta \in \widehat{K}_H} V_\delta^H \otimes (V_\delta^*)^H$$

From Lemma 8.1, we see that $(V_\delta^*)^H \simeq (V_\delta^H)^*$ naturally. This implies that

$$V_\delta^H \otimes (V_\delta^*)^H \simeq \mathrm{End}(V_\delta^H)$$

naturally. Through this natural isomorphism, the inclusion

$$V_\delta^H \otimes (V_\delta^*)^H \;\hookrightarrow\; V_\delta \otimes V_\delta^*$$

corresponds to a natural linear embedding

$$\mathrm{End}(V_\delta^H) \;\hookrightarrow\; \mathrm{End}(V_\delta). \tag{10}$$

We leave it to the reader to check that this embedding is given by

$$A \mapsto \iota \circ A \circ p,$$

where $\iota : V_\delta^H \to V_\delta$ is the inclusion map, and where $p : V_\delta \to V_\delta^H$ is the unique H-equivariant projection operator. Since $p \circ \iota$ is the identity of V_δ^H, it is readily seen that the embedding (10) is an embedding of algebras.

Lemma 8.3. *The subspaces $\mathscr{R}(H\backslash K/H)$ and $C(H\backslash K/H)$ of $C(K)$ are closed under the operation of convolution. Accordingly, these subspaces are subalgebras of the convolution algebra $(C(K),*)$.*

Proof. Let $f,g \in C(K)$. Then for $k \in K$ we have

$$L_k(f*g) = L_kL(f)g = L(L_kf)g = (L_kf)*g.$$

On the other hand,

$$R_k(f*g) = R_kL(f)g = L(f)R_kg = f*(R_kg).$$

From this we see that $C(H\backslash K/H)$ is closed under convolution. Since also $\mathscr{R}(K)$ is closed under convolution, it follows that $\mathscr{R}(H\backslash K/H) = \mathscr{R}(K)\cap C(H\backslash K/H)$ is closed under convolution. □

Proposition 8.4. *Let K be compact group and H a closed subgroup. Then the following assertions are equivalent:*

(a) *the algebra $(\mathscr{R}(H\backslash K/H),*)$ is commutative;*

(b) *the algebra $(C(H\backslash K/H),*)$ is commutative;*

(c) *for all $\delta \in \widehat{K}$, the space V_δ^H has dimension at most* 1.

Proof. '(a) $\Rightarrow$ (b)': The map $\beta : (f,g) \mapsto f*g$ is continuous bilinear from $C(K)\times C(K)$ to $C(K)$, hence from $C(H\backslash K/H)\times C(H\backslash K/H)$ to $C(H\backslash K/H)$. Since $\mathscr{R}(H\backslash K/H)$ is dense in $\mathbb{C}(H\backslash K/H)$, we see that (a) implies (b).

'(b) $\Rightarrow$ (c)': Assume (b) and let $\delta \in \widehat{K}$. Let $\iota : V_\delta^H \to V_\delta$ be the inclusion map, and let $p : V_\delta \to V_\delta^H$ be the H-equivariant projection map. Then $f \mapsto \delta(f)$ is a non-zero $K\times K$ equivariant map from $C(K)$ to $\mathrm{End}(V_\delta)$. Since $\delta\otimes\delta^\vee$ is an irreducible representation, this map is surjective. It follows that

$$T : C(K) \to \mathrm{End}(V_\delta^H),\ \ f \mapsto P\circ\delta(f)\circ\iota$$

is a surjective linear map. Let $P : C(K) \to C(H\backslash K/H)$ the unique $H\times H$-equivariant projection, which may be defined by $P = L(1_H)R(1_H)$. Then it readily seen that $T\circ P = T$. Hence, T maps $C(H\backslash K/H)$ onto $\mathrm{End}(V_\delta)^H$. We now note that for $f \in C(H\backslash K/H)$ the endomorphism $\delta(f)$ leaves V_δ^H invariant, so that the restriction T_0 of T to $C(H\backslash K/H)$ is given by

$$T_0(f) = \delta(f)|_{\mathrm{V}_\delta^{\mathrm{H}}}.$$

This implies that T_0 is a surjective algebra homomorphism. From (b) we now conclude that $\mathrm{End}(V_\delta^H)$ is commutative. This implies that $\dim(V_\delta^H) \leq 1$.

'$(c) \Rightarrow (a)$': Assume that (c) is valid and let $f,g \in \mathscr{R}(H\backslash K/H)$. Then it suffices to show that $\mathfrak{F}(f*g) = \mathfrak{F}(g*f)$. For this it suffices to show that $\delta(f)$ and $\delta(g)$ commute for any given $\delta \in V_\delta$. Let $p : V_\delta \to V_\delta^H$ be the H-equivariant projection operator. Then it is readily seen that $p\circ\delta(f)\circ p = \delta(f)$, so that $\delta(f)$ maps into V_δ^H and restricts to zero on the kernel of p. By condition (c) it follows that $\delta(f) = c_1p$ and $\delta(g) = c_2p$, for constants $c_1,c_2 \in \mathbb{C}$. It follows that $\delta(f)$ and $\delta(g)$ commute. □

Let G be a unimodular Lie group, and H a compact subgroup, hence unimodular as well. We fix a choice of Haar measure dx on G and normalized Haar measure dh on H. The quotient manifold G/H carries a positive G-invariant density, unique up to a positive scalar. We denote it by $d\bar{x}$. Let $\pi : G \to G/H$ denote the natural projection. For $f \in C_c(G)$ we define the function $\pi_* f : G \to \mathbb{C}$ by

$$\pi_*(f)(x) = \int_H f(xh)\, dh$$

Then it is readily verified that $\pi_* f \in C_c(G/H)$.

Lemma 8.5. *The normalization of $d\bar{x}$ may be fixed so that*

$$\int_G f(x)\, dx = \int_{G/H} \pi_*(f)(\bar{x})\, d\bar{x} \qquad (f \in C_c(G/H)).$$

If G is compact, then $d\bar{x}$ equals the normalized density.

Proof. First, let $d\bar{x}$ be chosen arbitrarily. Let $A \subset G$ be a compact subset. Then $B := AH$ is compact and right H-invariant. Let $\bar{B}$ be its compact image in G/H. Let $C_A(G)$ denote the space of functions $f \in C(G)$ with $\mathrm{supp} f \subset A$. Then it is readily seen that for $f \in C_A(G)$ we have $\pi_*(f) \in C_{\bar{B}}(G/H)$ and $\sup|\pi_*(f)| \leq \sup|f|$. It follows that π_* maps $C_B(G)$ continuous linearly into $C_{\bar{B}}(G/H)$. Likewise, it is readily checked that the map

$$I : C_c(G/H) \to \mathbb{C},\ \ \varphi \mapsto \int_{G/H} \varphi(\bar{x})\, d\bar{x}$$

maps $C_B(G/H)$ continuous linearly into $\mathbb{C}$. Thus, the composition $\mu := I \circ \pi_*$ defines a Radon measure on G. It is readily seen that π_* intertwines the left regular representation L of G on $C_c(G)$ with the similar representation L of G in $C_c(G/H)$. Furthermore, for every $x \in G$ we have $I \circ L_x = I$, by left invariance of $d\bar{x}$. It follows that the Radon measure μ is left invariant. It is readily verified that μ is positive, i.e., $\mu(f) > 0$ for $f \geq 0, f \neq 0$ hence corresponds to a positive multiple of dx. This means that there exists a positive $c > 0$ such that $\mu(f) = c\int_G f(x)dx$ for all $f \in C_c(G)$. Therefore, the density $c^{-1}d\bar{x}$ satisfies our requirements.

If G is compact, then $\pi_*(1_G) = 1_{G/H}$ and it is readily seen that the $c^{-1}d\bar{x}$ is normalized. $\square$

In the same setting, with $d\bar{x}$ normalized as in the above lemma, we define $L^2(G/H)$ as the corresponding space of L^2-functions on G/H.

Corollary 8.6. *The map $\pi^* : \varphi \mapsto \varphi \circ \pi$ is an isometry from $L^2(G/H)$ onto the space of right H-invariant functions in $L^2(G)$.*

Proof. By compactness of H, it follows that π^* maps $C_c(G/H)$ onto the space $C_c(G)^{R(H)}$ of right H-invariant functions on G. For $\varphi \in C_c(G/H)$ it is readily seen that $\pi_*\pi^*\varphi = \varphi$. Therefore,

$$\|\varphi\|^2_{L^2(G/H)} = \int_{G/H} |\varphi(\bar{x})|^2\, d\bar{x} = \int_{G/H} \pi_*\pi^*(|\varphi|^2)(\bar{x})\, d\bar{x} = \int_G |\pi^*\varphi(x)|^2 dx = \|\pi^*\varphi\|_{L^2(G)}.$$

It follows that $\pi^* : C_c(G/H) \to C_c(G)^{R(H)}$ is an isometric isomorphism for the L^2-norms. As the given spaces are in dense in $L^2(G/H)$ and $L^2(G)^{R(H)}$, the result follows. $\square$

In the above setting we shall use the isometry $\pi^* : L^2(G/H) \to L^2(G)^{R(H)}$ to identify the functions in these spaces. Then the following result is a consequence of the Peter-Weyl theorem for compact groups and the results of this section.

Proposition 8.7. *Let K be a compact group and H a closed subgroup. The Fourier transform $\mathfrak{F} : f \mapsto \hat{f}$ restricts to a isometric isomorphism*

$$L^2(K/H) \xrightarrow{\simeq} \widehat{\oplus}_{\delta \in \widehat{K}_H} V_\delta \otimes (V_\delta^*)^H,$$

which is equivariant for the left regular representation of K in $L^2(K/H)$ and the direct sum of the K-representations $\delta \otimes 1_{V_\delta^{*H}}$. In particular,

$$(L^2(K/H), L) \simeq \widehat{\bigoplus}_{\delta \in \widehat{K}_H} \dim(V_\delta^H) \cdot \delta.$$

This decomposition into irreducibles is multiplicity free if and only if the convolution algebra $C(H\backslash K/H)$ is commutative.

There is an interesting setting in which the above Plancherel decomposition is multiplicity free. If G is a Lie group, then by an involution of G we mean a Lie group automorphism $\sigma \in \mathrm{Aut}(G)$ with $\sigma^2 = \mathrm{I}_G$. The associated set of fixed points,

$$G^\sigma := \{x \in G \mid \sigma(x) = x\},$$

is readily seen to be a closed subgroup of G, hence a Lie group of its own right. We recall that an open subgroup H of G^σ is automatically closed in G^σ, hence in G. Such a subgroup H contains the component of the identity $(G^\sigma)_e$. Hence,

$$(G^\sigma)_e \subset H \subset G^\sigma. \tag{11}$$

Conversely, if H is a subgroup of G with (11), then H is a union of $(G^\sigma)_e$-cosets, hence open in G^σ, hence a closed subgroup of G.

We recall that a real Lie group G is called semisimple if and only if its Lie algebra $\mathfrak{g}$ is semisimple. This is equivalent to the condition that the Killing form B of $\mathfrak{g}$ is non-degenerate.

Proposition 8.8. *Let K be a compact connected semisimple Lie group, and σ an involution of K. Let H be an open subgroup of K^σ (hence closed in K). Then the convolution algebra $(C(H\backslash K/H), *)$ is commutative.*

The proof of this result will be given in the next section. It involves an application of the universal enveloping algebra.

The situation of the above proposition is geometrically interesting as the space K/H can be equipped with the structure of a compact Riemannian symmetric space on which K acts transitively by isometries.

Definition 8.9. A Riemannian symmetric space is defined to be Riemannian manifold (M, g) satisfying the following condition. For every point $a \in M$ the local point reflection $S_{a,\mathrm{loc}} : \mathrm{Exp}_a(X) \mapsto \mathrm{Exp}_a(-X)$ extends to a global isometry $S_a : M \to M$.

Remark 8.10. A Riemannian locally symmetric space is defined as above, but with the weaker condition that for every $a \in M$ the local reflection $S_{a,\mathrm{loc}}$ is an isometry in some neighborhood of a. The condition of local symmetry can be shown to be equivalent to the requirement that the associated Levi Civita connection ∇ has curvature R which is covariantly locally constant, i.e., $\nabla R = 0$. It is known that a locally symmetric space is globally symmetric if and only if it is geodesically complete.

Let K, σ, H be as in Proposition 8.8. We will indicate why K/H carries the structure of a symmetric space. Then Killing form B is negative definite on $\mathfrak{k}$, the Lie algebra of K. [1]

The differential $\sigma_* := d\sigma(e)$ is an automorphism of $\mathfrak{k}$. Furthermore, by using the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\sigma} & K \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{k} & \xrightarrow{\sigma_*} & \mathfrak{k} \end{array}$$

we see that $\sigma(\exp X) = \exp(\sigma_* X)$ for all $X \in \mathfrak{k}$. For this reason, we will briefly write σ for σ_*, if no confusion is caused.

We leave it as an exercise to the reader to check that K^σ has the fixed point set $\mathfrak{k}^\sigma$ as its Lie algebra.

Since H is an open subgroup of G^σ, its Lie algebra $\mathfrak{h}$ equals $\mathfrak{k}^\sigma$. Let $\mathfrak{q}$ be the minus one eigenspace of σ in $\mathfrak{k}$. We leave it to the reader to check that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q} \text{ en } [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k} \tag{12}$$

and that we have the following $\mathrm{Ad}(H)$-invariant direct sum decomposition.

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{q}. \tag{13}$$

As the Killing form is σ-invariant, it follows that the decomposition (13) is perpendicular for the Killing form. Let $g_e = -B|_{\mathfrak{q}}$. Then g_e is a positive definite inner product on $\mathfrak{q} \simeq T_{eH}(K/H)$ which is invariant under the action of $\mathrm{Ad}(H)$. This implies that g_e extends to a K-invariant Riemannian structure g on K/H which is given by the formula

$$g_{xH} = dl_x(e)^{*-1} g_e, \qquad (x \in K).$$

For this structure, K acts by isometries on K/H. We leave it to the reader to check that σ induces an isometry $\bar{\sigma} : K/K \to K/H$. It can be shown that the Riemannian exponential map $\mathrm{Exp}_{eH} : \mathfrak{q} \to K/H$ is given by

$$\mathrm{Exp}_{eH}(X) = \exp(X)H, \qquad (X \in \mathfrak{q}).$$

We leave it to the reader to check that the local geodesic reflection at e is given by $\bar{\sigma}$, hence extends to a global isometry. By homogeneity, it now follows that K/H is a Riemannian symmetric space.

[1] Lie groups will be denoted by Roman capitals, and their Lie algebras with the corresponding Gothic lower cases

Example 8.11. We consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$, equipped with the restriction of the Euclidean metric. Then the natural action of $SO(n+1)$ on S^n is transitive and isometric. We leave it to the reader to prove that the map

$$S : x \mapsto (x_1, -x_2, -x_3, \ldots, -x_{n+1})$$

restricts to an isometry of S^n, which equals the local geodesic reflection at the point $e_1 = (1,0,\ldots,0)$ of S^n. By homogeity this implies that S^n is a Riemannian symmetric space.

The stabilizer of e_1 in $\mathrm{SO}(n+1)$ is equal to the subgroup H consisting of the matrices of the form

$$\varphi(a) := \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \qquad (a \in \mathrm{SO}(n)).$$

Using φ to identify $\mathrm{SO}(n)$ with the closed subgroup H of $\mathrm{SO}(n+1)$, we see that

$$S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n).$$

We leave it to the reader to show that conjugation by the matrix of S defines an involution σ of $\mathrm{SO}(n+1)$ such that $\mathrm{SO}(n)$ is the connected component of $\mathrm{SO}(n+1)^\sigma$ and such that the induced diffeomorphism $\bar{\sigma}$ of $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ corresponds to $S|_{S^n}$. Finally, we leave it to the reader to check that the Riemannian metric on S^n constructed from the Killing form of $\mathrm{SO}(n+1)$ coincides with a scalar multiple of the Euclidean metric on S^n.

More generally, it can be shown that every compact Riemannian symmetric space can be realized as a quotient of the form K/H as above, with K compact and H compact. Furthermore, K is compact semisimple if and only if K/H has finite fundamental group. For this and other details on the geometry of compact symmetric spaces, we refer the reader to the standard reference [Hel78].

9 The universal enveloping algebra

For a given complex linear space V, we denote by $T(V)$ the tensor algebra of V. For the precise definition of this algebra, see for instance [Lan02]. Let $\nu : V \to T(V)$ be the canonical linear map, then the tensor algebra has the following universal property.

Universal property $T(V)$: Let A be any associative algebra (over $\mathbb{C}$) with unit, and let $\varphi : V \to A$ be any linear map. Then there exists a unique algebra homomorphism $\bar{\varphi} : T(V) \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & A \\ \nu\downarrow & \nearrow & \bar{\varphi} \\ T(V) & & \end{array} \tag{14}$$

In fact, the universal property characterizes the tensor algebra up to isomorphism. More precisely, let T' be an associative algebra with unit and $\nu' : V \to T'$ a linear map such that (ν', T') satisfies the above universal property, then there exists a unique isomorphism $\psi : T(V) \to T'$ of algebras such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\nu'} & T' \\ \nu\downarrow & \nearrow & \psi \\ T(V) & & \end{array} \tag{15}$$

We recall the precise argument. A homomorphism $\psi : T(V) \to T'$ exists by the universal property of $(\nu, T(V))$. By the universal property of (ν', T') there exists a similar algebra homomorphism $\psi' : T' \to T(V)$ in the opposite direction. The composition $\psi' \circ \psi$ must be the identity on $T(V)$ in view of the uniqueness assertion in the universal property for $T(V)$. Likewise, $\psi \circ \psi'$ is the identity on T'. Therefore, ψ is an isomorphism of associative algebras.

We list some properties of the tensor algebra. The product in $T(V)$ is denoted by $\otimes$. The map $V \to T(V)$ is injective, and allows us to identify V with a subspace of $T(V)$. This subspace generates the algebra $T(V)$. More precisely, if $\{v_i\}_{i \in I}$ is a basis for V, then a basis for $T(V)$ may be described as follows. Let $\mathscr{B}$ be the disjoint union of the Cartesian products I^n for $n \geq 1$. Then 1 together with the elements

$$v_{j_1} \otimes \cdots \otimes v_{j_n}, \qquad ((j_1, \ldots, j_n) \in \mathscr{B})$$

form a basis of $T(V)$.

Let $T^0(V) = \mathbb{C}1$ and for a fixed $n \geq 1$ let $T^n(V)$ be the span of the elements of the form $v_1 \otimes \cdots \otimes v_n$, with $v_j \in V$. Then

$$T(V) = \bigoplus_{n \geq 0} T^n(V)$$

and $T^p(V) \otimes T^q(V) \subset T^{p+q}(V)$, so that $T(V)$ is a graded algebra.

In terms of the tensor algebra, we may define the symmetric algebra $S(V)$ as follows. Let $\mathscr{I}$ be the two sided ideal of $T(V)$ generated by the elements $v \otimes w - w \otimes v$ for $v, w \in V$. The symmetric algebra $S(V)$ is defined to be the quotient algebra $T(V)/\mathscr{I}$. The ideal $\mathscr{I}$ is homogeneous in the sense that $\mathscr{I} = \oplus_{n\geq 0}(\mathscr{I} \cap T^n(X))$. It follows that $S(X)$ inherits a gradation from $T(V)$. As $\mathscr{I} \subset \oplus_{n\geq 2} T^n(V)$, the natural map $V \to S(V)$ is an injective linear map, via which we shall view V as a subspace. Clearly, the algebra $S(V)$ is commutative.

From the universal property of $T(V)$ we immediately obtain the following universal property of $S(V)$ with respect to the category of commutative associative algebras with unit. Let A be in this category, and let $\varphi : V \to A$ be linear. Then the unique algebra homomorphism $\bar{\varphi}$ as in (15) factors through an algebra homomorphism $\bar{\bar{\varphi}} : S(V) \to A$, which makes the following diagram commutative

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & A \\ \downarrow & \nearrow \bar{\bar{\varphi}} & \\ S(V) & & \end{array} \tag{16}$$

From the uniqueness of $\bar{\varphi}$ it follows that $\bar{\bar{\varphi}}$ is uniquely determined as well. By a similar argument as for the tensor algebra, the universal property determines $V \to S(V)$ up to isomorphism.

For us, the case that V is finite dimensional of dimension d will be of particular interest. If $v_1, \ldots, v_d$ is a basis of V then it is easily verified that the elements

$$v^\alpha := v_1^{\alpha_1} \cdots v_d^{\alpha_d} \tag{17}$$

for $\alpha \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n = n$, form a basis for the homogeneous component $S^n(V)$.

Let $P(V^*)$ be the graded algebra of polynomial functions $V^* \to \mathbb{C}$. Then we have the linear map $\iota : V \to P(V^*)$ given by $\iota(v) : \eta \mapsto \eta(v), V^* \to \mathbb{C}$. By the universal property of $S(V)$ this map gives rise to an algebra homomorphism $\bar{\bar{\iota}} : S(V) \to P(V^*)$. On the basis (17) this homomorphism is given by

$$\bar{\bar{\iota}}(v^\alpha) : \eta \mapsto \eta_1^{\alpha_1} \cdots \eta_d^{\alpha_d}.$$

As these images form a basis of $P(V^*)$ we see that $\bar{\bar{\iota}}$ defines a canonical algebra isomorphism

$$S(v) \xrightarrow{\simeq} P(V^*).$$

Let now $\mathfrak{g}$ be a finite dimensional complex Lie algebra. Then we define the universal Lie algebra $U(\mathfrak{g})$ to be the quotient of $T(\mathfrak{g})$ by the two sided ideal $\mathscr{J}$ generated by all elements of the form $X \otimes Y - Y \otimes X - [X,Y]$, for $X, Y \in \mathfrak{g}$. Then $U(\mathfrak{g})$ is an associative algebra with unit. The product of two elements $u, v \in U(\mathfrak{g})$ is denoted by uv. We note that the ideal $\mathscr{J}$ is not homogeneous, so that $U(\mathfrak{g})$ does in general not carry a natural gradation. However we may define a filtration on $U(\mathfrak{g})$ by

$$U(\mathfrak{g})_n := \text{image}\left(\oplus_{j=0}^n T^j(\mathfrak{g})\right)$$

The universal algebra has the following universal property. Let $j : \mathfrak{g} \to U(\mathfrak{g})$ be the composition of the canonical maps $\mathfrak{g} \to T(\mathfrak{g})$ and $T(\mathfrak{g}) \to U(\mathfrak{g})$.

Lemma 9.1. (Universal property). *Let A be any associative algebra with unit, and let $\varphi : \mathfrak{g} \to A$ be a linear map such that*

$$\varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) = \varphi([X,Y]), \qquad (X, Y \in \mathfrak{g}). \tag{18}$$

Then there exists a unique algebra homomorphism $\bar{\bar{\varphi}} : U(\mathfrak{g}) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & A \\ j \downarrow & \nearrow \bar{\bar{\varphi}} & \\ U(\mathfrak{g}) & & \end{array} \tag{19}$$

Remark. The associative algebra carries the commutator bracket given by $[a,b] = ab - ba$ for $a,b \in A$. Equipped with this bracket, A becomes a Lie algebra. The above requirement on φ is equivalent to the requirement that φ is a Lie algebra homomorphism for this Lie algebra structure on A.

Proof. Let φ be as given. Then by the universal property of $T(\mathfrak{g})$, the map φ has a unique lift to an algebra homomorphism $\bar{\varphi} : T(\mathfrak{g}) \to A$. By the condition on φ, we see that $\bar{\varphi}(X \otimes Y - Y \otimes X - [X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) - \varphi([X,Y]) = 0$. Hence $\mathscr{J} \subset \ker \bar{\varphi}$ and we see that $\bar{\varphi}$ factors through an algebra homomorphism $\bar{\bar{\varphi}} : U(\mathfrak{g}) \to A$. This homomorphism makes the diagram (19) commutative. Let $p : T(\mathfrak{g}) \to U(\mathfrak{g})$ denote the canonical projection. If $\bar{\bar{\varphi}}$ is an algebra homomorphism $U(\mathfrak{g}) \to A$ making the diagram (19) commutative, then $\bar{\bar{\varphi}} \circ p = \bar{\varphi}$ by the uniqueness part of the universal property of $T(\mathfrak{g})$. This implies that $\bar{\bar{\varphi}}$ is uniquely determined. □

As before, the universal property determines $\mathfrak{g} \to U(\mathfrak{g})$ up to isomorphism.

The following Poincaré – Birkhoff – Witt (or PBW) theorem will turn out to be an important tool.

Theorem 9.2. *Let $X_1, \cdots, X_d$ be a basis of $\mathfrak{g}$. Then the elements*

$$j(X_{\nu_1}) \cdots j(X_{\nu_k}) \tag{20}$$

with $k \geq 0$ and $1 \leq \nu_1 \leq \cdots \nu_k \leq d$ form a basis of $U(\mathfrak{g})$.

Proof. Inductively, we will prove that the elements (20) for $k \leq n$ span $U(\mathfrak{g})_n$. For $n = 0$ this statement is clear. Assume that the assertion has been established for all $n < m$, with m a positive integer. Let $i_1, \ldots, i_{m+1}$ be a sequence of indices from $\{1, \ldots, d\}$. Since every product $j(X_\nu)j(X_\mu)$ may be rewritten as a $j(X_\mu)j(X_\nu)$ modulo $j([X_\nu, X_\mu])$, which is an element of $U(\mathfrak{g})_1$, it follows that for every permutation $\sigma \in S_{m+1}$ we have

$$j(X_{i_1}) \cdots j(X_{i_{m+1}}) - j(X_{i_{\sigma(1)}}) \cdots j(X_{i_{\sigma(m+1)}}) \in U(\mathfrak{g})_m. \tag{21}$$

In particular this is valid for a permutation with

$$i_{\sigma(1)} \leq \cdots \leq i_{\sigma(m+1)}$$

so that $j(X_{i_{\sigma 1}}) \cdots j(X_{i_{\sigma(m+1)}})$ is a basis element. Applying the induction hypothesis we see that the difference in (21) is a linear combination of basis elements. Hence, so is the first term in (21). This establishes the spanning property for the elements (20).

The proof of their linear independence is more tricky. We refer to [Ser06] for a very clear account. See also [Hum78]. □

Remark 9.3. In particular it follows from the above result that the map $j : \mathfrak{g} \to U(\mathfrak{g})$ is injective. From now on we will use j to view $\mathfrak{g}$ as a subspace of $U(\mathfrak{g})$. Accordingly, given $X \in \mathfrak{g}$ we will write X for $j(X)$, and we will use the notation $X_{v_1} \cdots X_{v_k}$ for the element (20).

Corollary 9.4. *Let $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ be a subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ as a linear space. Then the natural map*

$$U(\mathfrak{g}_1) \otimes \cdots \otimes U(\mathfrak{g}_r) \longrightarrow U(\mathfrak{g})$$

determined by $u_1 \otimes \cdots \otimes u_r \mapsto u_1 \cdots u_r$ is a linear isomorphism.

Proof. Fix a basis of $\mathfrak{g}$ subordinate to the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ and apply the PBW theorem to each $\mathfrak{g}_j$ and to $\mathfrak{g}$. $\square$

We will now explain why the universal enveloping algebra is so important for the representation theory of Lie algebras. Let π be a Lie algebra representation of $\mathfrak{g}$ in a (possibly infinite dimensional) complex linear space V. Thus, π is a linear map from $\mathfrak{g}$ to the associative algebra $\mathrm{End}(V)$ of linear endomorphisms of V, satisfying $\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X,Y])$. By the universal property, the representation π extends to an algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}(V)$, which is also called a representation of the algebra $U(\mathfrak{g})$ in V. It is tradition to say that π turns V into a module for the Lie algebra $\mathfrak{g}$, and that this module structure extends to a module structure for the associative algebra $U(\mathfrak{g})$. Accordingly, given $x \in U(\mathfrak{g})$ and $v \in V$ we will write xv for $\pi(x)v$.

If V is a $U(\mathfrak{g})$-module, then the map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ defined by $\pi(x)v = xv$ satisfies

$$\begin{aligned} \pi(x)\pi(y)v - \pi(y)\pi(x)v &= x(yv) - y(xv) \\ &= (xy)v - (yx)v \\ &= xy - yx)v = [x,y]v = \pi([x,y])v, \end{aligned}$$

so that π is a representation of $\mathfrak{g}$ in V. Thus, V is a $\mathfrak{g}$-module in this way. The original $U(\mathfrak{g})$-module structure is retrieved by invoking the universal property of $U(\mathfrak{g})$. A linear map $T : V \to W$ to a second $\mathfrak{g}$-module is equivariant for the $\mathfrak{g}$-action if and only if it is so for the $U(\mathfrak{g})$-action. We thus see that the category of $\mathfrak{g}$-modules is isomorphic to the category of $U(\mathfrak{g})$-modules.

In the above we used that by the homomorphism property of $\pi : U(\mathfrak{g}) \to \mathrm{End}(V)$ we have $x(yv) = (xy)v$ for $x, y \in U(\mathfrak{g})$ and $v \in V$. From now on we will ignore the brackets and just write xyv.

We note that a subspace of a $\mathfrak{g}$-module V is $\mathfrak{g}$-invariant if and only if it is $U(\mathfrak{g})$-invariant. Thus, the $\mathfrak{g}$-module V is irreducible if and only if it is so as a $U(\mathfrak{g})$-module.

Given a (possible infinite dimensional) $\mathfrak{g}$-module V and a vector $v \in V$ we denote by $U(\mathfrak{g})v$ the image of $U(\mathfrak{g})$ in V under the map $x \mapsto xv$.

Lemma 9.5. *Let V be a finite dimensional $\mathfrak{g}$-module, and $v \in V$. Then $U(\mathfrak{g})v$ is the smallest $\mathfrak{g}$-submodule of V containing v.*

Proof. First of all, it is readily verified that $U(\mathfrak{g})v$ is a submodule of V. Let W be any submodule of $\mathfrak{g}$ containing v. Then by induction on n one sees that $W \supset U(\mathfrak{g})_n v$. Hence $W \supset U(\mathfrak{g})v$. $\square$

A vector $v \in V$ is said to be cyclic if $U(\mathfrak{g})v = V$.

Corollary 9.6. *Let V be an irreducible $\mathfrak{g}$-module. Then every vector $v \in V \setminus \{0\}$ is cyclic.*

Proof. Let $v \in V$ be a non-zero vector. Then $U(\mathfrak{g})v$ contains $1 \cdot v = v$ hence is different from 0. By irreducibility of V is follows that the $\mathfrak{g}$-invariant subspace $U(\mathfrak{g})v$ is V. $\square$

Let $\mathfrak{g}$ be a compact semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a maximal torus, R the associated root system and R^+ a choice of positive roots. We put

$$\mathfrak{g}_{\mathbb{C}}^+ := \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\mathbb{C}\alpha}.$$

Then $\mathfrak{g}_{\mathbb{C}}^+$ is a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$. It is normalized by $\mathfrak{t}_{\mathbb{C}}$, hence,

$$\mathfrak{b} := \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^+$$

is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. This subalgebra is called the Borel subalgebra associated with the positive system R^+. Let now $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$. We extend λ to a linear functional in $\mathfrak{b}^*$ which is trivial on $\mathfrak{g}_{\mathbb{C}}^+$. We define J_λ to be the left ideal of $U(\mathfrak{g})$ generated by the elements of the form $X - \lambda(X)$, for $X \in \mathfrak{b}$. Then $1_\lambda := 1 + J_\lambda$ is a cyclic vector in the natural left $U(\mathfrak{g})$-module

$$Z(\lambda) := U(\mathfrak{g})/J_\lambda.$$

We note that for $X \in \mathfrak{b}$ we have $X1_\lambda = \lambda(X)1_\lambda$. In particular, 1_λ is annihilated by $\mathfrak{g}_{\mathbb{C}}^+$. Thus, 1_λ is a cyclic highest-weight vector of weight λ.

From the basic master math course on Lie groups as well as that on Lie algebras, we recall the following result. If $\mathfrak{g}$ is a compact semisimple Lie algebra, then $\mathfrak{g}_{\mathbb{C}}$ is complex semisimple Lie algebra. It can be shown that every complex semisimple Lie algebra has a compact real form, hence arises in this way.

Theorem 9.7. *Let $\mathfrak{g}$ be compact semisimple as above. Let V and W be (not necessarily finite dimensional) $\mathfrak{g}_{\mathbb{C}}$-modules. Then the following are valid.*

(a) *If V has a cyclic highest weight vector, then it has a unique maximal proper submodule and a unique irreducible quotient. Furthermore, its highest weight is uniquely determined.*

(b) *If V is finite dimensional and irreducible, then V has a cyclic highest weight vector.*

(c) *If V and W are irreducible and have cyclic highest weight vectors of the same weight, then they are isomorphic.*

The module $Z(\lambda)$ defined above has a cyclic highest weight vector of weight λ. It follows that $Z(\lambda)$ has a unique maximal proper ideal $\mathfrak{M}_\lambda \triangleleft Z(\lambda)$. The associated quotient

$$V_\lambda := Z(\lambda)/\mathfrak{M}_\lambda$$

is irreducible.

We will now show that all $\mathfrak{g}_{\mathbb{C}}$-modules with a cyclic highest weight vectors can be obtained as quotients of these.

Proposition 9.8. *Let V be a non-trivial $\mathfrak{g}_{\mathbb{C}}$-module with a cyclic highest weight vector.*

(a) *There exists a unique $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ and a unique ideal $I_V \lhd Z(\lambda)$ such that $V \simeq Z(\lambda)/I_V$. Furthermore, λ is the highest weight of V.*

(b) *The module V is irreducible if and only if $V \simeq V_\lambda$.*

Proof. Let $v \in V$ be a cyclic highest weight vector. We know that the associated highest weight λ is uniquely determined. The map $\varphi : x \mapsto xv$ is a surjective $U(\mathfrak{g}_{\mathbb{C}})$-morphism from $U(\mathfrak{g})$ onto V, by cyclicity of v. Furthermore, by the highest weight property, $Xv = \lambda(X)v$ for $X \in \mathfrak{t}$ and $Yv = 0$ for $Y \in \mathfrak{g}_{\mathbb{C}}^+$. It follows that the left ideal $\ker\varphi \lhd U(\mathfrak{g}_{\mathbb{C}})$ contains the elements $X - \lambda(X)$, for $X \in \mathfrak{b}$. Hence, $\ker\varphi \supset J_\lambda$ and we see that φ factors through a surjective $U(\mathfrak{g}_{\mathbb{C}})$-homomorphism $\bar{\varphi} : Z(\lambda) \to V$. Let I_V be the kernel of $\bar{\varphi}$, then $V \simeq Z(\lambda)/I_V$.

We now turn to the uniqueness. Let $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ and $I_V \lhd Z(\lambda)$ be such that $V \simeq Z(\lambda)/I_V$. Then there exists a surjective $U(\mathfrak{g}_{\mathbb{C}}$-module homomorphism $Z(\lambda) \to V$ whose kernel equals I_V. As V is non-trivial, it follows that I_V is proper hence does not contain the image $[1_\lambda]$ of 1_λ in $Z(\lambda)$, which is a cyclic vector. It follows that $v_\lambda := \varphi([1_\lambda])$ is a cyclic highest weight vector of weight λ. Thus, λ is the (unique) highest weight of V. Let $\mathfrak{a}_\lambda$ be the annihilator of v_λ in $U(\mathfrak{g})$. Then I_V equals the image of $\mathfrak{a}_\lambda$ in $Z(\lambda)$, hence is uniquely determined. This establishes (a).

We turn to (b). The module V is irreducible if and only if the quotient $Z(\lambda)/I_V$ is irreducible. The latter is equivalent to the assertion that I_V equals the unique maximal ideal $\mathfrak{M})\lambda$ of $Z(\lambda)$. $\square$

It follows from the above that the V_λ form a complete set of representatives for the equivalence classes of irreducible highest weight modules. The finite dimensional modules among them form a complete set of representatives for the equivalence classes of irreducible finite dimensional $\mathfrak{g}$-modules. Thus, the following important result amounts to the classification of the irreducible finite dimensional $\mathfrak{g}$-modules.

Definition 9.9.

(a) We say that $\lambda \in \mathfrak{t}_{\mathbb{C}}$ is an integral weight if and only if

$$2\frac{\langle \lambda\,,\,\alpha \rangle}{\langle \alpha\,,\,\alpha \rangle} \in \mathbb{Z} \quad \text{for all} \quad \alpha \in R.$$

(b) A weight λ is said to be dominant (relative to R^+) if in addition $\langle \lambda\,,\,\alpha \rangle \geq 0$ for all $\alpha \in R^+$.

(c) The set of weights (which is a lattice) is denoted by Λ, the subset of dominant weights by Λ^+.

Theorem 9.10. *Let $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$. Then the irreducible highest weight module V_λ is finite dimensional if and only if λ is dominant integral.*

Proof. For the proof, which is based on the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, we refer to [Hum78], Section 21. $\square$

We will now apply the universal enveloping algebra to the setting of compact symmetric spaces introduced at the end of Section 8. In view of notation to be introduced later on, we change notation slightly.

Let $\mathfrak{u}$ be a compact semisimple Lie algebra, σ an involution of $\mathfrak{u}$ and

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$$

the associated decomposition into the $+1$ and -1 eigenspaces. Given a $\mathfrak{u}$-module V we denote by $V^{\mathfrak{k}}$ the subspace of vectors $v \in V$ which are annihilated by $\mathfrak{k}$, i.e., $Xv = 0$ for all $X \in \mathfrak{k}$.

Lemma 9.11. *Let V be a finite dimensional irreducible* $\mathfrak{u}$*-module. Then* $\dim_{\mathbb{C}} V^{\mathfrak{k}} \leq 1$.

To prepare for the proof, we fix a maximal abelian subspace

$$\mathfrak{b} \subset \mathfrak{q}.$$

Furthermore, we define $\mathfrak{m}$ to be the centralizer of $\mathfrak{b} \in \mathfrak{k}$. We fix a maximal torus $\mathfrak{t}$ of $\mathfrak{u}$ which contains $\mathfrak{b}$. Then $\mathfrak{t} \cap \mathfrak{k} = \mathfrak{t} \cap \mathfrak{m}$ obviously.

We claim that $\mathfrak{t}$ is stable under σ. Indeed, if $X \in \mathfrak{t}$ then $X - \sigma(X) \in \mathfrak{q}$. For $Y \in \mathfrak{b}$ we have

$$[X - \sigma X, Y] = [X,Y] - [\sigma X, Y] = [X,Y] - \sigma[X, \sigma Y] = [X,Y] + \sigma[X,Y] = 0,$$

since $\mathfrak{t}$ centralizes $\mathfrak{b}$. It follows that $X - \sigma(X)$ belongs to $\mathfrak{q}$ and centralizes $\mathfrak{b}$. Since $\mathfrak{b}$ is maximal abelian it follows that $X - \sigma(X) \in \mathfrak{b}$. Hence $\sigma(X)$ centralizes $\mathfrak{t}$ and we conclude that $\sigma(X) \in \mathfrak{t}$. This shows that σ leaves $\mathfrak{t}$ invariant, so that $\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{k}) \oplus (\mathfrak{t} \cap \mathfrak{q})$. The second summand is contained in $\mathfrak{q}$ and centralizes $\mathfrak{b}$, hence is contained in $\mathfrak{b}$. It follows that

$$\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{k}) \oplus \mathfrak{b} = (\mathfrak{t} \cap \mathfrak{m}) \oplus \mathfrak{b}.$$

Let R denote the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{u}_{\mathbb{C}}$, and let $R_{\mathfrak{b}}$ denote the subset of roots that do not vanish on $\mathfrak{b}$. We recall that all roots of R belong to $i\mathfrak{t}^*$. Hence the roots of $R_{\mathfrak{b}}$ restrict to non-zero elements of $i\mathfrak{b}^*$. For each $\alpha \in R_{\mathfrak{b}}$, the intersection $\ker \alpha \cap \mathfrak{b}$ is the kernel of $\alpha|_{\mathfrak{b}}$. Let $\mathfrak{b}^{\mathrm{reg}}$ denote the complement of these hyperplanes in $\mathfrak{b}$ and let $\mathfrak{b}^+$ be a connected component of $\mathfrak{b}^+$. Let $R_{\mathfrak{b}}^+$ denote the set of roots $\alpha \in R_{\mathfrak{b}}$ such that $-i\beta$ is strictly positive on $\mathfrak{b}^+$. Then it follows that

$$R_{\mathfrak{b}} = R_{\mathfrak{b}}^+ \cup (-R_{\mathfrak{b}}^+), \qquad \text{(disjoint union).}$$

We define

$$\mathfrak{n}_{\mathbb{C}} := \bigoplus_{\alpha \in R_{\mathfrak{b}}^+} \mathfrak{u}_{\mathbb{C}\alpha}, \quad \text{and} \quad \bar{\mathfrak{n}}_{\mathbb{C}} := \bigoplus_{\alpha \in -R_{\mathfrak{a}}^+} \mathfrak{u}_{\mathbb{C}\alpha}.$$

Lemma 9.12. *The spaces* $\mathfrak{n}_{\mathbb{C}}$ *and* $\mathfrak{n}_{\mathbb{C}}$ *are subalgebras of* $\mathfrak{u}_{\mathbb{C}}$*. Furthermore,*

$$\mathfrak{u}_{\mathbb{C}} = \bar{\mathfrak{n}}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}.$$

Proof. Let $\alpha, \beta \in R_{\mathfrak{b}}^{+}$. If $\alpha+\beta$ is a root then $\alpha+\beta \in R_{\mathfrak{b}}^{+}$. Hence

$$[\mathfrak{u}_{\mathbb{C}\alpha}, \mathfrak{u}_{\mathbb{C}\beta}] \subset \mathfrak{u}_{\mathbb{C}(\alpha+\beta)} \subset \mathfrak{n}_{\mathbb{C}}.$$

If $\alpha+\beta$ is not a root, then $[\mathfrak{u}_{\mathbb{C}\alpha}, \mathfrak{u}_{\mathbb{C}\beta}] = 0$. In all cases, $[\mathfrak{u}_{\mathbb{C}\alpha}, \mathfrak{u}_{\mathbb{C}\beta}] \subset \mathfrak{n}_{\mathbb{C}}$. Hence, $\mathfrak{n}_{\mathbb{C}}$ is a subalgebra. Likewise, $\bar{\mathfrak{n}}_{\mathbb{C}}$ is a subalgebra.

From the root space decomposition of $\mathfrak{u}_{\mathbb{C}}$ we see that

$$\mathfrak{u}_{\mathbb{C}} = \bar{\mathfrak{n}}_{\mathbb{C}} \oplus \mathfrak{z}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}, \tag{22}$$

where

$$\mathfrak{z}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R \setminus R_{\mathfrak{b}}} \mathfrak{u}_{\mathbb{C}\alpha}.$$

Since $R \setminus R_{\mathfrak{b}}$ consists of the roots vanishing on $\mathfrak{b}$, it follows that $\mathfrak{z}_{\mathbb{C}}$ equals the centralizer of $\mathfrak{b}$ in $\mathfrak{k}_{\mathbb{C}}$. Let $\tau : \mathfrak{u}_{\mathbb{C}} \to \mathbb{C}$ be the conjugation map determined by the real form $\mathfrak{u}$ of $\mathfrak{u}_{\mathbb{C}}$. As τ is an automorphism of $\mathfrak{u}_{\mathbb{C}}$ viewed as a real Lie algebra, and $\tau = I$ on $\mathfrak{b}$, it follows that $\mathfrak{z}_{\mathbb{C}}$ is τ-invariant, hence equals the complexification of the centralizer $\mathfrak{z}$ of $\mathfrak{b}$ in $\mathfrak{g}$. As $\sigma = -I$ on $\mathfrak{b}$, this centralizer is σ-invariant and decomposes as

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{u}) \oplus (\mathfrak{z} \cap \mathfrak{q}).$$

By definition the first summand equals $\mathfrak{m}$. Since $\mathfrak{b}$ is maximal abelian in $\mathfrak{q}$, the second summand equals $\mathfrak{a}$. Thus, $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{a}$ and the result follows from (22). $\square$

Lemma 9.13. *The complex linear extension $\sigma_{\mathbb{C}}$ of σ restricts to a linear isomorphism of $\mathfrak{n}_{\mathbb{C}}$ onto $\bar{\mathfrak{n}}_{\mathbb{C}}$. Furthermore,*

$$\mathfrak{u}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$$

as a direct sum of linear spaces.

Proof. The map $\sigma_{\mathbb{C}}$ is an automorphism of $\mathfrak{u}_{\mathbb{C}}$ which leaves $\mathfrak{t}$ invariant. Write σ for the map $R \to R, \alpha \mapsto \sigma^{-1*}\alpha$. Since $\sigma = -1$ on $\mathfrak{b}$, we see that σ maps $R_{\mathfrak{b}}^{+}$ bijectively onto $-R_{\mathfrak{b}}^{+}$. Furthermore, for $\alpha \in R_{\mathfrak{b}}^{+}$, we have

$$\sigma_{\mathbb{C}}(\mathfrak{u}_{\mathbb{C}\alpha}) = \mathfrak{u}_{\mathbb{C}\sigma(\alpha)}.$$

Taking the direct sum over the roots $\alpha \in \mathbb{R}_{\mathfrak{b}}^{+}$, we find that $\sigma_{\mathbb{C}}$ maps $\mathfrak{n}_{\mathbb{C}}$ bijectively onto $\bar{\mathfrak{n}}_{\mathbb{C}}$.

We consider the linear map $\varphi : \mathfrak{n}_{\mathbb{C}} + \mathfrak{m}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$ given by $\varphi(X) = X + \sigma(X)$ for $X \in \bar{\mathfrak{n}}_{\mathbb{C}}$ and by $\varphi|_{\mathfrak{m}_{\mathbb{C}}} = \mathrm{I}|_{\mathfrak{m}_{\mathbb{C}}}$. We claim that φ is bijective. If $X \in \mathfrak{n}_{\mathbb{C}}$, $Y \in \mathfrak{m}_{\mathbb{C}}$ and $\varphi(X+Y) = 0$ then $X + Y + \sigma_{\mathbb{C}}(X) = 0$ and we see that $X = 0$ and $Y = 0$ by directness of the sum (22). Thus, φ is injective. To see it is surjective, let $Z \in \mathfrak{u}_{\mathbb{C}}$ and decompose $Z = X + Y + U$ according to the decomposition (22). From $\sigma_{\mathbb{C}}(Z) = Z$, $\sigma_{\mathbb{C}}(Y) = Y$, and the first part of the proof, we conclude that $U = \sigma(X)$, hence $Z = \varphi(X+Y)$ and we see that φ is surjective, so that the claim holds. In particular, it follows that the spaces $\mathfrak{k}_{\mathbb{C}}$ and $\bar{\mathfrak{n}}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$ are of equal dimension.

By Lemma 9.12, the linear space $\bar{\mathfrak{n}}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$ is complementary to $\mathfrak{b}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ in $\mathfrak{u}_{\mathbb{C}}$. Since φ is an isomorphism from this space onto $\mathfrak{k}_{\mathbb{C}}$ and $\varphi - I$ maps into $\mathfrak{n}_{\mathbb{C}}$, it follows the $\mathfrak{k}_{\mathbb{C}}$ is complementary to $\mathfrak{b}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$. The result follows. $\square$

From the above lemma combined with Lemma 9.11 it follows that

$$U(\mathfrak{u}_\mathbb{C}) = U(\mathfrak{h}_\mathbb{C})U(\mathfrak{b}_\mathbb{C})U(\mathfrak{n}_\mathbb{C}).$$

Here the space on the right-hand side is defined to be the linear span of all products of the form uvw with $u \in U(\mathfrak{h}_\mathbb{C}), v \in U(\mathfrak{b}_\mathbb{C})$ and $w \in U(\mathfrak{n}_\mathbb{C})$. In other words, the space equals the image of $U(\mathfrak{h}_\mathbb{C}) \otimes U(\mathfrak{b}_\mathbb{C}) \otimes U(\mathfrak{n}_\mathbb{C})$ in $U(\mathfrak{g}_\mathbb{C})$.

We need a final preparation for the proof of Lemma 9.11.

Lemma 9.14. *There exists a positive system R^+ which contains $R^+_\mathfrak{b}$.*

Proof. Fix $X \in \mathfrak{b}^+$. There exists a $Y \in \mathfrak{t} \cap \mathfrak{k}$ such that the roots from $R \setminus R_\mathfrak{b}$ do not vanish on Y. For $t > 0$ sufficiently small, the roots of $R^+_\mathfrak{b}$ are positive on $i(X + tY)$. If $\alpha \in R \setminus R_\mathfrak{b}$ then $\alpha(X) = 0$, so α does not vanish on $i(X + tY)$. Let R^+ denote the set of roots that are positive on $i(X + tY)$, then $R^+_\mathfrak{b} \subset R^+$. □

Proof of Lemma 9.11. The module V^* is irreducible as well and has a unique highest weight $\lambda \in \mathfrak{t}^*_\mathbb{C}$. Let η be a non-zero highest weight vector. Then for $X \in \mathfrak{a}$ we have $\eta \circ X = -X\eta = -\lambda(X)$. Since $\mathfrak{n}_\mathbb{C}$ is a sum of positive root spaces, it follows that $\eta \circ Y = -Y\eta = 0$ for $Y \in \mathfrak{n}_\mathbb{C}$.

We will complete the proof by establishing the claim that the map $v \mapsto \eta(v)$ is injective from $V^\mathfrak{h}$ to $\mathbb{C}$. Indeed, assume that $\eta(v) = 0$. Then $U(\mathfrak{h})v \in \mathbb{C}v$ so $\eta = 0$ on $U(\mathfrak{h})v$. By the first part of the proof it follows that $\eta = 0$ on $U(\mathfrak{a})U(\mathfrak{h})v$ hence also on $U(\mathfrak{n})U(\mathfrak{a})U(\mathfrak{h})v = U(\mathfrak{g})v$.

Since $\eta \neq 0$ it follows that $U(\mathfrak{g})v$ is a proper invariant subspace of V and since V is irreducible, we see that $U(\mathfrak{g})v = 0$. Since v is contained in the latter space, it follows that $v = 0$. □

We now come to the proof of Proposition 8.8, which we recall in different notation.

Proposition 9.15. *Let U be a connected compact semisimple Lie group, σ an involution of U and K an open subgroup of U^σ. Then the convolution algebra $(C(K\backslash U/K), *)$ is commutative.*

Proof. Let (π, V) be an irreducible finite dimensional (continuous) representation of U. Then $\pi : U \to \mathrm{GL}(V)$ is a continuous homomorphism of Lie groups, hence smooth. Its derivative $\pi_* := d\pi(e), \mathfrak{u} \to \mathrm{End}(V)$ is a Lie algebra homomorphism, turning V into a $\mathfrak{u}$-module. By complex linear extension, V becomes a $\mathfrak{u}_\mathbb{C}$-module.

Since U is connected, V is in fact an irreducible $\mathfrak{u}$-module, hence also an irreducible $\mathfrak{u}_\mathbb{C}$ module. The Lie algebra of K equals the fixed point set $\mathfrak{k}$ of the infinitesimal involution σ. Hence, V^K is contained in the space $V^\mathfrak{k}$ which by Lemma 9.11 has dimension at most one. Now apply Proposition 8.4 to conclude that the convolution algebra is commutative. □

10 Symmetrization

In this section we will discuss an important linear map from the symmetric algebra to the universal enveloping algebra of a Lie algebra, called symmetrisation.

To prepare for this we first assume that V is a finite dimensional complex linear space. We will discuss symmetrisation of elements of the associated tensor algebra $T(V)$.

Let n be a positive integer. For σ an element of the permutation group S_n we define $\pi(\sigma) : T^n(V) \to T^n(V)$ to be the linear map given by

$$X_1 \otimes \cdots \otimes X_n \mapsto X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)}.$$

Then π defines a representation of S_n in the linear space $T^n(V)$. The associated subspace of S_n-invariants is called the space of symmetric tensors of order s and is denoted by $T^n_s(V)$. By the theory of isotypical components, this subspace of $T^n(V)$ has a unique S_n-invariant complement, which we denote by $T^n_c(V)$. Then

$$T^n(V) = T^n_s(V) \oplus T^n_c(V) \tag{23}$$

is a direct sum decomposition into S_n-invariant subspaces. The associated projection onto $T^n_s(V)$ is given by

$$P := \frac{1}{n!} \sum_{\sigma \in S_n} \pi(\sigma).$$

The kernel I of the canonical homomorphism $T(V) \to S(V)$ is homogeneous. The intersection $I_n := I \cap T^n(X)$ is equal to the linear span of the tensors of the form

$$\pi((j\, j+1))T, \qquad (1 \leq j < n,\ T \in T^n(V).$$

Here $(j j+1)$ denotes the transposition of the neighboring elements j and $j+1$. Such transpositions of neighboring elements will be called simple permutations. From now on we agree to also use the abbreviation $\sigma T := \pi(\sigma)(T)$, for $T \in T^n(V)$ and $\sigma \in S_n$.

Lemma 10.1. *Let $\sigma \in S_n$ and $T \in T^n(V)$. Then $\sigma T - T \in I_n$. In particular, I_n is S_n-invariant.*

Proof. The permutation σ admits a decomposition $\sigma = s_k \cdots s_1$ into simple permutations. We agree to write $\sigma_r = s_r \cdots s_1$, then

$$\sigma T - T = \sum_{r=1}^{k} s_{r+1}\sigma_{r-1}T - \sigma_{r-1}T \in I_n.$$

This proves the first assertion. The second assertion follows immediately from the first. □

Lemma 10.2. *Let I be the kernel of the canonical homomorphism $p : T(V) \to S(V)$. Then $I_n = I \cap T^n(V)$ equals the kernel of the projection $P : T^n(V) \to T^n_s(V)$. The map p restricts to a linear isomorphism $p_n : T^n_s(V) \to S^n(V)$.*

Proof. Let $\tau \in S_n$ then clearly
$$P \circ \tau = P \quad \text{on } T^n(V).$$
This implies that $P = 0$ on the generators of I_n, so that $I_n \subset \ker P$. On the other hand, if T is in $T^n(V)$ then
$$\sigma T - T \in I_n, \qquad (\sigma \in S_n).$$
Summing over all σ and dividing by $n!$ we find that $PT - T \in I_n$, so that
$$\ker P = \mathrm{im}(P - I) \subset I_n.$$
This establishes the first assertion. From (23) we now find that
$$T^n(X) = T^n_s(V) \oplus I_n. \tag{24}$$
Since I_n equals the kernel of $p|_{T^n(V)}$, the final assertion follows. □

We now assume that $\mathfrak{g}$ is a finite dimensional complex Lie algebra and use the above notation and results with $\mathfrak{g}$ in place of V.

Lemma 10.3. *Let $\pi : T(\mathfrak{g}) \to U(\mathfrak{g})$ denote the canonical projection. Let $T \in T^n(\mathfrak{g})$ and $\sigma \in S_n$. Then*
$$\pi(\sigma T - T) \in U(\mathfrak{g})_{n-1}.$$

Proof. The result is trivial for $n = 0, 1$. We may therefore assume that $n \geq 2$. As in the proof of Lemma 10.1 the element $\sigma T - T$ may be rewritten as a sum of tensors of the form $sS - S$, with $S \in T^n(\mathfrak{g})$ and s a simple permutation. Thus, it suffices to prove the result for $\sigma = (j\, j+1)$ and for T of the form $T = X_1 \otimes \cdots \otimes X_n$. Then
$$T - \sigma T - X_1 \otimes \cdots \otimes [X_j, X_{j+1}] \otimes \cdots X_n \in \ker(\pi)$$
so that
$$\pi(\sigma T - T) = -X_1 \cdots X_{j-1}[X_j, X_{j+1}]X_{j+1} \cdots X_n \in U(\mathfrak{g})_{n-1}.$$
□

Let $j : \mathfrak{g} \to U(\mathfrak{g})$ denote the inclusion map. We recall that $T(\mathfrak{g})$ is graded, and carries the associated filtration given by $T(\mathfrak{g})_n = \oplus_{k \leq n} T^k(\mathfrak{g})$. This gradation induces a gradation on $S(\mathfrak{g})$ and an induced image filtration on $U(\mathfrak{g})$, denoted $U(\mathfrak{g})_n$, $n \geq 0$. Thus, the space $U(\mathfrak{g})_n$ consists of the linear span of all products of at most n elements from $\mathfrak{g}$. We will refer to these gradations and associated filtrations as the standard ones.

Theorem 10.4. *There exists a unique linear map $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ such that*
$$s(X^m) = j(X)^m, \quad \text{for all} \quad X \in \mathfrak{g}.$$
This map is an isomorphism of filtered spaces (relative to the standard filtrations).

Remark. The uniquely defined map $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ is called the symmetrisation map.

We will give the proof of Theorem 10.4 by establishing a number of lemmas. In particular, the uniqueness assertion is an immediate consequence of the following lemma, applied to $\mathfrak{g}$ in place of V.

Lemma 10.5. *Let V be a finite dimensional complex linear space and let $n \geq 1$. Then the elements X^n, for $X \in V$, span $S^n(V)$ as a linear space.*

Proof. Let $X_1, \ldots, X_d$ be a basis of V as a linear space. For $c \in \mathbb{C}^d$ we write $Y_c := (c_1 X_1 + \cdots + c_d X_d)^n$. Then we may write

$$Y_c = \sum_{\alpha} c^{\alpha} X_1^{\alpha_1} \cdots X_d^{\alpha_d},$$

where the sum extends over the multi-indices $\alpha = (\alpha_1, \ldots \alpha_d)$ with $|\alpha| := \sum_j \alpha_j$ equal to n. As the appearing functions $c \mapsto c^{\alpha}$ are linearly independent over $\mathbb{C}$, we see that the Y_c, for $c \in \mathbb{C}^d$, span $S^n(\mathfrak{g})$. $\square$

The proof of the existence part of Theorem 10.4 will be given further on, based on a sequence of lemmas.

Let $n \in \mathbb{N}$. Then the canonical algebra homomorphism $T(\mathfrak{g}) \to U(\mathfrak{g})$ restricts to a linear map $T^n(\mathfrak{g}) \to U(\mathfrak{g})_n$ which by composition with the natural projection $U(\mathfrak{g})_n \to U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1}$ induces a linear map

$$T^n(\mathfrak{g}) \to U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1}. \tag{25}$$

Lemma 10.6. *For each $n \in \mathbb{N}$, the map (25) restricts to a linear isomorphism*

$$\pi_n : T_s^n(\mathfrak{g}) \to U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1}.$$

Proof. Let π_n' denote the map (25). By definition, the map $T(\mathfrak{g})_n \to U(\mathfrak{g})_n$ is surjective for every n. This implies that the map π_n' is surjective. It follows from Lemma 10.3 that $\pi_n'(\sigma T - T) = 0$ for every $T \in T^n(\mathfrak{g})$ and $\sigma \in S_n$. Hence, $\pi_n' = 0$ on I_n by Lemma 10.1, and in view of the decomposition (24) it follows that π_n is surjective.

In particular, $\dim U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1} \leq \dim T_s^n(\mathfrak{g}) = \dim S^n(\mathfrak{g})$. Summing over n we find that $\dim U(\mathfrak{g})_n \geq \dim S(\mathfrak{g})_n$ for all n.

On the other hand, from the PBW theorem it follows that $\dim U_n(\mathfrak{g}) \geq \dim S(\mathfrak{g})_n$ for all $n \geq 0$, so that $U(\mathfrak{g})_n$ and $S(\mathfrak{g})_n$ have the same dimension for all n. This in turn implies that domain and codomain of π_n have the same finite dimension. As π_n is linear and surjective, it must be injective as well. $\square$

Completion of the proof of Theorem 10.4. It remains to establish existence of the map $s : T(\mathfrak{g}) \to U(\mathfrak{g})$.

For $n \in \mathbb{N}$, we define $s_n : S^n(\mathfrak{g}) \to U(\mathfrak{g})$ by $s_n = \pi \circ p_n^{-1}$. Furthermore, we define the linear map $s : S^n(\mathfrak{g}) \to U(\mathfrak{g})$ by $s = s_n$ on $S^n(\mathfrak{g})$. If $X \in \mathfrak{g}$, then the element $T := X \otimes \cdots \otimes X$ (n factors) belongs to $T_s^n(\mathfrak{g})$ and has image X^n in $S^n(\mathfrak{g})$. Furthermore, the element T has image $j(X)^n$ in $U(\mathfrak{g})$. It follows that $s(X^n) = j(X)^n$. Thus, s satisfies the requirements.

It is clear that s preserves the filtrations. For s to be an isomorphism of filtered spaces, it suffices that $\mathrm{gr}\, s : S(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{n})$ be a linear isomorphism. Let $n \in \mathbb{N}$, then $(\mathrm{gr}\, s)_n = \pi_n \circ p_n^{-1}$, which is a linear isomorphism from $S^n(\mathfrak{g})$ onto $U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1} = (\mathrm{gr}\, U(\mathfrak{g}))^n$. $\square$

Corollary 10.7. *The symmetrisation map $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ preserves the standard filtrations and induces an isomorphism of graded spaces $S(\mathfrak{g}) \simeq \mathrm{gr}\, U(\mathfrak{g})$.*

Lemma 10.8. *Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras. Then the diagram*

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{S(\varphi)} & S(\mathfrak{g}) \\ s \downarrow & & \downarrow s \\ U(\mathfrak{h}) & \xrightarrow{U(\varphi)} & U(\mathfrak{h}) \end{array}$$

commutes.

Proof. Exercise for the reader. $\square$

Of particular interest is the situation that φ is an automorphism.

For instance, let L be a real Lie group with Lie algebra $\mathfrak{l}$. If $x \in L$, then the complexification of $\mathrm{Ad}(x)$ defines an automorphism of $\mathfrak{l}_{\mathbb{C}}$, which induces automorphisms of $S(\mathfrak{l}_{\mathbb{C}})$ and $U(\mathfrak{l}_{\mathbb{C}})$ These induced automorphisms will be denoted by $\mathrm{Ad}(x)$ as well. It follows from the above lemma that the following diagram commutes:

$$\begin{array}{ccc} S(\mathfrak{l}_{\mathbb{C}}) & \xrightarrow{\mathrm{Ad}(x)} & S(\mathfrak{l}_{\mathbb{C}}) \\ s \downarrow & & \downarrow s \\ U(\mathfrak{l}_{\mathbb{C}}) & \xrightarrow{\mathrm{Ad}(x)} & U(\mathfrak{l}_{\mathbb{C}}) \end{array} \tag{26}$$

In the above setting, let $X \in \mathfrak{l}$ and write $\mathrm{ad}(X)$ for the induced derivations of $S(\mathfrak{l}_{\mathbb{C}})$ and $U(\mathfrak{l}_{\mathbb{C}})$. Then from the above diagram with $x = \mathrm{Ad}(\exp tX)$ we find, by differentiating with respect to t at $t = 0$, that the following diagram commutes:

$$\begin{array}{ccc} S(\mathfrak{l}_{\mathbb{C}}) & \xrightarrow{\mathrm{ad}(X)} & S(\mathfrak{l}_{\mathbb{C}}) \\ s \downarrow & & \downarrow s \\ U(\mathfrak{l}_{\mathbb{C}}) & \xrightarrow{\mathrm{ad}(X)} & U(\mathfrak{l}_{\mathbb{C}}) \end{array} \tag{27}$$

More generally, such a commutative diagram is induced by any derivation δ of $\mathfrak{l}$. See exercises.

11 Invariant differential operators

In this section we will assume that G is a real Lie group, with Lie algebra $\mathfrak{g}$. The theory of the previous section will be used for the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

By a linear partial differential operator of order at most $k \in \mathbb{N}$ on a smooth manifold M we mean a linear operator $P : C^\infty(M) \to C^\infty(M)$ such that in local coordinates $x^1 \ldots, x^m$, the operator P takes the form

$$P = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| \leq k}} c_\alpha \partial^\alpha,$$

with c_α smooth functions. Here we have used the multi-index notation

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_m^{\alpha_m}}.$$

More precisely, for (U, κ) a chart of M, the pull-back map $f \mapsto \kappa^* f$ yields a (continuous) linear isomorphism from $C^\infty(\kappa(U))$ onto $C^\infty(U)$, which maps $C_c^\infty(\kappa(U))$ onto $C_c^\infty(U)$. By the above characterization of P we mean that for every coordinate chart (U, κ) of M, the operator P maps $C_c^\infty(U)$ into itself, and that $\kappa_*(P) := \kappa^{-1*} \circ P \circ \kappa^*$ takes the form

$$\kappa_*(P) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| \leq k}} c_\alpha \partial^\alpha,$$

on $C_c^\infty(\kappa(U))$, with smooth coefficients $c_\alpha \in C^\infty(\kappa(U))$.

We denote the complex vector space of such differential operators of order at most k on M by $\mathscr{D}(M)_k$. The union of these spaces of operators in $\mathrm{End}(C^\infty(M))$ is denoted by $\mathscr{D}(M)$. It is readily seen that with the indicated filtration by order, $\mathscr{D}(M)$ is a filtered algebra.

Given a vector field v on M, we define the first order differential operator ∂_v by

$$\partial_v(f)(x) = df(x)(v_x),$$

for $f \in C^\infty(M)$, $x \in M$.

Let now M be equipped with a smooth left G-action $l : G \times M \to M$, $(g, m) \mapsto l_g(m) = gm$. A differential operator $P \in \mathscr{D}(M)$ is said to be G-invariant for this left action if and only if P commutes with the pull-back $l_x^* : C^\infty(M) \to C^\infty(M)$ for every $x \in G$. An equivalent way of saying this is that P is equivariant for the representation L of G in $C^\infty(M)$ defined by

$$L_x f = l_{x^{-1}}^* : m \mapsto f(x^{-1} m)$$

for $f \in C^\infty(M)$ and $x \in G$.

Yet another way of saying this is that P is G-fixed for the representation of G in $\mathscr{D}(M)$ defined by

$$(x, P) \mapsto l_{x*}(P) := l_x^{-1*} \circ P \circ l_x^*.$$

We will now consider the situation with $M = G$, the action given by the usual left multiplication. Let $\mathfrak{g}$ be the Lie algebra of G. Then for a given $X \in \mathfrak{g}$ we define the vector field v_X on G by

$$v_X(x) = \frac{d}{dt}\bigg|_{t=0} x \exp tX$$

Then $v_X(e) = X$ and by an easy application of the chain rule it follows that

$$v_X(yx) = \frac{d}{dt}\bigg|_{t=0} l_y(x \exp tX) = dl_y(x)v_X(x),$$

for all $x, y \in G$. In other words, the vector field v_X is left invariant. The associated first order differential operator is denoted by ∂_{v_X}. Let $\mathbb{D}(G)$ denote the space of left invariant differential operators in $\mathscr{D}(G)$. It is readily verified that $\mathbb{D}(G)$ is a subalgebra of $\mathscr{D}(G)$. We equip it with the induced filtration by order. Then $\mathbb{D}(G)$ becomes a filtered algebra.

Lemma 11.1. *Let $X \in \mathfrak{g}$. Then the operator ∂_{v_X} belongs to $\mathbb{D}(G)_1$.*

Proof. It suffices to show that $\partial := \partial_{v_X}$ is left invariant. Let $f \in C^\infty(G)$, and $x, y \in G$. Then

$$\partial f(yx) = df(yx)v_X(yx) = df(yx) \circ dl_y(x)v_X(x) = d(f \circ l_y)(x)v_X(x) = \partial(l_y^* f)(x)$$

and the left invariance follows. □

For the introduction of the above left invariant first order differential operator we can also follow the following more direct representation theoretic approach.

For this we recall the following formula, in the settingof a continuous finite dimensional representation (π, V) of G. In this setting the map $\pi : G \to \mathrm{GL}(V)$ is a continuous homomorphism of Lie groups, hence smooth. The derived map $\pi_* = d\pi(e)$ is a Lie algebra homomorphism

$$\pi_* : \mathfrak{g} \to \mathrm{End}(V),$$

where the associative algebra $\mathrm{End}(V)$ is equipped with the commutator bracket. We also say that π_* is the Lie algebra representation associated with π. By using the chain rule one readily verifies that

$$\pi_*(X) = \frac{\partial}{\partial t}\bigg|_{t=0} \pi(\exp tX), \qquad (X \in \mathfrak{g}).$$

From now on we will write π for π_* unless confusion arises.

We now consider the right regular representation of G on $C^\infty(G)$ given by

$$R_x f(y) = f(yx).$$

In line with the above formula for $\pi_*(X)$, we define

$$R_X = \frac{\partial}{\partial t}\bigg|_{t=0} R_{\exp tX}, \qquad (X \in \mathfrak{g}),$$

which can be interpreted pointwise when applied to a function $f \in C^\infty(G)$, i.e.

$$R_X f(y) = \frac{\partial}{\partial t}\bigg|_{t=0} f(y \exp tX), \qquad (y \in G).$$

By an easy application of the chain rule we see that

$$R_X f(y) = df(y)[v_X(y)] = \partial_{v_x} f(y).$$

From now on we shall use the notation R_X for the differential operator ∂_{v_X}. In analogy with the above case of a finite dimensional representation, we now have the following result.

Lemma 11.2. *The map $X \mapsto R_X$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathscr{D}(X)$ equipped with the commutator bracket.*

Proof. First, let $x \in G$ and $Y \in \mathfrak{g}$. Then it is easily verified that

$$R_x R_{\exp tY} = R_{x \exp tY} = R_{\exp t \mathrm{Ad}(x) Y} R_x.$$

Applying this to functions from $C^\infty(G)$ and differentiating pointwise at $t = 0$ we find

$$R_x R_Y = R_{\mathrm{Ad}(x)Y} R_x.$$

Substituting $x = \exp sX$ and differentiating at $s = 0$ (again interpreted pointwise when applied to functions) we find that

$$R_X R_Y = \frac{\partial}{\partial s}\bigg|_{t=0} R_{\mathrm{Ad}(\exp sX)Y} + \frac{\partial}{\partial t}\bigg|_{s=0} R_Y R_{\exp sX} = R_{[X,Y]} + R_Y R_X.$$

This result follows. $\square$

Remark. We will later review the above calculation in the more general context of a continuous representation in a Fréchet space, involving the notion of a smooth vector.

Corollary 11.3. *The map $X \mapsto R_X$ from $\mathfrak{g}$ to $\mathbb{D}(X)$ has a unique extension to a homomorphism of associative algebras $U(\mathfrak{g}_\mathbb{C}) \to \mathbb{D}(X)$. This homomorphism preserves the filtrations.*

Proof. The map $X \mapsto R_X$ has a unique extension to a complex linear map $\mathfrak{g}_\mathbb{C} \to \mathbb{D}(G)$ which is also a Lie algebra homomorphism.

The existence and uniqueness of the extension to $U(\mathfrak{g}_\mathbb{C})$ now follows from the universal property of $U(\mathfrak{g}_\mathbb{C})$. The preservation of the filtrations follows from the fact that $\mathbb{D}(G)$ is a filtered algebra, and that $X \mapsto R_X$ maps $\mathfrak{g}$ into $\mathbb{D}(G)_1$. $\square$

We now have following result.

Theorem 11.4. *The extended map*

$$u \mapsto R_u, \;\; U(\mathfrak{g}) \to \mathbb{D}(G).$$

is an isomorphism of filtered algebras.

The proof will make use of the symmetrizer $s : S(\mathfrak{g}_\mathbb{C}) \to U(\mathfrak{g}_\mathbb{C})$, and the interpretation of $S(\mathfrak{g}_\mathbb{C})$ as constant coefficient partial differential operators on $\mathfrak{g}$ which we will first explain.

Let V be a finite dimensional real vector space. Then by $\mathbb{D}(V)$ we denote the algebra of translation invariant differential operators on V. Choosing a basis $e_1, \ldots, e_d$ of V and using the associated coordinate functions $x^j : V \to \mathbb{R}$ the partial differential operators on V of order at most k have a unique form

$$P = \sum_{|\alpha| \le k} c_\alpha \partial^\alpha,$$

with $c_\alpha \in C^\infty(V)$. It is readily seen that such an operator P is translation invariant if and only if each coefficient c_α is a constant function. We thus see that the algebra $\mathbb{D}(V)$ is a commutative associative algebra with unit.

Given $X \in V$ we define the first order differential operator ∂_X by

$$\partial_X f(x) = df(x)X$$

for $f \in C^\infty(V)$ and $x \in V$. We note that $\partial_{e_j} = \partial_j$ with respect to the given basis and coordinate functions. By the universal property of $S(V_\mathbb{C})$, the map $X \mapsto \partial_X$ has a unique extension to a homomorphism of algebras, denoted $u \mapsto \partial_u$. By using that the operators ∂^α form a basis for $\mathbb{D}(V)$, we obtain the following result.

Lemma 11.5. *The extended homomorphism $u \mapsto \partial_u$ is an isomorphism of filtered algebras*

$$S(V_\mathbb{C}) \xrightarrow{\simeq} \mathbb{D}(V).$$

We return to the setting of the Lie group G with Lie algebra $\mathfrak{g}$. If $f \in C^\infty(G)$ then $\exp^* f = f \circ \exp$ is a smooth function on $\mathfrak{g}$.

Theorem 11.6. *Let $u \in S(\mathfrak{g}_\mathbb{C})$. Then for all $f \in C^\infty(G)$ we have*

$$R_{s(u)} f(e) = \partial_u(\exp^* f)(0).$$

Proof. The expressions at both left and right hand side of the above equality depend linearly on $u \in S(\mathfrak{g}_\mathbb{C})$. The elements X^n, for $X \in \mathfrak{g}$ span the linear space $S(\mathfrak{g}_\mathbb{C})$, see the uniqueness part of the proof of Theorem 10.4. Therefore, it suffices to establish the identity for $u = X^n$. In this case, we have $s(u) = j(X)^n$ so that

$$\begin{aligned} R_{s(u)} f(e) &= \frac{\partial^n}{\partial t_1 \cdots \partial t_n} f(\exp t_1 X \cdots \exp t_n X)\bigg|_{t_j=0} \\ &= \frac{\partial^n}{\partial t_1 \cdots \partial t_n} (\exp^* f)[(t_1 + \cdots + t_n)X])\bigg|_{t_j=0}. \end{aligned}$$

This implies that

$$R_{s(u)} f(e) = (\partial_X)^n (\exp^* f)(0) = \partial_u(\exp^* f)(0). \tag{28}$$

$\square$

Proof of Theorem 11.4. There exists an open neighborhood Ω of 0 in $\mathfrak{g}$ which exp maps diffeomorphically onto an open neighborhood U of e in G. Let $\kappa : U \to \Omega$ be the inverse to this diffeomorphism.

For a differential operator $D \in \mathscr{D}(G)$ the push-forward $\kappa_*(D)$ is a differential operator on Ω. There exists a unique element $u = \lambda(D) \in S(\mathfrak{g})$ such that for all $\varphi \in C_c^\infty(\Omega)$,

$$\kappa_*(D)\varphi(0) = \partial_u \varphi(0). \tag{29}$$

This element $u = \lambda(D)$ is called the local expression of D at e. Substituting $\exp^* f$ for φ, we obtain

$$Df(e) = \partial_{\lambda(D)}(\exp^* f)(0), \qquad (f \in C^\infty(G)). \tag{30}$$

If $D \in \mathbb{D}(G)$, then for every $f \in C^\infty(G)$ and $x \in G$ we have

$$Df(x) = l_x^* Df(e) = D(l_x^* f) = \partial_{\lambda(D)}(\exp^*(l_x^* f))(e),$$

by left invariance of D. It follows that the map $D \mapsto \lambda(D)$ is injective $\mathbb{D}(G) \to S(\mathfrak{g})$. On the other hand, if $u \in S(\mathfrak{g})$ then clearly the operator $D_u \in \mathscr{D}(G)$ defined by

$$D_u f(x) = \partial_u[f(x\exp(\,\cdot\,))]$$

is left invariant, and $\lambda(D_u) = u$. It follows that $D \mapsto \lambda(D)$ is a linear isomorphism from $\mathbb{D}(G)$ onto $S(\mathfrak{g}_{\mathbb{C}})$.

From formulas (28) and (30) it follows that

$$\lambda(R_{s(u)}) = u \tag{31}$$

for all $u \in S(\mathfrak{g}_{\mathbb{C}})$. Since $\lambda : \mathbb{D}(G) \to S(\mathfrak{g}_{\mathbb{C}})$ is a filtered linear isomorphism, it follows that $u \mapsto R_{s(u)}$ is a filtered linear isomorphism $S(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}(G)$. Since $s : S(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})$ is a filtered linear isomorphism, it follows that the algebra homomorphism $U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}(G)$ is a filtered linear isomorphism, hence an isomorphism of filtered algebras. $\square$

Remark. It is good to notice that the map $D \mapsto \lambda(D)$ of the above proof defines a linear isomorphism $\mathbb{D}(G) \to S(\mathfrak{g}_{\mathbb{C}})$, but is not an algebra homomorphism in general.

12 Invariant differential operators on a homogeneous space

In this section we assume that G is a real Lie group, and H a closed subgroup.

We assume that there exists an $\mathrm{Ad}(H)$ invariant linear subspace $\mathfrak{q} \subset \mathfrak{g}$ which is complementary to the Lie algebra $\mathfrak{h}$ of H, i.e.,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

as a real linear space. Note that the above decomposition is $\mathrm{Ad}(H)$-invariant.

Exercise 12.1. Show that such a subspace $\mathfrak{q}$ exists if H is compact.

Let $\pi : G \to G/H$ denote the canonical projection and put $[e] = \pi(e)$. Then the derivative $\pi_* = d\pi(e) : \mathfrak{g} \to T_{[e]}(G/H)$ is surjective with kernel $\mathfrak{h}$ hence induces a linear isomorphism $\mathfrak{g}/\mathfrak{h} \to T_{[e]}(G/H)$ through which we shall identify these spaces.

The inclusion map $\mathfrak{q} \to \mathfrak{g}$ induces a linear isomorphism $\iota : \mathfrak{q} \to \mathfrak{g}/\mathfrak{h}$. Let $\mathrm{Exp} : \mathfrak{q} \to G/H$ be the map $X \mapsto \pi \circ \exp(X)$. Then via the mentioned identification, the tangent map of Exp at 0 equals $\iota : \mathfrak{q} \to \mathfrak{g}/\mathfrak{h}$. From now on we shall use ι to identify $\mathfrak{q}$ with $\mathfrak{g}/\mathfrak{h}$. Then $T_0\mathrm{Exp}$ corresponds to the identity map $\mathfrak{q} \to \mathfrak{q}$.

Lemma 12.2. *Let $h \in H$ and let $l_h : G/H \to G/H$ denote left multiplication by h. Then the tangent map $T_{[e]}l_h : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ equals the map induced by* $\mathrm{Ad}(h) : \mathfrak{g} \to \mathfrak{g}$.

Exercise 12.3. Derive this by differentiating the maps in the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathscr{C}_h} & G \\ \pi \downarrow & & \downarrow \pi \\ G/H & \xrightarrow{l_h} & G/H. \end{array}$$

Here $\mathscr{C}_h : x \mapsto hxh^{-1}$.

It follows that the identifications $\mathfrak{q} \simeq \mathfrak{g}/\mathfrak{h} \simeq T_{[e]}(G/H)$ are equivariant for the given actions by H.

We denote by $\mathbb{D}(G/H)$ the algebra of left G-invariant differential operators on G/H. The associated filtration by order is denoted by $\mathbb{D}(G/H)_k := \mathbb{D}(G/H) \cap \mathscr{D}(G/H)_k$. We will proceed in analogy with the definition of local expression in the previous section.

The following lemma serves as a preparation. Let V be a finite dimensional real linear space. If $\varphi : V \to V$is an invertible linear map, we define $\varphi^* : C^\infty(V) \to C^\infty(V)$ by $\varphi^* f = f \circ \varphi$. Furthermore, we define $\varphi_* : \mathscr{D}(V) \to \mathscr{D}(V)$ by $\varphi_*(P) = \varphi^{*-1} \circ P \circ \varphi^*$. Then clearly, φ^* preserves the filtration by order. On the other hand, φ induces a linear automorphism $S(\varphi)$ of $S(V_\mathbb{C})$, for which we shall use the abbreviated notation φ.

Lemma 12.4. *The map φ_* is an isomorphism of filtered algebras, which preservers the subalgebra $\mathbb{D}(V)$ of translation invariant differential operators. Furthermore, if $u \in S(\mathfrak{g})$ then*

$$\partial_{\varphi(u)} = \varphi_*(\partial_u).$$

Proof. It is readily seen φ_* preservers the filtration and has inverse equal to $(\varphi^{-1})_*$. This implies that φ_* is an isomorphism of filtered algebras.

Let $X \in \mathfrak{g}$. Then for $f \in C^\infty(V)$ and $v \in V$ we have

$$\begin{aligned}[][\varphi_* \partial_X f](v) &= \partial_X(f \circ \varphi)(\varphi^{-1}(v)) = d(f \circ \varphi)(\varphi^{-1}v)X \\ &= df(v)d\varphi(\varphi^{-1}(v))X = df(v)\varphi(X) \\ &= \partial_{\varphi(X)} f(v).\end{aligned}$$

Since $u \mapsto \partial_u$ is an algebra isomorphism from $S(V_\mathbb{C})$ onto $\mathbb{D}(V)$, whereas the elements of V generate the algebra $S(V)$, the equation follows. In particular this implies that φ_* preserves $\mathbb{D}(V)$. $\square$

Lemma 12.5. *Let $D \in \mathbb{D}(G/H)_k$. Then there exists a unique element $u \in S(\mathfrak{q})_k^H$ such that for all $f \in C^\infty(G/H)$ we have*

$$D(f)([e]) = \partial_u(\mathrm{Exp}^* f)(0) \tag{32}$$

Proof. Since $\mathrm{Exp} : \mathfrak{q} \to G/H$ is a local diffeomorphism at 0, there exists an open neighborhood Ω of 0 such that $\mathrm{Exp}|_\Omega$ is a diffeomorphism from Ω to an open neighbourhood U of $[e]$ in G/H. Let $\kappa : U \to \Omega$ be the inverse to this diffeomorphism, then $\kappa_*(D)$ is a differential operator on Ω. There exists a unique $u \in S(\mathfrak{q})$ such that

$$\kappa_*(D)(g)(0) = \partial_u g(0), \qquad (g \in C^\infty(\Omega)).$$

It follows that

$$D(f)([e]) = \kappa_*(D)[\mathrm{Exp}^*(f)](0) = \partial_u(\mathrm{Exp}^* f)(0).$$

Clearly, $u \in S(\mathfrak{q})$ is uniquely determined by this property. We now observe that D is left invariant under G hence under H, so that for all $h \in H$ we have

$$\begin{aligned}\partial_u(\mathrm{Exp}^* f)(0) &= Df([e]) = Df(l_h[e]) \\ &= l_h^* Df([e]) = D \circ (l_h^* f)([e] \\ &= \partial_u(\mathrm{Exp}^* l_h^* f)(0) = \partial_u(\mathrm{Ad}(h)^* \mathrm{Exp}^* f)(0) \\ &= (\mathrm{Ad}(h)_* \partial_u)(\mathrm{Exp}^* f)(0) = \partial_{\mathrm{Ad}(h)u}(\mathrm{Exp}^* f)(0).\end{aligned}$$

By the uniqueness assertion above, it follows that $\mathrm{Ad}(h)u = u$ for all $h \in u$, hence $u \in S(\mathfrak{q})^H$. $\square$

The element u specified in Lemma 12.5 will be denoted by $\lambda(D)$. Thus,

$$\lambda : D \mapsto \lambda(D), \quad \mathbb{D}(G/H) \to S(\mathfrak{q})^H \tag{33}$$

is a linear map of filtered spaces.

Lemma 12.6. *The map (33) is injective.*

Proof. Let $D \in \ker \lambda$. Then for $f \in C^\infty(G/H)$ and $x \in G$ we have

$$Df([x]) = Df(x[e]) = l_x^*(Df)([e]) = D(l_x^* f)([e]) = \partial_{\lambda(D)}(\mathrm{Exp}^* f)(0) = 0.$$

Hence $D = 0$. $\square$

Conversely, let $u \in S(\mathfrak{q}_{\mathbb{C}})^H$. Then we define the operator $D_u : C^\infty(G/H) \to C^\infty(G)$ by

$$D_u f(x) = \partial_u[(l_x^* f) \circ \mathrm{Exp}](0), \tag{34}$$

for $f \in C^\infty(G/H)$. From the definition we see that the operator is left equivariant. Furthermore, for all $x \in G$ and $h \in H$,

$$D_u f(xh) = \partial_u[(l_x^* f) \circ l_h \circ \mathrm{Exp}](0) = \partial_u[[(l_x^* f) \circ \mathrm{Exp}] \circ \mathrm{Ad}(h)](0) = D_u f(x)$$

by H-invariance of u, so that D_u maps into $C^\infty(G/H)$. Finally, $\mathrm{Exp} : \mathfrak{q} \to G/H$ is a local diffeomorphism at 0. There exists a smooth map $F : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$, locally defined at $(0,0)$ such that $F(0,0) = 0$ and

$$\exp(X)\mathrm{Exp}(Y) = \mathrm{Exp}\, F(X,Y)$$

for X and Y sufficiently close to 0. Thus, for X in a suitable neighborhood of 0 in $\mathfrak{q}$, we have

$$D_u f(\mathrm{Exp}X) = \partial_u[(l_{\exp X}^* f)\exp](0) = \partial_u[f \circ \mathrm{Exp}(X, \cdot)](0)$$

from which we see that D_u defines a linear partial differential operator on an open neighborhood of $[e]$ in G/H. By left equivariance it follows that D_u is a globally defined operator on G/H and that $D_u \in \mathbb{D}(G/H)$. Clearly, $D_u \in \mathbb{D}(G/H)_k$ if $u \in S^k(\mathfrak{q}_{\mathbb{C}})^H$.

Corollary 12.7. *The map $u \mapsto D_u$ defines an isomorphism $S(\mathfrak{q}_{\mathbb{C}})^H \to \mathbb{D}(G/H)$ of filtered linear spaces. Its inverse is the map λ given by (33).*

Proof. Since λ is an injective homomorphism of filtered linear spaces, it suffices to show that $\lambda(D_u) = u$ for all $u \in S(\mathfrak{q}_{\mathbb{C}})^H$. This is seen from (34) with $x = e$, combined with (32). □

In the following lemma, s denotes the symmetrizer map $S(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})$.

Lemma 12.8. *The linear map $\varphi : S(\mathfrak{q}_{\mathbb{C}}) \otimes S(\mathfrak{h}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})$, given by $u \otimes v \mapsto s(u)s(v)$ is a linear isomorphism.*

Proof. Let $S(\mathfrak{q}_{\mathbb{C}}) \otimes S(\mathfrak{h}_{\mathbb{C}}$ be equipped with the tensor product gradation. Then the multiplication map $m : S(\mathfrak{q}_i C) \otimes S(\mathfrak{h}_{\mathbb{C}}) \to S(\mathfrak{g}_{\mathbb{C}})$ is a graded isomorphism. It is an easy matter to verify that $m = \mathrm{gr}(\varphi)$. □

Corollary 12.9. *The map $S(\mathfrak{q}_{\mathbb{C}}) \otimes U(\mathfrak{h}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})$ given by $(u,v) \mapsto s(u)v$ is a linear isomorphism. It maps $S(\mathfrak{q}_{\mathbb{C}}) \otimes U(\mathfrak{h}_{\mathbb{C}})$ onto $U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}$.*

Proof. The first assertion follows from the previous lemma, since s maps $S(\mathfrak{h}_{\mathbb{C}})$ linearly isomorphically onto $U(\mathfrak{h}_{\mathbb{C}})$. The second assertion is an immediate consequence of the first one. □

Corollary 12.10. *The symmetrizer $s : S(\mathfrak{q}) \to U(\mathfrak{g})$ induces a linear isomorphism*

$$\bar{s} : \; S(\mathfrak{q}_{\mathbb{C}})^H \;\xrightarrow{\simeq}\; U(\mathfrak{g}_{\mathbb{C}})^H / U(\mathfrak{g}_{\mathbb{C}})^H \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}. \tag{35}$$

Proof. Let ψ denote the linear isomorphism of the previous corollary. We observe that by the PBW theorem

$$U(\mathfrak{h}_{\mathbb{C}}) \simeq \mathbb{C} \oplus U(\mathfrak{h}_{\mathbb{C}})\mathfrak{h}_{\mathbb{C}}.$$

This decomposition is stable under the adjoint action by H so that

$$[S(\mathfrak{q}_{\mathbb{C}}) \otimes U(\mathfrak{h})]^H = [S(\mathfrak{q}_{\mathbb{C}})^H \otimes \mathbb{C}] \oplus [S(\mathfrak{q}_{\mathbb{C}}) \otimes U(\mathfrak{h})\mathfrak{h}]^H$$

Since ψ is a linear isomorphism, with intertwines the adjoint H-actions, it maps the above sum isomorphically onto $U(\mathfrak{g})^H$ and the second summand onto $U(\mathfrak{g}_{\mathbb{C}})^H \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}$. Thus ψ induces a linear isomorphism from $S(\mathfrak{q}_{\mathbb{C}})^H \otimes \mathbb{C}$ onto the space $U(\mathfrak{g}_{\mathbb{C}})^H / U(\mathfrak{g}_{\mathbb{C}})^H \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}$ and the result follows. □

Clearly, $U(\mathfrak{g})^H$ is a subalgebra of $U(\mathfrak{g})$, with unit.

Lemma 12.11. *The space $U(\mathfrak{g}_{\mathbb{C}})^H \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}$ is a two-sided ideal in $U(\mathfrak{g}_{\mathbb{C}})^H$.*

Proof. This follows from the fact that $\mathfrak{h}$ and $U(\mathfrak{g}_{\mathbb{C}})^H$ are commuting subspaces of $U(\mathfrak{g}_{\mathbb{C}})$. □

In particular, it follows that the quotient on the right-hand side of (35) is an associative algebra with unit. We will show that in fact it is isomorphic to the algebra $\mathbb{D}(G/H)$.

Lemma 12.12. *Let $Z \in U(\mathfrak{g})^H_k$. Then there exists a unique partial differential operator $r_Z \in \mathscr{D}(G/H)_k$ such that $\pi^* \circ r_Z = R_Z \circ \pi^*$ on $C^\infty(G/H)$. The map $Z \mapsto r_Z$ defines an algebra homomorphism from $U(\mathfrak{g})^H$ into $\mathbb{D}(G/H)$.*

Proof. Uniqueness is obvious. We will establish existence. Let $Z \in U(\mathfrak{g}_{\mathbb{C}})^H_k$. Then there exists an element $u \in S(\mathfrak{q}_{\mathbb{C}})^H_k$ such that $s(u) - Z \in U(\mathfrak{g})\mathfrak{h}$. It follows that $R_s(u) = R_Z$ on the space $\pi^* C^\infty(G/H)$ of right H-invariant functions in $C^\infty(G)$. Let $f \in C^\infty(G/H)$, then we find that

$$R_Z \pi^*(f)(e) = R_{s(u)}\pi^*(f)(e) = \partial_u[\exp^* \pi^*](f)(0) = D_u f([e]).$$

By left-invariance it now follows that $R_Z \pi^* f(x) = D_u f(\pi(x))$ for all $x \in G$. This establishes the result with $r_Z = D_u$. By the uniqueness of r_Z the final assertion follows from the fact that $Z \mapsto R_Z$ is an algebra homomorphism $U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}(G)$. □

It is immediate from the characterisation in Lemma 12.12 that the algebra homomorphism $Z \mapsto r_Z$ factors through an algebra homomorphism

$$r : U(\mathfrak{g}_{\mathbb{C}})^H / U(\mathfrak{g}_{\mathbb{C}})^H \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h} \to \mathbb{D}(G/H). \tag{36}$$

Theorem 12.13. *The map (36) is an isomorphism of algebras. Furthermore,*

$$r_{s(u)} = D_u \tag{37}$$

for all $u \in S(\mathfrak{q}_{\mathbb{C}})^H$.

Proof. The equation follows from the proof of the previous lemma. Let $\mathbb{D}$ denote the algebra on the left-hand side of (36). Then $\bar{s} : S(\mathfrak{q}_{\mathbb{C}})^H \to \mathbb{D}$ is a linear isomorphism by Corollary 12.10 and $\mathrm{D} : u \mapsto D_u$ is linear isomorphism $S(\mathfrak{q}_{\mathbb{C}})^H \to \mathbb{D}(G/H)$ by Corollary 12.7. From the equation it now follows that the algebra homomorphism r is bijective, hence an isomorphism of algebras. $\square$

We now specialize further to the case of Riemannian homogeneous space G/H on which G acts by isometries. Here $\mathfrak{q} \simeq T_{[e]}(G/H)$ is equipped with an $\mathrm{Ad}(H)$-invariant positive definite inner product β. We leave it to the reader to check that for $x \in G$ the metric $g_x := dl_x([e])^{-1*}\beta$ on $T_{[x]}(G/H)$ depends on x through its class $[x]$ in G/H. Accordingly, we denote this metric by $g_{[x]}$. It is clear that $[x] \mapsto g_{[x]}$ defines a G-invariant smooth Riemannian metric on G/H.

Lemma 12.14. *Let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathfrak{q}$ relative to the inner product β. Then the Laplace operator on G/H is given by*

$$\Delta = r(X_1^2 + \cdots + X_n^2).$$

Proof. This is a somewhat elaborate exercise for the reader, see the exercise collection. $\square$

We finally consider the setting of a compact symmetric space K/H. Here K is a compact Lie group and H is an open subgroup of the fixed point group K^σ of an involution σ of K. Let $\mathfrak{q}$ be the minus one eigenspace of the infinitesimal involution σ of $\mathfrak{g}$. Then the decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{q}$ is $\mathrm{Ad}(H)$-invariant so that the theory of this section applies.

We fix a positive definite inner product β on $\mathfrak{q}$ which is $\mathrm{Ad}(H)$-invariant and extend it to a Riemannian metric on K/H. Then K/H with this metric is a compact symmetric space. We consider the linear isomorphism $u \mapsto D_u$ from $S(\mathfrak{q}_{\mathbb{C}})^H$ onto $\mathbb{D}(K/H)$, see Corollary 12.7.

Lemma 12.15. *Let K/H be a compact symmetric space as just specified.*

(a) *For each $\delta \in \widehat{K}$ such that $V_\delta^H \neq 0$ there exists a character $\xi_\delta : \mathbb{D}(G/H) \to \mathbb{C}$ such that $D = \xi_\delta(D)$ on $\mathscr{R}(K/H)[\delta]$.*

(b) *The algebra $\mathbb{D}(K/H)$ is commutative.*

Proof. Let δ be as in (a). Then $(V_\delta^*)^H$ is one dimensional, and is preserved by $U(\mathfrak{g})^H$. It follows that there exists a unique algebra homomorphism $\xi_\delta : U(\mathfrak{g})^H \to \mathbb{C}$ such that

$$u = \xi_\delta(u) I \quad \text{on} \quad (V_\delta^*)^H.$$

Let $f \in \mathscr{R}(K/H)[\delta]$ then it follows that there exist elements $v \in V_\delta$ and $\eta \in (V_\delta^*)^H$ such that

$$f(x) = T_\delta(v \otimes \eta)(x) = (\delta^\vee(x)\eta)(v), \qquad (x \in G).$$

We now note that $\eta \mapsto T_\delta(v \otimes \eta)$ intertwines $\delta^\vee$ with R so that $R_y f = T_\delta(v \otimes y\eta)$. Differentiating with respect to y at $y = e$ we find that $R_Y f = T_\delta(v \otimes Y\eta)$, for $Y \in \mathfrak{g}$. Replying this principle repeatedly, we see that this holds for $Y \in U(\mathfrak{k}_{\mathbb{C}})$, hence for $Y \in U(\mathfrak{k}_{\mathbb{C}})^H$. This implies that

$$R_Y f = \xi_\delta(Y) T_\delta(v \otimes \eta) = \xi_\delta(Y) f.$$

Clearly, the character ξ_λ factors through a character ξ_λ of $\mathbb{D}$ and we find that $r_Z = \xi_\lambda(Z)I$ on $\mathscr{R}(K/H)[\delta]$ for all $Z \in \mathbb{D}$. Put $\xi_\lambda = \xi_\lambda \circ r$. Using that r is an isomorphism $\mathbb{D} \to \mathbb{D}(G/H)$, we obtain assertion (a).

Since $\mathscr{R}(K/H)$ is the direct sum of the spaces $\mathscr{R}(K/H)[\delta]$ on which $\mathbb{D}(G/H)$ acts by a character, it follows that $PQ = QP$ on $\mathscr{R}(K/H)$ for all $P,Q \in \mathbb{D}(/H)$. This identity extends to $P,Q \in C^\infty(K/H)$ by density of $\mathscr{R}(K/H)$ in the latter space. [2] We thus obtain assertion (b). $\square$

Remark 12.16. In particular, it follows that all invariant differential operators on K/H commute with the Laplace operator Δ.

13 The Harish-Chandra isomorphism

If $\mathfrak{g}$ is a complex Lie algebra, we denote by $Z(\mathfrak{g})$ the center of the universal enveloping algebra $U(\mathfrak{g})$. We note that

$$Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}} := \{Z \in U(\mathfrak{g}) \mid [X,Z] = 0 \ \ (\forall X \in \mathfrak{g})\}.$$

Furthermore, if $\mathfrak{g}$ is the complexification of the Lie algebra $\mathfrak{g}_0$ of a connected real Lie group G_0, then it is readily seen that

$$Z(\mathfrak{g}) = U(\mathfrak{g})^{G_0} := \{Z \in U(\mathfrak{g}) \mid \mathrm{Ad}(x)Z = 0 \ \ (\forall x \in G_0)\}$$

In the rest of this section we assume that $\mathfrak{g}$ is semisimple. Our goal is to determine the structure of $Z(\mathfrak{g})$.

Lemma 13.1. *Let V be an irreducible finite dimensional $\mathfrak{g}$-module. Then there exists a unique character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ such that Z acts by the scalar $\chi(Z)$ on V, for every $Z \in Z(\mathfrak{g})$.*

Remark 13.2. The character $\chi = \chi_V$ is called the infinitesimal character of V. We will later see that V is completely determined by χ.

Proof. Let $Z \in \mathbb{Z}(\mathfrak{g})$. Then for $X \in \mathfrak{g}$ and $v \in V$ we have

$$ZXv - XZv = [Z,X]v = 0.$$

Thus, the linear operator $M_Z : V \to V, v \mapsto Zv$ is equivariant. By Schur's lemma, there exists a unique scaler $\chi(Z)$ such that $M_Z = \chi(Z)I_V$. The map $Z \mapsto M_Z$ is an algebra homomorphism from $Z(\mathfrak{g})$ to $\mathrm{End}(V)$. This implies that $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is an algebra homomorphism, i.e., a character of $Z(fg)$. $\square$

We will now determine the action of $Z(\mathfrak{g})$ on $\mathfrak{g}$-modules with a cyclic highest weight vector. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $R = R(\mathfrak{g},\mathfrak{h})$ the associated root system, R^+ a positive system.

[2] this will be proved at a later stage

Let S denote the associated set of simple roots, and W the Weyl group of R. We write $\mathfrak{g}^+$ for the sum of the positive root spaces and $\mathfrak{g}^-$ for the sum of the negative root spaces. Then

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+. \tag{38}$$

The subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{g}^+$ is called the standard Borel subgroup determined by R^+. From the direct sum decomposition it follows by the PBW theorem that

$$U(\mathfrak{g}) \simeq U(\mathfrak{g}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^+).$$

Lemma 13.3. *Let $Z \in Z(\mathfrak{g})$. Then there exists a unique element $Z_0 \in U(\mathfrak{h})$ such that $Z - Z_0 \in U(\mathfrak{g})\mathfrak{g}^+$.*

Proof. From the given decomposition of the universal enveloping algebra, it follows that there exists a unique element $Z_0 \in U(\mathfrak{h})$ such that

$$Z - Z_0 = U \oplus V \in \mathfrak{g}^- U(\mathfrak{g}^-) U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{g}^+. \tag{39}$$

Since $\mathfrak{h}$ centralizes Z and Z_0 if follows that $\mathfrak{h}$ centralizes $Z - Z_0$. The adjoint action by $\mathfrak{h}$ leaves the decomposition (38) and hence also the decomposition in (39) invariant. It follows that $\mathfrak{h}$ centralizes U. By applying the PBW we see that each non-zero element of $U(\mathfrak{g}^-)\mathfrak{g}^-$ can be written as a sum of linearly independent terms $X_1 \cdots X_n$ with $n \geq 1$, $X_j \in \mathfrak{g}_{\alpha_j}$ and $\alpha_j \in -R^+$. Such a term is a weight vector for $\mathfrak{h}$ with non-zero weight $\alpha_1 + \cdots + \alpha_n$. From this we conclude that the centralizer of $\mathfrak{h}$ in $\mathfrak{g}^- U(\mathfrak{g}^-) U(\mathfrak{h})$ equals 0. Hence, $U = 0$. □

Since $\mathfrak{h}$ is abelian we may use the symmetrizer to canonically identify $S(\mathfrak{h})$ with $U(\mathfrak{h})$. In view of the above lemma, we may now define the map

$${}^\backprime\gamma : Z(\mathfrak{g}) \to S(\mathfrak{h}) \tag{40}$$

by $Z - {}^\backprime\gamma(Z) \in U(\mathfrak{g})\mathfrak{g}^+$, for $Z \in Z(\mathfrak{g})$.

Lemma 13.4. *The map (40) is a homomorphism of algebras.*

Proof. Let $Z, W \in Z(\mathfrak{g})$ and write $Z = Z_0 + Z_1$ and $W = W_0 + W_1$, with $Z_0 = {}^\backprime\gamma(Z)$ and $W_0 = {}^\backprime\gamma(W)$. Then $Z_1 W = W Z_1 \in U(\mathfrak{g})\mathfrak{g}^+$ so that

$$ZW = (Z_0 + Z_1)W \in Z_0 W + U(\mathfrak{g})\mathfrak{g}^+.$$

We thus find that $ZW \in Z_0 W_0 + Z_0 W_1 + U(\mathfrak{g})\mathfrak{g}^+ \subset Z_0 W_0 + U(\mathfrak{g})\mathfrak{g}^+$. This implies that ${}^\backprime\gamma(ZW) = Z_0 W_0 = {}^\backprime\gamma(Z)\,{}^\backprime\gamma(W)$. □

If $\lambda \in h^*$ then $\lambda : \mathfrak{h} \to \mathbb{C}$ is linear hence extends to an algebra homomorphism (or character) $\lambda : S(\mathfrak{h}) \to \mathbb{C}$. Under the canonical embedding $\mathfrak{h} \subset P(\mathfrak{h}^*)$ the linear functional λ is given by the evaluation map $H \mapsto H(\lambda)$. It follows that the character $\lambda : S(\mathfrak{h}) \to \mathbb{C}$ corresponds to the evaluation map $p \mapsto p(\lambda)$, $P(\mathfrak{h}^*) \to \mathbb{C}$.

In view of the canonical identification $S(\mathfrak{h}) \simeq P(\mathfrak{h}^*)$ we may thus view ${}^\backprime\gamma(Z)$, for $Z \in Z(\mathfrak{g})$, as a polynomial function on $\mathfrak{h}^*$. This polynomial function is denoted by $\lambda \mapsto {}^\backprime\gamma(Z, \lambda)$.

Lemma 13.5. *Let V be a $\mathfrak{g}$-module with cyclic highest weight vector of weight λ. Then each $Z \in Z(\mathfrak{g})$ acts on V by the scalar ${}'\gamma(Z, \lambda)$.*

Proof. Let v_λ be a non-zero highest weight vector. If $H \in \mathfrak{h}$ then $Hv = \lambda(H)v$. This equality extends to all $H \in S(\mathfrak{h})$. In particular, for $Z \in Z(\mathfrak{g})$ we have ${}'\gamma(Z)v = {}'\gamma(Z, \lambda)v$. Since $Z - {}'\gamma(Z) \in U(\mathfrak{g})\mathfrak{g}^+$ and $\mathfrak{g}^+ v = 0$ the result follows. $\square$

Remark 13.6. In particular, the above lemma describes the action of $Z(\mathfrak{g})$ on the standard cyclic module $Z(\lambda)$.

If $\alpha \in R$ then by H_α we denote the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. In terms of this element, the reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by $\lambda \mapsto \lambda - \lambda(H_\alpha)\alpha$. It follows that the weight lattice is characterised by

$$\Lambda = \{\lambda \in \mathfrak{h}^* \mid \lambda(H_\alpha) \in \mathbb{Z}, (\forall \alpha \in \mathbb{R})\}.$$

For every simple root $\alpha \in S$ we define the element $\lambda_\alpha \in \mathfrak{h}^*$ by by

$$\lambda_\alpha(H_\alpha) = 1, \qquad \lambda_\alpha(H_\beta) = 0 \quad \text{for } \beta \in S \setminus \{\alpha\}. \tag{41}$$

Then $\lambda_\alpha \in \Lambda^+$, and it is readily seen that

$$\Lambda^+ = \bigoplus_{\alpha \in S} \mathbb{N}\lambda_\alpha.$$

In terms of the elements H_α we define the following real form of $\mathfrak{h}$,

$$\mathfrak{h}_\mathbb{R} := \sum_{\alpha \in R} \mathbb{R}H_\alpha.$$

It is readily seen that $\mathfrak{h}_\mathbb{R}$ equals the space of points in $\mathfrak{h}$ on which all roots of R attain a real value.

If $\alpha \in R$ there exist $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $H_\alpha, X_\alpha, Y_\alpha$ is a standard $\mathfrak{sl}(2)$-triple. The linear span of these elements constitutes the subalgebra

$$\mathfrak{s}_\alpha := \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha$$

of $\mathfrak{g}$. We recall that there exists a interior automorphism φ_α of $\mathfrak{g}$ that normalizes $\mathfrak{h}$ and satisfies

$$\varphi|_\mathfrak{h} = s_\alpha.$$

Lemma 13.7. *Let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be a finite dimensional irreducible representation of π. Then for each $\varphi \in \mathrm{Aut}(\mathfrak{g})_e$ the representation $\varphi^*\pi = \pi \circ \varphi$ is equivalent to π.*

Proof. The representation $\varphi^*\pi$ is irreducible and depends continuously on $\mathfrak{g}$. For each $H \in \mathfrak{h}$ the eigenvalues of $\pi(\varphi(H))$ depend continuously on φ and belong to the at most countable set $\{\lambda(H) \mid \lambda \in \Lambda\}$. It follows that the set of these eigenvalues is independent of $\varphi \in \mathrm{Aut}(\varphi)$. This implies that the set Λ_φ of weights of $\varphi^*\pi$ is independent of φ. It follows that the highest weight of $\varphi^*\pi$ is independent of φ. As $\varphi^*\pi$ is irreducible, this implies that all $\varphi^*\pi$ are equivalent. $\square$

Lemma 13.8. *Let V be a finite dimensional irreducible representation of $\mathfrak{g}$. Then the set Λ_V of weights of V is invariant under the Weyl group W. Furthermore, if $\mu \in \Lambda_V$ then the weight spaces V_μ and $V_{w\mu}$ have the same dimension.*

Proof. The equivalence class of V is determined by the highest weight λ of V.

Denote by $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be the Lie algebra homomorphism given by $\pi(X)v = Xv$ for $X \in \mathfrak{g}$ and $v \in V$. Let $\alpha \in R$ and let φ_α be an interior automorphism of $\mathfrak{g}$ such that φ_α normalizes $\mathfrak{h}$ and restricts to s_α on it (for its existence, see [Ban10, Lemma 36.7]. Let $\Lambda(\pi')$ denote the set of weights for the representation $\pi' = \varphi^*\pi(H)$. For $\mu \in \Lambda(\pi')$ we denote the associated π'-weight space by V'_μ. The representation π' is equivalent to π. Hence $\Lambda(\pi') = \Lambda_V$ and if $\mu \in \Lambda_V$ then $\dim(V_\mu) = \dim(V'_\mu)$.

On the other hand, $\pi'(H) = \pi(s_\alpha(H))$ for $H \in \mathfrak{h}$. Let $\mu \in \Lambda_V$ then it follows that $V_\mu = V'_{s_\alpha\mu}$. Hence $s_\sigma\mu \in \Lambda(\pi') = \Lambda_V$ and

$$\dim(V_{s_\alpha\mu}) = \dim(V'_{s_\alpha\mu}) = \dim(V_\mu).$$

As the reflections s_α generate the Weyl group, the result now follows. □

Lemma 13.9. *Let $\lambda \in \Lambda^+$, α a simple root and $\mu := s_\alpha(\lambda) - \alpha$. Then there exists a non-zero $\mathfrak{g}$-equivariant map $Z(\mu) \to Z(\lambda)$.*

Proof. Let $\bar{v}_\lambda$ denote the image of 1 in $Z(\lambda)$ and let v_λ denote the image of 1 in $V(\lambda)$, the unique irreducible quotient of $Z(\lambda)$.

The irreducible quotient $V(\lambda)$ is finite dimensional and of highest weight λ. Let $H_\alpha, X_\alpha, Y_\alpha$ be a standard triple as above. Then v_λ is a weight vector for H_α of weight $\lambda(H_\alpha)$. Since $X_\alpha v_\lambda = 0$, it follows from the representation theory of $\mathfrak{s}_\alpha$ that the vectors $Y_\alpha^k v_\lambda$ are non-trivial for $k = 0, \dots, m$ where $m = \lambda(H_\alpha)$; furthermore, these vectors become trivial for $k \geq m+1$. The $\mathfrak{h}$-weight of $Y_\alpha^m v_\lambda$ equals $\lambda - m\alpha = s_\alpha(\lambda)$.

By application of the PBW theorem one sees that the elements $Y_\alpha^k \bar{v}_\lambda$ are non-zero in $Z(\lambda)$. By induction one can show that $X_\alpha Y_\alpha^{k+1}\bar{v}_\lambda = c_k Y_\alpha^k \bar{v}_\lambda$ for uniquely determined constants $c_k \in \mathbb{C}$. Taking the images for $k = m$ in $V(\lambda)$ one sees that $c_m = 0$. This implies that

$$X_\beta Y_\alpha^{m+1}\bar{v}_\lambda = 0$$

for $\beta = \alpha$. If β is a simple root different from α it follows that $\beta - \alpha$ is not a root so that X_β and Y_α commute. This implies that the above equation is true for every simple root β. Therefore, the vector $\bar{v}_\mu := Y_\alpha^{m+1}\bar{v}_\lambda$ in $Z(\lambda)$ is non-zero, of weight μ and annihilated by $\mathfrak{g}^+$. It follows that the map $U(\mathfrak{g}) \to Z(\lambda)$, $U \mapsto U\bar{v}_\mu$ factors through a non-zero $U(\mathfrak{g})$-intertwining map $Z(\mu) \to Z(\lambda)$. □

We define $\delta \in \mathfrak{h}^*$ to be half the sum of the positive roots, i.e.,

$$\delta := \sum_{\alpha \in R^+} \alpha. \tag{42}$$

Lemma 13.10. *Let α be a simple root in R^+. Then $s_\alpha\delta = \delta - \alpha$.*

Proof. As s_α is simple, $\ker\alpha$ is a highest dimensional wall of the positive Weyl chamber $\mathscr{C}^+$ associated with R^+. Therefore the set of roots positive on the neighboring chamber $s_\alpha(\mathscr{C}^+)$ equals $(R^+\setminus\{\alpha\})\cup\{-\alpha\}$. It follows that $s_\alpha(R^+)=(R^+\setminus\{\alpha\})\cup\{-\alpha\}$. Hence,

$$s_\alpha\delta=\frac{1}{2}\sum_{\alpha\in s_\alpha(R^+)}\alpha=\delta-\alpha.$$

$\square$

Corollary 13.11. *In terms of the fundamental weights, δ is given by*

$$\delta=\sum_{\alpha\in S}\lambda_\alpha. \tag{43}$$

In particular, $\delta\in\Lambda^+$.

Proof. Let $\alpha\in S$. Then $\delta-\delta(H_\alpha)=s_\alpha(\delta)=\delta-\alpha$ by Lemma 13.10. We deduce that $\delta(H_\alpha)=-1$ for each $\alpha\in S$. In view of (41) this implies (43). $\square$

Corollary 13.12. *Let $Z\in Z(\mathfrak{g})$. Then the polynomial function $\lambda\mapsto{}'\gamma(Z,\lambda-\delta)$ is W-invariant.*

Proof. Denote the polynomial function by q and let α be a simple root. We will complete the proof by showing that $q(s_\alpha\nu)=q(\nu)$ for all $\nu\in\mathfrak{h}^*$ (recall that W is generated by the simple reflections).

Let $\lambda\in\Lambda^+$ and put $\mu=s_\alpha\lambda-\alpha$. Then it follows that Z acts on $Y(\mu)$ by the same scalar as on $Y(\lambda)$. Hence, $'\gamma(Z,\lambda)={}'\gamma(Z,\mu)$. This implies that

$$'\gamma(Z,(\lambda+\delta)-\delta)={}'\gamma(Z,s_\alpha(\lambda+\delta)-s_\alpha(\delta)-\alpha)={}'\gamma(Z,s_\alpha(\lambda+\delta)-\delta).$$

Hence, $q(\nu)=q(s_\alpha\nu)$ for all $\nu\in\Lambda^++\delta$. It follows that the polynomial $Q:\nu\mapsto q(\nu)-q(s_\alpha\nu)$ is zero on $\Lambda^++\delta$. Via the basis $\{\lambda_\alpha\mid\alpha\in S\}$ of fundamental weights, we may identify $\mathfrak{h}^*$ with $\mathbb{C}^r$, where $r=|S|$. Then $\Lambda^++\delta$ corresponds to $\mathbb{Z}^r_+$ and consequently, Q corresponds to a polynomial which vanishes on Λ^+. This implies that $Q=0$. Hence, $q(\nu)=q(s_\alpha\nu)$ for all $\nu\in\mathfrak{h}^*$. $\square$

We define the homomorphism $\gamma:Z\to S(\mathfrak{h})$ by $\gamma(Z,\lambda)={}'\gamma(Z,\lambda-\delta)$ for all $Z\in Z(\mathfrak{g})$ and $\lambda\in\mathfrak{h}^*$. Then it follows that γ is an algebra homomorphism from $Z(\mathfrak{g})$ to $S(\mathfrak{g})^W$.

Our first goal is to prove that γ is surjective. In the proof of this result we will compare γ with its analogue for $S(\mathfrak{g})$.

Let

$$S(\mathfrak{g})^{\mathfrak{g}}:=\{u\in S(\mathfrak{g})\mid \mathrm{ad}(X)u=0,\forall X\in\mathfrak{g}\}.$$

Since $\mathrm{ad}(\mathfrak{g})$ preserves the gradation of $S(\mathfrak{g})$ we see that $S(\mathfrak{g})^{\mathfrak{g}}$ is a graded subalgebra of $S(\mathfrak{g})$. Furthermore, by $\mathrm{ad}(\mathfrak{g})$-equivariance, the symmetrizer s restricts to an isomorphism $\bar{s}:I(\mathfrak{g})\to Z(\mathfrak{g})$ of filtered linear spaces. Here $Z(\mathfrak{g})$ is equipped with the filtration by order from $U(\mathfrak{g})$.

In view of the decomposition (38) we may define a projection map $p : \mathfrak{g} \to \mathfrak{h}$ with kernel $\mathfrak{g}^- \oplus \mathfrak{g}^+$. This map has a unique extension to an algebra homomorphism $p : S(\mathfrak{g}) \to S(\mathfrak{h})$. From the decomposition (38) we see that

$$S(\mathfrak{g}) \simeq S(\mathfrak{g}^-) \otimes S(\mathfrak{h}) \otimes S(\mathfrak{g}^+).$$

This implies

$$S(\mathfrak{g}) \simeq S(\mathfrak{h}) \oplus S(\mathfrak{g})(\mathfrak{g}^- \oplus \mathfrak{g}^+). \tag{44}$$

It is now readily seen that p equals the projection $S(\mathfrak{g}) \to S(\mathfrak{h})$ for this decomposition.

Any linear automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ extends to an algebra automorphism $S(\varphi) : S(\mathfrak{g}) \to S(\mathfrak{g})$.

Lemma 13.13. *Let φ be an interior automorphism of $\mathfrak{g}$. Then*

(a) *the induced automorphism $S(\varphi)$ centralizes $S(\mathfrak{g})^{\mathfrak{g}}$;*

(b) *the algebra homomorphism p maps $S(\mathfrak{g})^{\mathfrak{g}}$ into $S(\mathfrak{h})^W$.*

Proof. If $X \in \mathfrak{g}$ then it follows that

$$S(e^{\operatorname{ad} X}) = e^{S(\operatorname{ad} X)} = I \qquad \text{on } I(\mathfrak{g}).$$

The group $\operatorname{Int}(\mathfrak{g})$ of interior automorphisms is generated by the automorphisms $e^{\operatorname{ad} X}$. Hence, (a) is valid.

For (b) assume that $w \in W$. There exists an interior automorphism φ_w of $\mathfrak{g}$ which restricts to w on $\mathfrak{h}$, see [Ban10, Lemma 36.7]. It follows that φ_w preserves the decomposition (44) so that

$$p \circ S(\varphi_w) = S(\varphi_w) \circ p = S(w) \circ p.$$

Since $S(\varphi_w)$ equals the identity on $I(\mathfrak{g})$, it follows that

$$p = S(w) \circ p$$

on $I(\mathfrak{g})$. This proves (b). □

Lemma 13.14. *The algebra homomorphism $p : S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{h})^W$ is bijective.*

Proof. We consider the algebra $P(\mathfrak{g})$ of polynomial functions $\mathfrak{g} \to \mathbb{C}$ and the algebra $P(\mathfrak{h})$ of polynomial functions $\mathfrak{h} \to \mathbb{C}$. The restriction map $r : f \mapsto f|_{\mathfrak{h}}$ defines an algebra homomorphism $P(\mathfrak{g}) \to P(\mathfrak{h})$. Let G denote the group of interior automorphisms of $\mathfrak{g}$. Then by Chevalley's restriction theorem, see the appendix to this section, the map r maps $P(\mathfrak{g})^G$ isomorphically onto $P(\mathfrak{h})^W$.

The Killing form B of $\mathfrak{g}$ is non-degenerate and G-invariant, hence $B : X \mapsto B(X, \cdot)$ defines a linear isomorphism $\mathfrak{g} \to \mathfrak{g}^*$. This isomorphism lifts to an isomorphism $B : S(\mathfrak{g}) \to S(\mathfrak{g}^*) \simeq P(\mathfrak{g})$. The restriction of B to $\mathfrak{h}$ is non-degenerate as well, hence defines a linear isomorphism $b : \mathfrak{h} \to \mathfrak{h}^*$, which in turn lifts to a linear isomorphism $b : S(\mathfrak{h}) \to S(\mathfrak{h}^*) \simeq P(\mathfrak{h})$. We claim that the following diagram commutes,

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{B} & P(\mathfrak{g}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} \\ S(\mathfrak{h}) & \xrightarrow{b} & P(\mathfrak{h}) \end{array}$$

To establish the claim, let $X \in \mathfrak{g}$ and write $X = X_0 + X_1$, with $X_0 \in \mathfrak{h}$ and $X_1 \in \mathfrak{g}^- + \mathfrak{g}^+$. Then $rB(X) = rB(X_0)$ since $B(X_1, \mathfrak{h}) = 0$. Furthermore, $rB(X_0) = b(X_0) = b(p(X))$ so that $rB(X) = Bp(X)$ and the claim follows.

By G-equivariance of the Killing form and W-equivariance of its restriction to $\mathfrak{h}$, it follows that B maps $S(\mathfrak{g})^G$ isomorphically onto $P(\mathfrak{g})^G$ and that b maps $S(\mathfrak{h})^W$ isomorphically onto $P(\mathfrak{h})^W$. It now follows from the first part of the proof that p maps $S(\mathfrak{g})^{\mathfrak{g}} = S(\mathfrak{g})^G$ bijectively onto $S(\mathfrak{h})^W$. □

We now come to the main result of this section, which is due to Harish-Chandra.

Theorem 13.15. *The map $\gamma : Z(\mathfrak{g}) \to S(\mathfrak{a})^W$ is an isomorphism of filtered algebras.*

Proof. We equip $S(\mathfrak{g})$ and $U(\mathfrak{g})$ with the standard filtrations. Then the symmetrizer $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ becomes an isomorphism of filtered linear spaces. The subspaces $S(\mathfrak{g})^{\mathfrak{g}} \subset S(\mathfrak{g})$ and $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ are equipped with the induced filtrations. Then by $\mathrm{ad}(\mathfrak{g})$-equivariance of the symmetrizer it follows that s restricts to an isomorphism $s : S(\mathfrak{g})^{\mathfrak{g}} \to Z(\mathfrak{g})$ of filtered linear spaces. We consider the following diagram, which need not commute,

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\gamma} & S(\mathfrak{h})^W \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle I} \\ S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{p} & S(\mathfrak{h})^W \end{array}$$

All maps in the diagram preserve the filtrations. Furthermore, the vertical maps are isomorphisms of filtered spaces, whereas the map p at the bottom is surjective.

Let $k \geq 0$ and $U \in S(\mathfrak{g})^{\mathfrak{g}}_k$. We claim that we have $\gamma \circ s(U) - p(U) \in S(\mathfrak{h})^W_{k-1}$ (we agree that the latter space is zero for $k = 0$). If $k = 0$, U is constant, and the claim is obvious. Thus, assume $k \geq 1$.

In accordance with the decomposition (44) we write $U = U_0 + U_1$, with $U_0 \in S(\mathfrak{h})_k$ and $U_1 \in \mathfrak{g}^- S(\mathfrak{g}^-)_{k-1} \oplus U(\mathfrak{g})_{k-1}\mathfrak{g}^+$. If $Y \in \mathfrak{g}^-$ and $X \in S(\mathfrak{g}^-)$ then $s(YX) - s(Y)s(X) \in U(\mathfrak{g})_{k-1}$ and we see that $s(YX) \in \mathfrak{g}^- U(\mathfrak{g}^-)_{k-1} + U(\mathfrak{g}^-)_{k-1}$. Thus

$$s(\mathfrak{g}^- S(\mathfrak{g}^-)_{k-1}) \subset \mathfrak{g}^- U(\mathfrak{g}^-) + U(\mathfrak{g}^-)_{k-1}$$

By a similar argument we see that

$$s(S(\mathfrak{g})_{k-1}\mathfrak{g}^+) \subset U(\mathfrak{g})\mathfrak{g}^+ + U(\mathfrak{g})_{k-1}.$$

It follows from these inclusions that

$$s(U) = s(U_0) + s(U_1) \in s(U_0) + \mathfrak{g}^- U(\mathfrak{g}^-) + U(\mathfrak{g})\mathfrak{g}^+ + U(\mathfrak{g})_{k-1}.$$

This implies that

$$`\gamma(s(U)) \in U_0 + S(\mathfrak{h})_{k-1}.$$

Since $\gamma = T_{-\delta} \circ `\gamma$, where $T_{-\delta} : S(\mathfrak{h}) \to S(\mathfrak{h})$ preserves the filtrations it follows that

$$\gamma(s(U)) \in U_0 + S(\mathfrak{h})_{k-1} = p(U) + S(\mathfrak{h})_{k-1}.$$

This establishes the claim.

The bottom map in the above diagram is an isomorphism of graded linear spaces. It now follows from the claim that the diagram leads to the following commutative diagram of graded maps and graded linear spaces

$$\begin{array}{ccc} \mathrm{gr}\, Z(\mathfrak{g}) & \xrightarrow{\mathrm{gr}\,\gamma} & S(\mathfrak{h})^W \\ {\scriptstyle \mathrm{gr}\, s} \uparrow & & \uparrow {\scriptstyle I} \\ S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{p} & S(\mathfrak{h})^W \end{array}$$

Since the left, right and bottom maps are linear isomorphisms, it follows that $\mathrm{gr}\gamma$ is a linear isomorphism. This implies that γ is an isomorphism of filtered linear spaces. If we combine this with the established fact that γ is a homomorphism of algebras, the result follows. □

Lemma 13.16. *The isomorphism map $\gamma : Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ is independent of the choice of positive roots.*

Proof. We leave the proof to the reader, see exercises. □

The map γ is both called the canonical isomorphism and the Harish-Chandra isomorphism. It plays a fundamental role in the representation theory of real semisimple Lie groups.

A character of $S(\mathfrak{h}^*)^W$ is by definition a homomorphism $S(\mathfrak{h})^W \to \mathbb{C}$ of algebras with unit. The set of such homomorphisms is denoted by $[S(\mathfrak{h})^W]^\wedge$.

For $\lambda \in \mathfrak{h}^*$ we define the character ε_λ of $S(\mathfrak{h})^W \simeq P(\mathfrak{h}^*)^W$ by $\varepsilon_\lambda(p) = p(\lambda)$, for $p \in P(\mathfrak{h}^*)^W$.

Lemma 13.17. *The map $\lambda \mapsto \varepsilon_\lambda$ induces a bijection*

$$\mathfrak{h}^*/W \xrightarrow{\simeq} [S(\mathfrak{h})^W]^\wedge. \tag{45}$$

Proof. We will first show that $\varepsilon : \mathfrak{h}^* \to [S(\mathfrak{h})^W]^\wedge$ is surjective. Let $\xi : P(\mathfrak{h}^*)^W \to \mathbb{C}$ be a character. Then $\mathfrak{M} := \ker \xi$ is a maximal ideal in $P(\mathfrak{h}^*)^W$. It is now readily seen that $\mathscr{I} := P(\mathfrak{h}^*)\mathfrak{M}$ is an ideal in $P(\mathfrak{h}^*)$. From $1 \in \mathscr{I}$ it would follow from averaging over the W-action that $1 \in P(\mathfrak{h}^*)^W \mathfrak{M} \subset \mathfrak{M}$, contradiction. We conclude that $\mathscr{I}$ is a proper ideal, hence contained in a maximal ideal $\mathfrak{M}'$ of $P(\mathfrak{h}^*)$. It is well known that such a maximal ideal corresponds to a point λ of the affine space $\mathfrak{h}^*$, i.e, $\mathfrak{M}' = \{p \in P(\mathfrak{h}^*) \mid p(\lambda) = 0\}$. It follows that $\mathfrak{M} \subset \ker \varepsilon_\lambda$. Since $\mathfrak{M}$ is maximal, and ε_λ proper, we see that $\mathfrak{M} = \ker \varepsilon_\lambda$. It follows that ε_λ and ξ have the same kernel, hence both factor to a character of the unital algebra $\mathbb{C}$. Since $\mathbb{C}$ has only one character, it follows that $\xi = \varepsilon_\lambda$.

It is now clear that ε defines a surjective map $\mathfrak{h}^* \to [S(\mathfrak{h}^*)]^H$, which factors through the canonical projection $\mathfrak{h}^* \to \mathfrak{h}^*/W$. We will finish the proof by showing that the resulting map is injective. For this, assume that $\lambda, \mu \in \mathfrak{h}^*$ and assume that $\varepsilon_\lambda = \varepsilon_\mu$. Assume that $\mu \neq W\lambda$. Then there exists a $X \in \mathfrak{h}$ such that $\mu(X) = 0$ and $w\lambda(X) \neq 0$ for all $w \in W$. Define $P := \prod_{w \in W}(wX)$, then it follows that $P \in S(\mathfrak{h})^W$ and $P(\mu) = 0 \neq P(\lambda)$, contradiction. Hence, $W\mu = W\lambda$ and the injectivity of the map (45) follows. $\square$

For $\lambda \in \mathfrak{h}^*$ define the character χ_λ of $Z(\mathfrak{g})$ by

$$\chi_\lambda(Z) = \gamma(Z, \lambda) \qquad (Z \in Z(\mathfrak{g}).$$

Corollary 13.18. *The map $\lambda \mapsto \chi_\lambda$ induces a bijection*

$$\mathfrak{h}^*/W \xrightarrow{\simeq} Z(\mathfrak{g})^\wedge. \tag{46}$$

Proof. This follows from combining Lemma 13.17 with Theorem 13.15. $\square$

We recall the definition of infinitesimal character from Remark 13.2.

Corollary 13.19. *Let V, V' be irreducible $\mathfrak{g}$-modules with infinitesimal characters χ_V and $\chi_{V'}$. Then*

$$V \simeq V' \iff \chi_V = \chi_{V'}$$

Proof. The implication from left to right is obvious. Conversely, assume that V has highest weight λ and that V' has highest weight λ'. Then it follows that $\chi_V(Z) = \gamma(Z, \lambda + \delta)$ and that $\chi_{V'}(Z) = \gamma(Z, \lambda' + \delta)$ for all $Z \in Z(\mathfrak{g})$. From the previous corollary, it now follows that $\lambda + \delta$ and $\lambda' + \delta$ are conjugate under the Weyl group. Since both elements are strictly positive on H_α for α simple in R^+, it follows that $\lambda + \delta = \lambda' + \delta$ hence $\lambda = \lambda'$. This implies that $V \simeq V'$. $\square$

Finally, we consider the situation of a connected compact semisimple group K with Lie algebra $\mathfrak{k}$. We consider the algebra isomorphism $X \mapsto R_X$ from $U(\mathfrak{k}_\mathbb{C})$ onto the algebra $\mathbb{D}(K)$ of left invariant differential operators on K. It is readily checked that K acts on $\mathbb{D}(K)$ by

$$x\dot{D} = r_x^* \circ D \circ r_x^{-1^*}, \qquad (D \in \mathbb{D}(K),\ x \in K).$$

Furthermore, the isomorphism $R : U(\mathfrak{k}_\mathbb{C}) \to \mathbb{D}(K)$ is equivariant for this action of K on the image space and the adjoint action of K on $U(\mathfrak{k}_\mathbb{C})$ (we leave it to the reader to prove this, see exercises). Since $Z(\mathfrak{k}_\mathbb{C}) = U(\mathfrak{k}_\mathbb{C})^K$, it follows that R maps $Z(\mathfrak{k}_\mathbb{C})$ isomorphically onto the algebra $\mathbb{D}(K)^K$ of bi-invariant differential operators on K.

For a character $\chi \in Z(\mathfrak{k}_\mathbb{C})^\wedge$ we define the joint eigenspace

$$\mathscr{E}(K, \chi) := \{f \in C^\infty(K) \mid R_Z f = \chi(Z) f,\ \forall Z \in Z(\mathfrak{g})\}.$$

Lemma 13.20. *Let $\delta \in \widehat{K}$ and let χ be the infinitesimal character of $\delta^\vee$ (i.e., of the associated infinitesimal representation of $\mathfrak{k}_\mathbb{C}$). Then*

$$\mathscr{E}(K, \chi) = \mathscr{R}(K)_\delta. \tag{47}$$

Proof. We recall that $\mathscr{R}(K)_\delta$ is equal to the image of the map $T_\delta : V_\delta \otimes V_\delta^* \to C^\infty(K)$ given by

$$T_\delta(A)(x) = \mathrm{tr}(\delta(x)^{-1}A)(x),$$

for $A \in \mathrm{End}(V_\delta) \simeq V_\delta \otimes V_{\delta^*}$. This map intertwines the $K \times K$ representations $\delta \widehat{\otimes} \delta^\vee$ and $L \otimes R$. From for $A \in \mathrm{End}(V_\delta)$ it follows that

$$R_Z T_\delta(A) = T_\delta([1 \otimes \delta^\vee(Z)]A) = \chi(Z) T_\delta(A), \qquad (Z \in Z(\mathfrak{k}_\mathbb{C})),$$

hence $T_\delta(A) \in \mathscr{E}(K, \chi)$. If $\delta' \in \widehat{K}$ is not equivalent to δ, then the associated infinitesimal character χ' of $(\delta')^\vee$ is different from χ, in view of Corollary 13.19. It follows that $\mathscr{E}(K,\chi) \cap \mathscr{E}(K,\chi') = 0$. Hence,

$$\mathscr{R}(K)_K \cap \mathscr{E}(K,\chi) = \mathscr{R}(K)_\delta.$$

We will finish the proof by showing that the first of these spaces is dense in $\mathscr{E}(K,\chi)$ for the supnorm on K. Then by finite dimensionality of $\mathscr{R}(K)_\delta$ it follows that (47). To establish the density, let $f \in \mathscr{E}(K,\chi)$. Then for all $\varphi \in \mathscr{R}(K)_K$ and $Z \in Z(\mathfrak{k}_\mathbb{C})$ we have

$$R_Z(\varphi * f) = R_Z \circ L(\varphi) f = L(\varphi) \circ R_Z f = \chi(Z) \varphi * f$$

so that $\varphi * f \in \mathscr{E}(K,\chi)$. There exists a sequence φ_j in $\mathscr{R}(K)$ such that $\varphi_j * f \to f$ uniformly on K, for $j \to \infty$. Since $\varphi_j * f \in \mathscr{R}(K)_K \cap \mathscr{E}(K,\chi)$, density follows. $\square$

14 Appendix: Chevalley's theorem

In this section we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra, and that $\mathfrak{h}$ is a Cartan subalgebra. Let $P(\mathfrak{g})$ denote the algebra of polynomial functions $\mathfrak{g} \to \mathbb{C}$. Then the natural action of $\mathrm{Int}(\mathfrak{g})$ on $\mathfrak{g}$ induces a representation π of $\mathrm{Int}(\mathfrak{g})$ in $P(\mathfrak{g})$ given by

$$\pi(\varphi)p = p \circ \varphi^{-1}, \qquad (p \in P(\mathfrak{g}), \varphi \in \mathrm{Int}(\mathfrak{g})).$$

The homogeneous components $P^k(\mathfrak{h})$ are finite dimensional subspaces, on which the represena-tion π restricts to a smooth representation π^k. Moreover, for each $\varphi \in \mathrm{Int}(\mathfrak{g})$, the map $\pi(\varphi)$ is an automorphism of the graded algebra $P(\mathfrak{g})$.

Let π_* be the associated representation of $\mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Int}(\mathfrak{g}))$ in $P(\mathfrak{g})$ defined by $\pi_* = d\pi^k(I)$ on $P^k(\mathfrak{g})$. Then $\pi_*(\delta)$ is a derivation of $P(\mathfrak{g})^k$, for every $\delta \in \mathrm{Der}(\mathfrak{g})$. It follows that $\pi_* \circ \mathrm{ad}$ is a representation of $\mathfrak{g}$ in $P(\mathfrak{g})$ by derivations. Since $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is an isomorphism of Lie algebras, it follows from considering the representations on the finite dimensional homogeneous components that

$$P(\mathfrak{g})^{\mathfrak{g}} = P(\mathfrak{g})^{\mathrm{Int}(\mathfrak{g})}.$$

Before we proceed, we note that on each element $\xi \in \mathfrak{h}^*$ the representation $\pi_* \circ \mathrm{ad}$ is given by

$$[\pi_* \circ \mathrm{ad}](X) = -\xi \circ \mathrm{ad}(X) = \mathrm{ad}^\vee(X)\xi.$$

Thus, the representation $\pi_* \circ \mathrm{ad}$ on $P(\mathfrak{g}) \simeq S(\mathfrak{g}^*)$ is induced by the coadjoint representation of $\mathfrak{g}$ in $\mathfrak{g}^*$.

The action of the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ on $\mathfrak{h}$ naturally induces a representation of W in $P(\mathfrak{h})$, for which each homogeneous component $P^k(\mathfrak{h})$, for $k \geq 0$, is invariant. This action is given by the formula

$$wp(H) = p(w^{-1}H), \qquad (H \in \mathfrak{h}),$$

for $w \in W$ and $p \in P(\mathfrak{h})$.

If $w \in W$, then there exists a $\varphi \in \mathrm{Int}(\mathfrak{g})$ which normalizes $\mathfrak{h}$ and restricts to w on this space. It follows that

$$p \in P(\mathfrak{g})^{\mathfrak{g}} \Rightarrow p|_{\mathfrak{h}} \in P(\mathfrak{h})^W.$$

Our goal in this section is to prove the following result, due to Chevalley.

Theorem 14.1. *The restriction map $p \mapsto p|_{\mathfrak{h}}$ defines a isomorphism of algebras*

$$P(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\simeq} P(\mathfrak{h})^W.$$

Proof. Clearly, the given restriction map is a homomorphism of algebras. We will show that the map is bijective, and start with the injectivity.

Let R be the root system of $\mathfrak{h}$ in $\mathfrak{g}$, and let $\mathfrak{h}^{\mathrm{reg}}$ be the associated set of regular points of $\mathfrak{h}$. Thus,

$$\mathfrak{h}^{\mathrm{reg}} = \{H \in \mathfrak{h} \mid \alpha(H) \neq 0, \forall \alpha \in R\}.$$

We will first establish the claim that the natural action map $F : \mathrm{Int}(\mathfrak{g}) \times \mathfrak{h}^{\mathrm{reg}} \to \mathfrak{g}$, given by

$$F(\varphi, H) = \varphi(H)$$

has surjective differential everywhere. By equivariance, it suffices to prove this at each point (I, H_0), for $H_0 \in \mathfrak{h}^{\mathrm{reg}}$. Since $\mathfrak{g}$ is semisimple, the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a linear isomorphism from $\mathfrak{g}$ onto the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of $\mathrm{Int}(\mathfrak{g})$. Let $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$, then

$$dF(I,H_0)(\mathrm{ad}(X),H) = \left.\frac{d}{dt}\right|_{t=0} e^{t\,\mathrm{ad}(X)}(H_0+tH) = [X,H_0]+H = H - \mathrm{ad}(H_0)(X).$$

Since H_0 is regular, the map $\mathrm{ad}(H_0)$ maps the direct sum

$$\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$$

bijectively onto itself. Hence $\mathfrak{h} + \mathrm{ad}(H_0)(\mathfrak{g}) = \mathfrak{g}$ by the root space decomposition. This shows that $dF(I,H_0)$ is surjective.

It follows from the above claim that the image $\Omega := F(\mathfrak{g} \times \mathfrak{h}^{\mathrm{reg}})$ is a non-empty open subset of $\mathfrak{g}$. Now assume that $p \in P(\mathfrak{g})^{\mathfrak{g}}$ has zero restriction to $\mathfrak{h}$. Then p is invariant under $\mathrm{Int}(\mathfrak{g})$ and it follows that $p = 0$ on the non-empty open set Ω. This implies that $p = 0$.

Next, we turn to the surjectivity. We fix a choice R^+ of positive roots for $R := R(\mathfrak{g}, \mathfrak{h})$. Let $\lambda_1, \ldots, \lambda_r$ be an ordering of the corresponding fundamental weights. Then the elements $\lambda_1, \ldots, \lambda_r$ of $P(\mathfrak{h})$ are algebraicially independent over $\mathbb{C}$ and we have

$$P(\mathfrak{h}) = \mathbb{C}[\lambda_1, \ldots, \lambda_r]. \tag{48}$$

It is readily verified that the polyomials of the form $(\sum_j c_j \lambda_j)^k$, with $c_j \in \mathbb{N}$ and $k \in \mathbb{N}$ span (48) as a linear space. Equivalently, this means that the polyomial functions λ^k, for $\lambda \in \Lambda^+$ and $k \in \mathbb{N}$, span $P(\mathfrak{h})$ as a linear space. Now consider the W-equivariant projection $\mathrm{pr} : P(\mathfrak{h}) \to P(\mathfrak{h})^W$. Then it follows that the polynomials $\mathrm{pr}(\lambda^k)$ span $P(\mathfrak{h})^W$. Evidently, the projection map $\mathrm{pr} : P(\mathfrak{h}) \to P(\mathfrak{h})^W$ is given by averaging over the action of W so that

$$\mathrm{pr}(\lambda^k) = \frac{1}{|W|} \sum_{w\in W} (w\lambda)^k.$$

Fix $k \geq 0$. Then it suffices to show that for every $\lambda \in \Lambda^+$ there exists a polynomial $p \in P^k(\mathfrak{g})^{\mathfrak{g}}$ so that

$$p|_{\mathfrak{h}} = \mathrm{pr}(\lambda^k).$$

We will prove this by induction on the partial ordering $\preceq$ on Λ^+ given by

$$\mu \preceq \lambda \iff \lambda - \mu \in \mathbb{N}R^+.$$

Here $\mathbb{N}R^+$ denotes the subset of $\mathfrak{h}^*$ consisting of elements given by a sum of the form $\sum_{\alpha\in R^+} n_\alpha \alpha$, with coefficients $n_\alpha \in \mathbb{N}$.

Let $\lambda \in \Lambda^+$, and assume the assertion has already been established for elements $\mu \in \Lambda^+$ with $\mu \prec \lambda$. Let $V := V(\lambda)$ be the irreducible finite dimensional representation of $\mathfrak{g}$ of highest weight λ and let Λ_V be its set of weights. Then the function $P : \mathfrak{g} \to \mathbb{C}$ given by

$$P(X) = \mathrm{tr}(X^k : V \to V)$$

is readily seen to belong to $P(\mathfrak{g})^{\mathfrak{g}}$. Furthermore, using the decomposition of V into weight spaces, we see that its restriction to $\mathfrak{h}$ is given by

$$P|_{\mathfrak{h}} = \sum_{\mu \in \Lambda_V} \dim(V_\mu)\mu^k.$$

If $\mu \in \Lambda_V$ then $w\mu \in \Lambda_V$ and $\dim(V_{w\mu}) = \dim V_\mu$ for every $w \in W$. Since $W(\Lambda^+) = \Lambda$, it follows that

$$P|_{\mathfrak{h}} = \sum_{\mu \in \lambda_V \cap \Lambda^+ \setminus \{\lambda\}} \dim(V_\mu)\,\mathrm{pr}(\mu^k) + \mathrm{pr}(\lambda^k). \tag{49}$$

We now observe that all elements $\mu \in \Lambda_V \cap \Lambda^+ \setminus \{\lambda\}$ satisfy $\mu \prec \lambda$. By the induction hypothesis it follows that the sum on the right-hand side of (49) is a polynomial of the form $Q|_{\mathfrak{h}}$, with $Q \in P(\mathfrak{g})^{\mathfrak{g}}$. It follows that $\mathrm{pr}(\lambda^k)$ equals the restriction to $\mathfrak{h}$ of the polynomial $P - Q \in P(\mathfrak{g})^{\mathfrak{g}}$. The proof is complete. $\square$

15 Real groups and Cartan involutions

We assume that $\mathfrak{g}$ is a real semisimple Lie algebra. By an involution of $\mathfrak{g}$ we mean an automorphism of $\mathfrak{g}$ such that $\sigma^2 = I$. If σ is an involution of $\mathfrak{g}$, then

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \tag{50}$$

where $\mathfrak{g}_\pm$ are the plus and minus one eigenspaces, respectively. Since σ is an automorphism, it follows that the Killing form B of $\mathfrak{g}$ is invariant under σ, i.e.,

$$B(\sigma X, \sigma Y) = B(X, Y), \qquad (X, Y \in \mathfrak{g}).$$

This implies that $\mathfrak{g}_+ \perp \mathfrak{g}_-$, relative to the Killing form. Furthermore, since σ preserves the Lie brackets, it is readily seen that

$$[\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-, \quad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+.$$

In particular, $\mathfrak{g}_+$ is a subalgebra which stabilizes the decomposition (50). The following notion will turn out to be crucial for understanding the structure of $\mathfrak{g}$ and $\mathrm{Aut}(\mathfrak{g})$.

Definition 15.1. A *Cartan involution* of $\mathfrak{g}$ is an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that the Killing form B is positive definite on $\mathfrak{g}_+$ and negative definite on $\mathfrak{g}_-$.

Traditionally, a Cartan involution is denoted by θ. The associated eigenspaces are denoted by $\mathfrak{k} := \mathfrak{g}_+$ and $\mathfrak{p} := \mathfrak{g}_-$. Then $\mathfrak{k} \perp \mathfrak{p}$ relative to the Killing form and $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ which stabilizes the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

The following exhibits the standard example of a Cartan involution.

Example 15.2. Let $n \geq 2$. The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of the special linear group $\mathrm{SL}(n, \mathbb{R})$ equals the space of $A \in \mathrm{M}_n(\mathbb{R})$ with $\mathrm{tr} A = 0$, equipped with the commutator bracket. It is readily verified that the map

$$\theta : X \mapsto -X^{\mathrm{T}}$$

is an automorphism of $\mathfrak{sl}(n, \mathbb{R})$. The associated eigenspaces are given by

$$\mathfrak{k} = \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^{\mathrm{T}} = -X\} = \mathfrak{so}(n, \mathbb{R}), \qquad \mathfrak{p} = \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T = X\}.$$

We note that

$$\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$$

is the compact real form $\mathfrak{su}(n)$ of the complexified Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. The complex Killing form $B_{\mathbb{C}}$ restricts to the Killing form $B_{\mathfrak{u}}$ of $\mathfrak{u}$ which is negative definite. On the other hand, $B_{\mathbb{C}}$ restricts to the Killing form B on $\mathfrak{sl}(n, \mathbb{R})$. Thus, for $X \in \mathfrak{k} \setminus \{0\}$ we have

$$B(X, X) = B_{\mathbb{C}}(X, X) = B_{\mathfrak{u}}(X, X) < 0.$$

On the other hand, for $Y \in \mathfrak{p} \setminus \{0\}$ we have

$$-B(Y, Y) = B_{\mathbb{C}}(iY, iY) = B_{\mathfrak{u}}(iY, iY) < 0.$$

It follows that $B < 0$ on $\mathfrak{k}$ and $B > 0$ on $\mathfrak{p}$. Hence, θ is a Cartan involution.

Another related example comes from the theory of compact symmetric spaces.

Example 15.3. Let $\mathfrak{u}$ be a compact semisimple Lie algebra, and σ an involution of $\mathfrak{u}$. Let $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ the associated eigenspace decomposition for the eigenvalues $+1$ and -1, respectively. It is now readily verified that

$$\mathfrak{g} := \mathfrak{k} \oplus i\mathfrak{q}$$

is a real form of $\mathfrak{u}_{\mathbb{C}}$. The complex linear extension $\sigma_{\mathbb{C}}$ of σ to $\mathfrak{u}_{\mathbb{C}}$ restricts to an involution θ of $\mathfrak{g}$. By a similar reasoning as in the previous example, it follows that θ is a Cartan involution. In particular, we see that the -1-eigenspace $\mathfrak{p}$ of θ is given by $\mathfrak{p} = i\mathfrak{q}$.

On the other hand, if $\mathfrak{g}$ is a real semisimple Lie algebra with Cartan involution θ and associated decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then it is readily verified that

$$\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$$

is a real form of the complex semisimple algebra $\mathfrak{g}_{\mathbb{C}}$, on which the Killing form is negative definite. Thus $\mathfrak{u}$ is compact, and equipped with the involution $\sigma = \theta_{\mathbb{C}}|_{\mathfrak{u}}$. The associated decomposition into eigenspaces is given by $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$, where $\mathfrak{q} = i\mathfrak{p}$.

The one to one correspondence $(u, \mathfrak{k}) \leftrightarrow (\mathfrak{g}, \mathfrak{k})$ leads to the so called duality of Riemannian symmetric spaces of the compact type and those of the non-compact type.

If θ is a Cartan involution of $\mathfrak{g}$ and $\varphi \in \mathrm{Aut}(\mathfrak{g})$, then the conjugate $\varphi \circ \theta \circ \varphi^{-1}$ is a Cartan involution as well. The following result ensures that every real semisimple Lie algebra has a Cartan involution.

Proposition 15.4. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then $\mathfrak{g}$ has a Cartan involution. Any two Cartan involutions are conjugate by an element of* $\mathrm{Int}(\mathfrak{g})$.

Proof. For a proof we refer the reader to [Kna02, Cor. 6.18]. □

From now on we will assume that $\mathfrak{g}$ is a real semisimple Lie algebra, equipped with a Cartan involution θ. We define the bilinear form $\langle \cdot\,,\,\cdot \rangle$ on $\mathfrak{g}$ by

$$\langle X\,, Y \rangle = -B(X, \theta Y). \tag{51}$$

Lemma 15.5. *The form (51) is a positive definite inner product, for which the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal. Furthermore, with respect to this inner product,*

(a) $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ *is anti-symmetric for* $X \in \mathfrak{k}$;

(b) $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ *is symmetric for* $X \in \mathfrak{p}$;

Proof. Since B is invariant for the involution θ, the defined form is symmetric. Let $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, then

$$\langle X + Y\,, X + Y \rangle = -B(X + Y, X - Y) = -B(X, X) + B(Y, Y).$$

Since B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, it follows that $\langle \cdot\,,\,\cdot \rangle$ is positive definite on $\mathfrak{g}$. The orthogonality of $\mathfrak{k}$ and $\mathfrak{p}$ follow from the similar orthogonality for B. It remains to show (a) and (b). Let $X \in \mathfrak{g}$, then for $Y, Z \in \mathfrak{g}$ we have

$$\langle \mathrm{ad}(X)Y\,, Z \rangle = -B(\mathrm{ad}(X)Y, \theta Z) = B(Y, \mathrm{ad}(X)\theta Z) = -\langle Y\,, \mathrm{ad}(\theta X)Z \rangle$$

so that

$$(\mathrm{ad}X)^{\mathrm{T}} = -\mathrm{ad}(\theta X). \tag{52}$$

Assertion (a) and (b) now follow. $\square$

Assume that G is a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let K be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$.

Lemma 15.6. *The map $K \times \mathfrak{p} \to G$, $(k,X) \mapsto k\exp(X)$ has bijective differential everywhere.*

Proof. Denote the given map by φ, let $X \in \mathfrak{p}$ and put $x = \exp X$. Then by left K-equivariance, it suffices to prove that $S := dr_x(e)^{-1} d\varphi(e,X) : \mathfrak{k} \times \mathfrak{p} \to \mathfrak{g}$ is bijective. Let $Y \in \mathfrak{p}$ and $U \in \mathfrak{k}$. Then

$$S(U,Y) = \left.\frac{d}{dt}\right|_{t=0} \exp(tU)\exp(X+tY)\exp(-X) = U + T_X(Y),$$

where

$$T_X = \frac{e^{\mathrm{ad}X} - I}{\mathrm{ad}(X)} := \sum_{k=1}^{\infty} \frac{1}{k!}\,\mathrm{ad}(X)^{k-1}.$$

Indeed, this follows by application of [Ban10, Lemma 8.2] It suffices to show that S is injective $\mathfrak{k} \times \mathfrak{p} \to \mathfrak{g}$. For this it suffices to show that the composition

$$\mathrm{pr}_{\mathfrak{p}} \circ T_X|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$$

is injective, with $\mathrm{pr}_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}$ the projection along $\mathfrak{k}$. Since the odd powers in the power series for T_X map $\mathfrak{p}$ to $\mathfrak{k}$ it follows that

$$\mathrm{pr}_{\mathfrak{p}} \circ T_X|_{\mathfrak{p}} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} A^n,$$

where $A = (\mathrm{ad}X)^2|_{\mathfrak{p}}$. Since $\mathrm{ad}(X)$ is symmetric, it follows that $\mathrm{ad}(X)^2$ is symmetric with non-negative eigenvalues. As $\mathrm{ad}(X)^2$ leaves $\mathfrak{p}$ invariant, it follows that $A|_{\mathfrak{p}}$ is an endomorphism of $\mathfrak{p}$ which diagonalizes with non-negative eigenvalues. Let $a \geq 0$ be such an eigenvalue, and $\mathfrak{p}(a)$ the corresponding eigenspace for A, then $\mathrm{pr}_{\mathfrak{p}} \circ T_X = t(a)\mathrm{I}$ on $\mathfrak{p}(a)$, with

$$t(a) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} a^n \geq 1.$$

This implies that $\mathrm{pr}_{\mathfrak{p}} \circ T_X|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ diagonalizes with non-zero eigenvalues, hence is bijective. $\square$

Let $\mathfrak{s}_n$ be the space of symmetric matrices in $\mathfrak{sl}(n,\mathbb{R})$, and let P denote the set of positive definite symmetric matrices in $\mathrm{SL}(n,\mathbb{R})$.

Lemma 15.7. *The set P is a closed submanifold of* $\mathrm{SL}(n,\mathbb{R})$ *and* $\exp : \mathfrak{s}_n \to \mathrm{SL}(n,\mathbb{R})$ *is an embedding onto P.*

Proof. If X is a symmetric matrix in $\mathrm{M}_n(\mathbb{R})$, then the set $\sigma(X)$ of eigenvalues of X is contained in $\mathbb{R}$. For every $\lambda \in \sigma(X)$ we denote by $P_{X,\lambda}$ the orthogonal projection onto the eigenspace $\ker(X - \lambda I)$. Then

$$X = \sum_{\lambda \in \sigma(X)} \lambda P_{X,\lambda}.$$

The exponential of X is given by

$$\exp X = e^X = \sum_{\lambda \in \sigma(X)} e^\lambda P_{X,\lambda}.$$

If $Y \in \mathrm{M}_n(\mathbb{R})$ is symmetric, then $\exp(X) = \exp(Y)$ implies

$$\sum_{\lambda \in \sigma(X)} e^\lambda P_{X,\lambda} = \sum_{\mu \in \sigma(Y)} e^\mu P_{Y,\mu}.$$

Since the exponential function $t \mapsto e^t$ is injective, this is readily seen to imply that $X = Y$. It follows from this that the map $\exp : \mathfrak{s}_n \to \mathrm{SL}(n,\mathbb{R})$ maps $\mathfrak{s}_n$ injectively into P. If $x \in P$, then $\mathrm{spec}(x) \subset]0,\infty[$. Put

$$X := \sum_{\lambda \in \sigma(x)} \log(\lambda) P_{x,\lambda},$$

then it follows that $X \in \mathfrak{s}_n$ and $\exp(X) = x$. Thus, $\exp : \mathfrak{s}_n \to P$ is a bijection. It follows from Lemma 15.6 that exp is an immersion. If $\|X\|_{\mathrm{op}} \to \infty$, then it follows that $\max \sigma(X) \to \infty$, so that $\max \sigma(\exp X) = e^{\max \sigma(X)} \to \infty$ and $\|x\|_{\mathrm{op}} \to \infty$. This implies that the map $\exp : \mathfrak{s}_n \to \mathrm{SL}(n,\mathbb{R})$ is proper. It follows that $\exp : \mathfrak{s}_n \to \mathrm{SL}(n,\mathbb{R})$ is an embedding onto a closed submanifold. This submanifold is $\exp(\mathfrak{s}_n) = P$. $\square$

Lemma 15.8. *Let $\mathfrak{s}_n$ denote the space of $n \times n$ symmetric matrices of trace zero. Then the map*

$$\varphi : \mathrm{SO}(n) \times \mathfrak{s}_n \to \mathrm{SL}(n,\mathbb{R}), \quad (k,X) \mapsto k \exp X$$

is a diffeomorphism onto.

Proof. It follows from Lemma 15.6 that φ is a local diffeomorphism. Thus, it suffices to show that φ is bijective.

We first show that φ is injective. Assume that $x = k_1 \exp(X_1) = k_2 \exp(X_2)$, for $k_j \in \mathrm{SO}(n)$ and $X_j \in \mathfrak{s}_n$. Then $x^{\mathrm{T}} x = \exp(2X_1) = \exp(2X_2)$, and from Lemma 15.7 we conclude that $X_1 = X_2$ and hence also $k_1 = k_2$. This establishes the injectivity.

For the surjectivity, let $x \in \mathrm{SL}(n,\mathbb{R})$. Then $x^{\mathrm{T}} x$ belongs to the set P of positive definite symmetric matrices in $\mathrm{SL}(n,\mathbb{R})$ hence can be written as $\exp(2X)$ for some $X \in \mathfrak{s}_n(\mathbb{R})$. It follows that

$$\exp(-X) x^{\mathrm{T}} = \exp X x^{-1}.$$

Put $k = x \exp(-X)$, then $k^{\mathrm{T}} = \exp(-X) x^{\mathrm{T}}$, whereas $k^{-1} = \exp(X) x^{-1}$, and we see that $k^{-1} = k^{\mathrm{T}}$. Hence, $k \in \mathrm{O}(n)$. Since x and $\exp X$ have determinant one, so has k and we see that $x = k \exp(X) = \varphi(k,X)$. This establishes the surjectivity. $\square$

Corollary 15.9. *Let $G \subset \mathrm{SL}(n,\mathbb{R})$ be a connected closed subgroup, invariant under transposition. Then*

$$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{so}(n)) \oplus (\mathfrak{g} \cap \mathfrak{s}_n) \tag{53}$$

and the map $(k,X) \mapsto k\exp(X)$ defines a diffeomorphism

$$(G \cap \mathrm{SO}(n)) \times (\mathfrak{g} \cap \mathfrak{s}_n) \xrightarrow{\simeq} G.$$

Proof. The map $\Theta : x \mapsto (x^{\mathrm{T}})^{-1}$ is readily checked to be an involution of the group $\mathrm{SL}(n,\mathbb{R})$ which leaves G invariant. The associated infinitesimal involution is the standard Cartan involution $\theta : X \mapsto -X^{\mathrm{T}}$, $\mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R})$, see Example 15.2. If $X \in \mathfrak{g}$ then $\exp(t\theta X) = \Theta(\exp tX) \in G$ for all $t \in \mathbb{R}$. By differentiating at $t = 0$ we find that $\theta(X) \in \mathfrak{g}$, hence $\mathfrak{g}$ is invariant under θ and (53) follows.

Since G is closed, $G \cap \mathrm{SO}(n)$ is a compact subgroup, with Lie algebra $\mathfrak{g} \cap \mathfrak{so}(n)$. In particular, $G \cap \mathrm{SO}(n)$ is a closed submanifold of $\mathrm{SO}(n)$. It follows from Lemma 15.8 that the map $(k,X) \mapsto k\exp X$ defines an embedding of $(G \cap \mathrm{SO}(n)) \times (\mathfrak{g} \cap \mathfrak{s}_n)$ onto a closed submanifold of S of $\mathrm{SL}(n,\mathbb{R})$. The submanifold S contained in the submanifold G of $\mathrm{SL}(n,\mathbb{R})$, hence a closed submanifold of G. Since $\dim(S) = \dim(\mathfrak{g})$ by (53) it follows that S is an open subset of G. Since G is connected, we infer that $S = G$. The result follows. ☐

From now on we assume that G is a connected semisimple Lie group with Lie algebra $\mathfrak{g}$.

Lemma 15.10. $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})_e$.

Proof. The image $\mathrm{Ad}(G)$ is the connected Lie subgroup of $\mathrm{GL}(\mathfrak{g})$, with Lie algebra $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$. The latter is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. The result follows. ☐

Lemma 15.11. *The algebra $\mathrm{Der}(\mathfrak{g})$ is contained in $\mathfrak{sl}(\mathfrak{g})$. The group $\mathrm{Aut}(\mathfrak{g})_e$ is contained in $\mathrm{SL}(\mathfrak{g})$.*

Proof. The map $\mathrm{tr}\,\mathrm{ad} : \mathfrak{g} \to \mathbb{R}$ satisfies $\mathrm{tr}\,\mathrm{ad}([X,Y]) = \mathrm{tr}[\mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X)] = 0$. Since $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, it follows that $\mathrm{tr} \circ \mathrm{ad} = 0$. Since $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$, the first assertion now follows. Since $\mathrm{Aut}(\mathfrak{g})$ has Lie algebra $\mathrm{Der}(\mathfrak{g})$ which is contained in $\mathfrak{sl}(\mathfrak{g})$ it follows that $\mathrm{Aut}(\mathfrak{g})_e$ is contained in $\mathrm{SL}(\mathfrak{g})$. ☐

We assume that θ is a Cartan involution of $\mathfrak{g}$, that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the corresponding decomposition into eigenspaces and that K is the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$.

Theorem 15.12. *The group K is closed in G. Furthermore, the map*

$$\varphi : K \times \mathfrak{p} \to G,\ (k,X) \mapsto k\exp(X)$$

is diffeomorphism onto.

Proof. In Lemma 15.6 we already proved that φ is a local diffeomorphism. It therefore suffices to show that φ is bijective.

By semisimplicity, G has discrete kernel, so that so that $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is a covering homomorphism, i.e., a surjective Lie group homomorphism which is a covering. Now $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})_e \subset \mathrm{SL}(\mathfrak{g})$ and we see that $\mathrm{Ad}(G)$ is a connected closed subgroup of $\mathrm{SL}(\mathfrak{g})$.

We equip $\mathfrak{g}$ with the inner product (51) and denote the corresponding transposition map $\mathrm{End}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$ by $X \mapsto X^{\mathrm{T}}$. We now note that for all $X \in \mathfrak{g}$ we have

$$\mathrm{ad}(X)^{\mathrm{T}} = -\,\mathrm{ad}(\theta X), \tag{54}$$

by (52). It follows that $\mathrm{ad}(\mathfrak{g})$ is stable under transposition. From this it immediately follows that the subgroup generated by $e^{\mathrm{ad}X}$, for $X \in \mathfrak{g}$, is stable under transposition. This subgroup equals $\mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_e = \mathrm{Ad}(G)$. From Corollary 15.9 we now conclude that the map $(k,Y) \to ke^Y$ defines a diffeomorphism

$$(\mathrm{Ad}(G) \cap \mathrm{SO}(n)) \times (\,\mathrm{ad}(\mathfrak{g}) \cap \mathfrak{s}_n) \xrightarrow{\simeq} \mathrm{Ad}(G). \tag{55}$$

Since $\mathrm{Ad}(G)$ is connected, we infer that $\mathrm{Ad}(G) \cap \mathrm{SO}(n)$ is connected as well. We now claim that

$$\mathrm{ad}(\mathfrak{g}) \cap \mathfrak{so}(n) = \mathrm{ad}(\mathfrak{k}) \quad \text{and} \quad \mathrm{ad}(\mathfrak{g}) \cap \mathfrak{s}_n = \mathrm{ad}(\mathfrak{p}). \tag{56}$$

To see this, let $X \in \mathfrak{g}$. Then from (54) and the injectivity of ad, it follows that $X \in \mathfrak{k} \iff \mathrm{ad}(X) \in \mathfrak{so}(n)$ and $X \in \mathfrak{p} \iff \mathrm{ad}(X) \in \mathfrak{s}_{\mathfrak{g}}$. This implies the claim.

It follows from (56) that $\mathrm{Ad}(K)$ is an open subgroup of $\mathrm{Ad}(G) \cap \mathrm{SO}(n)$. Since the latter group is connected, we find that

$$\mathrm{Ad}(G) \cap \mathrm{SO}(n) = \mathrm{Ad}(K).$$

It thus follows that the map (55) coincides with the multiplication map $\psi : (k,Y) \mapsto ke^Y$, $\mathrm{Ad}(K) \times \mathrm{ad}(\mathfrak{p}) \to \mathrm{Ad}(G)$ which is therefore a diffeomorphism.

If $k \in K$ and $X \in \mathfrak{p}$, then $\mathrm{Ad}(k \exp X) = \mathrm{Ad}(k)e^{\mathrm{ad}(X)}$ and we see that the following diagram commutes

$$\begin{array}{ccc} K \times \mathfrak{p} & \xrightarrow{\varphi} & G \\ {\scriptstyle \mathrm{Ad} \times \mathrm{ad}} \downarrow & & \downarrow {\scriptstyle \mathrm{Ad}} \\ \mathrm{Ad}(K) \times \mathrm{ad}(\mathfrak{p}) & \xrightarrow{\psi} & \mathrm{Ad}(G) \end{array}$$

If $(k,X),(k',X') \in K \times \mathfrak{p}$ and $k\exp(X) = k'\exp(X')$ then it follows from the commutativity of the diagram that

$$\mathrm{Ad}(k)e^{\mathrm{ad}(X)} = \mathrm{Ad}(k')e^{\mathrm{ad}(X')}$$

and since the bottom horizontal map is injective, this implies that $\mathrm{ad}(X) = \mathrm{ad}(X')$. Since ad is injective, this implies that $X = X'$, hence also $k = k'$. We conclude that φ is injective as well.

The surjectivity of φ follows from a covering argument as follows. $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is a covering homomorphism, and so is $\mathrm{Ad}|_K : K \to \mathrm{Ad}(K)$. Put $X = K \times \mathfrak{p}$, then the composition $p : \psi \circ (\mathrm{Ad} \times \mathrm{ad}) : X \to \mathrm{Ad}(G)$ is a covering. We thus have a diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & G \\ & {\scriptstyle p} \searrow & \downarrow {\scriptstyle \mathrm{Ad}} \\ & & \mathrm{Ad}(G) \end{array}$$

where p and Ad are coverings. We claim that this implies that φ must be surjective. Indeed, consider the basepoints $x_0 = (e,0) \in X$, $e \in G$ and $I \in \mathrm{Ad}(G)$. Then $\varphi(x_0) = e$ and $p(x_0) = I$. Let $g \in G$. Then the exists a continuous curve $\gamma\colon [0,1] \to G$ with $\gamma(0) = e$ and $\gamma(1) = g$. The image $\mathrm{Ad} \circ \gamma$ in $\mathrm{Ad}(G)$ connects I with $\mathrm{Ad}(g)$. It has a unique lift to a continuous curve $\widetilde{\gamma}\colon [0,1] \to X$ with $\widetilde{\gamma}(0) = x_0$. By commutativity of the diagram, $\varphi \circ \tilde{\gamma}$ is a lift of $q \circ \gamma$ with initial point e. By uniqueness of lifting, we conclude that $\varphi \circ \widetilde{\gamma} = \gamma$. Hence, φ is surjective. $\square$

We can now show that any Cartan involution on $\mathfrak{g}$ comes from a unique involution on G.

Lemma 15.13. *There exists a unique involution Θ on G such that $d\Theta(e) = \theta$. Furthermore, K is the fixed point group of Θ.*

Proof. We will first establish uniqueness. If Θ fulfills the condition, then locally at e, $\Theta \circ \exp = \exp \circ \theta$. Since $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at 0, this implies that Θ is uniquely determined at e. If Θ' is a second involution with the stated property, then the set $H := \{x \in G \mid \Theta(x) = \Theta'(x)\}$ contains an open neighborhood of e in G. On the other hand, H is a subgroup of G, and we see that H is an open subgroup. Since G is connected, it follows that $H = G$ hence $\Theta = \Theta'$.

For the existence, we define $\Theta : G \to G$ by

$$\Theta(k \exp X) := k \exp(-X)$$

for $k \in K$ and $X \in \mathfrak{p}$. Then Θ is a diffeomorphism of G with $\Theta^2 = I$ and $d\Theta(e) = \theta$. Furthermore, K is the set of fixed points for Θ. To finish the proof, it suffices to show that Θ is a group homomorphism. Since $\Theta(e) = e$, it suffices to show that $\Theta(xy) = \Theta(x)\Theta(y)$ for all $x, y \in G$. Define

$$F : G \times G \to G, \ \ (x,y) \mapsto \Theta(x)^{-1}\Theta(xy)\Theta(y)^{-1}.$$

Since G is connected and $F(e,e) = e$, it suffices to show that F is locally constant.

Let $\vartheta : \mathrm{SL}(\mathfrak{g}) \to \mathrm{SL}(\mathfrak{g})$ be the standard Cartain involution relative to (51), defined by $\vartheta(\varphi) = \varphi^{-1\mathrm{T}}$. Then ϑ restricts to an involution of $\mathrm{Ad}(G)$ and it is readily checked that for all $k \in K$ and $X \in \mathfrak{p}$ we have

$$\begin{aligned} \mathrm{Ad}(\Theta(k \exp X)) &= \mathrm{Ad}(k \exp(-X)) = \mathrm{Ad}(k)\mathrm{Ad}(\exp(X))^{-1} \\ &= [\mathrm{Ad}(k)\mathrm{Ad}(\exp X)]^{-1\mathrm{T}} = \vartheta \mathrm{Ad}(k \exp X). \end{aligned}$$

Thus, $\mathrm{Ad}(\theta(x)) = \vartheta \mathrm{Ad}(x)$, for all $x \in G$. Since $\vartheta \circ \mathrm{Ad}$ is a group homorphism, it now follows that

$$\mathrm{Ad} F(x,y) = I$$

for all $(x,y) \in G \times G$. Since F is continuous and Ad a covering, it follows that F is locally constant. $\square$

An involution Θ of G that arises from a Cartan involution of its Lie algebra $\mathfrak{g}$ in the above fashion, is called a Cartan involution of G. From now on we will denote it by the symbol θ as well.

Lemma 15.14. *Let G be a connected semisimple Lie group, θ a Cartan involution of G, and $K := G^\theta$. Then K is connected and the center of G is contained in K. Furthermore, the following assertions are equivalent.*

(a) *The group K is compact.*

(b) *The center of G is finite.*

Proof. Let $g \in G$ be an element of the center $Z(G)$ of G. Write $g = k \exp X$, with $k \in K$ and $X \in \mathfrak{p}$. Then it follows that $I = \mathrm{Ad}(g) = \mathrm{Ad}(k) e^{\mathrm{ad}(X)}$. By the Cartan decomposition of $\mathrm{Ad}(G)$ it follows that $\mathrm{ad} X = 0$ hence $X = 0$ and we see that $g \in K$.

Since $\mathrm{Ad}(K) = \mathrm{Ad}(G) \cap \mathrm{SO}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_e \cap \mathrm{SO}(\mathfrak{g})$ as in the proof of Theorem 15.12, it follows that $\mathrm{Ad}(K)$ is compact. Since $\mathrm{Ad} : K \to \mathrm{Ad}(K)$ is a covering, we see that K is compact if and only if $\ker(\mathrm{Ad}) \cap K$ is finite. The latter group equals $Z(G) \cap K = Z(G)$. □

Corollary 15.15. *Let G be a connected semisimple Lie group with finite center. Let θ be a Cartan involution of G and K the associated group of fixed points. Then K is a maximal compact subgroup of G.*

Proof. Let K' be a compact subgroup of G containing K. Then K' is left K-invariant, hence equal to $K \exp S$ where S is the set of points $X \in \mathfrak{p}$ such that $\exp X \in K'$. Let $X \in S$. Then it follows that the set of points $\exp nX = (\exp X)^n$, for $n \in \mathbb{Z}$ is contained in the compact subset K' of G. This implies that the set of points

$$e^{n\,\mathrm{ad}(X)} = \mathrm{Ad}(\exp nX)$$

is contained in a compact subset of $\mathrm{Ad}(G)$, hence is bounded in $\mathrm{SL}(\mathfrak{g})$. Since $\mathrm{ad}(X)$ is symmetric for the Cartan inner product on $\mathrm{SL}(\mathfrak{g})$, it follows that the eigenvalues of $\mathrm{ad}(X)$ are real. If λ is such an eigenvalue, then it follows that $e^{n\lambda}$ must be a bounded function of $n \in \mathbb{Z}$. This implies that $\lambda = 0$, hence $\mathrm{ad} X = 0$ and we see that $X = 0$. We conclude that $S = \{0\}$, so that $K' = K$. The maximality follows. □

Remark 15.16. Let G be a connected semisimple Lie group with finite center. It can be shown that every maximal compact subgroup of G comes from a Cartan involution in the above fashion. In particular, this implies that all maximal compact subgroups of G are conjugate.

Let G be a connected semisimple Lie group with finite center, θ a Cartan involution of G and $K = G^\theta$. The derivative of θ, denoted by the same symbol, is an involution of $\mathfrak{g}$. As before, we write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

for the associated decomposition into eigenspaces for θ. This decomposition is invariant under $\mathrm{Ad}(K)$. Let β be any $\mathrm{Ad}(K)$-invariant positive definite inner product on $\mathfrak{p}$ (we could take $\beta = B|_{\mathfrak{p}\times\mathfrak{p}}$). Then β may be viewed as a metric on $T_{[e]}(G/K) \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$ which extends to a G-invariant measure on G/K. It can be shown as before that G/K thus becomes equipped with the structure of a Riemannian symmetric space (and has non-positive sectional curvature).

From the Cartan decomposition it follows by inversion that the map $\mathfrak{p} \times K \to G$ given by $(X,k) \mapsto \exp Xk$ is a diffeomorphism onto G. This implies that the map

$$\mathrm{Exp} : X \mapsto \exp(X)K, \quad \mathfrak{p} \to G/K$$

is a diffeomorphism onto. Thus, G/K is diffeomorphic to a finite dimensional vector space. It can be shown that the map Exp coincides with the Riemannian exponential map $T_{[e]}(G/K) \to G/K$ if we use the identification $T_{[e]}(G/K) \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$ induced by the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Lemma 15.17. *The algebra $\mathbb{D}(G/K)$ of invariant differential operators is commutative.*

Proof. The proof is based on the duality of symmetric spaces of the non-compact type, with those of the compact type.

First, we recall from Theorem 12.13 that the map $U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}} \to \mathbb{D}(G/K)$ induces an isomorphism of algebras

$$r : U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}} / U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}} \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k} \xrightarrow{\simeq} \mathbb{D}(G/K).$$

We will now compare with the algebra of invariant differential operators for the dual compact symmetric space. Let $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$, then $\mathfrak{u}$ is a compact semisimple Lie algebra. Let U be a connected compact group with Lie algebra (isomorphic to) $\mathfrak{u}$ (e.g. we may take the group $\mathrm{Aut}(\mathfrak{u})_e$). Let K be the connected subgroup of U with Lie algebra $\mathfrak{k}$. Then U/K is a compact symmetric space, and we have seen that $\mathbb{D}(U/K)$ is commutative. On the other hand,

$$\mathbb{D}(U/K) \simeq U(\mathfrak{u}_{\mathbb{C}})^{\mathfrak{k}} / U(\mathfrak{u}_{\mathbb{C}})^{\mathfrak{k}} \cap U(\mathfrak{u}_{\mathbb{C}})\mathfrak{k}.$$

Since $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, we see that $\mathbb{D}(G/K) \simeq \mathbb{D}(U/K)$, as algebras. The result now follows from the commutativity of $\mathbb{D}(U/K)$, see Lemma 12.15 (b). □

16 The restricted root system

Throughout this section, we assume that $\mathfrak{g}$ is a real semisimple Lie algebra, equipped with a Cartan involution θ. As before, the associated Cartan decomposition is denoted by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

If $H \in \mathfrak{p}$, then $\mathrm{ad}(H) : \mathfrak{g} \to \mathfrak{g}$ is symmetric with respect to the associated positive definite inner product $\langle \cdot \, , \, \cdot \rangle$, defined by (51). This implies that $\mathrm{ad}(H)$ has real eigenvalues, and admits a diagonalization with respect to a suitable orthonormal basis of $\mathfrak{g}$. Using this fact, we will introduce a kind of root space decomposition as follows.

By a maximal abelian subspace of $\mathfrak{p}$ we mean a commutative subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is maximal subject to the condition that $\mathfrak{a} \subset \mathfrak{p}$. We fix such a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Given $\lambda \in \mathfrak{a}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$ we define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X, \ \ \forall H \in \mathfrak{a}\}.$$

Lemma 16.1. *Let $\lambda, \mu \in \mathfrak{a}^*$. Then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.*

The proof, based on the fact that $\mathrm{ad}(H)$ is a derivation of $\mathfrak{g}$ for every $H \in \mathfrak{a}$, is standard and left to the reader.

Lemma 16.2. *Let $\lambda, \mu \in \mathfrak{a}^*$ be such that $\lambda + \mu \neq 0$. Then $\mathfrak{g}_\lambda \perp \mathfrak{g}_\mu$ relative to the Killing form.*

Proof. There exists $H \in \mathfrak{a}$ such that $\lambda(H) \neq -\mu(H)$. Let $X \in \mathfrak{g}_\lambda$ and $Y \in \mathfrak{g}_\mu$. Then

$$\lambda(H)B(X,Y) = B(\,\mathrm{ad}(H)X, Y) = -B(X, \mathrm{ad}(H)Y) = -\mu(H)B(X,Y).$$

It follows that $B(X,Y) = 0$. $\square$

Definition 16.3. A root of $\mathfrak{a}$ in $\mathfrak{g}$ is a real linear functional $\alpha \in \mathfrak{a}^*$ such that $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The set of these roots is denoted by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$.

The following result amounts to the mentioned root space decomposition.

Lemma 16.4. *The set Σ is finite. Furthermore,*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha. \tag{57}$$

as a direct sum of real linear spaces.

Proof. If H_1, H_2 then the endomorphisms $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$ of $\mathfrak{g}$ are diagonalizable with real eigenvalues, and commute with each other, hence preserve each others eigenspaces. It follows that $\mathfrak{g}$ admits a decomposition as a finite direct sum of spaces $\mathfrak{g}[j]$, $1 \leq j \leq k$, such that $\mathrm{ad}(H)$ acts by a scalar on $\mathfrak{g}[j]$ for each j and every $H \in \mathfrak{a}$. Hence, for each j there exists a unique linear functional $\lambda_j \in \mathfrak{a}^*$ sucht that $\mathrm{ad}(H) = \lambda_j(H)I$ on $\mathfrak{g}[j]$. Let $\Sigma' := \{\lambda_j \mid 1 \leq j \leq k\} \setminus \{0\}$. Then $\Sigma = \Sigma'$ and we obtain (57) with $\mathfrak{g}_\alpha = \oplus_{\lambda_j = \alpha} \mathfrak{g}[j]$, for $\alpha \in \Sigma \cup \{0\}$. $\square$

One may now wonder whether the pair $(\mathfrak{a}^*, \Sigma)$ is a root system according to the definition given before in the context of the theory of compact (or complex) semisimple Lie algebras. At a later stage will see that this is almost true, but not completely.

Definition 16.5. A possibly non-reduced root system is a pair (E,Σ) consting of a finite dimensional real linear space E together with a finite subset $\Sigma \subset E\setminus\{0\}$ such that the following assertions are valid.

(a) The set Σ spans E.

(b) For every $\alpha\in\Sigma$ there exists a (necessarily unique) reflection $s_\alpha : E\to E$ with $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\Sigma)\subset\Sigma$.

(c) For all $\alpha,\beta\in\Sigma$ we have $s_\alpha(\beta)\in\beta+\mathbb{Z}\alpha$.

Remark 16.6. We recall that a reflection in E in a point $\alpha\in E\setminus\{0\}$ is a linear map $s: E\to E$ such that $s(\alpha) = -\alpha$ and $\mathbb{R}\alpha\oplus\ker(s-I) = E$.

Let F be the group of $\varphi\in\mathrm{GL}(E)$ with $\varphi(\Sigma)\subset\varphi$. Since Σ spans E, the map $\varphi\mapsto\varphi|_\Sigma$ is an embedding of F into the finite group of permutations of Σ. In particular, F is finite, and there exists a positive definite inner product on E for which the elements of F are orthogonal. It follows that every reflection of E which preserves Σ is orthogonal, hence completely determined by its -1-eigenspace. Accordingly, there is at most one reflection s in a point $\alpha\in E\setminus\{0\}$ with $s(\Sigma)\subset\Sigma$.

Lemma 16.7. *Let (E,Σ) be a possibly non-reduced root system and let $\alpha\in\Sigma$. Then $-\alpha\in\Sigma$. Furthermore, there exists a $\beta\in\Sigma$ such that*

$$\mathbb{R}\alpha\cap\Sigma\subset\{\pm\frac{1}{2}\beta,\pm\beta\}.$$

Proof. Let s_α be the uniquely determined reflection of condition (b) of Definition 16.5. Then $-\alpha = s_\alpha(\alpha)\in\Sigma$. Thus we see that Σ is invariant under the map $\gamma\mapsto-\gamma$.

Let $\beta\in\mathbb{R}_{>0}\alpha\cap\Sigma$ be such that $t\beta\notin\Sigma$ for $t>1$. Then $s_\beta = -I$ on $\mathbb{R}\beta = \mathbb{R}\alpha$. Thus, if $\gamma\in\mathbb{R}\alpha\cap\Sigma$ then $-\gamma = s_\beta(\gamma)\in\gamma+\mathbb{Z}\beta$ from which we infer that $\gamma\in\frac{1}{2}\mathbb{Z}\beta$ hence $\gamma\in\{\frac{1}{2}\beta,\beta\}$. We thus see that $\mathbb{R}_{>0}\alpha\cap\Sigma\subset\{\frac{1}{2}\beta,\beta\}$. The result follows. □

Remark 16.8. A possibly non-reduced root system in the above sense is said to be reduced if it satisfies the following familiar additional condition

(d) For every $\alpha\in\Sigma$ we have $\mathbb{R}\alpha\cap\Sigma = \{-\alpha,\alpha\}$.

The system Σ is said to be non-reduced if it is not reduced. It is clear that the above definition of reduced root system coincides with the old definition of root system, that we encountered in the contex of compact semisimple Lie algebras.

In the course of this section, we will show that $(\mathfrak{a}^*,\Sigma)$ (with $(\Sigma = \Sigma(\mathfrak{g},\mathfrak{a}))$ is a possibly non-reduced root system in the above sense. In the exercises, we will encounter examples which show that Σ need not be reduced.

Given $\alpha\in\Sigma$ we define the associated root hyperplane $\ker\alpha\subset\mathfrak{a}$ by

$$\ker\alpha := \{H\in\mathfrak{a}\mid\alpha(H) = 0\}.$$

Lemma 16.9.

(a) $\cap_{\alpha\in\Sigma}\ker\alpha = \{0\}$;

(b) *the set* Σ *spans* $\mathfrak{a}^*$;

(c) *if* $\alpha\in\Sigma$ *then* $-\alpha\in\Sigma$ *and* $\theta(\mathfrak{g}_\alpha)=\mathfrak{g}_{-\alpha}$.

Proof. Let $H\in\mathfrak{a}$ be such that $\alpha(H)=0$ for all $\alpha\in\Sigma$. Then $\mathrm{ad}H$ is zero on each of the root spaces $\mathfrak{g}_\alpha$, for $\alpha\in\Sigma$. In view of the root space decomposition this implies that $\mathrm{ad}(H)=0$. Since $\mathfrak{g}$ has trivial center, it follows that $H=0$. Thus, (a) is valid.

We turn to (b). Assume to the contrary the span $\mathfrak{s}$ of Σ is a proper subspace of $\mathfrak{a}^*$. Then there exists a non-zero linear functional $\xi\in\mathfrak{a}^{**}$ such that $\xi=0$ on $\mathfrak{s}$. Let $H\in\mathfrak{a}$ be the canonical image of ξ for the canonical isomorphism $\mathfrak{a}\simeq\mathfrak{a}^{**}$. Then $H\neq 0$ and $\alpha(H)=0$ for all $\alpha\in\Sigma$, contradicting (a).

Let $\alpha\in\Sigma$. Then for every $H\in\mathfrak{a}$ and $X_\alpha\in\mathfrak{g}_\alpha$ we have

$$[H,\theta(X_\alpha)]=\theta[\theta(H),X_\alpha]=-\theta[H,X_\alpha]=-\alpha(H)\theta(X_\alpha).$$

This implies that $\{0\}\subsetneq\theta(\mathfrak{g}_\alpha)\subset\mathfrak{g}_{-\alpha}$. Hence $-\alpha\in\Sigma$. By the same reasoning it follows that $\theta(\mathfrak{g}_{-\alpha})\subset\mathfrak{g}_\alpha$ and the result follows. $\square$

Corollary 16.10. *The root space decomposition (57) is orthogonal with respect to the Cartan inner product* $\langle\,\cdot\,,\,\cdot\,\rangle$.

Proof. Let $\lambda,\mu\in\mathfrak{a}^*$, $\lambda\neq\mu$. Let $X\in\mathfrak{g}_\lambda$ and $Y\in\mathfrak{g}_\mu$. Then $\theta Y\in\mathfrak{g}_{-\mu}$ by Lemma 16.9 (c), hence it follows from Lemma 16.2 that $\langle X\,,\,Y\rangle=-B(X,\theta Y)=0$. $\square$

Lemma 16.11. $\mathfrak{a}=\mathfrak{g}_0\cap\mathfrak{p}$.

Proof. Let $Z\in\mathfrak{g}_0\cap\mathfrak{p}$. Then $\mathfrak{b}=\mathfrak{a}\oplus\mathbb{R}Z$ is an abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$. By maximality of $\mathfrak{a}$ it follows that $\mathfrak{b}=\mathfrak{a}$ hence $Z\in\mathfrak{a}$ and we see that $\mathfrak{g}_0\cap\mathfrak{p}\subset\mathfrak{a}$. The converse inclusion is obvious. $\square$

We define $\mathfrak{m}$ to be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, i.e.,

$$\mathfrak{m}:=\mathfrak{k}\cap\mathfrak{g}_0.$$

Lemma 16.12. $\mathfrak{g}_0=\mathfrak{m}\oplus\mathfrak{a}$.

Proof. Since θ preserves $\mathfrak{a}$, it follows that $\theta(\mathfrak{g}_0)=\mathfrak{g}_0$. This implies that

$$\mathfrak{g}_0=(\mathfrak{g}_0\cap\mathfrak{k})\oplus(\mathfrak{g}_0\cap\mathfrak{p})=\mathfrak{m}\oplus\mathfrak{a}.$$

$\square$

Lemma 16.13. *Let* $\alpha\in\Sigma$, $X\in\mathfrak{g}_\alpha$ *and* $Y\in\mathfrak{g}_{-\alpha}$. *Then* $[X,Y]\in\mathfrak{m}\oplus\mathfrak{a}$ *and*

$$B([X,Y],H)=\alpha(H)B(X,Y),\qquad (H\in\mathfrak{a}).\tag{58}$$

Remark 16.14. Only in case $\mathfrak{m} = 0$ it can be concluded that $[X,Y] \in \mathfrak{a}$, so that $[X,Y]$ is completely determined by (58).

Proof. We have $[X,Y] \in \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. Let $H \in \mathfrak{a}$. Then

$$B([X,Y],H) = -B(Y,[X,H]) = B(Y,[H,X]) = \alpha(H)B(Y,X) = \alpha(H)B(X,Y).$$

□

Given $\alpha \in \Sigma$ we denote by H_α the element of $\mathfrak{a}$ characterized by the properties

$$H_\alpha \perp \ker\alpha \quad \text{and} \quad \alpha(H_\alpha) = 2.$$

Lemma 16.15. *Let $X \in \mathfrak{g}_\alpha \setminus \{0\}$ then there exists $X_\alpha \in \mathbb{R}^+X$ such that H_α, X_α and $Y_\alpha := -\theta X_\alpha$ form a standard $\mathfrak{sl}(2)$-triple.*

Proof. Put $Y = -\theta X$. Then $[X,Y] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. For $H \in \ker\alpha$ we have, by (58),

$$B([X,Y],H) = \alpha(H)B(X,Y) = 0,$$

so that $[X,Y] \perp \ker\alpha$. It follows that $[X,Y] = cH_\alpha$ for a constant $c \in \mathbb{R}$. Substituting $H = H_\alpha$ in (58) we find

$$cB(H_\alpha, H_\alpha) = -2B(X, \theta X) > 0$$

so that $c > 0$. Taking $X_\alpha = c^{-1/2}X_\alpha$, we find $[X_\alpha, Y_\alpha] = H_\alpha$. The remaining commutator relations are obvious. □

Lemma 16.16. *Let $\varphi \in \mathrm{Aut}(\mathfrak{g})$ be such that $\varphi(\mathfrak{a}) \subset \mathfrak{a}$. Then $\varphi^{*-1} : \mathfrak{a}^* \to \mathfrak{a}^*$ preserves Σ. Furthermore, writing $\varphi(\alpha) := \varphi^{*-1}\alpha = \alpha \circ \varphi^{-1}$, we have*

$$\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\varphi\alpha} \tag{59}$$

Proof. This is a straightforward consequence of the definitions. □

Because of the obvious notational convenience of (59), we agree to use the notation φ for $(\varphi|_\mathfrak{a})^{*-1} : \mathfrak{a}^* \to \mathfrak{a}^*$ is φ is an automorphism of $\mathfrak{g}$ that leaves $\mathfrak{a}$ invariant.

Lemma 16.17. *Let $\alpha \in \Sigma$. Then there exists an interior automorphism $\varphi = \varphi_\alpha \in \mathrm{Int}(\mathfrak{g})$ such that*

(a) $\varphi(\mathfrak{a}) \subset \mathfrak{a}$;

(b) $\varphi(\alpha) = -\alpha$.

(c) $\varphi = I$ *on* $\ker\alpha$.

Proof. Let $X_\alpha \in \mathfrak{g}_\alpha$ be such that H_α, X_α and $Y_\alpha = -\theta X_\alpha$ form a standard $\mathfrak{sl}_2$-triple. Then as in the theory of root systems for a compact semisimple algebra, we put

$$U_\alpha = \frac{\pi}{2}(X_\alpha + Y_\alpha), \quad \varphi_\alpha := e^{\operatorname{ad}(U_\alpha)}.$$

As in the mentioned theory it is verified that φ_α satisfies all assertions. For details, including motivation, see [Ban10, Lemma 36.7]. $\square$

Corollary 16.18. *There exists a reflection $s_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$ in α such that $s_\alpha(\Sigma) = \Sigma$. If φ is an automorphism as in Lemma 16.17 then $s_\alpha = \varphi$ on $\mathfrak{a}^*$.*

Remark 16.19. Since Σ spans $\mathfrak{a}^*$, the reflection s_α is uniquely determined by the mentioned property. For details, see Remark 16.6 or [Ban10, Lemma 36.10].

Lemma 16.20. *Let $\alpha, \beta \in \Sigma$. Then $s_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$.*

Proof. Let $X_\alpha \in \mathfrak{g}_\alpha$ be such that H_α, X_α and $Y_\alpha = -\theta X_\alpha$ form a standard $\mathfrak{sl}_2$-triple. Let $\mathfrak{s}_\alpha$ be the linear span of H_α, X_α and Y_α, then $\mathfrak{s}_\alpha$ is a subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Put

$$V := \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$$

Then V is a $\operatorname{ad}(\mathfrak{s}_\alpha$-invariant, and thus becomes a $\mathfrak{s}_\alpha$ module. Define U_α and $\varphi := \varphi_\alpha$ as in the proof of Lemma 16.17. Then $s_\alpha(\beta) = \varphi(\beta)$. The endomorphism $\operatorname{ad}(U_\alpha)$ preserves V and therefore, so does $\varphi := \varphi_\alpha$. Hence,

$$\mathfrak{g}_{s_\alpha(\beta)} = \mathfrak{g}_{\varphi\beta} = \varphi(\varphi_\beta) \subset V$$

and we see that $s_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$. $\square$

In the course of this section we have shown that the pair $(\mathfrak{a}^*, \Sigma)$ satisfies all conditions of Definition 16.5, hence is a (possibly non-reduced) root system.

We end this section with some remarks that show for possibly non-reduced root systems, how the associated theory of positive systems, fundamental systems and Weyl groups can immediately be obtained from the analogous theory for reduced root systems.

Let (E, Σ) be a possibly non-reduced root system.

Lemma 16.21. *Let $\alpha, 2\alpha \in \Sigma$. Then $s_\alpha = s_{2\alpha}$.*

Proof. The linear map $s_\alpha : E \to E$ is uniquely determined by the requirements that it is a reflection which preserves Σ and maps α to $-\alpha$. By linearity, s_α maps 2α to 2α hence is a reflection in 2α as well. By uniqueness we see that $s_\alpha = s_{2\alpha}$. $\square$

A root $\alpha \in \Sigma$ is said to be reduced if $\alpha/2 \notin \Sigma$. The set of reduced roots is denoted by $\Sigma_\circ$. The easy proof of the following result is left to the reader.

Lemma 16.22. *The pair $(E, \Sigma_\circ)$ is a reduced root system.*

We define the Weyl group $W = W(\Sigma)$ of the pair (E, Σ) to be the subgroup of $\mathrm{GL}(E)$ generated by the reflections s_α for $\alpha \in \Sigma$. It follows from Lemmas 16.21 and 16.7 that W is already generated by the reflections s_α for $\alpha \in \Sigma_\circ$, hence equal to the Weyl group $W(\Sigma_\circ)$ of the reduced system (E, Σ_0). In particular, W is finite.

Definition 16.23. A *system of positive roots* for Σ is a subset $\Pi \subset \Sigma$ such that $\Sigma = \Pi \cap (-\Pi)$ and such that the Π and $-\Pi$ are separated by a hyperplane in E.

The following lemma is obvious.

Lemma 16.24. *Let Π be a positive system for Σ. Then $\Pi_\circ := \Pi \cap \Sigma^+$ is a positive system for $\Sigma_\circ$. The map $\Pi \mapsto \Pi_\circ$ is a bijection of the set of positive systems for Σ to the set of positive systems for $\Sigma_\circ$.*

Finally, a fundamental system of Σ is a subset $\Delta \subset \Sigma$ such that Δ is a basis of the real linear space E and such that

$$\Sigma \subset \mathbb{N}\Delta \cup (-\mathbb{N}\Delta).$$

Here $\mathbb{N}\Sigma$ denotes the set of linear combinations of the form $\sum_{\alpha \in \Delta} k_\alpha \alpha$ with coefficients $k_\alpha \in \mathbb{N} = \{0, 1, \dots\}$. Clearly, the fundamental systems of Σ coincide with the fundamental systems of $\Sigma_\circ$.

17 The Iwasawa decomposition

We retain the notation of the previous section. A point in $\mathfrak{a}$ is said to be regular, if it is not contained in any of the root hyperplanes $\ker \alpha$, for $\alpha \in \Sigma$. The set of regular points is denoted by $\mathfrak{a}^{\mathrm{reg}}$. Being the complement of a finite union of hyperplanes, $\mathfrak{a}^{\mathrm{reg}}$ is an open dense subset of $\mathfrak{a}$. Its connected components are called the (open) Weyl chambers in $\mathfrak{a}$.

If C is such a Weyl chamber, then every root $\alpha \in \Sigma$ is either positive or negative on the entire chamber C. We put

$$\Pi_C := \{\alpha \in \Sigma \mid \alpha > 0 \text{ on } C\}.$$

Clearly, $\Sigma = \Pi_C \cup (-\Pi_C)$. Furthermore, if $H \in C$, then

$$\Pi_C = \{\alpha \in \Sigma \mid \alpha(H) > 0\}, \text{ and } -\Pi_C = \{\alpha \in \Sigma \mid \alpha(H) < 0\}.$$

Viewing H as an element of $\mathfrak{a}^{**}$ we see that the sets Π_C and $-\Pi_C$ are separated by the hyperplane $\ker H$. Thus, Π_C is a system of positive roots. Conversely, if Π is a system of positive roots, then

$$C_\Pi := \{H \in \mathfrak{a} \mid \alpha(H) > 0, \forall \alpha \in \Pi\}$$

is a Weyl chamber. We thus see that the set of Weyl chambers is finite and that the map $C \to \Pi_C$ is a bijection from the set of Weyl chambers to the set of positive systems for Σ. Furthermore, we see that each Weyl chamber is the intersection of a finite number of open half spaces, hence an open polyhedral cone.

We fix a positive system Σ^+ for Σ. The associated Weyl chamber is referred to as the positive Weyl chamber and is denoted by $\mathfrak{a}^+$. Thus, $\mathfrak{a}^+$ equals the set of $H \in \mathfrak{a}$ such that $\alpha(H) > 0$ for all $\alpha \in \Sigma^+$.

Lemma 17.1. *The space* $\oplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ *a subalgebra of* $\mathfrak{g}$.

Proof. We denote the above space by $\mathfrak{v}$. Let $\alpha,\beta\in\Sigma^+$ and let $X_\alpha\in\mathfrak{g}_\alpha$ and $X_\beta\in\mathfrak{g}_\beta$. Then by linearity it suffices to show that $Y:=[X_\alpha,X_\beta]\in\mathfrak{v}$. From Lemma 16.2 we see that $Y\in\mathfrak{g}_{\alpha+\beta}$. If $\alpha+\beta\notin\Sigma$ then $Y=0\in\mathfrak{v}$. On the other hand, if $\alpha+\beta\in\Sigma$ then $\alpha+\beta>0$ on $\mathfrak{a}^+$ and we see that $\alpha+\beta\in\Sigma^+$. Hence $Y\in\mathfrak{v}$. $\square$

We define the following subalgebras of $\mathfrak{g}$,

$$\mathfrak{n}:=\bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha,\qquad \bar{\mathfrak{n}}=\bigoplus_{\alpha\in-\Sigma^+}\mathfrak{g}_{-\alpha}.$$

The in view of Lemma 16.9 (c) it follows that $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$. Furthermore, from the root space decomposition it follows that

$$\mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n},\tag{60}$$

as a direct sum of real linear spaces. Finally, this decomposition is orthogonal in view of Corollary 16.10.

Lemma 17.2. *The map* $\varphi:\bar{\mathfrak{n}}\to\mathfrak{k},\,Y\mapsto Y+\theta(Y)$ *induces a linear isomorphism* $\bar{\varphi}:\bar{\mathfrak{n}}\to\mathfrak{k}/\mathfrak{m}$.

Proof. Let $Y\in\bar{\mathfrak{n}}$. Then $Y+\theta Y\in\mathfrak{g}^\theta=\mathfrak{k}$, so that φ is indeed a linear map $\bar{\mathfrak{n}}\to\mathfrak{k}$. Assume $\varphi(Y)=X_0\in\mathfrak{m}$. Then

$$0=Y-X_0+\theta(Y),\quad\text{with}\quad Y\in\bar{\mathfrak{n}},\,X_0\in\mathfrak{m},\,\theta(Y)\in\mathfrak{n}.$$

By directness of the sum in (60) it follows that $Y=0$. Thus, the induced map $\bar{\varphi}$ is injective. For its surjectivity, assume that $U\in\mathfrak{k}$. Then we may decompose $U=Y+V+H+X$ with $Y\in\bar{\mathfrak{n}}$, $V\in\mathfrak{m}$, $H\in\mathfrak{a}$, $X\in\mathfrak{n}$. From $U=\theta U$ it follows that

$$Y+V+H+X=\theta(X)+V-H+\theta(Y),\quad\text{with}\ \theta(X)\in\bar{\mathfrak{n}},\,\theta(Y)\in\mathfrak{n}.$$

By directness of the sum in (60) we infer that $H=0$ and $X=\theta Y$. It follows that

$$U+\mathfrak{m}=Y+\theta(Y)+V+\mathfrak{m}=\bar{\varphi}(Y),$$

whence the surjectivity. $\square$

The decomposition described in the following lemma is known as the infinitesimal Iwasawa decomposition.

Lemma 17.3. *The Lie algebra* $\mathfrak{g}$ *admits the following decomposition as a direct sum of real linear spaces,*

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}\tag{61}$$

Proof. Let $U \in \mathfrak{g}$. Then according to (60) we may write

$$U = Y + V + H + X, \quad \text{with} \quad Y \in \bar{\mathfrak{n}},\ V \in \mathfrak{m},\ H \in \mathfrak{a},\ X \in \mathfrak{n}.$$

It follows that

$$U = (Y + \theta(Y)) + V + H + (X - \theta Y)$$

with $Y + \theta(Y) + V \in \mathfrak{g}^\theta = \mathfrak{k}$, $H \in \mathfrak{a}$ and $X - \theta(Y) \in \mathfrak{n}$. This implies that (61) holds as a vector sum, i.e., with everywhere $+$ in place of $\oplus$. We will conclude the proof by counting dimensions.

It follows from Lemma 17.2 that $\dim(\mathfrak{k}) = \dim(\bar{\mathfrak{n}}) + \dim(\mathfrak{m})$. Using directness of the sum in (60) we now infer that

$$\dim(\mathfrak{g}) = \dim(\mathfrak{k}) + \dim(\mathfrak{a}) + \dim(\mathfrak{n})$$

The proof is complete. $\square$

In the remainder of this section we will show that the infinitesimal Iwasawa decomposition (61) has a global version. Let G be a connected semisimple Lie group with algebra $\mathfrak{g}$. Let K be the connected Lie subgroup of G with algebra $\mathfrak{k}$. From the Cartan decomposition for G it follows that K is a closed subgroup. We have seen it is compact if and only if G has finite center.

Let A be the connected Lie subgroup of G with Lie algebra $\mathfrak{a}$. Since $\mathfrak{a}$ is abelian it follows that $A = \exp(\mathfrak{a})$.

Lemma 17.4. *The group A is closed in G and* $\exp : \mathfrak{a} \to A$ *is a diffeomorphism.*

Proof. According to the Cartan decomposition $(k, X) \mapsto k \exp X$ is a diffeomorphism of $K \times \mathfrak{p}$ onto G. Since $\{e\} \times \mathfrak{a}$ is a closed submanifold of $K \times \mathfrak{p}$ it follows that $\exp : \mathfrak{a} \to G$ is a diffeomorphism onto a closed submanifold of G. The result follows. $\square$

Since $\mathfrak{a}$ is abelian, the map $\exp : \mathfrak{a} \to A$ is in fact an isomorphism of the additive Lie group $(\mathfrak{a}, +, 0)$ onto A. We define $\log : A \to \mathfrak{a}$ to be inverse to this isomorphism. Let $\lambda \in \mathfrak{a}^*$. The we define

$$a^\lambda := e^{\lambda(\log a)}, \qquad (a \in A).$$

Since $\lambda : \mathfrak{a} \to \mathbb{R}$ is a character of $(\mathfrak{a}, +, 0)$, it follows that $(\,\cdot\,)^\lambda$ is a character $A \to]0, \infty[$.

Lemma 17.5. *The adjoint action of A on $\mathfrak{g}$ preserves the root space decomposition. Furthermore, if $\alpha \in \Sigma \cup \{0\}$, then*

$$\mathrm{Ad}(a) = a^\alpha I \quad \text{on} \ \mathfrak{g}_\alpha.$$

In particular, $\mathrm{Ad}(a)$ *preserves the subalgebra* $\mathfrak{n}$.

Proof. Let $H \in \mathfrak{a}$. Then $\mathrm{Ad}(\exp H) = e^{\mathrm{ad}(H)}$. Since $\mathrm{ad}(H)$ preserves the root space decomposition, so does $a = \exp(H)$. Furthermore, $\mathrm{ad}(H)$ acts on $\mathfrak{g}_\alpha$ by the scalar $\lambda(H)$. It follows that on $\mathfrak{g}_\alpha$ we have

$$\mathrm{Ad}(a) = e^{\mathrm{ad}(H)} = e^{\alpha(H)} I = e^{\alpha(\log a)} I = a^\alpha I.$$

The result follows. $\square$

Let N be the connected Lie subgroup with G with Lie algebra $\mathfrak{n}$.

Theorem 17.6. *The map*

$$\varphi : (k,a,n) \mapsto kan, \;\; K \times A \times N \to G \tag{62}$$

is a diffeomorphism.

Remark 17.7. In particular, the theorem implies that N is a closed subgroup of G.

The first important step towards the proof is presented by the following lemma.

Lemma 17.8. *The map (62) has bijective derivative everywhere.*

Proof. By left K-equivariance, and right N-equivariance, it suffices to show that $d\varphi(e,a,e) : \mathfrak{k} \times T_aA \times \mathfrak{n} \to \mathfrak{g}$ is bijective. Let $U \in \mathfrak{k}$, $H \in \mathfrak{a}$ and $V \in \mathfrak{n}$. Then for $t \in \mathbb{R}$ we have

$$\varphi(\exp tU, \exp tHa, \exp tV)a^{-1} = \exp(tU)\exp(tH)\exp(t\mathrm{Ad}(a)V)$$

Differentiating at $t = 0$ we find

$$dr_{a^{-1}}(a) \circ d\varphi(U, dr_a(e)H, V) = U + H + \mathrm{Ad}(a)V.$$

We thus see that it suffices to show that the map $(U,H,V) \mapsto U + H + \mathrm{Ad}(a)V$ is a linear isomorphism from $\mathfrak{k} \times \mathfrak{a} \times \mathfrak{n}$ onto $\mathfrak{g}$. Since $\mathrm{Ad}(a)|_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$ is a linear isomorphism, this follows from the infinitesimal Iwasawa decomposition. $\square$

Our next step will be to prove Theorem 17.6 for the group $G = \mathrm{SL}(n,\mathbb{R})$, $n \geq 2$. To prepare for this, we need the following result.

Lemma 17.9. *Let $n \geq 1$ and let $\mathfrak{u}$ be the space of upper triangular matrices in $\mathrm{M}(n,\mathbb{R})$ with diagonal entries equal to zero. Let U be the subset of upper triangular matrices in $\mathrm{M}(n,\mathbb{R})$ with diagonal entries equal to 1. Then the following assertions are valid.*

(a) *U is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ with Lie algebra equal to $\mathfrak{u}$.*

(b) *The exponentional map $\exp : X \mapsto e^X$ maps $\mathfrak{u}$ diffeomorphically onto (the submanifold) U.*

(c) *There exists a polynomial map $\mu : \mathfrak{u} \times \mathfrak{u} \to \mathfrak{u}$ such that $\exp \mu(X,Y) = \exp(X)\exp(Y)$ for all $X, Y \in \mathfrak{u}$.*

(d) *Let $\mathfrak{n} \subset \mathfrak{u}$ be a sub Lie algebra. Then $N := \exp(\mathfrak{n})$ is a connected closed subgroup of U. Its Lie algebra equals $\mathfrak{n}$.*

Proof. We note that $\mathfrak{u}$ is a subalgebra of $\mathrm{M}(n,\mathbb{R})$, equipped with matrix multiplication. Hence, it is also a Lie subalgebra for the commutator bracket. Furthermore, $\mathfrak{u}$ is nilpotent in the sense that $X^n = 0$ for all $X \in \mathfrak{u}$. The set $U \subset \mathrm{M}(n,\mathbb{R})$ may be viewed as the translate $I + \mathfrak{u}$ of $\mathfrak{u}$. In particular, U is a closed submanifold of $\mathrm{M}(n,\mathbb{R})$, with tangent space everywhere equal to $\mathfrak{u}$.

If $x_1, x_2 \in U$ then $x_j = I + X_j$, with $X_j \in \mathfrak{u}$ and we see that $x_1x_2 = I + X_1 + X_2 + X_1X_2 \in I + \mathfrak{u} = U$. Furthermore, if $x \in U$ then $x = I - (I - x)$, so that

$$x^{-1} = I + \sum_{k \geq 1} (I - x)^k.$$

By nilpotency of $I-x$ the terms become zero for $k \geq 0$ and we see that $x^{-1} \in U$. It follows that U is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$. We observed already that its tangent space at I equals $\mathfrak{u}$, so that $\mathfrak{u}$ equipped with the commutator bracket is the Lie algebra of U. This establishes (a).

We turn to (b). By the usual power series expansion for exp we see that $\exp(\mathfrak{u}) \subset I+\mathfrak{u} = U$, and the exponential map is given by

$$\exp X = \sum_{k=0}^{n-1} \frac{1}{k!} X^k, \qquad (X \in \mathfrak{u}).$$

In particular, $\exp|_{\mathfrak{u}}$ is polynomial with respect to the (associative) algebra structure of $\mathfrak{u}$.

Let $F : \mathfrak{u} \to \mathfrak{u}$ be the polynomial map given by

$$F(Y) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} Y^k,$$

and define the map $L : U \to \mathfrak{u}$ by $L(x) = F(x-I)$. We will show that the smooth map L is the two sided inverse to $\exp : \mathfrak{u} \to U$.

First, we claim that $L(\exp X) = X$ for $X \in \mathfrak{u}$. To see this, let $X \in \mathfrak{u}$. Then

$$\begin{aligned} \frac{d}{dt} L(\exp(tX)) &= \frac{d}{dt} \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} (e^{tX} - I)^k \\ &= \sum_{k=1}^{n} (-1)^{k-1} (e^{tX} - I)^{k-1} e^{tX} X \\ &= \sum_{k=1}^{n} (I - e^{tX})^{k-1} e^{tX} X \end{aligned}$$

Write $Y = I - e^{tX}$, then $Y^n = 0$, hence

$$\sum_{k=1}^{n} Y^{k-1} = (I - Y^n)(I-Y)^{-1} = e^{-tX}$$

and we see that

$$\frac{d}{dt} L(\exp(tX)) = X.$$

As $L(I) = F(0) = 0$, this implies that $L(\exp tX) = tX$ for all $X \in \mathfrak{u}$ and $t \in \mathbb{R}$. We conclude that $L \circ \exp = I$ on $\mathfrak{u}$. This implies that exp is injective on $\mathfrak{u}$ and a local diffeomorphism everywhere. Furthermore, L is a local inverse to exp at I and we find that $\exp \circ L = I$ in an open neighborhood of I in U. Equivalently,

$$\exp(F(Z)) = I + Z \tag{63}$$

for Z in an open neighborhood of 0 in $\mathfrak{u}$. As both F and exp are polynomial maps, (63) is an identity of polynomials in $Y \in \mathfrak{u}$. Being valid in an open neighborhood of 0 in $\mathfrak{u}$, it must be valid for all $Y \in \mathfrak{u}$. This implies that $\exp \circ L = I$ and establishes (b).

We turn to (c). If $X, Y \in \mathfrak{u}$ then $\mu(X,Y) = L(\exp(X)\exp(Y))$ depends in a polynomial fashion on (X,Y) and $\exp(\mu(X,Y)) = \exp(X)\exp(Y)$.

For (d), assume that $\mathfrak{n}$ is as stated, and let N be the connected Lie subgroup of U with Lie algebra $\mathfrak{n}$. Then since $\exp|_{\mathfrak{n}} : \mathfrak{n} \to N$ is a local diffeomorphism at 0 we infer that there exists an open neighborhood Ω of 0 in $\mathfrak{n}$ such that $\exp(\Omega)\exp(\Omega) \subset \exp(\mathfrak{n})$. By definition of μ this implies that μ maps $\Omega \times \Omega$ to $\mathfrak{n}$. Since μ is polynomial it follows that μ maps $\mathfrak{n} \times \mathfrak{n}$ into $\mathfrak{n}$. It follows that $\exp : \mathfrak{n} \to N$ maps onto a subgroup of N containing $\exp(\mathfrak{n})$ hence $N_e = N$ and we conclude that $N = \exp(\mathfrak{n})$. Finally, since $\exp : \mathfrak{u} \to U$ is a diffeomorphism and $\mathfrak{n}$ a closed subset of $\mathfrak{u}$, it follows that N is closed in U hence in $\mathrm{GL}(n,\mathbb{R})$. □

Example 17.10. We consider the group $G = \mathrm{SL}(n,\mathbb{R})$. Its Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ of consists of the matrices $X \in \mathrm{M}(n,\mathbb{R})$ of trace 0. The standard Cartan involution of $\mathfrak{sl}(n,\mathbb{R})$ is given by $\theta : X \mapsto -X^{\mathrm{T}}$. Accordingly, $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ equals $\mathfrak{s}_n$, the space of $X \in \mathfrak{sl}(n,\mathbb{R})$ with $X^T = -X$. The space $\mathfrak{a}$ of traceless diagonal matrices is abelian and contained in $\mathfrak{p}$. Let E_i^j be the matrix with every entry equal to zero, except for the entry in the j-th column and i-th row, which equals 1. Then it it readily seen that

$$[H, E_i^j] = (H_{ii} - H_{jj})E_i^j,$$

for all $1 \leq i, j \leq n$ and $H \in \mathfrak{a}$. It follows that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and that the associated root space decomposition is given by

$$\mathfrak{sl}(n,\mathbb{R}) := \mathfrak{a} \oplus \bigoplus_{i \neq j} \mathbb{R}E_i^j.$$

Furthermore, the associated set $\Sigma = \Sigma(\mathfrak{g},\mathfrak{a})$ of roots equals $\Sigma = \{\alpha_{ij} \mid 1\ leqi, j \leq n, i \neq j\}$, where $\alpha_{ij} \in \mathfrak{a}^*$ is given by

$$\alpha_{ij}(H) = H_{ii} - H_{jj}, \qquad (H \in \mathfrak{a}).$$

The set $\Sigma^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ is readily seen to be a positive system for Σ, and the associated subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is the linear span of the matrices E_i^j with $j > i$, i.e., the set of upper triangular matrices, with zero diagonal entries.

On the level of the group, $K = \mathrm{SO}(n)$ and $A = \exp \mathfrak{a}$ consists of the diagonal matrices with positive entries and determinant 1. The group N of upper triangular matrices with diagonal entries equal to one is the connected Lie subgroup of $\mathrm{SL}(n,\mathbb{R})$ with Lie algebra $\mathfrak{n}$. In fact, it is a connected closed subgroup, and the exponential map $\exp : \mathfrak{n} \to N$ is given by $\exp : X \mapsto e^X$. By Lemma 17.9 this map is a diffeomorphism from $\mathfrak{n}$ onto N.

Example 17.11. Flag manifold. By a (full) flag in $\mathbb{R}^n$ we mean an increasing sequence $0 \subset F_1 \subset \cdots \subset F_n = \mathbb{R}^n$ of linear subspaces in $\mathbb{R}^n$ with $\dim(F_j) = j$, for $1 \leq j \leq n$. The set of these flags is denoted by $\mathscr{F}$. The group $\mathrm{SL}(n,\mathbb{R})$ acts on $\mathscr{F}$ in a natural way; if $x \in \mathrm{SL}(n,\mathbb{R})$ and $F = (F_j) \in \mathscr{F}$, then

$$xF := (x(F_1), \ldots, x(F_n)).$$

We claim that this action is transitive. Indeed, let E be the standard flag in $\mathscr{F}$, i.e., $E = (E_j)$, where E_j is spanned by the first j standard basisvectors $e_1, \ldots, e_j$. Indeed, let $F = (F_j) \in \mathscr{F}$, then we may chose an orthonormal and positively oriented basis $f_1, \ldots, f_n$ of $\mathbb{R}^n$ such that F_j is spanned by $f_1, \dots, f_j$. Let $k : \mathbb{R}^n \to \mathbb{R}^n$ be the unique linear map determined by $ke_j = f_j$ for all $1 \leq j \leq n$. Then $k \in K = \mathrm{SO}(n)$ and we see that $F = k(E)$ for a suitable element $k \in \mathrm{SO}(n)$. Thus $\mathrm{SO}(n)$ acts hence the bigger group $\mathrm{SL}(n,\mathbb{R})$ acts transitively on $\mathscr{F}$. Let P be the stabilizer of E in $\mathrm{SL}(n,\mathbb{R})$. Then P is a closed subgroup. Furthermore, the action map $x \mapsto xE$ induces a bijection

$$\mathrm{SL}(n,\mathbb{R})/P \xrightarrow{\simeq} \mathscr{F}$$

through which we equip $\mathscr{F}$ with the structure of a smooth manifold. This manifold is called the manifold of full flags in $\mathbb{R}^n$.

We observe that P exists of all upper triangular matrices with determinant 1. Let M be the group of diagonal matrices with determinant 1 all of whose diagonal matrices are ± 1. Then it is readily seen that $P = MAN$, with A and N as in Example 17.10.

From the above it follows that $\mathrm{SO}(n)$ acts smoothly and transitively on $\mathscr{F}$. We now note that the stabilizer in $\mathrm{SO}(n)$ of the standard flag E equals $\mathrm{SO}(n) \cap P = M$.

Accordingly, the action map $k \mapsto kE$ induces a diffeomorphism

$$\mathrm{SO}(n)/M \xrightarrow{\simeq} \mathscr{F}.$$

In particular we see that the flag manifold is compact.

Lemma 17.12. *Let $K = \mathrm{SO}(n)$, $A \subset \mathrm{SL}(n,\mathbb{R})$ the group of diagonal matrices with positive entries and determinant 1 and $N \subset \mathrm{SL}(n,\mathbb{R})$ the group of upper triangular matrices with diagonal entries equal to 1. Then the map $\varphi : K \times A \times N \to \mathrm{SL}(n,\mathbb{R})$, $(k,a,n) \mapsto kan$ is a diffeomorphism.*

Proof. According to Example 17.10 the groups K, A and N are the connected Lie subgroups with algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an infinitesimal Iwasawa decomposition. By Lemma 17.8 it follows that φ is a local diffeomorphism everywhere, so that it suffices to show that φ is bijective.

To establish the surjectivity, we will relate the above groups to the manifold $\mathscr{F}$ of (full) flags in $\mathbb{R}^n$, defined in Example 17.11. Let E be the standard flag in $\mathbb{R}^n$. Then $\mathrm{SL}(n,\mathbb{R})E = \mathscr{F} = \mathrm{SO}(n)E$. Thus, if $x \in \mathrm{SL}(n,\mathbb{R})$, there exists $k \in \mathrm{SO}(n)$ such that $xE = kE$. This implies $k^{-1}xE = E$ from which we see that $k^{-1}x \in P$. We infer that

$$\mathrm{SL}(n,\mathbb{R}) = KP \subset KMAN = KAN.$$

Hence the map φ is surjective.

To see that the map is injective, assume that $k_j \in K$, $a_j \in A$ and $n_j \in N$, for $j = 1,2$ and that $k_1a_1n_1 = k_2a_2n_2$. The set T of upper triangular matrices with positive diagonal entries and determinant 1 is a subgroup of $\mathrm{SL}(n,\mathbb{R})$ to which both a_1n_1 and a_2n_2 belong. It follows that

$$k_2^{-1}k_1 = a_2n_2(a_1n_1)^{-1} \in \mathrm{SO}(n) \cap T.$$

By the orthogonality relations the latter intersection consists of only the identity matrix, and we find that $k_1 = k_2$ and $a_1n_1 = a_2n_2$. From the latter identity, we find that

$$a_2^{-1}a_1 = n_2n_1^{-1} \in A \cap N = \{I\},$$

hence $a_1 = a_2$ and $n_1 = n_2$. We infer that φ is injective. □

We are now prepared to prove Theorem 17.6. Let G be a connected real semisimple Lie group and let notation be as in the mentioned theorem.

Proof of Theorem 17.6. We equip $\mathfrak{g}$ with the Cartan inner product defined by (51). In the proof of Theorem 15.12 we have seen that $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})_e$ is a surjective homomorphism with discrete kernel, hence a covering. Its restriction to K has the same kernel and defines a covering homomorphism onto $\mathrm{Ad}(K) = \mathrm{Aut}(\mathfrak{g})_e \cap \mathrm{SO}(\mathfrak{g})$.

We fix an element $H_0 \in \mathfrak{a}$ in the positive Weyl chamber and in the complement of the finitely many hyperplanes $\ker(\alpha - \beta)$, for α, β distinct elements of $\Sigma \cup \{0\}$. There is now a unique numbering $\alpha_1, \dots, \alpha_d$ of the elements of $\Sigma \cup \{0\}$ such that $\alpha_1(H) < \cdots < \alpha_d(H)$. We agree to write $\mathfrak{g}_j := \mathfrak{g}_{\alpha_j}$, so that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_d. \tag{64}$$

Then for all $U \in \mathfrak{a}$ the endomorphism $\mathrm{ad}(X)$ preserves the decomposition (64) and acts by a real scalar on each summand. Furthermore, if $\alpha \in \Sigma^+$, $X \in \mathfrak{g}_\alpha$, and $Y \in \mathfrak{g}_k$, then

$$[X, \mathfrak{g}_k] \subset \mathfrak{g}_{\alpha + \alpha_k} \subset \bigoplus_{j>k} \mathfrak{g}_j,$$

since $(\alpha + \alpha_k)(H_0) > \alpha_k(H_0)$. It follows that

$$\mathrm{ad}(\mathfrak{n})\mathfrak{g}_k \subset \bigoplus_{j>k} \mathfrak{g}_j, \qquad (1 \le k \le d).$$

We now fix an orthonormal basis $e_1, \dots e_n$ of $\mathfrak{g}$ which is subordinate to the decomposition (64). Let $\underline{\mathfrak{k}}$ be the algebra of elements in $\mathfrak{sl}(\mathfrak{g})$ whose matrix is skew symmetric with respect to this basis. Let $\underline{\mathfrak{a}}$ be the subalgebra of elements of $\mathfrak{sl}(\mathfrak{g})$ which diagonalize with respect to this basis. Finally, let $\underline{\mathfrak{n}}$ be the subalgebra of elements of $\mathfrak{sl}(\mathfrak{g})$ whose basis is strict upper triangular with respect to this basis. Then

$$\mathfrak{sl}(\mathfrak{g}) = \underline{\mathfrak{k}} \oplus \underline{\mathfrak{a}} \oplus \underline{\mathfrak{n}} \tag{65}$$

is an infinitesimal Iwasawa decomposition for $\mathfrak{sl}(\mathfrak{g})$, see Example 17.10 for details. It follows from the above that

$$\mathrm{ad}(\mathfrak{k}) \subset \mathrm{ad}(\mathfrak{g}) \cap \underline{\mathfrak{k}}, \quad \mathrm{ad}(\mathfrak{a}) \subset \mathrm{ad}(\mathfrak{g}) \cap \underline{\mathfrak{a}}, \quad \mathrm{ad}(\mathfrak{n}) \subset \mathrm{ad}(\mathfrak{g}) \cap \underline{\mathfrak{n}}. \tag{66}$$

Since ad is injective, it follows from the infinitesimal Iwasawa decomposition of $\mathfrak{g}$ that

$$\mathrm{ad}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{k}) \oplus \mathrm{ad}(\mathfrak{a}) \oplus \mathrm{ad}(\mathfrak{n}). \tag{67}$$

Combining this with (65) and (66) we see that the decompositions (67) and (65) are compatible, and that the inclusions of (66) are in fact equalities.

Let $\underline{K} = \mathrm{SO}(\mathfrak{g})$, let $\underline{A}$ be the group of $x \in \mathrm{SL}(\mathfrak{g})$ with $xe_j \in \mathbb{R}_{>0}e_j$ for all j, and let $\underline{N}$ be the group of $x \in \mathrm{SL}(\mathfrak{g})$ such that $x - I$ is strict upper triangular with respect to this basis. Then the groups $\underline{K}$, $\underline{A}$ and $\underline{N}$ are connected closed subgroups of $\mathrm{SL}(n, \mathbb{R})$ with Lie algebras $\underline{\mathfrak{k}}, \underline{\mathfrak{a}}$ and $\underline{\mathfrak{n}}$, respectively. Since he inclusions (66) are equalities, it follows that $\mathrm{Ad}(K), \mathrm{Ad}(A)$ and $\mathrm{Ad}(N)$ equal the connected components of $\mathrm{Ad}(G) \cap \underline{K}$, $\mathrm{Ad}(G) \cap \underline{A}$ and $\mathrm{Ad}(G) \cap \underline{N}$ respectively. As

$\mathrm{Ad}(G = \mathrm{Aut}(\mathfrak{g})_e$ is closed in $\mathrm{SL}(\mathfrak{g})$, it follows that $\mathrm{Ad}(K)$, $\mathrm{Ad}(A)$ and $\mathrm{Ad}(N)$ are closed subgroups of $\underline{K}$, $\underline{A}$ and $\underline{N}$, respectively. By Lemma 17.12 it follows that the multiplication map

$$\underline{\varphi} : \underline{K} \times \underline{A} \times \underline{N} \to \mathrm{SL}(\mathfrak{g})$$

is a diffeomorphism. Its restriction to the product $\mathrm{Ad}(K) \times \mathrm{Ad}(A) \times \mathrm{Ad}(N)$ is an embedding onto a closed submanifold of $\mathrm{SL}(\mathfrak{g})$ hence of $\mathrm{Ad}(G)$. On the other hand, by Lemma 17.8 it follows that KAN is open in G and since $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is a covering we see that $\mathrm{Ad}(K)\mathrm{Ad}(A)\mathrm{Ad}(N) = \mathrm{Ad}(KAN)$ is open in $\mathrm{Ad}(G)$. Since $\mathrm{Ad}(G)$ is connected, we conclude that

$$\mathrm{Ad}(G) = \mathrm{Ad}(K)\mathrm{Ad}(A)\mathrm{Ad}(N).$$

Thus, the restriction of $\underline{\varphi}$ maps $\mathrm{Ad}(K) \times \mathrm{Ad}(A) \times \mathrm{Ad}(N)$ diffeomorphically onto the closed submanifold $\mathrm{Ad}(G)$ of $\mathrm{SL}(\mathfrak{g})$.

To complete the proof, we look at the commutative diagram

$$\begin{array}{ccc} K \times A \times N & \xrightarrow{\varphi} & G \\ {\scriptstyle \mathrm{Ad}\times\mathrm{Ad}\times\mathrm{Ad}} \downarrow & & \downarrow {\scriptstyle \mathrm{Ad}} \\ \mathrm{Ad}(K) \times \mathrm{Ad}(A) \times \mathrm{Ad}(N) & \xrightarrow{\underline{\varphi}} & \mathrm{Ad}(G) \end{array}$$

In this diagram, the vertical maps and the bottom horizontal map are coverings. From this we infer that φ is surjective, using unique lifting of curves in a similar fashion as in the proof of Theorem 15.12.

It remains to be shown that φ is injective.

For this we first note that by Lemma 17.9 the exponential map $\exp : \underline{\mathfrak{n}} \to \underline{N}$ maps $\mathrm{ad}(\mathfrak{n})$ diffeomorphically onto an open subgroup of $\mathrm{Ad}(N)$, hence onto the connected group $\mathrm{Ad}(N)$. In particular, it follows that $\mathrm{Ad}(N)$ is simply connected. Since $\mathrm{Ad} : N \to \mathrm{Ad}(N)$ is a covering homomorphism, it now follows that $\mathrm{Ad} : N \to \mathrm{Ad}(N)$ is a diffeomorphism. In particular, this map is injective.

Furthermore, since $\exp : \mathfrak{a} \to A$ a diffeomorphism $\exp : \underline{\mathfrak{a}} \to \underline{A}$ is a diffeomorphism, and $\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}$, it follows that $\mathrm{Ad} : A \to \mathrm{Ad}(A)$ is a diffeomorphism.

Finally, for the injectivity of φ, let $(k_j, a_j, n_j) \in K \times A \times N$ have the same image under φ, for $j = 1,2$. Then it follows that $\mathrm{Ad}(k_j)\mathrm{Ad}(a_j)\mathrm{Ad}(n_j)$ is independent of $j = 1,2$. By the injectivity of the map $\underline{\varphi}$ of () it follows that $\mathrm{Ad}(a_1) = \mathrm{Ad}(a_2)$ and $\mathrm{Ad}(n_1) = \mathrm{Ad}(n_2)$ so that $a_1 = a_2$ and $n_1 = n_2$. Since $k_1 a_1 n_1 = k_2 a_2 n_2$, it follows that also $k_1 = k_2$. Hence, φ is injective. $\square$

From the final part of the above proof, we can deduce the following result.

Lemma 17.13. *Let G be a connected real semisimple Lie group and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ an infinitesimal Iwasawa decomposition. Let N be the connected Lie subgroup of G with Lie algebra $\mathfrak{n}$. Then N is a closed subgroup and* $\exp : \mathfrak{n} \to N$ *is a diffeomorphism.*

Proof. Since N is the image of the closed subset $\{e\} \times \{e\} \times N$ of $K \times A \times N$ under the diffeomorphism given by the Iwasawa product map, it follows that N is closed in G. For the remaining statement, we consider the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\mathrm{Ad}} & \mathrm{Ad}(N) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{n} & \xrightarrow{\mathrm{ad}} & \mathrm{ad}(\mathfrak{n}) \end{array}$$

where the bottom horizontal map is a diffeomorphism. In the above proof of Theorem 17.6 we proved that $\mathrm{Ad} : N \to \mathrm{Ad}(N)$ and $\exp : \mathrm{ad}(\mathfrak{n}) \to \mathrm{Ad}(N)$ are diffeomorphisms as well. It follows that $\exp : \mathfrak{n} \to N$ is a diffeomorphism onto. $\square$

18 Harmonic analysis on real semisimple groups

Let G be a real semisimple Lie group with finite center. We have seen that $\mathrm{Ad}(G) \subset \mathrm{SL}(\mathfrak{g})$, so that G is unimodular. It follows that G can be equipped with a bi-invariant Haar measure dx. Furthermore, the left times right regular action $L \times R$ of $G \times G$ on $L^2(G, dx)$ is unitary. Harmonic analysis on G deals with the problem of decomposing this represention into a superposition of irreducible unitary representations, which is intuitively formulated in integral notation as follows:

$$L \times R \simeq \int_{\widehat{G}}^{\oplus} \pi \otimes \pi^{\vee} \; d\mu(\pi). \tag{68}$$

Here $\widehat{G}$ denotes the unitary dual of G, i.e., the set of equivalence classes of irreducible unitary representations of G. This set carries a particular locally compact Hausdorff topology. The formula involves an (essentially unique) Borel measure $d\mu$ on $\widehat{G}$, called the Plancherel measure. Generalizing the Peter-Weyl formula, the given decomposition for a function $f \in C_c^\infty(G)$ goes accompagnied with a Fourier transform

$$\pi \mapsto \hat{f}(\pi) \in \mathrm{End}(\pi)_{\mathrm{HS}}$$

such that the following Parseval identity is valid

$$\|f\|^2_{L^2(G)} = \int_{\widehat{G}} \|\pi(f)\|^2_{\mathrm{HS}} \, d\mu(\pi).$$

The precise description of this Plancherel formula was obtained by the Indian mathematician Harish-Chandra (1923 - 1983).

A fundamental difficulty in the theory of unitary representations of G is that finite dimensional ones are all supported by the compact normal subgroups of G. We can express this as follows.

Lemma 18.1. *Let π be a finite dimensional continuous unitary representation of G. Then $G_0 := \ker(\pi)$ is a closed normal subgroup of G and $G = KG_0$. In particular, the group G/G_0 is compact.*

Remark 18.2. It follows from the lemma that for $\mathfrak{g}$ simple and not compact, the group G has no finite dimensional unitary representations besides the trivial one. Thus, the Plancherel formula must contain infinite dimensional irreducible representation of G. In order to create sufficiently many, we will use the process of induction, described in the next section.

Proof. Let $\mathscr{H}$ be the finite dimensional complex Hilbert space in which π is realized. Then the map $\pi : G \to \mathrm{GL}(\mathscr{H})$ is continuous, hence a smooth homomorphism of Lie groups. It maps G into the compact group $U(\mathscr{H})$. The kernel G_0 of π is a closed normal subgroup of G.

Let $\pi_* := d\pi(e) : \mathfrak{g} \to \mathfrak{u}(\mathscr{H})$ be the associated Lie algebra representation of $\mathfrak{g}$. Then the Lie algebra $\mathfrak{g}_0$ of G_0 equals the kernel of π_*. This kernel is an ideal of $\mathfrak{g}$, hence a direct sum of a collection of simple ideals of $\mathfrak{g}$. Let $\mathfrak{c}$ be the unique complementary ideal, consisting of the direct sum of the remaining simple ideals. We claim that $\mathfrak{c} \subset \mathfrak{k}$.

To see that the claim is valid, let $\mathfrak{c}_0$ be a simple ideal contained in $\mathfrak{c}$. Then

$$\mathfrak{c}_0 = (\mathfrak{c}_0 \cap \mathfrak{k}) \oplus (\mathfrak{c}_0 \cap \mathfrak{p}) \tag{69}$$

by the lemma below. The trace form $\tau : (S,T) \mapsto \mathrm{tr}(ST)$ is real valued, symmetric, negative definite and $U(\mathscr{H})$-conjugation invariant on the algebra $\mathfrak{u}(\mathscr{H})$. Its pull-back b to $\mathfrak{c}_0$ under the map $\pi_*|_{\mathfrak{c}_0}$ is therefore symmetric and negative definite. Furtheremore, it is under $\mathrm{Ad}(G)$ hence under $\mathrm{ad}(\mathfrak{c}_0)$. The Killing form of $\mathfrak{c}_0$ equals the restriction $B_0 := B|_{\mathfrak{c}_0\times\mathfrak{c}_0}$ of the Killing form of $\mathfrak{g}$. Consider B_0 and b as linear maps $\mathfrak{c}_0 \to \mathfrak{c}_0^*$. Then $b^{-1}B_0$ is an invertible map $\mathfrak{c}_0 \to \mathfrak{c}_0$ which intertwines the $\mathrm{ad}(\mathfrak{c}_0)$-actions. As $\mathfrak{c}_0$ is simple, it follows that $b^{-1}B_0$ is a non-zero real scalar. Since b is definite, this implies that B_0 is definite as well. From $B_0 > 0$ it would follow that $\mathfrak{c}_0 \subset \mathfrak{p}$, hence $\mathfrak{c}_0 = [\mathfrak{c}_0, \mathfrak{c}_0] \subset \mathfrak{k} \cap \mathfrak{p} = 0$, contradicting that $\mathfrak{c}_0$ is simple. Therefore, we must have that $B_0 < 0$ and infer that $\mathfrak{c}_0 \subset \mathfrak{k}$. This establishes the claim. Let C be the connected subgroup of G with Lie algebra $\mathfrak{c}$. Then the natural map $j : C \to G/G_0$ is a smooth Lie group homomorphism whose differential is bijective. Since G/G_0 is connected, it follows that $j(C) = G/G_0$ hence $G = CG_0$. Since $\mathfrak{c} \subset \mathfrak{k}$, it follows that $C \subset K$. □

Lemma 18.3. *Let σ be a Cartan involution of $\mathfrak{g}$. Then θ leaves each ideal of $\mathfrak{g}$ invariant.*

Proof. As every ideal is a sum of simple ones, it suffices to establish the assertion for the simple ideals. Let $\mathfrak{s}$ be such a simple ideal, and let $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ be a numbering of the simple ideals such that $\mathfrak{s} = \mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$, as a Lie algebra. The Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is compact semisimple, hence a direct sum of its simple ideals $\mathfrak{u}_1, \ldots, \mathfrak{u}_t$. Now $\mathfrak{u}_{j\mathbb{C}}$ are the simple ideals of the complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and so are $\mathfrak{g}_{j\mathbb{C}}$. It follows that $s = t$ and after a renumbering we may assume that $\mathfrak{u}_j$ and $\mathfrak{g}_j$ have the same complexification.

We will finish the proof by showing that

$$\mathfrak{g}_1 = (\mathfrak{g}_1 \cap \mathfrak{k}) \oplus (\mathfrak{g}_1 \cap \mathfrak{p}). \tag{70}$$

θ leaves $\mathfrak{g}_1$ invariant. Let $X \in \mathfrak{g}_1$. Then $X = U + V$ with $U \in \mathfrak{k}$ and $V \in \mathfrak{p}$. Now $U \in \mathfrak{u}$ and $V \in i\mathfrak{u}$ whereas $U + V \in \mathfrak{g}_1 \subset \mathfrak{u}_{1\mathbb{C}}$. It follows that $U \in \mathfrak{u}_1$ and $V \in i\mathfrak{u}_1$. Hence $U \in \mathfrak{u}_1 \cap \mathfrak{k} = \mathfrak{u}_1 \cap \mathfrak{g} \cap \mathfrak{k} \subset \mathfrak{g}_1 \cap \mathfrak{k}$ and, likewise, $V \in \mathfrak{g}_1 \cap \mathfrak{p}$. This establishes (70). □

19 Induced representations

In this section we assume that G is an arbitrary Lie group, and that H is a closed subgroup. We assume that ξ is a finite dimensional continuous representation of H in a (finite dimensional) complex vector space V_ξ. In terms of these data, we define the so called *induced representation* of G, denoted

$$\pi_\xi = \mathrm{ind}_H^G(\xi).$$

The representation space is defined to be the space of continuous functions $\varphi : G \to V_\xi$ transforming according to the rule

$$\varphi(xh) = \xi(h)^{-1}\varphi(x), \qquad (x \in G, h \in H). \tag{71}$$

The space of these functions is denoted by $C(G:H:\xi)$. It is a closed linear subspaceof $C(G,V_\xi)$, relative to the usual Fréchet topology. Accordingly, $C(G:H:\xi)$ is a Fréchet space for the restricted topology. The induced representation of G in this space is defined by restriction of the left regular representation. Thus,

$$[\pi_\xi(g)\varphi](x) = \varphi(g^{-1}x), \qquad (\varphi \in C(G:H:\xi),\, x,g \in G).$$

There is a natural way to view $(\pi_\xi, C(G:H:\xi))$ as represenation of G in the space of continuous sections of a vector bundle on G/H. To be able to give a transparent definition of this vector bundle, we will first discuss the general notion of an equivariant vector bundle.

Let M be a smooth manifold equipped with a smooth left action by the Lie group G. By a *G-equivariant vector bundle* on M, we mean a vector bundle $p:\mathscr{V}\to M$ together with a smooth left action of G on $\mathscr{V}$ such that the following diagram commutes, for every $g\in G$,

$$\begin{array}{ccc} \mathscr{V} & \xrightarrow{g\cdot} & \mathscr{V} \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{g\cdot} & M \end{array} \tag{72}$$

The requirement that the above diagram be commutative for all $g\in G$ is equivalent to the requirement that G acts by vector bundle automorphisms, consistent with the action on M. Another way to say this, is that the action map $g\cdot:\mathscr{V}\to\mathscr{V}$ maps each fiber $\mathscr{V}_m$, for $m\in M$, linearly isomorphically onto $\mathscr{V}_{gm}$.

Now assume that $p:\mathscr{V}\to G/H$ is an equivariant vector bundle over the quotient manifold G/H, where the latter space is equipped with the natural G-action by left translation. Then the group H fixes the origin $[e]:=eH$, hence its action on $\mathscr{V}$ retricts to the smooth action on the fiber $V:=\mathscr{V}_{[e]}$ by linear automorphisms. In other words, this action of H on V corresponds to a representation ξ of H in V.

Lemma 19.1. *The map $\psi: G\times V\to\mathscr{V}$ defined by $(g,v)\mapsto gv$ is a smooth surjective submersion. Its fibers are the orbits of the right H-action on $G\times V$ defined by $(g,v)h=(gh,\xi(h)^{-1}v)$. The induced map $\bar\psi:(G\times V)/H\to\mathscr{V}$ is a diffeomorphism.*

Proof. The map ψ is obviously smooth. To establish its surjectivity, let $v\in\mathscr{V}$. Then v belongs to the fiber $\mathscr{V}_m$ where $m=p(v)\in G/H$. Let $g\in G$ be such that $m=[g]$. Then $w:=g^{-1}v\in\mathscr{V}_{[e]}=V$ and $v=gw=\psi(g,w)$. Hence, ψ is surjective, and (g,w) is an element of the fiber $\psi^{-1}(v)$. Clearly, every element of the form $(gh,\xi(h)^{-1}w)$ belongs to the same fiber. Conversely, if (g',w') belongs to the fiber, then $[g']=p(\psi(g',w'))=p(v)=[g]$, so that $g'=gh$ for a unique element $h\in H$. From $gw=g'w'=ghw'$ we now conclude that $w=hw'$, hence $(g',w')=(g,w)h$.

We will now show that ψ is a submersion. By left G-equivariance, it suffices to do this at the point (e,v), for every $v\in V$. For this we note that the map $u\mapsto\psi(e,v+u)$ is a translation $V\to V=\mathscr{V}_{[e]}$, so that $\operatorname{im}(d\psi(e,v))$ contains $T_v\mathscr{V}_{[e]}=\ker dp(v)$. Since $p\circ\psi(g,v)=[g]$, for $g\in G$, it follows $dp(v)\circ d\psi(e,v)$ is surjective, and we find that $d\psi(e,v)$ is surjective.

It follows from the above arguments that the induced map $\bar\psi$ is a bijection. Furthermore, since ψ is submersive, it follows that $\bar\psi$ is a smooth submersion as well. For dimensional reasons, it now follows that $\bar\psi$ is a local diffeomorphism, hence a diffeomorphism. $\square$

Example 19.2. Typical examples of equivariant vector bundles on the homogeous manifold G/H are the various geometric vector bundles over this manifold. Let us first consider the tangent bundle $T(G/H)$. The action $l : G \times G/H \to G/H$ given by $l_g(m) = gm$ lifts to a smooth map $G \times T(G/H) \to T(G/H)$, $(g, \xi) \mapsto g \cdot \xi$ given by

$$g \cdot \xi := dl_g(x)\xi, \qquad (g \in G,\, x \in G/H,\, \xi \in T_x(G/H)).$$

By application of the chain rule, one sees that this is indeed an action, which is fiberwise linear. We thus see that $T(G/H)$ equipped with this action is an equivariant vector bundle.

If we use the natural identification $T_{[e]}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$, the action of H on $T_{[e]}(G/H)$ given by $(h, v) \mapsto dl_h([e])$ corresponds to the action on $\mathfrak{g}/\mathfrak{h}$ induced by the adjoint action of H on $\mathfrak{g}$. In view of the above remark, the map

$$G \times (\mathfrak{g}/\mathfrak{h}),\ (g, v) \mapsto dl_g([e])v$$

factors through a diffeomorphism

$$[G \times (\mathfrak{g}/\mathfrak{h})]/H \xrightarrow{\simeq} T(G/H).$$

Let $p : \mathscr{V} \to G/H$ be an equivariant vector bundle. Then there is a natural representation π of G in the space of continuous sections $\Gamma(M, \mathscr{V})$. Given an element $g \in G$ the map $\pi(g) : \Gamma(M, \mathscr{V}) \to \Gamma(M, \mathscr{V})$ is defined by

$$[\pi(g)s](m) = g \cdot s(g^{-1}m), \qquad (s \in \Gamma(M, \mathscr{V}),\, m \in M).$$

It is readily verified that this is a continuous representation of G in $\Gamma(M, \mathscr{V})$, equipped with the usual Fréchet topology.

Let ξ be the natural representation of H in the fiber $V = \mathscr{V}_{[e]}$, defined above Lemma 19.1. We will show that the representation π is equivalent to $\pi_\xi = \mathrm{ind}_H^G(\xi)$ in a precise sense.

Given a section $s \in \Gamma(G/H, \mathscr{V})$ we define the continous function $\varphi_s : G \to V$ by

$$\varphi_s(x) = x^{-1}s([x]) \in \mathscr{V}_{[e]} = V.$$

Then for $h \in H$ we have

$$\varphi_s(xh) = h^{-1}x^{-1}s([xh]) = \xi(h)^{-1}x^{-1}s([x]) = \xi(h)^{-1}\varphi_s(x).$$

Thus we see that $\varphi_s \in C(G : H : \xi)$.

Lemma 19.3. *The map $\Phi : s \mapsto \varphi_s$ defines a continuous linear isomorphism $\Gamma(G/H, \mathscr{V}) \to C(G : H : \xi)$, which intertwines π with π_ξ.*

Proof. For $\varphi \in C(G : H : \xi)$ we define the continuous function $\grave{s}_\varphi : G \to \mathscr{V}$ by

$$\grave{s}_\varphi(x) = x \cdot \varphi(x).$$

Then ${}^\backprime s_\varphi(xh) = xh \cdot \xi(h)^{-1}\varphi(x) = x \cdot \varphi(x) = {}^\backprime s_\varphi(x)$. It follows that ${}^\backprime s_\varphi$ is right H-invariant hence factors through a continuous function $s_\varphi : G/H \to \mathscr{V}$. Since ${}^\backprime s_\varphi$ maps $x \in G$ into $\mathscr{V}_{[x]}$, it follows that $s_\varphi \in \Gamma(G/H, \mathscr{V})$. It is readily verified that $\varphi \mapsto s_\varphi$ is a continuous linear map $C(G : H : \xi) \to \Gamma(G/H, \mathscr{V})$, and a two-sided inverse to Φ.

It remains to establish the intertwining property. Let $s \in \Gamma(G/H, \mathscr{V})$ and $g \in G$, then

$$\begin{aligned}(\Phi \circ \pi(g)s)(x) &= x^{-1}(\pi(g)s)([x]) = x^{-1}g \cdot s(g^{-1}[x]) \\ &= (g^{-1}x)^{-1}s([g^{-1}x]) = \Phi(s)(g^{-1}x) \\ &= \pi_\xi(g) \circ \Phi(s)(x)\end{aligned}$$

and the required property follows. □

We thus see that an equivariant vector bundle gives rise to an induced representation. We will see that every induced representation arises in this way.

More precisely, let (ξ, V_ξ) be a smooth representation of H in a finite dimensional complex (or real) vector space V_ξ. Then motivated by Lemma 19.1 we consider the right H-action on $G \times V_\xi$ by

$$(x, v) \cdot h := (xh, \xi(h)^{-1}v), \qquad (x \in G,\ v \in V_\xi,\ h \in H).$$

This action is proper and free, since the right action of H on G is already proper and free. The associated smooth quotient manifold is denoted by

$$G \times_H V_\xi := G \times V_\xi / H.$$

Then the projection $\mathrm{pr}_1 : G \times V_\xi \to G$ is right H-equivariant, hence induces a smooth map

$$p : G \times_H V_\xi \to G/H,$$

so that the following diagram commutes

$$\begin{array}{ccc} G \times V_\xi & \xrightarrow{\tilde{\pi}} & G \times_H V_\xi \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{\pi} & G/H. \end{array} \tag{73}$$

Lemma 19.4. *The map $p : G \times_H V_\xi \to G/H$ has a unique structure of vector bundle for which the pair of horizontal maps is a vector bundle morphism.*

Remark 19.5. The above construction is known as the associated vector bundle construction for the principal bundle $G \to G/H$ and the H-representation ξ. It works more generally for any H-principal bundle $\pi : P \to M \simeq P/H$. The above definition and lemma are valid if G and G/H are everywhere replaced by P and M.

Proof. It is actually easiest to prove the lemma in the generality of an H-principal bundle $\pi : P \to M$, as this allows localization to trivial bundles. We will do this.

Let $x \in P$ and put $m = \pi(x)$. Then the map $j_x : V \to P \times_H V$ given by $v \mapsto [(x, v)]$ is readily checked to be a bijection onto the fiber $(P \times_H V)_m := p^{-1}(m)$. If p is equipped with a vector bundle structure satisfying the requirements, the map j_x must be a linear isomorphism. On the other hand, if $x' \in P$ is a second point in P, then $\pi(x') = m$ if and only if $x' = xh$ for an element $h \in H$, and in this case it is readily seen that the following diagram commutes

$$\begin{tikzcd} V \arrow[d, "\xi(h)^{-1}"'] \arrow[dr, "j_x"] & \\ & (P \times_H V)_m \\ V \arrow[ur, "j_{x'}"'] & \end{tikzcd}$$

It follows that the fiber $p^{-1}(m)$ carries a unique structure of linear space for which all maps $j_{x'}$ with $\pi(x') = m$ become linear isomorphisms. We equip every fiber of p with this structure of linear space.

Then it remains to be shown that for every $m_0 \in M$ there exists an open neighborhood $\mathscr{O} \ni m_0$ and a diffeomorphism $\tau : p^{-1}(\mathscr{O}) \to \mathscr{O} \times V$ such that the following diagram commutes

$$\begin{array}{ccc} p^{-1}(\mathscr{O}) & \xrightarrow{\tau} & \mathscr{O} \times V \\ {\scriptstyle p} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\ \mathscr{O} & \xrightarrow{\mathrm{I}} & \mathscr{O} \end{array}$$

and so that for every $m \in \mathscr{O}$, the map $\mathrm{pr}_2 \circ \tau$ restricts to a linear isomorphism $\tau_m : p^{-1}(m) \to V$.

We put $P_{\mathscr{O}} := \pi^{-1}(\mathscr{O})$ and may choose $\mathscr{O}$ such that the principal fiber bundle π allows a trivialization $P_{\mathscr{O}} \simeq \mathscr{O} \times H$ over $\mathscr{O}$. In that case, the natural map $P_{\mathscr{O}} \times V \to P \times V$ factors through a diffeomorphism from $P_{\mathscr{O}} \times_H V$ onto the open subset $p^{-1}(\mathscr{O})$ of $P \times_H V$ under which the natural projections onto $\mathscr{O}$ correspond.

We thus see that we may reduce to the case of a trivial principal bundle, and we may assume that $P = M \times H$ from the start. Then

$$P \times_H V = (M \times H \times V)/H = M \times H \times_H V$$

and we infer that it suffices to prove the existence of a trivialisation $\varphi : M \times H \times_H V \to M \times V$ over M such that the map $\mathrm{pr}_2 \circ \varphi \circ j_{(m,h)} : V \to V$ is a linear isomorphism, for every $(m,h) \in P$.

We define φ by $[(m,h,v)] \mapsto (m, \xi(h)v)$. In the converse direction, we define $\psi : (m,v) \mapsto [(m,e,v)]$. Then φ and ψ are smooth and two-sided inverses of each other, hence φ is a diffeomorphism. Furthermore, the following diagram commutes

$$\begin{array}{ccc} M \times H \times_H V & \xrightarrow{\varphi} & M \times V \\ {\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\ M & \xrightarrow{\mathrm{I}} & M \end{array}$$

and we see that φ is a trivialisation. Finally,

$$\mathrm{pr}_2 \circ \varphi \circ j_{(m,h)}(v) = \mathrm{pr}_2 \circ \varphi([(m,h,v)]) = \mathrm{pr}_2(m, \xi(h)v) = \xi(h)v,$$

so that $\mathrm{pr}_2 \circ \varphi \circ j_{(m,h)}$ equals the linear automorphism $\xi(h)$ of V. $\square$

Lemma 19.6. *Let notation be as in Lemma 19.4. Then the natural left action of G on $G \times V_\xi$ factorizes through a smooth action of G on $G \times_H V_\xi$. For this action, $p : G \times_H V \to G/H$ is a G-equivariant vector bundle.*

Remark 19.7. From now on we shall also write $\mathscr{V}_\xi := G \times_H V_\xi$.

Proof. The natural left action of G on $G \times V$ commutes with the right H-action, hence induces a smooth map $G \times (G \times_H V) \to (G \times_H)$ such that the following diagram is commutative

$$\begin{array}{ccc} G \times (G \times_H V) & \longrightarrow & G \times_H V \\ \downarrow & & \downarrow \\ G \times G/H & \longrightarrow & G/H. \end{array}$$

Since the vertical map on the left is a surjective submersion, it follows that the horizontal map at the bottom is smooth. We thus see that $p : G \times_H V \to G/H$ is a G-equivariant vector bundle. $\square$

Corollary 19.8. *Let $p : \mathscr{V} \to G/H$ be a G-equivariant bundle. Let $\bar{\psi} : G \times_H V \to \mathscr{V}$ be defined as in Lemma 19.1. Then $\bar{\psi}$ is a (natural) isomorphism of G-equivariant vector bundles.*

Proof. By the mentioned lemma, $\bar{\psi}$ is a diffeomorphism. Since $\psi : G \times V \to \mathscr{V}$ intertwines the left G-actions, it follows that $\bar{\psi}$ does as well. It now remains to be shown that $\bar{\psi}$ restricts to a linear map $(G \times_H V)_m \to \mathscr{V}_m$ for every $m \in G/H$. By left G-equivariance, it suffices to show this for $m = [e]$.

Let $i : V \to \mathscr{V}$ be the natural inclusion map, with image $\mathscr{V}_{[e]}$, and let $j = j_e : V \to (G \times_H V)_{[e]}$ be the linear isomorphism induced by $v \mapsto [(e, v)]$. Then $\bar{\psi} \circ j = i$ and we find that $\bar{\psi}$ restricts to a linear isomorphism $(G \times_H V)_{[e]} \to \mathscr{V}_{[e]}$. $\square$

Example 19.9. For later use, we consider the example of the bundle $\mathscr{D}^\alpha T(G/H)$ of (complex valued) α-densities, with α a positive real number. We will especially be interested in the bundles for the values $\alpha = 1$ and $\alpha = 1/2$.

If V is a finite dimensional real linear space of dimension n, then $\mathscr{D}^\alpha V$ is defined to be the space of functions $\lambda : \wedge^n V \to \mathbb{C}$ transforming according to the rule

$$\lambda(tv) = |t|^\alpha \lambda(v), \quad (u \in \wedge^n V,\, t \in \mathbb{R}).$$

Let $\omega \in \wedge^n V^* \setminus \{0\}$, then $|\omega|^\alpha : u \mapsto |\omega(u)|^\alpha$ is an example of such a density, and the map

$$z \mapsto z|\omega|^\alpha,\ \mathbb{C} \to \mathscr{D}^\alpha V$$

is a linear isomorphism.

Given another real linear space W of dimension n and a linear map $A : V \to W$ we define the linear map $A^* : \mathscr{D}^\alpha W \to \mathscr{D}^\alpha V$ by

$$A^* \lambda = \lambda \circ (\wedge^n A), \qquad (\lambda \in \mathscr{D}^\alpha W).$$

In case $W = V$, it is readily verified that

$$A^* \lambda = |\det A|^\alpha \lambda. \tag{74}$$

Accordingly, the natural action of $\mathrm{GL}(V)$ on V induces an action on $\mathscr{D}^\alpha V$ given by the formula $(A,\lambda) \mapsto \lambda := A^{-1*}\lambda$. From (74) we see that this action is given by the character

$$A \mapsto |\det A|^{-\alpha}.$$

The functor assigning to a finite dimensional real linear space its space of α-densities depends smoothly on parameters, hence may be applied to the tangent bundle of a smooth manifold M to give the bundle of α-densities $\mathscr{D}^\alpha TM$.

If $\varphi : M \to N$ is a diffeomorphism of manifolds of equal dimension n, then there is a natural isomorphism of vector bundles $\tilde{\varphi} : \mathscr{D}^\alpha TM \to \mathscr{D}^\alpha TN$ defined by

$$\tilde{\varphi} : (\lambda_x) = d\varphi(x)^{-1*}\lambda_x \in \mathscr{D}T_{\varphi(x)}N, \qquad (x \in M,\ \lambda_x \in \mathscr{D}^\alpha T_x M).$$

We now turn to the setting of $\mathscr{D}^\alpha T(G/H)$. The action of G on G/H by left multiplication induces a smooth action on the bundle $\mathscr{D}^\alpha T(G/H)$ by automorphisms of vector bundles. Accordingly, the bundle is G-equivariant. It follows that we have a natural isomorphism

$$G \times_H \mathscr{D}^\alpha T_{[e]}(G/H) \xrightarrow{\simeq} \mathscr{D}^\alpha T(G/H).$$

The H-space $\mathscr{D}^\alpha T_{[e]}(G/H)$ is one dimensional over $\mathbb{C}$. Hence H acts by a character which we will now compute.

We recall that the derivative of the projection $\pi : G \to G/H$ induces a natural isomorphism $\mathfrak{g}/\mathfrak{h} \simeq T_{[e]}(G/H)$, through which we identify these spaces. The action of H on $T_{[e]}(G/H)$ is given by $(h,v) \mapsto dl_h([e])$. On $\mathfrak{g}/\mathfrak{h}$ this action coincides with the action $h \mapsto \overline{\mathrm{Ad}}(h)$, $H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ induced by $\mathrm{Ad}|_H : H \to \mathrm{GL}(\mathfrak{g})$. We now infer that the action of $h \in H$ on an element $\lambda \in \mathscr{D}^\alpha(\mathfrak{g}/\mathfrak{h})$ is given by

$$h \cdot \lambda = \overline{\mathrm{Ad}}(h)^{-1*}\lambda = |\det_{\mathfrak{g}/\mathfrak{h}} \overline{\mathrm{Ad}}(h)|^{-\alpha}\lambda = \Delta^\alpha(h)\lambda,$$

where

$$\Delta(h) = |\det_{\mathfrak{g}/\mathfrak{h}} \overline{\mathrm{Ad}}(h)|^{-1} = \frac{|\det \mathrm{Ad}(h)|_{\mathfrak{h}}|}{|\det \mathrm{Ad}(h)|_{\mathfrak{g}}|}, \qquad (h \in H). \tag{75}$$

Given a character δ of H, let Let $\mathbb{C}_\delta$ denote $\mathbb{C}$ equipped with the H-module structure $(h,z) \mapsto \xi(h)z$. Then it follows that

$$G \times_H \mathbb{C}_{\Delta^\alpha} \xrightarrow{\simeq} \mathscr{D}^\alpha T(G/H). \tag{76}$$

The isomorphism is determined up to a choice of normalization. Fix a non-zero top order alternating form $\omega \in \wedge^{\mathrm{top}}((\mathfrak{g}/\mathfrak{h}^*)$. Then $z \mapsto z|\omega|^\alpha$ defines a linear isomorphism from $\mathbb{C}$ onto $\mathscr{D}^\alpha(\mathfrak{g}/\mathfrak{h})$. The corresponding linear isomorphism (76) is induced by

$$G \times \mathbb{C} \to \mathscr{D}^\alpha T(G/H),\ \ (g,z) \mapsto z \cdot dl_g([e])^{-1*}|\omega|^\alpha.$$

Remark 19.10. Before we proceed with our study of induced representations, we note that the assignment $\mathscr{F} : \mathscr{V} \rightsquigarrow \mathscr{V}_{[e]}$ is a functor from the category $\mathrm{VB}_G(G/H)$ of finite rank equivariant smooth complex vector bundles on G/H to the category $\mathrm{Rep}(H)$ of finite dimensional continuous representations of H. In the converse direction, $\mathscr{G} : (\xi, V_\xi) \rightsquigarrow G \times_H V_\xi$ is a functor from

$\mathrm{Rep}(H)$ to $\mathrm{VB}_G(G/H)$. Corollary 19.8 essentially asserts that the composed functor $\mathscr{G} \circ \mathscr{F}$ is naturally equivalent to the identity functor on $\mathrm{VB}_G(G/H)$. Likewise, it can be shown that $\mathscr{F} \circ \mathscr{G}$ is naturally equivalent to the identity functor on $\mathrm{Rep}(H)$. We conclude that the two functors set up an equivalence of the categories $\mathrm{VB}_G(G/H)$ and $\mathrm{Rep}(H)$.

The realization of $\mathrm{ind}_H^G(\xi)$ as the representation in $C(G:H:\xi)$ obtained by restriction of the left regular representation, is sometimes called the 'induced picture.' The realization as the representation in $\Gamma(G/H, \mathscr{V}_\xi)$ is called the 'vector bundle picture'. For most of our purposes, it will be sufficient to refer to the 'induced picture' of $\mathrm{ind}_H^G(\xi)$. However, for understanding the concept of unitary and normalized induction, a proper understanding of the bundle of densities will turn out to be very useful.

The idea of unitary induction arises from the question whether the described process of induction preserves unitarity. For investigating this, we assume that V_ξ is a finite dimensional Hilbert space, equipped with a Hermitian inner product $\langle \cdot\,,\,\cdot \rangle_\xi$, and that ξ is a unitary representation. We consider the subspace $C_c(G:H:\xi)$ of $C(G:H:\xi)$ consisting of functions φ with support $\pi(\mathrm{supp}\varphi)$ compact in G/H. Given functions $\varphi, \psi \in C_c(G:H:\xi)$, we define the function

$$\langle \varphi\,,\,\psi \rangle_\xi : G \to \mathbb{C}, \quad x \mapsto \langle \varphi(x)\,,\,\psi(x) \rangle_\xi. \tag{77}$$

Then for $x \in G$ and $h \in H$ we see that

$$\langle \varphi\,,\,\psi \rangle_\xi(xh) = \langle \xi(h)^{-1}\varphi(x)\,,\,\xi(h)^{-1}\psi(x) \rangle_\xi = \langle \varphi\,,\,\psi \rangle_\xi(x),$$

and we see that the function (77) is right H-invariant, hence factors through a compactly supported continuous function $G/H \to \mathbb{C}$.

Now assume that G/H admits a G-invariant positive density $d\bar{x}$. Then we may define a Hermitian inner product $\langle \cdot\,,\,\cdot \rangle$ on $C_c(G:H:\xi)$ by

$$\langle \varphi\,,\,\psi \rangle := \int_{G/H} \langle \varphi\,,\,\psi \rangle_\xi(\bar{x})\, d\bar{x}. \tag{78}$$

Then

$$\begin{aligned}
\langle \pi_\xi(g)\varphi\,,\,\pi_\xi(g)\psi \rangle &= \int_{G/H} \big(\langle \varphi\,,\,\psi \rangle_\xi(g^{-1}\bar{x}) \big)\; d\bar{x} \\
&= \int_{G/H} l_g^{-1*} \big(\langle \varphi\,,\,\psi \rangle_\xi d\bar{x} \big) \\
&= \int_{G/H} \big(\langle \varphi\,,\,\psi \rangle_\xi d\bar{x} \big) = \langle \varphi\,,\,\psi \rangle
\end{aligned}$$

where we used left-invariance of the density in the third equality, and invariance of integration of densities (substitution of variables) in the next.

In other words, the given pre-Hilbert structure on $C_c(G:H:\xi)$ is invariant for the representation π_ξ. Let $L^2(G:H:\xi)$ be the Hilbert completion of $C_c(G:\mathscr{V}:\xi)$ with respect to the given pre Hilbert structure. Then π_ξ has a unique extension to a unitary representation of G in $L^2(G:H:\xi)$, in view of Lemma 1.8.

Before proceeding we observe that we can also define an equivariant pairing if ξ is a possibly non-unitary representation in a finite dimensional Hilbert space V_ξ. We may define the conjugate adjoint of ξ to be the representation ξ^* of H in V_ξ by the formula

$$\xi^*(h) = \xi(h^{-1})^*$$

where the star on the right-hand side indicates that the Hermitian adjoint with respect to the inner product is taken. Then ξ is unitary if and only if $\xi^* = \xi$. The above scheme may be used to define an equivariant sesquilinear pairing

$$C_c(G:H:\xi) \times C_c(G:H:\xi^*) \to \mathbb{C}$$

by the same formula (78).

The condition that G/H carries a positive invariant density is equivalent to the condition that

$$|\det \mathrm{Ad}(h)|_{\mathfrak{h}}| = |\det \mathrm{Ad}(h)|_{\mathfrak{g}}|, \quad (h \in H),$$

which is fulfilled in case G is compact, or more generally in case both G and H are unimodular, but not in general.

In the general case, we introduce the following definition of normalized induction to circumvent the choice of a possible invariant density on G/H. We assume that ξ is a continuous representation in a finite dimensional Hilbert space V_ξ. Then the H-module $V_\xi \otimes \mathbb{C}_{\Delta^{1/2}}$ is naturally isomorphic to V_ξ equipped with the representation $\xi \otimes \Delta^{1/2}$ given by

$$(\xi \otimes \Delta^{1/2})(h)v = \Delta(h)^{1/2}\xi(h)v, \qquad (v \in V_\xi, h \in H).$$

We agree to write $V_{\xi\otimes\Delta^{1/2}}$ for the Hilbert space V_ξ equipped with this representation. It is then readily verified that the inner product of V_ξ defines the H-equivariant sequilinear pairing

$$\langle \cdot\,,\,\cdot \rangle_\xi : V_{\xi\otimes\Delta^{1/2}} \times V_{\xi^*\otimes\Delta^{1/2}} \to \mathbb{C}_\Delta.$$

For $\varphi \in C_c(G:H:\xi\otimes\Delta^{1/2})$ and $\psi \in C_c(G:H:\xi^*\otimes\Delta^{1/2})$, we define the function $\langle \varphi\,,\,\psi \rangle : G \to \mathbb{C}$ by

$$\langle \varphi\,,\,\psi \rangle_\xi := \langle \varphi(x)\,,\,\psi(x) \rangle_\xi.$$

Then it follows that $\langle \varphi\,,\,\psi \rangle_\xi \in C_c(G:H:\Delta)$.

Let $\omega \in \wedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})^* \setminus \{0\}$. Then it follows as in Example 19.9 that the map

$$\langle \varphi\,,\,\psi \rangle_{\xi,\omega} :\ x \mapsto \langle \varphi\,,\,\psi \rangle_\xi \; dl_x([e])^{-1*}|\omega|, \quad G \to \mathscr{D}T(G/H), \tag{79}$$

is right H-invariant and defines a compactly supported section of the density bundle $\mathscr{D}T(G/H)$ on G/H. We may thus define $\langle \varphi\,,\,\psi \rangle \in \mathbb{C}$ by

$$\langle \varphi\,,\,\psi \rangle := \int_{G/H} \langle \varphi\,,\,\psi \rangle_{\xi,\omega} \tag{80}$$

Lemma 19.11. *The sesquilinear pairing $C_c(G:H:\xi\otimes\Delta^{1/2})\times C_c(G:H:\xi^*\otimes\Delta^{1/2})\to\mathbb{C}$ defined by (80) is G-equivariant.*

Proof. We put $\pi=\pi_{\xi\otimes\Delta^{1/2}}$ and $\pi^*=\pi_{\xi^*\otimes\Delta^{1/2}}$. Then

$$\begin{aligned}\langle\pi(g)\varphi\,,\,\pi^*(g)\psi\rangle_{\xi,\omega}(x) &= \langle L_g\varphi\,,L_g\psi\rangle(x)dl_x(e)^{-1*}|\omega|\\ &= \langle\varphi\,,\,\psi\rangle(g^{-1}x)dl_x(e)^{-1*}|\omega|\\ &= dl_{g^{-1}}(x)^*\langle\varphi\,,\,\psi\rangle(g^{-1}x)dl_{g^{-1}x}(e)^{-1*}|\omega|\\ &= l^*_{g^{-1}}(\langle\varphi\,,\,\psi\rangle_{\xi,\omega})(x).\end{aligned}$$

It follows that

$$\begin{aligned}\langle\pi(g)\varphi\,,\,\pi(g)\psi\rangle &= \int_{G/H} l^*_{g^{-1}}(\langle\varphi\,,\,\psi\rangle_{\xi,\omega})\\ &= \int_{G/H}\langle\varphi\,,\,\psi\rangle_{\xi,\omega}\\ &= \langle\varphi\,,\,\psi\rangle\end{aligned}$$

by invariance of the integration of densities. This establishes the equivariance. □

Lemma 19.12. *Assume that ξ is unitary (so that $\xi^*=\xi$). Then the pairing (80) defines a G-equivariant pre-Hilbert structure on $C_c(G:H:\xi\otimes\Delta^{1/2})$. The representation $\mathrm{ind}_H^G(\xi\otimes\Delta^{1/2})$ has a unique extension to a unitary representation in the Hilbert space completion $L^2(G:H:\xi\otimes\Delta^{1/2})$.*

Proof. The given pairing is sesquilinear and G-equivariant by Lemma 19.11. We will show it is positive definite. Let $\varphi\in C_c(G:H:\xi\otimes\Delta^{1/2})$. Then

$$\langle\varphi\,,\,\varphi\rangle_\xi(x)=\langle\varphi(x)\,,\,\varphi(x)\rangle\geq 0$$

for all $x\in G$. It follows that $\langle\varphi\,,\,\varphi\rangle_{\xi,\omega}$ is a compactly supported continuous density on G/H, which is everywhere nonnegative. This implies that

$$\langle\varphi\,,\,\varphi\rangle=\int_{G/H}\langle\varphi\,,\,\varphi\rangle_{\xi,\omega}\geq 0,$$

with equality if and only if $\langle\varphi\,,\,\varphi\rangle_{\xi,\omega}=0$ everywhere. This in turn is equivalent to $\varphi=0$. It follows that the pairing defines a pre-Hilbert structure. The final assertion follows by application of Lemma 1.8. □

From now on we agree to write

$$\mathrm{Ind}_H^G(\xi):=\mathrm{ind}_H^G(\xi\otimes\Delta^{1/2}),$$

for every continuous representation ξ of H in a finite dimensional Hilbert space. This adapted induction procedure will be called *normalized induction*. If the representation ξ is unitary, we also speak of *unitary induction*.

20 The principal series

In the present section we will apply the theory of the previous section to define the principal series of representations of a connected real semisimple Lie group G with finite center.

We assume that θ is a Cartan involution of its Lie algebra $\mathfrak{g}$, with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a system Σ^+ of positive roots for the associated root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$. Accordingly, the sum of the positive root spaces is denoted by $\mathfrak{n}$ and the sum of the associated negative root spaces by $\bar{\mathfrak{n}}$. The connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are denoted by K, A and N, respectively. The group K is compact, by Lemma 15.14. Recall that $\mathfrak{m}$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The centralizer of $\mathfrak{a}$ in K is denoted by M. This is is a closed subgroup of K, hence compact. We note that it may be highly disconnected. This is apparent in the standard example of $\mathrm{SL}(n,\mathbb{R})$ with the standard Iwasawa decomposition. In that case, M consists of all diagonal matrices with diagonal entries equal to ± 1 and with product of diagonal entries equal to one.

We define the so called minimal parabolic subgroup P of G by

$$P = MAN.$$

Lemma 20.1. *The group P is closed in G. The map $(m,a,n) \mapsto man$ is a diffeomorphism from $M \times A \times N$ onto P.*

Proof. This is an immediate consequence of the Iwasawa decomposition. □

We now assume that ξ is a unitary representation of M in a finite dimensional Hilbert space V_ξ and that $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Then $a \mapsto a^\lambda$ is a character of A, which is unitary if and only if $\lambda \in i\mathfrak{a}$. Since M commutes with $\mathfrak{a}$, hence with A, the set MA is a subgroup of G, which is closed by the Iwasawa decomposition. Furthermore, $M \times A \to MA$, $(m,a) \mapsto ma$ is an isomorphism of Lie groups. For $(m,a) \in M \times A$ we may now put

$$(\xi \otimes \lambda)(ma) : V_\xi \to V_\xi, \ \ v \mapsto a^\lambda \xi(m)v.$$

Then $\xi \otimes \lambda$ defines a continuous representation of MA in the Hilbert space V_ξ, which is unitary if and only if $\lambda \in i\mathfrak{a}^*$.

Lemma 20.2.

(a) *The group MA normalizes N.*

(b) *N is a normal subgroup of P.*

Proof. For (a) we note that $\mathrm{Ad}(MA)$ centralizes $\mathfrak{a}$, hence normalizes $\mathfrak{n}$. Since $N = \exp(\mathfrak{n})$ it follows that MA normalizes N. Assertion (b) follows from (a). □

In view of the above lemma, we may define a representation $(\xi \otimes \lambda \otimes 1)$ of P in V_ξ by

$$(\xi \otimes \lambda \otimes 1)(man) = (\xi \otimes \lambda)(ma) = a^\lambda \xi(m), \qquad ((m,a,n) \in M \times A \times N).$$

We note that

$$(\xi\otimes\lambda\otimes 1)^* = \xi\otimes(-\bar{\lambda})\otimes 1.$$

In particular, this representation is unitary if and only if $\lambda\in i\mathfrak{a}^*$.

The principal series of representations is the series of all induced representations $\mathrm{ind}_P^G(\xi\otimes\lambda\otimes 1)$ with $\xi\in\widehat{M}$ (irreducible) and $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$. However, we will see that the manifold G/P does not carry a G-invariant density, so that it is more convenient to use normalized induction, and a corresponding shift of parameter, as explained in the previous section.

In accordance with (75) we define the positive character Δ of P by

$$\Delta(y) = |\det{}_{\mathfrak{g}/(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})}\overline{\mathrm{Ad}}(y)|^{-1}, \quad (y\in P).$$

Let $\rho\in\mathfrak{a}^*$ be defined by

$$\rho(H) = \frac{1}{2}\mathrm{tr}(\,\mathrm{ad}(H)|_{\mathfrak{n}}).$$

Then by the root space decomposition of $\mathfrak{n}$, we see that

$$\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\alpha.$$

where $m_\alpha = \dim_{\mathbb{R}}(\mathfrak{g}_\alpha)$. Thus, ρ is half the sum of the positive roots, counted with multiplicities.

Lemma 20.3. *The character Δ of P is given by*

$$\Delta(man) = a^{2\rho}, \qquad (m\in M,\ a\in A,\ n\in N).$$

Proof. Since M is compact, it follows that $\Delta = 1$ on M. Furthermore, if $y=\exp Y$ with $Y\in\mathfrak{n}$, then $\mathrm{ad}(Y)$ is a nilpotent element of $\mathfrak{sl}(\mathfrak{g})$, so that $\mathrm{Ad}(y) = \exp\mathrm{ad}(Y)$ and its restriction to $\mathrm{Lie}(P) = \mathfrak{m}+\mathfrak{a}+\mathfrak{n}$ has determinant 1. Hence $\Delta = 1$ on N. Finally, if $a\in A$, then $\mathrm{Ad}(a)$ preserves the root space decomposition, so that

$$\begin{aligned}|\det{}_{\mathfrak{g}/(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})}\overline{\mathrm{Ad}}(a)| &= |\det(\mathrm{Ad}(a)|_{\bar{\mathfrak{n}}})| = |\det e^{\mathrm{ad}(\log a)|_{\bar{\mathfrak{n}}}}| \\ &= e^{\mathrm{tr}(\,\mathrm{ad}(\log a)|_{\bar{\mathfrak{n}}})} = e^{-2\rho(\log a)} \\ &= a^{-2\rho}.\end{aligned}$$

The result now follows. □

It follows from the above that

$$(\xi\otimes\lambda\otimes 1)\otimes\Delta^{1/2} = \xi\otimes(\lambda+\rho)\otimes 1,$$

so that

$$\pi_{\xi,\lambda} := \mathrm{Ind}_P^G(\xi\otimes\lambda\otimes 1) = \mathrm{ind}_P^G(\xi\otimes(\lambda+\rho)\otimes 1).$$

The associated space $C(G:P:\xi\otimes(\lambda+\rho)\otimes 1)$ in which $\pi_{\xi,\lambda}$ is realized, will now be denoted by $C(P:\xi:\lambda)$. Thus, this space consists of the continuous functions $\varphi: G\to V_\xi$ transforming according to the rule

$$\varphi(xman) = a^{-\lambda-\rho}\xi(m)^{-1}\varphi(x), \qquad (x\in G, (m,a,n)\in M\times A\times N\}.$$

Furthermore, $\pi_{\xi,\lambda}$ is the representation of G in $C(P:\xi:\lambda)$ obtained by restricting the left regular representation.

Since $G = KP$, the coset space G/P is compact. Let ω be a non-zero top order alternating form on $\mathfrak{g}/(\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n})$, so that $|\omega|$ defines a positive density on this space. Following the procedure of the previous section, we obtain for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ a non-degenerate sesquilinear pairing

$$C(P:\xi:\lambda)\times C(P:\xi:-\bar{\lambda}) \to \mathbb{C}, \quad (\varphi,\psi)\mapsto \langle\varphi\,,\,\psi\rangle \tag{81}$$

defined as in (80), with P in place of H. By Lemma 19.11 this pairing is G-equivariant for the respective representations $\pi_{\xi,\lambda}$ and $\pi_{\xi,-\bar{\lambda}}$ on these spaces. In particular, if $\lambda \in i\mathfrak{a}^*$, then $\lambda = -\bar{\lambda}$ and this pairing defines a pre-Hilbert structure on $C(P:\xi:\lambda)$. By Lemma 19.12 the representation $\pi_{\xi,\lambda}$ then uniquely extends to a unitary representation in the Hilbert completion of $C(P:\xi:\lambda)$.

It turns out that in the present setting the pairing (81) has a very appealing description. The following lemma prepares for this.

Lemma 20.4. *The inclusion map $K \to G$ induces a diffeomorphism*

$$\iota : K/M \xrightarrow{\simeq} G/P.$$

Proof. Let $k \in K\cap P$. Then $k = man$ for $(m,a,n) \in M\times A\times N$. By the uniqueness part of the Iwasawa decomposition it follows that $k = m \in M$. Thus, we see that $K\cap P = M$. It follows that M is the stabilizer of $[e]$ in K for the natural action of K on G/P. The result follows. □

We return to the principal series representation $\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi\otimes\lambda\otimes 1)$, with $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Let dk be normalized Haar measure on K.

Theorem 20.5. *The form ω can be normalized such that the G-equivariant pairing (81) is given by*

$$(\varphi,\psi)\mapsto \int_K \langle\varphi(k)\,,\,\psi(k)\rangle_\xi\, dk, \tag{82}$$

where $\langle\cdot\,,\,\cdot\rangle_\xi$ denotes the inner product of V_ξ.

Proof. Let φ and ψ be as in (82). Then

$$\langle\varphi(km)\,,\,\psi(km)\rangle_\xi = \langle\varphi(k)\,,\,\psi(k)\rangle_\xi,$$

by unitarity of ξ. It follows that the function $\langle\varphi\,,\,\psi\rangle_\xi$ is right M-invariant. Hence,

$$\int_K \langle\varphi(k)\,,\,\psi(k)\rangle_\xi\, dk = \int_{K/M} \langle\varphi\,,\,\psi\rangle_\xi(\bar{k})\, d\bar{k} \tag{83}$$

where $d\bar{k}$ denotes the the normalized left K-invariant density on K/M.

There exists an alternating form $\omega_M \in \wedge^{\mathrm{top}}(\mathfrak{k}/\mathfrak{m})^*$ such that $d\bar{k}([k]) = dl_k([e])^{-1*}|\omega_M|$ for all $k\in K$. It follows that the integral in (83) equals the integral

$$\int_{K/M}\langle\varphi\,,\,\psi\rangle_{\xi,\omega_M} \tag{84}$$

where the integrand is the density on K/M given by

$$\langle \varphi\,,\,\psi\rangle_{\xi,\omega_M}(k) := \langle \varphi\,,\,\psi\rangle_{\xi}(k)\; dl_k([e])^{-1*}|\omega_M|, \qquad (k\in K).$$

Let $\iota : K/M \to G/P$ be the diffeomorphism induced by inclusion. Then $d\iota([e])$ is the linear map $\mathfrak{k}/\mathfrak{m} \to \mathfrak{g}/(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})$ induced by inclusion. Since this map is a linear isomorphism we may adapt the normalization of $\omega \in \wedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{m}+\mathfrak{a}+\mathfrak{n})^*$ to arrange that

$$d\iota([e])^*|\omega| = |\omega_M|.$$

As in the previous section, Eqn. (79) with P in place of H, we define the density $\langle \varphi\,,\,\psi\rangle_{\xi,\omega}$ on G/P by

$$\langle \varphi\,,\,\psi\rangle_{\xi,\omega} = \langle \varphi\,,\,\psi\rangle_{\xi}(x)\; dl_x([e])^{-1*}|\omega|.$$

Since $\iota : K/M \to G/P$ intertwines the natural K-actions, it follows that

$$\iota^*(\langle \varphi\,,\,\psi\rangle_{\xi,\omega} = \langle \varphi\,,\,\psi\rangle_{\xi,\omega_M}.$$

In view of Lemma 20.4 it now follows by invariance of integration that (84) equals

$$\int_{K/M} \iota^*(\langle \varphi\,,\,\psi\rangle_{\xi,\omega}) = \int_{G/P} \langle \varphi\,,\,\psi\rangle_{\xi,\omega} = \langle \varphi\,,\,\psi\rangle,$$

see (80) for the final equality. □

We will now describe the so-called 'compact picture' of the principal series. We consider the space

$$C(K:M:\xi) := \{\varphi \in C(K,V_\xi) \mid \varphi(km) = \xi(m)^{-1}\varphi(k), \quad \forall k\in K, m\in M\}.$$

The representation of K in this space obtained by restricting the left regular representation is just the induced representation $\mathrm{ind}_M^K(\xi)$, which equals the unitarily induced representation $\mathrm{Ind}_M^K(\xi)$, by compactness of K.

Lemma 20.6. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the restriction map $\varphi \mapsto \varphi|_K$ defines a K-equivariant topological linear isomorphism*

$$r_\lambda : C(P:\xi:\lambda) \to C(K:M:\xi). \tag{85}$$

Proof. If $\varphi : G \to V_\xi$ belongs to $C(P:\xi:\lambda)$, then it is clear that $r_\lambda(\varphi) = \varphi|_K$ belongs to $C(K:M:\xi)$. Thus, (85) is a well-defined continuous linear map. We define the map $i_\lambda : C(K:M:\xi) \to C(G,V_\xi)$ by

$$i_\lambda(f)(kan) = a^{-\lambda-\rho} f(k).$$

This map is well defined and continuous linear in view of the Iwasawa decomposition. Furthermore, it is readily checked that it maps into $C(P:\xi:\lambda)$. Finally, it is straightforward that r_λ and i_λ are inverse to each other. This shows that (85) is a topological linear isomorphism. The K-equivariance is obvious. □

Via the above isomorphism, we may transfer the representation $\pi_{\xi,\lambda}$ to a continuous representation of G in $C(K:\xi)$, which will denoted by $\pi_{\xi,\lambda}$ as well. We will derive a formula for it which expresses the dependence on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

In view of the Iwasawa decomposition and since $\exp : \mathfrak{a} \to A$ is a diffeomorphism with inverse $\log : A \to \mathfrak{a}$, we may define smooth maps $\kappa : G \to K$ and $H : G \to \mathfrak{a}$ by

$$\kappa(kan) = k, \;\; H(kan) = \log a, \qquad ((k,a,n) \in K \times A \times N).$$

Lemma 20.7. *The representaton $\pi_{\xi,\lambda}$ of G in $C(K:\xi)$ is given by*

$$[\pi_{\xi,\lambda}(x)\varphi](k) = e^{-(\lambda+\rho)H(x^{-1}k)}\,\varphi(\kappa(x^{-1}k)), \tag{86}$$

for $\varphi \in C(K:M:\xi)$, $x \in G$ and $k \in K$.

Proof. Write $x^{-1}k = k'a'n'$ in according to the Iwasawa decomposition. Then $k' = \kappa(x^{-1}k)$ and $a = \exp H(x^{-1}k)$. The expression on the left-hand side of (86) equals by definition

$$\begin{aligned} (r_\lambda \circ L_x \circ i_\lambda)(\varphi)(k) &= L_x \circ i_\lambda(\varphi)(k) \\ &= i_\lambda(\varphi)(x^{-1}k) \\ &= i_\lambda(\varphi)(k'a'n') = (a')^{-\lambda-\rho}\,\varphi(k'). \end{aligned}$$

The latter expression equals the expression on the right-hand side of (86). □

Remark 20.8. A special feature of the compact picture is that the representation space for $\pi_{\xi,\lambda}$ has become independent of the parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Moreover, the representation $\pi_{\xi,\lambda}$ depends on λ in a holomorphic fashion. At a later stage this will allow analytic continution of certain identities in the parameter λ.

We denote by $L^2(K:M:\xi)$ the space of functions $\varphi \in L^2(K,V_\xi)$ satisfying the transformation rule

$$\varphi(km) = \xi(m)^{-1}\varphi(k),$$

for $k \in K$ and $m \in M$. This space is a closed subspace of the Hilbert space $L^2(K,V_\xi)$, hence a Hilbert space of its own right.

It is readily seen from the description of the representation $\pi_{\xi,\lambda}$ in Lemma 20.7 that the representation has a unique extension to a continuous linear representation of G in $L^2(K:M:\xi)$. Furthermore, the pairing (83) is the restriction of the inner product of $L^2(K:M:\xi)$. By continuity and density of $C(K:M:\xi)$ in $L^2(K:M:\xi)$ it follows that the inner product on $L^2(K:M:\xi)$ gives a sesquilinear pairing

$$L^2(K:M:\xi) \times L^2(K:M:\xi) \to \mathbb{C}$$

which is equivariant for the represention $\pi_{\xi,\lambda}$ and $\pi_{\xi,-\bar{\lambda}}$, respectively. Equivalently, this means that

$$\pi_{\xi,\lambda}^* = \pi_{\xi,-\bar{\lambda}}.$$

In particular, if $\lambda \in i\mathfrak{a}^*$ then the representation $\pi_{\xi,\lambda}$ of G in $L^2(K:M:\xi)$ is unitary.

Finally, via the isomorphism r_λ, the L^2-inner product on $C(K:M:\xi)$ transfers to a pre-Hilbert structure on $C(P:\xi:\lambda)$. Let $L^2(P:\xi:\lambda)$ denote the completion of $C(K:M:\xi)$. Then the map r_λ extends to an isometry $L^2(P:\xi:\lambda) \simeq L^2(K:M:\xi)$ which intertwines the restriction of L with $\pi_{\xi,\lambda}$.

We conclude this section with a result on the K-isotypical components of the representations of the principal series. We agree to write

$$C^\infty(P:\xi:\lambda) := C(P:\xi:\lambda) \cap C^\infty(G, V_\xi).$$

This is a closed subspace of $C^\infty(G, V_\xi)$, hence Fréchet in a natural way. Furthermore, it is invariant under left transflation, and $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ restricts to a continuous representation on it.

Proposition 20.9. *Let $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the space $C(P:\xi:\lambda)_K$ of left K-finite functions in $C(P:\xi:\lambda)$ is a dense subspace of $C^\infty(P:\xi:\lambda)$. Furthermore, if $\delta \in \widehat{K}$, then the associated K-isotypical component in $C(P:\xi:\lambda)$ is finite dimensional, of dimension*

$$\dim C(P:\xi:\lambda)[\delta] = \dim \mathrm{Hom}_M(\delta|_M, \xi) \cdot \dim \delta.$$

Proof. Since $r_\lambda : C(P:\xi:\lambda) \to C(K:M:\xi)$ is a K-equivariant topological linear isomorphism which restricts to a topological linear isomorphism from $C^\infty(P:\xi:\lambda)$ onto $C^\infty(K:M:\xi) := C(K:M:\xi) \cap C^\infty(K, V_\xi)$, with inverse i_λ, it suffices to prove the analogous statements for $C(K:M:\xi)$ and $C^\infty(K:M:\xi)$.

Now $C(K:M:\xi)$ is a left K-invariant subspace of $C(K, V_\xi)$. Using the canonical identification of the latter space with $C(K) \otimes V_\xi$, we see that

$$C(K:M:\xi)_K \subset C(K, V_\xi)_K = \mathscr{R}(K) \otimes V_\xi \subset C^\infty(K, V_\xi).$$

This implies that $C(K:M:\xi)_K = C^\infty(K:M:\xi)$. Since the left regular representation of K in $C^\infty(K:M:\xi)$ is a continuous representation in a Fréchet space, it follows by application of Proposition 7.4 that $C(K:M:\xi)_K$ is dense in $C^\infty(K:M:\xi)$. This establishes the first assertion.

For the assertion on the dimension, we note that

$$C(K:M:\xi)[\delta] \simeq \mathrm{Hom}_K(V_\delta, C(K:M:\xi)) \otimes V_\delta.$$

Now

$$\mathrm{Hom}_K(V_\delta, C(K:M:\xi)) = \mathrm{Hom}_K(\delta, \mathrm{ind}_M^K(\xi)) \simeq \mathrm{Hom}_M(\delta|_M, \xi),$$

by Frobenius reciprocity (see exercises). The result follows. $\square$

21 Smooth vectors in a representation

We assume that G is a connected real semisimple Lie group with finite center, that $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$ and that K is the associated maximal compact subgroup of G. Let V be a complete locally convex space. If Ω is an open subset of $\mathbb{R}^n$, then we denote by $C(\Omega, V) = C^0(\Omega, V$ the space of continuous functions $f : \Omega \to V$. This space carries the locally convex topology induced by the fundamental system of seminorms

$$\| \cdot \|_{K,s} : f \mapsto \sup_{x \in K} s(f(x)),$$

for $K \subset \Omega$ compact and s a continuous seminorm on V.

If $\Omega \subset \mathbb{R}^n$ is an open subset then a function $f : \Omega \to V$ is said to be C^1 if the partial derivatives $\partial_j f$, $(1 \leq j \leq n)$, exist and define continuous functions $\Omega \to V$. By recursion, we say that f is C^p for $p \geq 1$ if the partial derivatives $\partial_j f$, $(1 \leq j \leq n)$, exist and define C^{p-1}-functions $\Omega \to V$. The space of these functions is denoted by $C^p(\Omega, V)$. Finally, the space of smooth functions $\Omega \to V$ is defined by

$$C^\infty(\Omega, V) := \bigcap_{p \geq 1} C^p(\Omega, V).$$

By the usual proof one shows that the partial derivatives ∂_i and ∂_j (for $1 \leq i, j \leq n$) commute on $C^2(\Omega, V)$. Accordingly, on $C^p(\Omega, V)$ the general mixed partial derivative is of the form

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

for $\alpha \in \mathbb{N}^n$ a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ at most p.

The space $C^p(\Omega, V)$ can be equipped with the continuous seminorms

$$\| \cdot \|_{K,s,p} : f \mapsto \max_{|\alpha| \leq p} \sup_{x \in K} s(\partial^\alpha f(x)),$$

for $K \subset \Omega$ compact and s a continuous seminorm of V. This turns $C^p(\Omega, V)$ into a complete locally convex space. Furthermore, it is Fréchet if V is Fréchet. The space $C^\infty(\Omega, V)$, equipped with all seminorms $\| \cdot \|_{K,p,s}$, for $K \subset \Omega$ compact, $p \geq 1$ and σ a continuous seminorm on V is a complete locally convex space as well. If V is Fréchet, then so is $C^\infty(\Omega, V)$.

If $\varphi : \Omega' \to \Omega$ is a diffeomorphism of open subsets of $\mathbb{R}^n$, then by making iterated use of the chain rule one sees that $\varphi^* : f \mapsto f \circ \varphi$ defines a continuous linear map $C^p(\Omega) \to C^p(\Omega')$. This allows to define the spaces $C^p(M, V)$, for a smooth manifold M. The spaces can be equipped with the coarsest locally convex topology for which every restriction map $f \mapsto f|_\Omega$, with Ω a coordinate chart, becomes continuous.

In other words, let $\{\Omega_j\}_{j \in J}$ be a countable open cover of M with coordinate charts. Let $0 \leq p \leq \infty$. Then

$$\Pi := \Pi_{i \in I} C^p(\Omega_i, V),$$

equipped with the product topology, is a locally convex space. Indeed the topology is determined by the collection of seminorms

$$\varphi \mapsto \max_{i \in I_0} s_i(\varphi_i),$$

with $I_0 \subset I$ a finite subset, and with s_i continuous seminorms on $C^p(\Omega_i)$, for $i \in I_0$. It is readily seen that Π is complete, and Fréchet as soon as V is Fréchet.

We consider the map $r : C^p(M,V) \to \Pi$ defined by

$$r(f)_i = f|_{\Omega_i}.$$

Then r is injective with image equal to

$$\mathrm{im}(r) = \{\varphi \in \Pi \mid \varphi_i|_{\Omega_i \cap \Omega_j} = \varphi_j|_{\Omega_i \cap \Omega_j}, \ \forall i,j \in I\}.$$

This is readily seen to be a closed subspace of Π, hence a complexe locally convex space, which is Fréchet as soon as V is. The topology of $C^p(M,V)$ is such that r is an isomorphism of locally convex spaces.

We now turn to the situation of $C^p(G,V)$ with G a Lie group. Given a compact subset $K \subset G$, a continuous seminorm s of V a finite subset $F \subset U(\mathfrak{g})_p$ we define the seminorm

$$\|\cdot\|_{K,s,F} : C^p(G,V) \to [0,\infty[, \quad f \mapsto \max_{u \in F} \sup_{x \in K} s(R_u f(x)).$$

Lemma 21.1. *The above system of seminorms $\|\cdot\|_{K,s,F}$ determines the locally convex topology of $C^p(G,V)$. The space $C^p(G,V)$ is complete. If V is Fréchet, then so is $C^p(G,V)$.*

Proof. It suffices to prove the result for a fixed $p < \infty$.

Let Ω be an open neighborhood of 0 in $\mathfrak{g}$ such that $\exp|_\Omega$ is a diffeomorphism onto an open neighborhood Ω_G of e in G. Let $C^p(\Omega_G, V)$ be equipped with the seminorms $\|\cdot\|_{\mathscr{K},s,F}$ as above, with $\mathscr{K} \subset \Omega_G$. Then by left equivariance, it suffices to show that the bijective linear map $\exp^* : C^p(\Omega_G,V) \to C^p(\Omega,V)$ is a topological linear isomorphism for the previously defined locally convex topology on the second space.

For every $u \in U(\mathfrak{g})_p$, the operator $A(u) := \exp_*(R_u) = \exp^{-1*} \circ R_u \circ \exp^*$ is a uniquely defined partial differential operator on Ω, hence can be viewed as a smooth function $A(u) : \Omega \to S(\mathfrak{g})_p$ with the property that

$$A(u)\varphi(X) = \partial_{A(u)(X)}\varphi(X), \qquad (X \in \Omega).$$

Since A is linear, it follows that $A(u)(X) = A(X)u$, with $A : \Omega \to \mathrm{Hom}(U(\mathfrak{g})_p, S(\mathfrak{g})_p)$ a smooth function. We observe that

$$\exp^*(R_u f)(X) = \partial_{A(X)u}(\exp^* f)(X), \qquad (X \in \Omega),$$

for $f \in C^p(\Omega_G,V)$ and $u \in U(\mathfrak{g})_p$. This expression implies that that for every $\mathscr{K} \subset \Omega$ there exists a constant $C_{\mathscr{K}} > 0$ such that

$$\|R_u f\|_{\exp(\mathscr{K}),s} \leq C_K \|\exp^* f\|_{\mathscr{K},s,p}, \qquad (f \in C^p(\Omega,V)).$$

Hence, the operator $\exp^{-1*}$ is continuous.

On the other hand, let $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrizer. Then $\bar{A}(X) := s \circ A(X)$ is a linear endomorphism of $U(\mathfrak{g})_p$ depending smoothly on $X \in \Omega$. Furthermore, $\bar{A}(0) = I$ by Theorem

11.6. Replacing Ω by a smaller neighborhood if necessary, we may assume that $\det \bar{A}(X) \neq 0$ for $X \in \Omega$. By Cramer's rule, it now follows that $X \mapsto \bar{A}(X)^{-1}$ is smooth, hence $X \mapsto A(X)^{-1}$ is smooth as well. Writing $B(X) = A(X)^{-1}$, we obtain that

$$\exp^*(R_{B(X)u} f)(X) = \partial_u(\exp^* f)(X), \qquad (X \in \Omega),$$

for all $u \in U(\mathfrak{g})_p$ and $f \in C^p(\Omega_G)$. Let F be a basis of $U(\mathfrak{g})_p$, then we see that for every compact subset $\mathscr{K} \subset \Omega$ there exists a constant $C_{\mathscr{K}} > 0$ such that

$$\|\exp^* f\|_{\mathscr{K},s,p} \leq C_{\mathscr{K}} \|f\|_{\exp(\mathscr{K}),s,F}$$

for all $f \in C^p(\mathscr{K}, V)$. This implies that $\exp^*$ is a continuous linear map. $\square$

Definition 21.2. Let (π, V) be a continuous representation of G in a Fréchet space V. A vector $v \in V$ is said to be C^∞ or smooth if $x \mapsto \pi(x)v$ is a smooth map $G \to V$. The space of these vectors is denoted by V^∞.

Lemma 21.3. *V^∞ is a G-invariant subspace of V.*

Proof. Let $v \in V^\infty$ and $y \in G$. Then

$$\pi(x)\pi(y)(v) = \pi(xy)v = [R_y \varphi](x)$$

where $\varphi : x \mapsto \pi(x)v$ is a smooth function $G \to V$. It follows that $\pi(y)v \in V^\infty$. $\square$

Given $X \in \mathfrak{g}$ and $v \in V^\infty$, we define

$$\pi_*(X)v := \left.\frac{d}{dt}\right|_{t=0} \pi(\exp tX)v.$$

Lemma 21.4. *If $X \in \mathfrak{g}$ and $v \in V^\infty$ then $R_X v \in V^\infty$. The map $\mathfrak{g} \times V^\infty \to V^\infty$, $(X, v) \mapsto R_X v$ defines a representation of $\mathfrak{g}$ in V^∞.*

Proof. Given $v \in V^\infty$ we consider the function $\varphi_v : G \to V$, $(x, v) \mapsto \pi(x)v$. Then it follows that

$$\pi(x)\pi_*(X)v = \pi(x) \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tX)) = \left.\frac{d}{dt}\right|_{t=0} \pi(x \exp tX)v$$

since $\pi(x) : V \to V$ is continuous linear. It follows from the above that

$$\pi(x)\pi_*(X)v = R_X \varphi_v(x), \qquad (x \in G).$$

Since $R_X \varphi_v$ is a smooth function, we see that $\pi_*(X)v \in V^\infty$ and

$$\varphi_{\pi_*(X)v} = R_X \varphi_v.$$

In particular, it follows from the above that

$$\pi_*(X)v = (R_X \varphi_v)(e).$$

from this we infer that the map $\mathfrak{g} \times V^\infty \to V^\infty$ given by $(X, v) \mapsto \pi_*(X)v$ is bilinear. To finish the proof we observe that for $X, Y \in \mathfrak{g}$ and $v \in V^\infty$ we have

$$\pi_*(X)\pi_*(Y)v - \pi_*(Y)\pi_*(X)v = [R_X R_Y \varphi_v - R_Y R_X \varphi_v](e) = R_{[X,Y]}\varphi_v(e) = \pi_*([X,Y])v.$$

This implies that $\pi_*([X,Y]) = [\pi_*(X), \pi_*(Y)]$ on V^∞. $\square$

Lemma 21.5. *Let $v \in V^\infty$, $x \in G$ and $X \in \mathfrak{g}$. Then*

$$\pi(x)\pi_*(X)v = \pi_*(\mathrm{Ad}(x)X)\pi(x)v.$$

Proof. From the first display in the proof of the previous lemma, it follows that

$$\pi(x)\pi_*(X)v = \frac{d}{dt}\bigg|_{t=0} \pi(x \exp tX)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp t\mathrm{Ad}(x)X)\pi(x)v.$$

Since $\pi(x)v \in V^\infty$, the result follows. $\square$

It follows from the above, that V^∞ equipped with the representions π_* of $\mathfrak{g}$ and $\pi|_K$ of K is a so called $(\mathfrak{g}, K)$-module.

Before proceeding we note that V^∞ may be equipped with a locally convex topology for which the restriction

$$\pi^\infty = \pi|_{V^\infty}$$

becomes a continuous representation of G.

To see this, let $C^\infty(G,V)^G$ denote the space of smooth functions $\varphi : G \to V$ transforming according to the rule $L_x\varphi = \pi(x)^{-1}v$ for all $x \in G$. Then clearly, $C^\infty(G,V)^G$ is a closed subspace of $C^\infty(G,V)$, hence a complete locally convex space for the restriction topology. Furthermore, if V is a Fréchet space, then so is $C^\infty(G,V)^G$.

We define $\Phi : V^\infty \to C^\infty(G,V)$ by $\Phi(v)(x) = \pi(x)v$. Then it is readily seen that this map defines a linear isomorphism

$$\Phi : V^\infty \xrightarrow{\simeq} C^\infty(G,V)^G$$

for which the evaluation map $\mathrm{ev}_e : \varphi \mapsto \varphi(e)$ is a two-sided inverse. It is readily checked that $C^\infty(G,V)^G$ is invariant under the right regular action R of G and that Φ intertwines π^∞ with R.

We equip V^∞ with the locally convex topology for which Φ is a topological linear isomorphism. Obviously π^∞ becomes a continuous representation for this topology.

Lemma 21.6. *Assume that (π, V) is a continuous representation in a Fréchet space (or more generally, a complete barreled space). Then the topology of V^∞ is generated by the seminorms*

$$v \mapsto \|v\|_{F,s} := \max_{X \in F} s(\pi_*(X)v),$$

with $F \subset U(\mathfrak{g})$ a finite subset and s a continuous seminorm of V.

Proof. Let $\mathscr{T}$ denote the locally convex topology on V^∞ induced by the given collection of seminorms. It is clear that that the map $\mathrm{ev}_e : C^\infty(G,V)^G \to V^\infty$ is continuous for this topology on V^∞. To see that $\Phi : V^\infty \to C^\infty(G,V)^G$ is continuous relative to $\mathscr{T}$, let $\mathscr{K} \subset G$ be compact, $F \subset U(\mathfrak{g})$ a finite subset and s a continuous seminorm of V.

By the principle of uniform boundedness, the set of continuous linear maps $\{\pi(k) \mid k \in \mathscr{K}\}$ in $\mathrm{End}(V)$ is equicontinuous. Hence, there exists a continuous seminorm s' on V such that $s(\pi(k)v) \leq s'(v)$ for all $k \in \mathscr{K}$ and $v \in V$. This implies that

$$\|\Phi(v)\|_{\mathscr{K},F,s} = \max_{X\in F} \sup_{k\in\mathscr{K}} s(\pi(k)\pi_*(X)v) \leq \|v\|_{s',F}.$$

Thus, the map $\Phi : (V^\infty, \mathscr{T}) \to C^\infty(G,V^\infty)^G$ is continuous. $\square$

Lemma 21.7. *Let $\psi \in C_c^\infty(G)$. Then $\pi(\psi)$ defines a linear map $V \to V^\infty$. If V is a Fréchet space (or, more generally, complete and barreled), this map is continuous.*

Proof. Let ω be a relatively compact open subset of G containing e. Then there exists a compact subset $\mathscr{K}$ of G (only depending on ω and $\mathrm{supp}\psi$) such that the map $y \mapsto L_y\psi$ is smooth as a map $\omega \to C_{\mathscr{K}}(G)$. Let $v \in V$ and let s be a continous seminorm on V. Then $s(\pi(x)v)$ is bounded by a constant $C > 0$ for $x \in \mathscr{K}$. It follows that

$$s(\pi(\psi)v) \leq C\|\psi\|_{\mathscr{K}}. \tag{87}$$

We infer that the map $\psi \mapsto \pi(\psi)v$, $C_{\mathscr{C}}(G) \to V$ is continuous. Combining this with the first observation of the proof, we see that $y \mapsto \pi(L_y\psi)v$ is a smooth map $\omega \to V$. This implies that

$$y \mapsto \pi(y)\pi(\psi)v = \pi(L_y\psi)v$$

is smooth on ω, and we conclude that $\pi(\psi)v \in V^\infty$. Furthermore, if $X \in \mathfrak{g}$ then $(L_{\exp tX}\psi - \psi)/t \to L_X\psi$ in $C_{\mathscr{C}}(G)$, so that

$$\pi_*(X)\pi(\psi)v = \left.\frac{d}{dt}\right|_{t=0} \pi(L_{\exp tX}\psi)v = \pi(L_X\psi)v. \tag{88}$$

Applying this argument repeatedly, we see that the equality between the expressions on the left and right hold for all $X \in U(\mathfrak{g})$.

We will now prove the final assertion about continuity. Let s be a continuous seminorm on V. Then by the principle of uniform boundedness, tthe family $\{\pi(k) \mid k \in \mathscr{C}\}$ of linear maps $V \to V$ is equicontinuous. Hence, there exists a continuous seminorm s' on V such that $s \circ \pi(x) \leq s'$ for all $x \in \mathscr{K}$. It follows that

$$s(\pi_*(X)\pi(\psi)v) = s(\pi(L_X\psi)v) \leq \|L_X\psi\|_{\mathscr{K}} \cdot s'(v), \qquad (v \in V).$$

In view of Lemma 21.6 this establishes the continuity. $\square$

Lemma 21.8. *The space V^∞ is dense in V.*

Proof. Let $\{\psi_j\}_{j\geq 0}$ be an approximation of the identity on G, consisting of smooth compactly supported functions. Let $v \in V$. Then by Lemma 2.7 it follows that

$$\pi(\psi_j)v \to v \quad (j \to \infty)$$

in V. The elements of this sequence belong to V^∞, by Lemma 21.7. $\square$

22 Admissible representations

We assume that G is a connected real semisimple Lie group with finite center, and that $(\pi, \mathscr{H})$ is a unitary representation of G. Let $\mathrm{Hom}_G(\mathscr{H})$ denote the algebra of continous linear intertwining operators $\mathscr{H} \to \mathscr{H}$. In the exercises, we mentioned the following version of Schur's lemma, which can be proved by using the spectral theorem for bounded self-adjoint operators.

Lemma 22.1. *The representation π is irreducible if and only if $\mathrm{End}_G(\mathscr{H}) = \mathbb{C}\mathrm{I}_{\mathscr{H}}$.*

Based on this lemma, one can prove the following density theorem.

Lemma 22.2. *Let $(\pi, \mathscr{H})$ be an irreducible unitary representation of G and let $\mathscr{A}$ be the subalgebra of $\mathrm{End}(\mathscr{H})$ spanned by elements $\pi(g)$, for $g \in G$. Then for every linearly independent set $v_1, \dots, v_k$ in $\mathscr{H}$, every collection of elements $w_1, \dots, w_k$ in $\mathscr{H}$, and every $\varepsilon > 0$ there exists an operator $A \in \mathscr{A}$ such that*

$$\|Av_j - w_j\| < \varepsilon \qquad (1 \le j \le k).$$

Proof. This follows from an observation of von Neumann, see [Wal88, §1.2] for details. $\square$

We recall that $Z(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. The following may be viewed as a version of Schur's lemma.

Theorem 22.3. *Let $(\pi, \mathscr{H})$ be an irreducible unitary representation of G in a Hilbert space $\mathscr{H}$. Then $Z(\mathfrak{g})$ acts by scalars on $\mathscr{H}^\infty$.*

Proof. If $Z \in Z(\mathfrak{g})$ then $\mathrm{Ad}(x)Z = Z$ for all $x \in G$, since G is connected. It now follows from Lemma 21.5 that

$$\pi(x)\,\pi_*(Z) = \pi_*(Z)\,\pi(x) \qquad \text{on } V^\infty$$

for all $x \in G$. From the unitarity of π, it follows by differentiation and extension that

$$\langle \pi(X)v\,,\, w\rangle = \langle v\,,\, \pi(X^\vee)\rangle$$

for all $X \in U(\mathfrak{g})$ and $v, w \in \mathscr{H}^\infty$. We will now finish the proof by showing that $\pi_*(Z)v \in \mathbb{C}v$ for all $Z \in Z(\mathfrak{g})$ and $v \in V^\infty$. Arguing by contradiction, assume that $v \in V^\infty$ and $Z \in Z(\mathfrak{g})$ are such that v and $\pi_*(Z)v$ are linearly indendent. Let $\mathscr{A}$ be as in Lemma 22.2. Then the elements of $\mathscr{A}$ commute with $\pi_*(Z)$ on V^∞. There exists a sequence $(A_j) \subset \mathscr{A}$ such that

$$A_j v \to v, \quad A_j \pi_*(Z)v \to v.$$

Let $w \in \mathscr{H}^\infty$, then it follows that

$$\begin{aligned} \langle v\,,\, w\rangle &= \lim \langle A_j \pi_*(Z)v\,,\, w\rangle = \lim\,\langle \pi_*(Z)A_j v\,,\, w\rangle \\ &= \lim\,\langle A_j v\,,\, \pi_*(Z^\vee)w\rangle = \langle v\,,\, \pi_*(Z^\vee)w\rangle = \langle \pi_*(Z)v\,,\, w\rangle. \end{aligned}$$

Since $\mathscr{H}^\infty$ is dense in $\mathscr{H}$, this implies that $v = \pi_*(Z)v$, contradiction. $\square$

It follows from the above result that for an irreducible unitary representation $(\pi, \mathscr{H})$ the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by a character on $\mathscr{H}^\infty$.

Definition 22.4. Let (π, V) be a continuous representation in a locally convex space V. The representation π is said to be quasi-simple if there exists a (unique) character $\chi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ such that

$$\pi_*(Z) = \chi(Z)I \quad \text{on } V^\infty.$$

In this case, the uniquely determined character χ_π is called the infinitesimal character of π.

The following result, due to Harish-Chandra, is an important step towards the algebraization of the representation theory of G. We say that a continuous representation (π, V) of G in a complete locally convex space is K-finite finitely generated if and only if there exists a finite subset $F \subset V_K$ such that the closed span of the elements $\pi(g)v$ for $g \in G$ and $v \in F$ equals V.

Theorem 22.5. (Harish-Chandra) *Let (π, V) be a continuous representation of G in a complete locally convex space V. Assume that π is quasi-simple and K-finite finitely generated. Then for every $\delta \in \widehat{K}$ the K-isotypical component $V[\delta]$ is finite dimensional.*

Proof. (Sketch). There exists a K-invariant finite dimensional subspace V_0 of V such that V is the closed span of $\pi(G)V_0$. Let $\delta \in \widehat{K}$ and let $P_\delta : V \to V[\delta]$ be the unique K-equivariant projection operator. We define the function $f : G \to \mathrm{Hom}(V_0, V[\delta])$ defined by

$$f(x)(v) = P_\delta(\pi(x)v), \qquad (x \in G, v \in V_0).$$

Then the function f satisfies the differential equations

$$R_Z f = \chi(Z) f, \qquad (Z \in Z(\mathfrak{g})),$$

together with finitely many initial value conditions at any given point $a_0 \in A^{\mathrm{reg}}$. It can be shown that the solution space to this initial value problem is finite dimensional. Moreover, there exists a finite dimensional subspace $H_1 \subset \mathrm{Hom}_K(V_\delta, V)$ such that the initial values are contained in $\mathrm{Hom}(V_0, V_1)$, where V_1 is the (finite dimensional) canonical image of $V_\delta \otimes H_1$ in $V[\delta]$. It follows from the nature of the differential equations that f must have all its values in $\mathrm{Hom}(V_0, V_1)$. Since V equals the closed span of the elements $\pi(g)v$ for $v \in V_0$ and $g \in G$, this implies that $V[\delta] = P_\delta(V) \subset \overline{V_1} = V_1$. It follows that $V[\delta]$ is finite dimensional. We refer the reader to [Var77, p. 312, Thm. 12], for further details. □

Definition 22.6. A continuous representation (π, V) of G in a complete locally convex space is said to be admissible if $\dim V[\delta] < \infty$ for all $\delta \in \widehat{K}$.

It follows from the above result that any K-finite finitely generated quasi-simple representation of G is admissible. In particular, this applies to irreducible unitary representations.

Lemma 22.7. *Let (π, V) be an admissible representation of G in a complete locally convex space. Then for all $\delta \in \widehat{K}$ we have the inclusion $V[\delta] \subset V^\infty$. The space of K-finite vectors is a $\mathfrak{g}$-invariant subspace of V^∞.*

Proof. The unique equivariant projection operator $P_\delta : V \to V[\delta]$ is surjective and is given by $P_\delta = \dim(\delta)\,\pi|_K(\chi_{\delta^\vee})$. From this formula one readily sees that it maps V^∞ into V^∞, hence into $V^\infty \cap V[\delta]$. By density of V^∞ in V it now follows that $P_\delta(V^\infty)$ hence $V[\delta] \cap V^\infty$ is dense in $V[\delta]$. By finite dimensionality of the latter space it follows that these two spaces are equal so that $V[\delta] \subset V^\infty$.

For the latter assertion it suffices to show that for a given $\delta \in \widehat{K}$ we have $\pi_*(\mathfrak{g})V[\delta] \subset V_K$. In view of Lemma 21.5 the natural map $\mathfrak{g} \otimes V[\delta] \to V$ is a K-module map; here $\mathfrak{g}$ is equipped with the adjoint representation by K. Since the tensor product is a finite dimensional K-module, it follows that the image is contained in V_K. $\square$

Definition 22.8. A $(\mathfrak{g}, K)$-module is a complex linear space V equipped with the structure of $\mathfrak{g}$-module and of a K representation π such that

(a) For all $k \in K$ and $X \in \mathfrak{g}$, we have $\pi(k) \circ X = [\mathrm{Ad}(k)X] \circ \pi(k)$ on V.

(b) For all $v \in V$ the linear span W_v of the elements $\pi(k)v$, $(k \in K)$, is finite dimensional and the restricted representation $\pi|_{W_v}$ of K in W_v is continuous.

(c) For all $X \in \mathfrak{k}$ and $v \in V$,
$$Xv = \frac{d}{dt}\bigg|_{t=0} \pi(\exp tX)v.$$

For a $(\mathfrak{g}, K)$-module we may define isotypical components $V[\delta]$, for $\delta \in \widehat{K}$ as before. Then $V[\delta]$ is the canonical image of $\mathrm{Hom}(V_\delta, V) \otimes V_\delta$ in V. Furthermore,
$$V = \bigoplus_{\delta \in \widehat{K}} V[\delta].$$

The associated projections $P_\delta : V \to V[\delta]$ are given by the formula
$$P_\delta v := \dim(\delta) \int_K \chi_\delta^\vee(k)\pi(k)v\, dk.$$

Indeed this formula makes sense because of condition (b) in the definition above.

Definition 22.9. Let V be a $(\mathfrak{g}, K)$-module. Then V is said to be admissible if $\dim V[\delta] < \infty$ for all $\delta \in \widehat{K}$.

Let (V, π) be an admissible continuous representation of G in a complete locally convex space, then it follows that V_K is an admissible $(\mathfrak{g}, K)$-module.

Lemma 22.10. *Let (π, V) be an admissible representation of G in a complete locally convex space V. Then $W \mapsto W_K$ defines a bijective correspondence between the set of closed G-invariant subspaces of V and the set of $(\mathfrak{g}, K)$-submodules of V.*

Proof. Let W be a closed submodule. Then clearly, $W^\infty \subset V^\infty$ and it follows that W^∞ is a $\mathfrak{g}$-submodule of V^∞. It follows that W_K is a $\mathfrak{g}$-submodule of V_K. Furthermore, W_K is dense in W, so that W is the closure of W_K in W. This establishes the injectivity of the mentioned map. To prove its surjectivity, assume that W_0 is a $(\mathfrak{g}, K)$-submodule of V. We claim that its closure W is invariant under G. Assuming the claim we find that W_0 is dense in W, hence for every $\delta \in \widehat{K}$ it follows that $W_0[\delta] = P_\delta(W_0)$ is dense in $W[\delta] = P_\delta(W)$. By using the finite dimensionality of $V[\delta]$ we conclude that $W[\delta] = W_0[\delta]$. Hence $W_K = W_0$.

To see that the claim holds, let $\lambda \in V'$ be a continuous linear functional which vanishes on W_0. Given $v \in W_0$ the function $f : x \mapsto \lambda(\pi(x)v)$ can be shown to be analytic on G (this follows from application of the elliptic regularity theorem, see [Var77] for details). By differentiating with respect to x at e we find that $R_X f = \lambda(\pi_*(u)v) = 0$ for all $X \in U(\mathfrak{g})$. By analyticity this implies that f vanishes on the connected group G. Hence λ vanishes on the linear span of the elements $\pi(g)v$. By the Hahn-Banach theorem it follows that $\pi(g)v \in W$ for all $v \in W_0$ and $g \in G$. Thus, if $g \in G$ then $\pi(g)$ maps W_0 into W. Since $\pi(g)$ is continuous, and W is the closure of W_0 it follows that $\pi(g)(W) \subset W$. The result follows. □

Definition 22.11. Let V be a $(\mathfrak{g}, K)$-module. Then V is irreducible if 0 and V are the only subspaces which are invariant under both $\mathfrak{g}$ and K.

Corollary 22.12. *Let (π, V) be an admissible representation of G in a complete locally convex space, and let V_K be the associated $\mathfrak{g}, K)$-module. Then the following assertions are equivalent.*

(a) *π is irreducible;*

(b) *V_K is an irreducible $(\mathfrak{g}, K)$-module.*

Proof. This follows from Lemma 22.10. □

Definition 22.13. A Harish-Chandra module is a $(\mathfrak{g}, K)$-module which is finitely generated.

We have the following Schur's lemma for irreducible Harish-Chandra modules. Given a $(\mathfrak{g}, K)$-module V, we denote by $\mathrm{End}_{\mathfrak{g},K}(V)$ the space of linear maps $T : V \to V$ that intertwine the actions of both $\mathfrak{g}$ and K.

Lemma 22.14. *Let V be an irreducible Harish-Chandra module. Then* $\mathrm{End}_{\mathfrak{g},K}(V) = \mathbb{C}\mathrm{I}_V$.

Proof. Let $T \in \mathrm{End}_{\mathfrak{g},K}(V)$. We may assume that $V \neq 0$. Fix $\delta \in \widehat{K}$ such that $V[\delta] \neq 0$. Then T restricts to a linear endomorphism $V[\delta] \to V[\delta]$. Hence there exists $\lambda \in \mathbb{C}$ such that $\ker(T - \lambda \mathrm{I}_V)$ has non-trivial intersection with $V[\delta]$. It follows that $\ker(T - \lambda \mathrm{I}_V)$ is a non-trivial subspace of V, which is $(\mathfrak{g}, K)$-invariant. As V is irreducible, we infer that $T = \lambda \mathrm{I}_V$. □

Let $(\pi, \mathscr{H})$ be an irreducible unitary representation. Then the associated space $\mathscr{H}_K$ of K-finite vectors is an irreducible Harish-Chandra module. If $X \in \mathfrak{g}$ then the unitarity of π implies that for every $v, w \in \mathscr{H}_K$ we have

$$\langle \pi(\exp tX)v, \pi(\exp tX)w \rangle = \langle v, w \rangle.$$

Since $V_K \subset\in V^\infty$, it follows by differentiating the above expression at $t = 0$ that

$$\langle Xv, w \rangle + \langle v, Xw \rangle, \qquad (v, w \in \mathscr{H}_K,\ X \in \mathfrak{g}). \tag{89}$$

Definition 22.15. Let V be a $(\mathfrak{g},K)$-module. The module V is said to be unitary for a given pre-Hilbert structure $\langle\cdot\,,\,\cdot\rangle$ on V if (89) is valid and K leaves the pre-Hilbert structure invariant. Two such pre-Hilbert structures are said to be equivalent if they differ by a positive scalar factor.

Lemma 22.16. *Let V be an irreducible Harish-Chandra module. If V is unitary for two pre-Hilbert structures , then these pre-HIlbert structures are equivalent.*

Proof. Let $\langle\cdot\,,\,\cdot\rangle_j$ for $j = 1,2$ be two pre-Hilbert structures on V for which V is unitary. For $\delta \in \widehat{K}$ we define $T_\delta : V[\delta] \to V[\delta]$ by

$$\langle v\,,\,w\rangle_2 = \langle T_\delta v\,,\,w\rangle_1, \qquad (v,w \in V[\delta]).$$

We define $T : V \to V$ to be the direct sum of the maps T_δ. Then T is an invertible endomorphism of V.

It follows from the unitarity of the K-actions that the projections P_δ are symmetric, hence orthogonal with respect to both pre-HIlbert structures. It follows that the spaces $V[\delta]$ are mutually orthogonal for both pre-Hilbert structures. Hence,

$$\langle v\,,\,w\rangle_2 = \langle Tv\,,\,w\rangle_1, \qquad (v,w \in V).$$

By non-degeneracy of the pre-Hilbert structures, T is uniquely defined by this identity. Furthermore, by $(\mathfrak{g},K)$-equivariance of the pre-Hilbert structures it follows that T is a $(\mathfrak{g},K)$-module homomorphism. Since V is irreducible, we conclude that $T = \lambda \mathrm{I}_V$ for a constant $\lambda \in \mathbb{C}$. By positive definiteness of the inner products, $\lambda > 0$. The equivalence follows. $\square$

The following result, due to Harish-Chandra, is given without proof. It implies that the classification of irreducible unitary representations comes down to the classification of irreducible unitary Harish-Chandra modules.

Theorem 22.17. *Let V be a unitary irreducible Harish-Chandra module. Then there exists an irreducible unitary representation $(\pi,\mathscr{H})$ of G such that $\mathscr{H}_K \simeq V$, unitarily. If $(\pi',\mathscr{H}')$ is a second such representation, then π and π' are unitarily equivalent.*

We now come to a result that provides a severe limitation on the possible irreducible Harish-Chandra modules. We assume that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition and $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. Let $G = KAN$ be an associated Iwasawa decomposition, and $P = MAN$ the associated minimal parabolic subgroup.

We recall that every principal series representation $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ for $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is admissible. It can be shown that the associated $(\mathfrak{g},K)$-module, denoted $C(P:\xi:\lambda)_K$ is finitely generated, hence a Harish-Chandra module.

Theorem 22.18. (Subrepresentation Theorem) *Let V be an irreducible Harish-Chandra module. Then there exist $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and a $(\mathfrak{g},K)$-equivariant embedding*

$$V \hookrightarrow C(P:\xi:\lambda)_K.$$

We give a sketch of the proof. By using differential equations one proves that $V/\mathfrak{n}V$ is a non-trivial finite dimensional $(M,\mathfrak{a})$-module, different from 0. The existence of the embedding is then a consequence of the following result, which we shall prove.

Theorem 22.19. (Casselman's reciprocity) *Let V be a Harish-Chandra module. Then for all $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have*

$$\mathrm{Hom}_{\mathfrak{g},K}(V, C(P:\xi:\lambda)_K) \simeq \mathrm{Hom}_{M,\mathfrak{a}}(V/\mathfrak{n}V, V_{\xi,\lambda+\rho}).$$

Proof. Let $T \in \mathrm{Hom}_{\mathfrak{g},K}(V, C(P:\xi:\lambda)_K)$, then we define $\varepsilon(T): V \to V_{\xi,\lambda}$ by $\varepsilon(T)(v) = T(v)(e)$. For $X \in \mathfrak{n}$ and $v \in \mathfrak{n}$ we have

$$\varepsilon(T)(Xv) = [T(Xv)](e) = [L_X[T(v)](e)] = 0,$$

since Tv is a right N-invariant function. It follows that $\varepsilon(T)$ induces a map $V/\mathfrak{n}V \to V_{\xi,\lambda}$. Furthermore, if $m \in M$, then for $v \in V$ we have

$$\varepsilon(T)(mv) = [T(mv)](e) = L_m[T(v)](e)] = T(v)(m^{-1}) = \xi(m)\varepsilon(T).$$

By a similar calculation one sees that for $H \in \mathfrak{a}$,

$$\varepsilon(T)(Hv) = [\lambda+\rho](H)\varepsilon(T)(v).$$

We thus see that $\varepsilon(T) \in \mathrm{Hom}_{M,\mathfrak{a}}(V/\mathfrak{n}V, V_{\xi,\lambda+\rho})$. Thus, ε defines a linear map from the space $\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{ind}_P^G(\omega))$ to $\mathrm{Hom}_{M,\mathfrak{a}}(V/\mathfrak{n}V, V_{\xi,\lambda})$. If $\varepsilon(T) = 0$, then it follows that $(Tv)(e) = 0$ for all $v \in V$ so that

$$Tv(k) = T(k^{-1}v)(e) = 0$$

for all $v \in V$ and $k \in K$. By the compact picture, this implies that $Tv = 0$ for all $v \in V$ hence $T = 0$. We thus see that ε is injective.

To see that ε is surjective, let $\psi \in \mathrm{Hom}_{M,\mathfrak{a}}(V/\mathfrak{n}V, V_{\xi,\lambda+\rho})$. Define $S: V \to C(G, V_\xi)$ by $[Sv](kan) = a^{-\lambda}\psi(k^{-1}v)$. Then it is readily verified that S maps V equivariantly into the space $C(P:\xi:\lambda)_K$. Furthermore,

$$\varepsilon(S)(v) = [Sv](e) = \psi(v)$$

so that $\varepsilon(S) = \psi$. □

Completion of proof of subrepresentation theorem. We can now complete the proof of the subrepresentation theorem as follows. The space $V/\mathfrak{n}V$ is non-trivial and a finite dimensional $(M,\mathfrak{a})$-module on which $\mathfrak{n}$ acts trivially. We may select an $\mathfrak{a}$-weight $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ of $(V/\mathfrak{n}V)^*$ and an irreducible M-submodule of the associated weight space. This leads to an embedding $j: V_\sigma \otimes \mathbb{C}_\mu \hookrightarrow (V/\mathfrak{n}V)^*$ of $(M,\mathfrak{a})$-modules, with σ an irreducible representation of M. The transpose of j defines a non-trivial projection of $(M,\mathfrak{a})$-modules

$$j^*: V/\mathfrak{n}V \to (V_\sigma \otimes \mathbb{C}_\mu)^* \simeq V_\xi \otimes \mathbb{C}_{\lambda+\rho} \simeq V_{\xi,\lambda+\rho}$$

with $\xi = \sigma^\vee$ and $\lambda = -\mu - \rho$. Let $T = \varepsilon^{-1}(j^*)$, then $T: V \to C(P:\xi:\lambda)_K$ is a non-trivial $(\mathfrak{g},K)$-module morphism. Since V is irreducible, the kernel of T is trivial. Thus, T is the desired embedding. □

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