Fredholm index and spectral flow

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1 The finite dimensional theory

A note on Sobolev space

We consider the Sobolev space $W^{1,2} = W^{1,2}(\mathbb{R})$ of L^2-functions $u \in \mathcal{D}'(\mathbb{R})$ for which the distributional derivative u' belongs to $L^2(\mathbb{R})$ as well. This space is a Hilbert space with norm given by

$$\|u\|_{1,2}^2 = \|u\|_2^2 + \|u'\|_2^2.$$

Lemma 1.1 $W^{1,2} \subset C(\mathbb{R})$. *Moreover, for every* $T > 0$, *the restriction map* $\rho_T \colon f \mapsto f|_{[-T,T]}$ *is compact from* $W^{1,2}$ *to* $C([-T,T])$.

Proof. By a standard convolution argument it follows that $C_0 \colon= C(\mathbb{R}) \cap W^{1,2}$ is dense in $W^{1,2}$. Let $f \in W^{1,2}$. Then there exists a sequence f_k in C_0 such that $f_k \to f$ in $W^{1,2}$. In particular the sequence is bounded in $W^{1,2}$. For every $s, t \in \mathbb{R}$ with $s < t$ we have

$$|f_k(s) - f_k(t)| \leq \int_s^t |f_k'(\tau)| \, dt \leq \|f_k'\|_2 \sqrt{|s-t|} \leq \|f_k\|_{1,2} \sqrt{|s-t|}$$

for all k. This shows that the sequence of functions is uniformly equicontinuous. Let now $t \in \mathbb{R}$ and assume that the sequence $(|f_k(t)|)$ is unbounded. Then it follows from the estimate that the sequence $(\min_{[t-1,t+1]} |f_k|)$ is unbounded and therefore also that $(\|f_k\|_{L^2})$ is unbounded, contradiction. We conclude the the sequence $\|f_k(t)\|$ is bounded, for every $t \in \mathbb{R}$. Let now $T > 0$. Then it follows by Ascoli's theorem that the sequence has a subsequence f_{k_j} which converges to a continuous function $g \in C[-T,T]$, uniformly on $[-T,T]$. It follows that also $f_{k_j} \to g$ in $L^2([-T,T])$, so that $g = f$ on $[-T,T]$. This establishes the continuity of f. The final statement follows by exactly the same argument. $\square$

The Fredholm property

We assume that V is a finite dimensional real linear space equipped with a positive definite inner product $\langle \cdot \,,\, \cdot \rangle$ and denote by $\mathrm{End}(V)$ the space of linear endomorphisms of V, equipped with the operator norm. For $B \in \mathrm{End}(V)$ we put

$$\begin{aligned} E^s(B) &:= \{x \in V \mid \lim_{t\to\infty} e^{tB}x = 0\} \\ E^u(B) &:= \{x \in V \mid \lim_{t\to-\infty} e^{tB}x = 0\}. \end{aligned}$$

Then $E^s(B)$ is the sum of the generalized eigenspaces for the eigenvalues of B with real part strictly negative, and $E^u(B)$ is the sum of the generalized eigenspaces for the eigenvalues with real part strictly positive. The linear map $B \in \mathrm{End}(B)$ is said to be hyperbolic if non of its eigenvalues is purely imaginary, or equivalently, if

$$V = E^s(B) \oplus E^u(B).$$

By $C_b(\mathbb{R}, \mathrm{End}(V))$ we denote the space of bounded continuous functions $\mathbb{R} \to \mathrm{End}(V)$. Equipped with the norm

$$\|A\|_\infty := \sup_{t\in\mathbb{R}} \|A(t)\|_{\mathrm{op}}$$

this space is a Banach space.

For $A \in C_b(\mathbb{R}, \mathrm{End}(V))$ we define the continuous linear map $D_A: W^{1,2}(\mathbb{R}, V) \to L^2(\mathbb{R}, V)$ by

$$D_A\xi = \frac{d}{dt}\xi - A\xi.$$

Lemma 1.2 *The map $A \mapsto D_A$ is Lipschitz-continuous, with Lipschitz-constant* 1, *from $C_b(\mathbb{R}, V) \otimes \mathrm{M}_n(\mathbb{R})$ to $L(W^{1,2}(\mathbb{R}, V), L^2(\mathbb{R}, V))$.*

Proof. Straightforward. □

The purpose of this section is to prove the following result.

Theorem 1.3 *Let $A \in C_b(\mathbb{R}, \mathrm{End}(V))$ and assume that the limits $\lim_{t\to\pm\infty} A(t) = A_\pm$ exist and are hyperbolic matrices. Then the operator $D_A: W^{1,2}(\mathbb{R}, V) \to L^2(\mathbb{R}, V)$ is Fredholm.*

We start by proving a stronger result in case A is constant.

Lemma 1.4 *Let A be constant and hyperbolic. Then the operator D_A is a topological linear isomorphism from $W^{1,2}(\mathbb{R}, V)$ onto $L^2(\mathbb{R}, V)$.*

Proof. We will give a proof based on Fourier transform. Let $\mathcal{F}: L^2(\mathbb{R}, V) \to L^2(\mathbb{R}, V)$ be the (isometric) Fourier transform. Then the image $\widehat{W}$ of $W^{1,2}(\mathbb{R}, V)$ under $\mathcal{F}$ consists of the measurable functions $\varphi: \mathbb{R} \to V$ with $\omega \mapsto (1+|\omega|)\varphi(\omega)$ an L^2-function. We equip $\widehat{W}$ with the obvious Hilbert norm, so that $\mathcal{F}$ becomes a topological linear isomorphism from $W^{1,2}(\mathbb{R}, V)$ onto $\widehat{W}$.

If $\xi \in W^{1,2}(\mathbb{R})$, then

$$\mathcal{F}(D_A\xi)(\omega) = (i\omega I - A)\mathcal{F}\xi(\omega).$$

Put $\nu(\omega) = \omega$. Then it suffices to show that the map $i\nu I - A$ is a topological linear isomorphism from $L^2(\mathbb{R}, V)$ onto $\widehat{W}$.

As A is hyperbolic, there exists a constant $c > 0$ such that

$$\|(A - i\omega I)^{-1}\|_{\mathrm{op}} \le C(1 + |\omega|)^{-1}$$

for every $\omega \in \mathbb{R}$. This implies that there exists a constant $C_2 > 0$ such that for all $\varphi \in L^2(\mathbb{R}, V)$,

$$\begin{aligned}
\|(1 + |\nu|)\varphi\|_{L^2(\mathbb{R},V)} &= \|(1 + |\nu|)(i\nu I - A)^{-1}(i\nu I - A)\varphi\|_{L^2(\mathbb{R},V)} \\
&\le C\,\|(i\nu I - A)\varphi\|_{L^2(\mathbb{R},V)} \\
&\le C_2\,\|(1 + |\nu|)\varphi\|_{L^2(\mathbb{R},V)}.
\end{aligned}$$

From this we see that $(i\nu I - A)$ is a topological linear isomorphism from $L^2(\mathbb{R}, V)$ onto $\widehat{W}$. □

We will need the following consequence of the above result.

Lemma 1.5 *Let A be as in Theorem 1.3. Then there exist constants $T > 0$ and $C > 0$ such that for all $\xi \in W^{1,2}(\mathbb{R}, V)$ with $\operatorname{supp} \xi \cap [-T, T] = \emptyset$ we have*

$$\|\xi\|_{L^2} \le \|D_A \xi\|_{L^2}.$$

Proof. For reasons of symmetry, we may restrict ourselves to ξ with $\operatorname{supp} \xi \subset [0, \infty[$, as we shall do from now on.

Let $A^+ = \lim_{t\to\infty} A(t)$ and let A_0 be constant and equal to A^+. There exists a constant $C_0 > 0$ such that for all $\xi \in W^{1,2} \otimes \mathbb{R}^n$ we have

$$\|\xi\|_{L^2} \le C_0 \|D_{A_0} \xi\|_{L^2}.$$

Let $\delta > 0$ be such that $\delta C_0 < 1$. There exists a constant $T > 0$ such that for $t \ge T$ we have $\|A(t) - A_0(t)\| < \delta$. Let $\xi \in W^{1,2}(\mathbb{R}, V)$ have support contained in $[T, \infty[$. Then

$$\|D_A(\xi) - D_{A_0}(\xi)\|_{L^2} = \|A\xi - A_0\xi\|_{L^2} \le \delta \|\xi\|_{L^2}.$$

It follows that

$$\|\xi\|_{L^2} \le C_0(\|D_A(\xi)\|_{L^2} + \delta\|\xi\|_{L^2})$$

from which we conclude that

$$\|\xi\|_{L^2} \le (1 - \delta C_0)^{-1} C_0 \|D_A(\xi)\|_{L^2}.$$

The result follows. □

Proposition 1.6 *Let A be as in Theorem 1.3. Then D_A has closed range and finite dimensional kernel.*

Proof. We select $T > 1$ and C as in the above lemma. Moreover, we select a cut off function $\chi \in C_c^\infty(\mathbb{R})$ with $0 \le \chi \le 1$, $\chi = 1$ on $[-T, T]$ and $\chi = 0$ on $\mathbb{R} \setminus [-S, S]$, where $S = T + 1$. Then one readily sees that for all $\xi \in W^{1,2}(\mathbb{R}, V)$ we have

$$\|\xi\|_{L^2} \le \|\chi\xi\|_{L^2} + \|(1-\chi)\xi\|_{L^2} \le \|\xi\|_{L^2([-T,T])} + \|(1-\chi)\xi\|_{L^2}. \tag{1.1}$$

Moreover, from the above lemma it follows that

$$\begin{aligned} \|(1-\chi)\xi\|_{L^2} &\le C\|D_A((1-\chi)\xi)\|_{L^2} \\ &= C\|(1-\chi)D_A\xi - \chi'\xi\|_{L^2} \\ &\le C\|D_A\xi\|_{L^2} + C\|\chi'\|_\infty \|\xi\|_{L^2[-S,S]}. \end{aligned}$$

Combining this estimate with (1.1) we see that there exists a constant $C_1 > 0$ such that for all $\xi \in W^{1,2} \otimes \mathbb{R}^n$ we have

$$\|\xi\|_{W^{1,2}} \le C_1(\|\xi\|_{L^2([-S,S])} + \|D_A\xi\|_{L^2}.$$

In view of Lemma 1.1 it follows that there exists a compact operator $\rho\colon W^{1,2}(\mathbb{R}, V) \to L^2(\mathbb{R}, V)$ such that

$$\|\xi\|_{W^{1,2}} \le C_1(\|\rho\xi\|_{L^2} + \|D_A\xi\|_{L^2}.$$

The result now follows by application of the lemma below. □

Lemma 1.7 *Let W, V_1, V_2 be Banach spaces, let $K\colon W \to V_1$ and $D\colon W \to V_2$ be bounded linear operators, and assume that K is compact. Assume moreover that there exists a constant $C > 0$ such that*

$$\|x\|_W \leq C(\|Kx\|_{V_1} + \|Dx\|_{V_2}) \tag{1.2}$$

for all $x \in W$. Then D has closed range and finite dimensional kernel.

Proof. Let (x_n) be a bounded sequence in W with $Dx_n \to y \in V_2$. Then Kx_n has a converging subsequence. From the estimate (1.2) it now follows that (x_n) has a subsequence which is Cauchy, hence converges.

In particular, it follows from the above argument that that every bounded sequence in $\ker D$ has a converging subsequence. Hence $\ker D$ is finite dimensional.

Let $W_0 \subset W$ be a closed subspace such that $W = \ker D \oplus W_0$. Then $D_0 := D|_{W_0}$ is bounded and injective, $K_0 := K|_{W_0}$ is compact, and it suffices to show that D_0 has closed image. Thus, without loss of generality we may assume that D is injective.

Let (x_n) now be a sequence in W such that Dx_n converges in V_2 with limit y. We claim that (x_n) must be bounded. For assume not, then passing to a subsequence we may arrange that $\|x_n\| \to \infty$. Put $\xi_n = x_n/\|x_n\|$. Then $D\xi_n \to 0$, and by the first part of the proof (ξ_n) has a converging subsequence whose limit ξ belongs to $\ker D$. On the other hand, the limit must have norm 1, contradicting the injectivity of D.

Thus, (x_n) is bounded. By the first part of the proof, (x_n) has a convergent subsequence (x_{n_k}) with limit $x \in W$. It follows that $Dx = y$, hence $y \in \operatorname{im} D$. □

Lemma 1.8 *Let A be as in Theorem 1.3. Then $\ker D_A$ equals the space of functions $\xi \in C^1(\mathbb{R}, V)$ such that $\xi' = A\xi$ and $\xi \in L^2$. Moreover, $\operatorname{im}(D_A)^\perp$ equals the kernel of D_{-A^*}.*

Proof. If $\xi \in \ker D_A$, then $\xi \in C(\mathbb{R}, V)$ and the differential equation holds in the sense of distributions, from which we conclude that $\xi \in C^1(\mathbb{R}, V)$. Conversely, if $\xi \in C^1(\mathbb{R}, V)$ satisfies the equation and belongs to $L^2(\mathbb{R}, V)$, then from $\xi' = Ab\xi$ and the uniform boundedness of A, it follows that $\xi' \in L^2$. Hence $\xi \in W^{1,2}(\mathbb{R}, V)$ and it follows that $\xi \in \ker D_A$.

Now assume that $\eta \in L^2(\mathbb{R}, V)$ is orthogonal to $\operatorname{im}(D_A)$. Then for all $\xi \in C_c^\infty(\mathbb{R}) \otimes V$ we have

$$\langle \eta\,,\, D_A\xi \rangle = \langle \eta\,,\, \frac{d}{dt}\xi \rangle - \langle A^*\eta\,,\, \xi \rangle = 0,$$

which implies that $\frac{d}{dt}\eta = -A^*\eta$ in distribution sense. From the fact that $-A^*$ is uniformly bounded, we conclude that $\eta \in W^{1,2}(\mathbb{R}, V)$ and $D_{-A^*}\eta = 0$. Thus, $\eta \in \ker D_{-A^*}$. Conversely, if $\eta \in \ker D_{-A^*}$, then it follows that $\eta \perp D_A(C_c^\infty(\mathbb{R}, V))$. By density and continuity, this implies that $\eta \in \operatorname{im}(D_A)$. We have proved that $(\operatorname{im} D_A)^\perp = \ker(D_{-A^*})$. Since D_A has closed image, the result follows. □

Completion of the proof of Theorem 1.3 It follows from the previous lemma that D_A has a closed range of finite codimension, which equals $\dim\ker D_{-A^*}$. Hence, D_A is Fredholm, and

$$\operatorname{index}(D_A) = \dim\ker D_A - \dim\ker D_{-A^*}.$$

Spectral flow, computation of the index

We consider the fundamental matrix $\Phi(t,\tau) \in \mathrm{End}(V)$ associated with the differential operator D_A. For each $\tau \in \mathbb{R}$ the function $t \mapsto \Phi(t,\tau)$ is C^1 and uniquely determined by

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau), \quad (t \in \mathbb{R}), \qquad \Phi(\tau,\tau) = I_V.$$

The existence and uniqueness of Φ corresponds to the existence and uniqueness for the initial value problem for the differential operator D_A. It follows from the uniqueness and existence that

$$\Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau) \tag{1.3}$$

for all $t, s, \tau \in \mathbb{R}$. In particular, we see that

$$\Phi(t,s)\Phi(s,t) = I.$$

From this equation it follows that $s \in \Phi(t,s)$ is C^1 as well, with derivative given by

$$\frac{d}{ds}\Phi(t,s) = -\Phi(t,s)A(s).$$

Taking conjugates on both sides of the equation, we see that the fundamental matrix $\widetilde{\Phi}$ associated with $D_{\widetilde{A}}$, where $\widetilde{A} = -A^*$, is given by

$$\widetilde{\Phi}(s,t) = \Phi(t,s)^*.$$

For every $t \in \mathbb{R}$ we define

$$\begin{aligned} E^s(t) &= \{\xi \in V \mid \lim_{\tau \to \infty} \Phi(\tau,t)\xi = 0\} \\ E^u(t) &= \{\xi \in V \mid \lim_{\tau \to -\infty} \Phi(\tau,t)\xi = 0\}. \end{aligned}$$

Then it is immediate from (1.3) that

$$\Phi(\tau,t)E^s(t) = E^s(\tau), \tag{1.4}$$

and we have analogous relations for $E^u(t)$.

We now come to the determination of the index of D_A. First of all, by homotopy invariance of the index, we may assume that A is locally constant outside a compact interval of the form $[-T,T]$. Thus, from now on we will assume that

$$A(t) = \begin{cases} A_+ & \text{for} \quad t \geq T, \\ A_- & \text{for} \quad t \leq -T. \end{cases}$$

In this setting we have that

$$\Phi(t,s) = e^{(t-s)A_+} \qquad \text{for} \qquad s,t \geq T,$$

and a similar equation with A_-, for $s,t \leq T$. Accordingly,

$$E^s(t) = E^s(A_+), \qquad \text{and} \qquad E^u(t) = E^u(A_+) \qquad \text{for} \qquad t \geq T,$$

and similar relations with A_-, for $t \leq -T$. Combining this with (1.4), we find that, for all $t \in \mathbb{R}$,

$$E^s(t) = \Phi(t,T)E^s(A_+), \qquad \text{and} \qquad E^u(t) = \Phi(t,T)E^u(A_+).$$

Moreover, similar relations hold with A_- and $-T$ in place of A_+ and T. Let $\widetilde{E}_s(t)$ denote the analogue of $E^s(t)$ for $\widetilde{A}$. Then

$$\widetilde{E}^s(t) = \Phi(t,T)E^s(-A_+^*), \qquad \text{and} \qquad \widetilde{E}^u(t) = \Phi(t,T)E^u(-A_+^*)$$

Lemma 1.9 *Let $T: V \to V$ be a linear map. Then for each linear subspace $E \subset V$ we have $T^*(E)^\perp = T^{-1}(E^\perp)$. In particular, $(\mathrm{im}\, T^*)^\perp = \ker T$.*

Proof. Straightforward. □

Lemma 1.10 *Let $B \in \mathrm{End}(V)$ be hyperbolic. Then*

$$E^s(-B^*) = E^s(B)^\perp, \qquad \textit{and} \qquad E^u(-B^*) = E^u(B)^\perp.$$

Proof. We may replace V by its complexification equipped with the Hermitian continuation of the given inner product, and prove a similar statement there. Let $n = \dim V$. Let Λ denote the set of eigenvalues of B and $\Lambda_+\pm$ the subset of eigenvalues λ with $\pm\mathrm{Re}\,\lambda > 0$.

For $\lambda \in \Lambda$, the map $(B - \lambda I)^n$ has image equal to the sum S_λ of the remaining generalized eigenspaces, and kernel equal to the generalized eigenspace V_λ of eigenvalue λ. It follows by application of the previous lemma that $(B^* - \bar{\lambda} I)^n$ has kernel equal to the orthocomplement of the image S of $(B - \lambda I)^n$. This shows that $-\bar{\lambda}$ is an eigenvalue of $-B^*$, with associated generalized eigenspace equal to $S_\lambda^\perp$. We thus see that

$$E^s(-B^*) = \oplus_{\lambda \in \Lambda_-} S_\lambda^\perp = (\cap_{\lambda \in \Lambda_-} S_\lambda)^\perp = [\oplus_{\lambda \in \Lambda_+} V_\lambda]^\perp = E^s(B)^\perp.$$

The proof of the second assertion is similar. □

Corollary 1.11 *For every $t \in \mathbb{R}$, we have*

$$\tilde{E}^s(t) = E^s(t)^\perp, \qquad \textit{and} \qquad \tilde{E}^u(t) = E^u(t)^\perp.$$

Proof. We have

$$\widetilde{E}^s(t) = \widetilde{\Phi}(t,T)E^s(-A_+^*) = \Phi(T,s)^*[E^s(A_+)^\perp] = [\Phi(T,s)^{-1}E^s(A_+)]^\perp = E^s(t)^\perp.$$

The proof of the remaining identity is similar. □

Lemma 1.12 *Let $t \in \mathbb{R}$. Then the map $\xi \mapsto \xi(t)$ is a linear isomorphism from $\ker D_A$ onto $E^s(t) \cap E^u(t)$.*

Proof. Let S be the space of $\xi \in C^1(\mathbb{R}, V)$ with $\xi' = A\xi$. Then the map $\mathrm{ev}_t: S \to V, \xi \mapsto \xi(t)$ is a linear isomorphism. For $\xi \in S$, the condition $\xi|_{[T,\infty[} \in L^2$ is equivalent to $\xi(T) \in E^s(A_+)$, which in turn is equivalent to $\xi(t) \in E^s(t)$. Similarly, the condition $\xi|_{]-\infty,-T]} \in L^2$ is equivalent to $\xi(-T) \in E^u(A_-)$, hence to $\xi(t) \in E^u(t)$. We conclude that for $\xi \in S$, the condition $\xi \in L^2(\mathbb{R}, V)$ is equivalent to $\xi(t) \in E^s(t) \cap E^u(t)$. The result follows. □

Theorem 1.13 *Let A be as before. Then the index of D_A is given by*

$$\operatorname{index} D_A = \dim E^s(A_+) - \dim E^s(A_-).$$

Proof. In view of the previous results, we have, with $\tilde{A} = -A^*$, for every $t \in \mathbb{R}$,

$$\begin{aligned}
\operatorname{index} D_A &= \dim\ker D_A - \dim\ker D_{\tilde{A}} \\
&= \dim(E^s(t) \cap E^u(t)) - \dim(\tilde{E}^s(t) \cap \tilde{E}^u(t)) \\
&= \dim(E^s(t) \cap E^u(t)) - \dim(E^s(t) + E^u(t))^{\perp} \\
&= \dim E^s(t) + \dim E^u(t) - \dim(E^s(t) + E^u(t)) - \operatorname{codim}(E^s(t) + E^u(t))] \\
&= \dim E^s(t) - \operatorname{codim} E^u(t) \\
&= \dim E^s(A_+) - \dim E^s(A_-).
\end{aligned}$$

The last number may be interpreted as minus the spectral flow of the family $t \mapsto A(t)$, in the sense that it gives minus the net change of the number of eigenvalues with negative real part into eigenvalues with positive real part as t goes from $-\infty$ to ∞.

2 Self-adjoint operators in Hilbert space

Let H be a complex Hilbert space. By a densely defined operator on H we mean a linear map $A: D(A) \to H$ with $D(A)$ a dense subset of H. The set $D(A)$ is called the domain of A. The adjoint A^* of A is defined in two steps, as follows. First of all, its domain $D(A^*)$ is defined to be the set of $y \in H$ such that the linear functional $x \mapsto \langle Ax\,,\, y\rangle$ extends to a continuous linear functional on H. As $D(A)$ is dense, this continuous linear functional is unique. Let $y \in D(A^*)$. Then by the Riesz representation theorem there exists a unique $z \in H$ such that $\langle Ax\,,\, y\rangle = \langle x\,,\, z\rangle$ for all $x \in D(A)$. We put $A^*y = z$. Thus, for $y \in D(A^*)$, the image $A^*y \in H$ is uniquely defined by the requirement that

$$\langle Ax\,,\, y\rangle = \langle x\,,\, A^*y\rangle \qquad \text{for all} \quad x \in D(A).$$

Lemma 2.1 *Let $A: D(A) \to H$ be a densily defined operator. Then A^* has closed graph.*

Proof. Let (y_n, x_n) be a sequence in the graph of A^* which converges to a point $(y, x) \in H \times H$. Then for every $v \in D(A)$ we have $\langle Av\,,\, y\rangle = \lim\langle Av\,,\, y_n\rangle = \lim\langle v\,,\, x_n\rangle = \langle v\,,\, x\rangle$. This shows that $y \in D(A^*)$ and $x = A^*y$. Hence (y, x) belongs to the graph of A^*. □

The operator A is called symmetric if

$$\langle Ax\,,\, y\rangle = \langle x\,,\, Ay\rangle$$

for alle $x, y \in D(A)$. Equivalently, this means that $D(A) \subset D(A^*)$ and $A^* = A$ on $D(A)$. Identifying A with its graph $\{(x, Ax) \mid x \in D(A)\} \subset H \times H$ this may in turn be reformulated as $A \subset A^*$.

Definition 2.2 A densely defined operator A on H with domain $D(A)$ is said to be self-adjoint if $A = A^*$.

Thus, a densily defined operator is self-adjoint if and only if it is symmetric and $D(A) = D(A^*)$.

Remark 2.3 It follows from this definition combined with Lemma 2.1 that a self-adjoint operator has closed graph. This fact plays an important role in the spectral theory of such operators.

Remark 2.4 If $A: H \to H$ is a bounded linear operator, then $D(A^*) = D(A) = H$ and it follows that A is self-adjoint if and only if A is symmetric. A bounded symmetric operator is also called a bounded Hermitean operator.

A complex number $\lambda \in \mathbb{C}$ is said to belong to the resolvent set of the densily defined operator A if $A - \lambda I$ has dense image, and $(A - \lambda I)^{-1}$ extends to a bounded operator on H. This extension $R(A, \lambda)$ is called the resolvent of A at λ. It is not hard to show that the resolvent set Ω is open in $\mathbb{C}$ and that the map $\lambda \mapsto R(A, \lambda)$ is holomorphic from Ω to $L(H, H)$, equipped with the operator norm. The complement of the resolvent set is called the spectrum of A and denoted by $\sigma(A)$.

A bounded operator A on H is said to be normal if A and its adjoint A^* commute.

Lemma 2.5 *Let $A: H \to H$ be a bounded normal operator. Let $V \subset H$ be a linear subspace such that $AV \subset V$. Then $AV^\perp \subset V^\perp$.*

Proof. Easy. □

In case H is finite dimensional, it follows from the above lemma by induction on the dimension that any operator $A \in L(H, H)$ is normal if and only if A diagonalizes on an orthonormal basis of H.

This result has a rather straightforward generalization to compact normal operators in Hilbert space.

Lemma 2.6 *Let H be a Hilbert space, and $A: H \to H$ a bounded operator. Then A is compact and normal if and only if the following conditions are fulfilled, with Λ be the set of eigenvalues of A.*

(a) 0 *is the only possible accumulation point of* Λ;

(b) *for every $\lambda \in \Lambda \setminus \{0\}$ the associated eigenspace H_λ is finite dimensional;*

(c) *the eigenspaces H_λ, for $\lambda \in \Lambda$ are mutually orthogonal;*

(d) *the direct sum $\oplus_{\lambda \in \Lambda} H_\lambda$ is dense in H.*

Proof. See, for instance, [3]. □

3 The spectral theorem for densely defined self-adjoint operators

There is also a spectral theory for densely defined self-adjoint operators, which we shall now briefly review. The key ingredient of this theory is the notion of a projection valued or spectral measure in a Hilbert space H.

We recall that the Borel σ-algebra $\mathcal{B}$ of a locally compact Hausdorf space M is the σ-algebra generated by the open subsets of M.

Definition 3.1 Let M be a locally compact Hausdorf space and H a Hilbert space. A *projection-valued Borel measure* in H based on M is a map P from the Borel σ-algebra $\mathcal{B}$ on M to the set of orthogonal projections in $B(\mathcal{H})$ with the following properties,

(a) for all $U, V \in \mathcal{B}$, $P(U \cap V) = P(U)P(V)$;

(b) $P(M) = I$;

(c) P is countably additive, i.e., for each countable sequence (U_n) of mutually disjoint measurable subsets,

$$P(\cup_n U_n) = \sum_n P(U_n).$$

The sum on the right-hand side is required to converge in the strong operator topology.

The requirement in (c) of convergence in the strong operator topology means that $\sum_n P(U_n)x$ should converge in H, for any $x \in H$. It follows from (c) that P is finitely additive, by letting $U_n = 0$ after some index. From combining (c) and (b) we see that $P(M) = P(M) + P(\emptyset)$, so that $P(\emptyset) = 0$. If we combine this with (a), we see that $P(U)$ and $P(V)$ have perpendicular images if U and V are disjoint measurable sets.

Let $x, y \in \mathcal{H}$. Then it follows from the above definition that

$$\mu_{x,y} \colon U \mapsto \langle P(U)x \,,\, y\rangle$$

defines a bounded Borel measure.

Lemma 3.2 *Let P be a projection valued measure based on M. Then, for every $x \in \mathcal{H}$ and each Borel set $U \subset M$,*

$$\mu_{x,x}(U) = \|P(U)x\|^2.$$

In particular, $\mu_{x,x}$ has all its values in $[0, \|x\|]$.

Proof. $\mu_{x,x}(U) = \langle P(U)x \,,\, x\rangle = \langle P(U)x \,,\, P(U)x\rangle = \|P(U)x\|^2$. □

If P is a spectral measure on the locally compact Hausdorff space M, with values in H, then we have an associated map I_P (integration against P) from the space $\mathcal{M}_b(M)$ of bounded Borel measurable functions on M to the space $L(H, H)$ of bounded linear operators on H, such that

$$\langle x \,,\, I_P(f)y\rangle = \int_M f(m)\, d\mu_{x,y}(m).$$

For obvious reasons, we also write

$$I_P(f) = \int_M f(m)\, dP(m).$$

Theorem 3.3 (Spectral theorem for self-adjoint operators) *Let A be a densely defined self-adjoint operator in H. Then there exists a unique projection-valued measure P based on $\mathbb{R}$, such that*

$$Tv = \int_{\mathbb{R}} \lambda \, dP(\lambda)v$$

for all $v \in D_T$. Moreover, D_T consists of the elements $v \in \mathcal{H}$ with

$$\int_{\mathbb{R}} \lambda^2 \, \langle dP(\lambda)v \,,\, v\rangle < \infty.$$

4 The space $\mathcal{S}(W, H)$

Let H be a separable Hilbert space, and let $W \subset H$ be a dense linear subspace. We assume that W is equipped with a norm $\|\cdot\|_W$ for which it is a Banach space and for which the inclusion map $\iota_W : W \to H$ is compact.

Let $L(W, H)$ denote the space of continuous linear maps $W \to H$. Equipped with the operator norm, this space is a Banach space. An element $A \in L(W, H)$ will be viewed as a linear operator on H with dense domain $D(A) = W$.

Lemma 4.1 *Let $A \in L(W, H)$. Then the following conditions are equivalent.*

(a) *The graph Γ_A of A is closed in $H \times H$.*

(b) *There exists a constant $c > 0$ such that*

$$\|x\|_W \leq c(\|x\|_H + \|Ax\|_H), \qquad (x \in W).$$

Proof. Assume (a). The natural Hilbert norm on $H \times H$ is given by $\|(x, y)\|^2_{H\times H} = \|x\|^2_H + \|y\|^2_H$. It is readily seen that this norm is equivalent to the sum norm $(x, y) \mapsto \|x\|_H + \|y\|_H$. The graph $\Gamma = \Gamma_A$ is a Banach space for the restriction of the sum norm. Moreover, the map $x \mapsto (x, Ax)$ is a bijective and a continuous linear map from the Banach space W onto the closed Banach space Γ. By the closed graph theorem for Banach spaces, the map is a topological linear isomorphism. This implies (b).

Now assume (b). Let (x_n) be a sequence in W such that (x_n, Ax_n) converges in Γ, say to a point (x, y). Then (x_n, Ax_n) is a Cauchy-sequence for the sum-norm, and it follows from (b) that (x_n) is Cauchy in the Banach space W. Hence (x_n) converges in W with a limit $\tilde{x}$. By continuity of the embedding $W \hookrightarrow H$, it follows that $x_n \to \tilde{x}$ in H, hence $\tilde{x} = x$. By continuity of A it follows that $Ax_n \to Ax$. Hence $Ax = y$ and we see that $(x, y) \in \Gamma$. It follows that Γ is closed. □

Corollary 4.2 *Let $A \in L(W, H)$ have a graph which is closed in $H \times H$. Let $V \subset W$ be a subspace such that $A|_V : V \to H$ is continuous. Then V is finite dimensional.*

Proof. There exists a constant $C_1 > 0$ such that $\|Ax\|_H \leq C_1\|x\|_H$ for all $x \in V$. In view of the above lemma it follows that $\|\cdot\|_H$ and $\|\cdot\|_W$ define the same topologies on V.

Let now (x_n) be a bounded sequence in V. Then by compactness of the embedding $W \subset H$, it follows that (x_n) has a subsequence which converges in H. This subsequence is Cauchy in the $\|\cdot\|_H$ norm, hence also in the $\|\cdot\|_W$ norm, hence converges to an element of W. It follows that every $\|\cdot\|_W$-bounded sequence (x_n) in V has a converging subsequence in W. Let now (y_n) be a bounded sequence in the closure $\bar{V}$ of V in W. Choose $x_n \in V$ with $\|x_n - y_n\| < 2^{-n}$. Then (x_n) is bounded hence has a subsequence that converges to an element $y \in \bar{V}$. The corresponding subsequence of (y_n) converges to y as well. It follows that every closed and bounded subset of $\bar{V}$ is compact. Hence $\bar{V}$ is finite dimensional and we conclude that $V = \bar{V}$ is finite dimensional. □

The space $L_s(W, H)$ is defined to be the subspace of $A \in L(W, H)$ that are symmetric, i.e.,

$$\langle Ax\,,\, y\rangle_H = \langle x\,,\, Ay\rangle_H, \qquad \text{for all} \quad x, y \in W.$$

Clearly, this is a closed subspace of $L(W, H)$, hence a Banach space of its own right. We define $\mathcal{S}(W, H)$ to be the subset of self-adjoint elements in $L_s(W, H)$, i.e., the set of $A \in L_s(W, H)$ with $A = A^*$. The goal of this section is to investigate the structure of the set $\mathcal{S}(W, H)$.

Let $A \in L_s(W, H)$. Any complex number $\lambda \in \mathbb{C}$ such that $A - \lambda I$ has a non-trivial kernel in W is called an eigenvalue for A. The set of eigenvalues of A is denoted by $\Lambda(A)$ and for $\lambda \in \Lambda(A)$ we denote the associated eigenspace by

$$E(\lambda) = \ker(A - \lambda I).$$

Thus, $E(\lambda)$ is defined to be a subspace of W.

Lemma 4.3 *Let $A \in L_s(W, H)$. Then every eigenvalue of A real. If $\lambda, \mu \in \Lambda(A)$ are distinct, then $E(\lambda) \perp E(\mu)$. If the graph of A is closed in $H \times H$, then every eigenspace is finite dimensional.*

Proof. Let $\lambda, \mu \in \Lambda(A)$ and $x \in E(\lambda)$, $y \in E(\mu)$ Then $x, y \in W$, hence $\lambda\langle x\,,\, y\rangle = \langle Ax\,,\, y\rangle = \langle x\,,\, Ay\rangle = \bar{\mu}\langle x\,,\, y\rangle$, from which we see that

$$(\lambda - \bar{\mu})\langle x\,,\, y\rangle = 0.$$

Taking $\lambda = \mu$ and $x = y \neq 0$ we see that λ is real. On the other hand, if λ, μ are different, it follows that $\langle x\,,\, y\rangle = 0$, from which we see that $E(\lambda) \perp E(\mu)$.

The restriction $A|_{E(\lambda)}$ equals $x \mapsto \lambda x$ hence is continuous linear from $E(\lambda)$ to H. If A has a closed graph we may apply Cor. 4.2 with $V = E(\lambda)$. □

If $E(\lambda)$ is finite dimensional, we agree to write P_λ for the associated orthogonal projection $H \to E(\lambda)$.

Theorem 4.4 *Let $A \in L_s(W, H)$. Then the following conditions are equivalent.*

(a) *$A \in \mathcal{S}(W, H)$.*

(b) *All eigenspaces of A are finite dimensional,*

$$H = \widehat{\bigoplus}_{\lambda \in \Lambda(A)} E(\lambda), \tag{4.5}$$

and

$$W = \{x \in H \mid \sum_{\lambda \in \Lambda(A)} \lambda^2 \|P_\lambda x\|_H^2 < \infty \ \}.$$

Moreover, if any of these conditions is satisfied, then the norm $\|\cdot\|_W$ on W is equivalent to the norm $\|\cdot\|_g$ given by

$$\|x\|_g^2 = \sum_{\lambda \in \Lambda(A)} (1 + \lambda^2)\|P_\lambda x\|^2.$$

Proof. First, assume (a). By the spectral theorem for unbounded self-adjoint operators, there exists a spectral measure P such that

$$A = \int_{\mathbb{R}} \lambda dP(\lambda).$$

Let $R > 0$ and let $\Omega \subset \mathbb{R}$ be a Borel subset contained in $[-R, R]$. Then $H(\Omega) = P(\Omega)H$ is a linear subspace of W. If $x \in E(\Omega)$ then

$$Ax = \int_\Omega \lambda \, dP(\lambda)x$$

and

$$\|Ax\|^2 = \int_\Omega \lambda^2 \langle dP(\lambda)x \, , \, x\rangle \leq 2R^3 \|x\|^2,$$

from which we see that $A|_{H(\Omega)}: H(\Omega) \to W$ is bounded with respect to the restriction of $\| \cdot \|_H$ on these spaces. Since A is self-adjoint, its graph is closed, and it follows by application of Corollary 4.2 that $H(\Omega)$ is finite dimensional and that $P(\Omega)$ is an orthogonal projection of finite rank. This implies that the restriction of the spectral measure P to $[-R, R]$ has finite support S_R. Hence, the full spectral measure P has discrete support S. We note that $S_R = S \cap [-R, R]$. If $\lambda \in S$, we put $P_\lambda = P(\{\lambda\})$. Then P_λ is an orthogonal projection of finite rank. Moreover, if $\lambda, \mu \in S$ are distinct, then $P_\lambda P_\mu = 0$. The full spectral measure is given by

$$P(\Omega) = \sum_{\lambda \in \Omega \cap S} P_\lambda, \tag{4.6}$$

with absolute convergence in the strong operator topology. Equivalently, the image of $P(\Omega)$ equals the closure of the orthogonal direct sum of the spaces $H(\{\lambda\})$, for $\lambda \in S$. On the other hand, $P(\mathbb{R}) = I$, so that

$$H = H(\mathbb{R}) = \widehat{\bigoplus}_{\lambda \in S} H(\lambda).$$

Using (4.6) with $\Omega = \{\lambda\}$, we see that $H(\lambda) \subset E(\lambda)$, for every $\lambda \in S$. We conclude that $S = \Lambda(A)$, and that $H(\lambda) = E(\lambda)$ for every $\lambda \in S$. Hence, all eigenspaces are finite dimensional, and (4.5) follows.

Finally, it follows from the spectral theorem that $W = D(A)$ consists of the vectors $x \in H$ with

$$\int_\mathbb{R} \lambda^2 \langle dP(\lambda)x \, , \, x\rangle < \infty.$$

The given integral equals

$$\sum_{\lambda \in \Lambda(A)} \lambda^2 \langle P_\lambda \, x \, , \, x\rangle_H = \sum_{\lambda \in \Lambda(A)} \lambda^2 \|P_\lambda \, x\|_H^2,$$

whence the last assertion of (b).

We will now prove the converse implication. If $x \in W$ and $y \in H$ then it follows that

$$\langle Ax \, , \, y\rangle = \sum_{\lambda \in \Lambda(A)} \langle \lambda P_\lambda x \, , \, y\rangle = \sum_\lambda \langle x \, , \, \lambda P_\lambda \, y\rangle.$$

Thus, $x \mapsto \langle Ax \, , \, y\rangle$ extends to a continuous linear map on H if and only if $\sum_\lambda \lambda^2 \|P_\lambda y\|^2 < \infty$. It follows that $D(A^*) \subset W = D(A)$, hence A is self-adjoint.

The final assertion follows from the fact that there exist constants $C_1, C_2 > 0$ such that

$$\|x\|_W^2 \leq C_1(\|x\|_H^2 + \|Ax\|_H^2) \leq C_2 \|x\|_W^2.$$

Indeed, this follows from () and the fact that $i_W: W \to H$ and $A: W \to H$ are bounded. $\square$

Using the above result we shall now derive some other useful characterizations of $\mathcal{S}(W, H)$.

Lemma 4.5 *Let $A \in \mathcal{S}(W, H)$. Then for every $\mu \in \mathbb{C} \setminus \Lambda(A)$, the map $A - \mu I$ extends to a topological linear isomorphism from W onto H. In particular, the spectrum $\sigma(A)$ of A equals $\Lambda(A)$.*

Proof. Assume that $\mu \in \mathbb{C} \setminus \Lambda(A)$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 < \frac{|\lambda - \mu|^2}{1 + \lambda^2} < C_2.$$

for all $\lambda \in \Lambda(A)$. If $x \in W$ then for every $\lambda \in \Lambda(A)$ we have

$$C_1(1 + \lambda^2)\|P_\lambda x\|^2 \leq \|(A - \mu I)P_\lambda x\|^2 \leq C_2(1 + \lambda^2)\|P_\lambda x\|^2.$$

Summing up over $\lambda \in \Lambda(A)$ we obtain

$$C_1\|x\|_g^2 \leq \|(A - \mu I)x\|^2 \leq C_2\|x\|_g^2,$$

where $\|x\|_g$ is the norm defined in the last part of Theorem 4.4. Since this norm is equivalent to $\| \cdot \|_W$, the result follows. □

Theorem 4.6 *Let $A \in L_s(W, H)$. Then the following statements are equivalent.*

(a) *$A \in \mathcal{S}(W, H)$*

(b) *There exists a $\mu \in \mathbb{R}$ such that $A - \mu I$ is a topological linear isomorphism from W onto H.*

Proof. Assume (a). Then (b) follows from Thm. and the fact that $\sigma(A)$ is a discrete subset of $\mathbb{R}$.

Conversely, assume (b). As $W \hookrightarrow H$ is a compact embedding, the operator $T = \iota_W \circ (A - \mu I)^{-1} \colon H \to H$ is compact. We claim that T is symmetric. Indeed, let $x', y' \in H$. Then there exist $x, y \in W$ such that $x' = (A - \mu I)x$ and $y' = (A - \mu I)y$. We note that $Tx' = x$ and $Ty' = y$ so that

$$\langle Tx' \, , \, y' \rangle = \langle x \, , \, y' \rangle = \langle x \, , \, (A - \mu I)y \rangle$$

and

$$\langle x' \, , \, Ty' \rangle = \langle x' \, , \, y \rangle = \langle (A - \mu I)x \, , \, y \rangle.$$

Thus the symmetry of A implies that T is symmetric as well. By the spectral theorem for compact symmetric operators on H, it follows that T has a set $\Lambda(T)$ of real eigenvalues with only 0 as a possible accumulation point. Moreover, as T has trivial kernel, zero is not an eigenvalue, and it follows that all eigenspaces are finite dimensional and mutually orthogonal. Finally, the vector sum of the eigenspaces is dense in H.

For $\lambda \in \Lambda(T)$ we denote the associated eigenspace by $H(T, \lambda)$ and the associated orthogonal projection by $P_{T,\lambda}$. If $x \in H(T, \lambda)$, then $x = T(\lambda^{-1}x) \in T(H) = W$. It follows that $(A - \mu I) = \lambda^{-1}I$ on $H(T, \lambda)$. Hence $H(T, \lambda) \subset H(A, \mu + \lambda^{-1})$. Conversely, let $x \in W$ be an eigenvector for A, with eigenvalue ν. then $(A - \mu I)x = (\nu - \mu)x$ hence $x = T((\nu - \mu)x)$. It follows that $\nu \neq \mu$ and that $(\nu - \mu)^{-1}$ is an eigenvalue for T. Thus we see that $\Lambda(A) = \mu + \Lambda(T)^{-1}$ is a discrete subset of $\mathbb{R}$ and that for every $\lambda \in \lambda(T)$,

$$H(A, \mu + \lambda^{-1}) = H(T, \lambda).$$

It follows that all eigenspaces for A are finite dimensional and mutually orthogonal. Finally, H is the closure of the direct sum of these. It remains to be shown that the characterization of W is valid. From now on, for $\nu \in \Lambda(A)$ we put $P_\nu = P_{A,\nu}$. Similarly, we put $H(\nu) = H(A,\nu)$.

From the spectral representation of T it follows that $W = \operatorname{im} T$ consists of all vectors of the form

$$x = \sum_{\lambda \in \Lambda(A)} (\lambda - \mu)^{-1} v_\lambda,$$

with $v_\lambda \in H(\lambda)$ and $\sum_{\lambda \in \Lambda(A)} \|v_\lambda\|^2 < \infty$. We note that for such an expression we have $P_\lambda x = (\lambda - \mu)^{-1} v_\lambda$. Therefore, W consists of all $x \in H$ with

$$\sum_{\lambda \in \Lambda(A)} |\lambda - \mu|^2 \|P_\lambda x\|^2 < \infty.$$

We now observe that there exist constants $C_1, C_2 > 0$ such that

$$C_1 < |\lambda - \mu|^{-2}(1 + \lambda^2) < C_2$$

for all $\lambda \in \Lambda(A)$. Hence W consists of all vectors $x \in H$ with

$$\sum_{\lambda \in \Lambda(A)} (1 + \lambda^2) \|P_\lambda x\|^2 < \infty.$$

Since $\sum_\lambda \|P_\lambda x\|^2 = \|x\|^2$ for all $x \in H$, the desired characterization of W follows. We conclude that $A \in \mathcal{S}(W, H)$. □

Corollary 4.7 *The set $\mathcal{S} = \mathcal{S}(W, H)$ is open in $L_s(W, H)$ with respect to the operator norm topology.*

Proof. Let $A \in \mathcal{S}(W, H)$. Then there exists a $\mu \in \mathbb{C}$ such that $A - \mu I$ extends to a topological linear isomorphism from W onto H. Let $B \in L_s(W, H)$. Then

$$(A - \mu I)^{-1}(B - \mu I) - I_W = (A - \mu I)^{-1}(B - A).$$

It follows that the continuous linear operator $(A - \mu I)^{-1}(B - \mu I) - I_W \colon W \to W$ has operator norm at most

$$\|(A - \mu I)^{-1}\|_{\mathrm{op}} \|B - A\|_{\mathrm{op}}.$$

This implies that $(A - \mu I)^{-1}(B - \mu I)$ is a topological linear isomorphism from W onto itself if

$$\|B - A\|_{\mathrm{op}} < \|(A - \mu I)^{-1}\|_{\mathrm{op}}^{-1}.$$

The result follows. □

5 Perturbations in $\mathcal{S}(H, W)$.

Due to the special spectral properties of operators from $\mathcal{S}(H, W)$, there is a decent perturbation theory for these operators.

We will need the following well known fact.

Lemma 5.1 *Let V_1, V_2 be isomorphic Banach spaces, and let $L_0(V_1, V_2)$ denote the space of topological linear isomorphisms from V_1 onto V_2. Then $L_0(V_1, V_2)$ is open in $L(V_1, V_2)$. Moreover, the map $T \mapsto T^{-1}$ is analytic from $L_0(V_1, V_2)$ to $L_0(V_2, V_1)$.*

Proof. It suffices to establish the result for $V_1 = V_2 = V$ in which case $L_0 = \mathrm{GL}(V)$. By homogeneity it suffices to show that I is an interior point, and that $T \mapsto T^{-1}$ is analytic in a neighborhood of I. This in turn follows from the convergence of the power series for $H \mapsto (I - H)^{-1}$ in the ball $\|H\|_{\mathrm{op}} < 1$. □

In what follows, we assume that Ω is a non-empty open subset of a real or complex Banach space, and that $F\colon \Omega \to \mathcal{S}(H, W)$ is a C^p map, where $p \in \mathbb{N} \cup \{\infty, \omega\}$.

Lemma 5.2 *Let $t_0 \in \Omega$ and let $\Lambda_0 = \sigma(A(t_0))$.*

(a) *Let U be an open subset of $\mathbb{C}$ with closure compact and disjoint from Λ_0. Then there exists an open neighborhood Ω_0 of t_0 in Ω such that $\Lambda_t := \sigma(A(t))$ is disjoint from U for every $t \in \Omega_0$.*

(b) *If U is any open subset of $\mathbb{C}$ such that $\sigma(A(t)) \cap U = \emptyset$ for all $t \in \Omega$, then*

$$(z, t) \mapsto (zI - A(t))^{-1}$$

is a C^p-map $U \times \Omega \to L(H, W)$, which in addition is holomorphic in z.

Proof. From the assumption it follows that $A(t_0) - zI \in L_0(W, H)$, for $z \in U$. By continuity and compactness, we may select Ω_0 such that $zI - A(t) \in L_0(W, H)$, for $(z, t) \in U \times \Omega$. This implies the assertion about the spectrum of $A(t)$.

Assume now that the hypothesis of (b) is fulfilled. Then $zI - A(t) \in L_0(W, H)$, satisfies the stated properties as a function of (z, t). By analyticity of the inversion map, the same holds for $(zI - A(t)^{-1} \in L(H, W)$. □

Lemma 5.3 *Let $t_0 \in \Omega$ and let γ be a closed oriented C^1-curve in $\mathbb{C} \setminus \sigma(A(t_0))$. Then there exists an open neighborhood Ω_0 of t_0 such that γ is disjoint from $\sigma(A(t))$, for every $t \in \Omega_0$. Moreover, if Ω_0 is any open neighborhood with this property, then*

$$M(t) := \frac{1}{2\pi i} \int_\gamma (zI - A(t))^{-1}\, dz$$

defines a C^p-map $\Omega_0 \to L(H, W)$. Moreover, for every $t \in \Omega_0$ we have

$$M(t) = \sum_{\lambda \in \sigma(A(t))} W(\gamma, \lambda) P_{A(t), \lambda}. \tag{5.7}$$

Here $W(\gamma, \lambda)$ denotes the winding number of γ with respect to λ and $P_{A(t),\lambda}$ the orthogonal projection onto the (finite dimensional) eigenspace $\ker(A(t) - \lambda I)$.

Proof. The first assertion follows by compactness of γ. By the previous result on holomorphy and by differentiation under the integral sign, the first assertion on M follows. For the second assertion, fix t. As $\sigma(A(t))$ is finite, the expression on the right-hand side defines a continuous linear map, so that the equation needs only be checked on a dense subspace of H. Thus, it suffices to check the equation on each eigenspace $E_\mu = \ker(A(t) - \mu I_)$, for $\mu \in \sigma(A(t))$. Fix such an eigenvalue μ. Then on E_μ we have

$$M(t) = \frac{1}{2\pi i} \int_\gamma (zI - \mu I)^{-1}\, dz = W(\gamma; \mu) I,$$

by Cauchy's integral formula. On the other hand, the map on the right-hand side of (5.7) restricts on the space $E\mu$ to $W(\gamma; \mu)I$. The identity follows. □

Proposition 5.4 *Let $t_0 \in \Omega$, let $\mu \in \sigma(A(t_0))$ and let $m = \dim\ker(A - \mu I)$. Let U be an open disc around μ whose closure is disjoint from $\Sigma(A(t_0)) \setminus \{\mu\}$. Then there exists an open neighborhood Ω_0 of t_0 such that the following holds.*

(a) *for every $t \in \Omega_0$ the map $A(t)$ has precisely m eigenvalues $\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t)$ contained in U, counting multiplicities;*

(b) *the functions λ_j may be chosen in such a fashion that they are continuous on $t \in \Omega_0$;*

(c) *the orthogonal projection $P(t)$ onto the sum of the eigenspaces for the eigenvalues $\lambda_j(t)$ is a $L(H, W)$-valued continuous function of $t \in \Omega_0$.*

(d) *if $m = 1$, then λ_1 is a C^p-function.*

Remark 5.5 Note that $\lambda_j(t_0) = \mu$ for all j.

Proof. We select a ball neighborhood Ω_0 of t_0 such that $\sigma(A(t))$ is disjoint from the boundary ∂U for $t \in \Omega_0$. Then by the above result,

$$P(t) = \frac{1}{2\pi i} \int_{\partial U} (zI - A(t))^{-1}\, dz$$

depends C^p on t and defines the orthogonal projection onto the sum of the eigenspaces for the eigenvalues of $A(t)$ contained in U. The projection $P(t)$ depends continuously on t and has finite rank which therefore must be constant. This proves (a) and (c). We now order the eigenvalues $\lambda_1(t) \leq \cdots \leq \lambda_m(t)$. Then the eigenvalues are continuous at $t = t_0$. Applying the same reasoning at any other point of Ω_0, we see that the λ_j, thus ordered, are continuous functions. Hence (b).

Finally, assume that $m = 1$. We select a unit vector $v \in \ker(A(t_0) - \mu I)$ and define

$$v(t) = \frac{P(t)v}{\|P(t)v\|}.$$

Then $v(t)$ depends C^p on t, for t in a sufficiently small neighborhood Ω_1 of t_0 in Ω_0. Now $\lambda_1(t) = \langle A(t)v(t)\,,\, v(t)\rangle$, from which we see that λ_1 is C^p on Ω_1. Treating the other points of Ω_0 in a similar fashion, we conclude (d). □

Remark 5.6 If Ω is an open interval in $\mathbb{R}$ and $p = 1$, then using techniques from [1] it can be proved that the eigenvalues $\lambda_1, \ldots, \lambda_m$ can be chosen in C^1 fashion. This fact is quite hard to prove (of course one should not teke the eigenvalues in increasing order).

Surprisingly, the following result is also true. Let $\Omega \subset \mathbb{R}$ be an interval and assume that $t \mapsto A(t)$ is analytic on Ω. Then the eigenvalues may be chosen to depend analytically on t. Another surprise: this is in fact easier to prove than the previous assertion, by using complex function theory.

Thus prepared, we are now able to establish a few useful facts about the geometry of $\mathcal{S}(W,H)$. For $k \in \mathbb{N}$ we define $\mathcal{S}_k = \mathcal{S}_k(W,H)$ to be the subset of operators $A \in \mathcal{S}(W,H)$ with $\dim \ker A = k$.

Lemma 5.7 *Let $k \in \mathbb{N}$. Then $\mathcal{S}_k$ is a smooth Banach submanifold of $L_s(W,H)$ of codimension equal to $\frac{1}{2}k(k+1)$.*

Proof. Let $A \in \mathcal{S}_k$. Then $N := \ker A$ is k-dimensional. Let P denote the orthogonal projection onto N. We consider the analytic family $A(X) = A + X \in L_s(W,H)$, for $X \in L_s(W,H)$.

Select $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \cap \sigma(A) = \{0\}$. Let Ω_0 be a sufficiently small neighborhood of 0 in $L_s(W,H)$ with the properties of Proposition 5.4, relative $\mu = 0$ and $U =]-\varepsilon, \varepsilon[$. For $X \in \Omega_0$, let $P(X)$ denote the orthogonal projection onto the sum of the eigenspaces of $A(X)$, associated with the eigenvalues of $A(X)$ in $]-\varepsilon, \varepsilon[$. Then $P(X)$ has rank k and depends analytically on X.

Let i_N denote the inclusion map $N \to H$. Then we may adapt Ω_0 to achieve that $P \circ P(X) \circ i_N$ is an invertible element of $L_s(N,N)$ for every $X \in \Omega_0$. It follows that $P(X) \circ i_N$ has rank at least k hence is surjective onto $\mathrm{im}\, P(X)$ for every $X \in \Omega_0$. Moreover, $P(X)$ is injective on $\mathrm{im}\,(P)$ for every $X \in \Omega_0$. Since $A(X)$ maps $\mathrm{im}\, P(X)$ onto itself, it follows that $A(X)P(X) = 0$ if and only if $PA(X)P(X)i_N = 0$.

We define the function $f : \Omega_0 \to L_s(N,N)$ by

$$f(X) = P \circ A(X) \circ P(X) \circ i_N.$$

In the above we showed that $f^{-1}(0)$ consist of $X \in \Omega_0$ for which $A(X)P(X) = 0$. We observe that $\ker A(X) \subset \mathrm{im}\, P(X)$. Hence if $\ker A(X)$ is k-dimensional, then $\ker A(X) = \mathrm{im}\, P(X)$. Conversely, if $A(X)P(X) = 0$, then 0 is the only eigenvalue of $A(X)$ contained in $]-\varepsilon, \varepsilon[$, and it follows that $\dim \ker A(X) = k$. Thus, $A(X)$ belongs to $\mathcal{S}_k$ if and only if $A(X)P(X) = 0$. We see that

$$A(X) \in \mathcal{S}_k \iff f(X) = 0.$$

To show that $\mathcal{S}_k$ is a smooth Banach submanifold of $L_s(W,H)$, it suffices to show that the total derivative $Df(0) : L_s(W,H) \to L_s(N,N)$ is surjective.

By application of the chain rule it follows that the derivative is given by

$$Df(0)X = P\, DA(0)(X)\, P(0) i_N + PA(0)\, DP(0)(X)\, i_N.$$

We now observe that $A(0) = A$ and $PA = 0$, so that the second term vanishes. On the other hand, $DA(0)(X) = X$ and $P(0) = P$, so that

$$Df(0)X = PXi_N.$$

From this it is readily seen that the differential is onto $L_s(N,N)$. The latter space has dimension $\frac{1}{2}k(k+1)$, hence $\mathcal{S}_k$ is a smooth Banach submanifold at A of codimension k. Moreover, the tangent space $T_A\mathcal{S}_k$ equals $\ker Df(0)$, so that

$$\begin{aligned} T_A\mathcal{S}_k &= \{X \in L_s(W,H) \mid PXi_N = 0\} \\ &= \{X \in L_s(W,H) \mid X(N) \subset N^\perp\}. \end{aligned}$$

□

From the last part of the above proof, the following result is an easy consequence.

Lemma 5.8 *Let $A \in \mathcal{S}_k$ and let $N = \ker A$. Then the map $(X, Y) \mapsto X + i_N \circ Y \circ P_N$ defines a topological linear isomorphism from $T_A\mathcal{S}_k \oplus L_s(N, N)$ onto $L_s(W, H)$.*

Proof. Since $\dim L_s(N, N) = \frac{1}{2}k(k+1) = \operatorname{codim} T_A\mathcal{S}_k$, it suffices to show, for $Y \in L_s(N, N)$, that $i_N \circ Y \circ P_N \in T_A\mathcal{S}_k \Rightarrow Y = 0$. For this we note: if $i_N \circ Y \circ P_N \in T_A\mathcal{S}_k$ then $P_N \circ (i_N \circ Y \circ P_N) \circ i_N = 0$. The expression on the right-hand side of this equation equals Y. □

Lemma 5.9 *The manifold $\mathcal{S}_1$ comes equipped with a natural orientation.*

Proof. We now consider the map $\nu: \mathcal{S}_1 \to L_s(W, H)'$ defined by $\nu_A(T) = \operatorname{tr}(P_{\ker A} \circ T \circ P_{\ker A})$. As $\ker A$ is one dimensional, it follows that $\nu_A(T) = 0$ if and only if $P_{\ker A} \circ T \circ i_{\ker A} = 0$, which in turn is equivalent to $T \in T_A\mathcal{S}_1$. Thus, ν is a smooth nowhere vanishing one form defined along $\mathcal{S}_1$, with $\ker \nu_A = T_A\mathcal{S}_1$ for all $A \in \mathcal{S}_1$. This defines an orientation of $\mathcal{S}_1$. □

Remark 5.10 Alternatively, the orientation may be defined by means of a vector field along $\mathcal{S}_1$. For $A \in \mathcal{S}_1$ we define $v(A) \in L_s(W, H)$ to be the orthogonal projection onto $\ker A$. Let $A \in \mathcal{S}_1$ and put $N = \ker A$. Then $v(A) = i_N \circ \mathrm{id}_N \circ P_N$, and by the above lemma we see that $L_s(W, H) = T_A\mathcal{S}_1 \oplus \mathbb{R}v(A)$. Thus v defines a vector field along the manifold $\mathcal{S}_1$ which is everywhere transversal to it. We observe that $\nu_A(v(A)) = 1$ for $A \in \mathcal{S}_1$.

6 Fredholm operators associated with curves in $\mathcal{S}(W, H)$.

Following [2] we introduce the space $\mathcal{B} = \mathcal{B}(\mathbb{R}, W, H)$ of continuous (with respect to the operator norm topology) curves $A: \mathbb{R} \to L_s(W, H)$ for which the limits

$$A^{\pm} = \lim_{t \to \pm\infty} A(t)$$

exist in the operator norm topology. Equipped with the norm

$$\|A\| = \sup_{t \in \mathbb{R}} \|A(t)\|_{\mathrm{op}}$$

this space is a Banach space. Moreover, we define $\mathcal{A} = \mathcal{A}(\mathbb{R}, W, H)$ to be the subset of elements $A \in \mathcal{B}$ such that

(a) $A(t) \in \mathcal{S}(W, H)$ for each $t \in \mathbb{R}$;

(b) $A^{\pm}$ are topological linear isomorphisms $W \to H$.

Observe that condition (b) is equivalent to $A^{\pm} \in \mathcal{S}(W, H)$ and $\ker A^{\pm} = \{0\}$. Since for $B \in L_s(W, H)$ the conditions $B \in \mathcal{S}(W, H)$ and B invertible are open, it is readily seen that $\mathcal{A}$ is an open subset of the Banach space $\mathcal{B}$.

Lemma 6.1 *Let A in $L(W,H)$ and assume that there exists a constant $c > 0$ such that*

$$\|\xi\|_W \leq c(\|A\xi\|_H + \|\xi\|_H)$$

for all $\xi \in W$. Then for every $C > 0$ there exists a constant $\delta > 0$ such that for all $A' \in L(W,H)$ with $\|A - A'\| < \delta$ we have

$$\|\xi\|_W \leq C(\|A'\xi\|_H + \|\xi\|_H), \qquad (\xi \in W).$$

Proof. We fix $\delta > 0$ such that $c < C(1 - c\delta)$. Then for all $\xi \in W$,

$$\begin{aligned} \|\xi\|_W &\leq \|A\xi\|_H + \|\xi\|_H) \\ &\leq c\|A'\xi\|_H + \|(A - A')\xi\|_H + \|\xi\|_H) \\ &\leq c\|A'\xi\|_H + \|\xi\|_H) + c\delta\|\xi\|_W. \end{aligned}$$

This implies the desired estimate. □

Corollary 6.2 *If $A \in \mathcal{A}$, then A is uniformly self-adjoint in the sense that there exists a constant $c > 0$ such that*

$$\|\xi\|_W \leq c(\|A(t)\xi\|_H + \|\xi\|_H)$$

for all $t \in \mathbb{R}$.

Remark 6.3 Note that this is condition (A-2) of [2]. In view of Remark 3.1, which we will prove further down, this shows that the condition is really superfluous!

Proof. In view of Lemma 4.1 the estimate holds for each t separately, with a constant c_t possibly depending on t. A similar estimate holds for the operators $A^{\pm}$. Using the above lemma and a compactness argument, we see that the constant may in fact be chosen uniformly bounded as a function of t. □

It follows from the corollary that the space of families of bounded linear operators $A(t): W \to H$ satisfying (A-1), (A-2) and (A-3) of [2], page 8, coincides with the space $\mathcal{A}^{w1}$, which we shall now introduce. First of all, we define $\mathcal{B}^{w1} = \mathcal{B}^{w1}(\mathbb{R}, W, H)$ to be the space of $A \in \mathcal{B}$ with the property that $t \mapsto A(t)$ is C^1 for the weak operator topology, and such that $\|A'(t)\|_{\text{op}}$ is uniformly bounded in t. This space carries an obvious locally convex topology, for which it is complete.

Next, we define

$$\mathcal{A}^{w1}(\mathbb{R}, W, H) = \mathcal{A} \cap \mathcal{B}^{w1}.$$

It is convenient to work with the even smaller space

$$\mathcal{A}^1 = \mathcal{A} \cap \mathcal{B}^1,$$

where $\mathcal{B}^1$ is the space of C^1-maps $A: \mathbb{R} \to L_s(W,H)$ with the property that $\|A(t)\|$ and $\|A'(t)\|$ are uniformly bounded in t. This space is Banach for the norm

$$\sup_{t \in \mathbb{R}} (\|A(t)\|_{\text{op}} + \|A'(t)\|_{\text{op}}).$$

It is easy to verify that $\mathcal{B}^1 \subset \mathcal{B}^{w1} \subset \mathcal{B}$, hence also

$$\mathcal{A}^1 \subset \mathcal{A}^{w1} \subset \mathcal{A}.$$

Let $\varphi \in C_0^1(\mathbb{R})$, $\varphi \geq$ and $\int \varphi(t)\, dt = 1$. For $\delta > 0$ we put $\varphi_\delta(t) = \delta^{-1}\varphi(\delta^{-1}t$. Then φ_δ is an approximation of the Dirac measure for $\delta \to 0$, and the following lemma can be proved in a standard fashion.

Lemma 6.4 *Let $A \in \mathcal{B}$ and put $A_\delta := \varphi_\delta * A$. Then*

(a) $A_\delta \in \mathcal{B}^1$ *for all* $\delta > 0$;

(b) $\sup_{\mathbb{R}} \|A'_\delta\|$ *is uniformly bounded as* $\delta \to 0$

(c) *if* $A \in \mathcal{B}^{w1}$, *then* $A'_\delta(t) \to A'(t)$, *in the weak operator topology, locally uniformly with respect to* t.

Since $\mathcal{A}$ is open in $\mathcal{B}$, the above lemma implies in particular that the elements of $\mathcal{A}$ may be approximated by elements from $\mathcal{A}^1$.

For $A \in \mathcal{A}^{w1}$ we may define a differential operator D_A as in the finite dimensional case. However, we have to be a bit careful with our function spaces.

First of all, we define the Hilbert spaces $\mathcal{H} = L^2(\mathbb{R}, H)$ and $W^{1,2}(\mathbb{R}, H)$ in the usual fashion (for the latter space the definition is most easily given by using Fourier transform). Secondly, we define

$$\mathcal{W} = L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, W).$$

This space naturally is a Hilbert space with norm

$$\|\xi\|^2_{\mathcal{W}} = \int_{\mathbb{R}} (\|\xi(t)\|^2_W + \|\xi'(t)\|^2_H)\, dt.$$

For $A \in \mathcal{B}$, we define the differential operator $D_A: \mathcal{W} \to \mathcal{H}$ by

$$D_A\xi(t) = \frac{d}{dt}\xi(t) - A(t)\xi(t).$$

It is readily checked that $D_A: \mathcal{W} \to \mathcal{H}$ is continuous. Moreover, the map $A \mapsto D_A$ is continuous from $\mathcal{B}$ to $L(\mathcal{W}, \mathcal{H})$. More precisely,

$$\|D_A - D_B\|_{\text{op}} \leq \sup_{t\in\mathbb{R}} \|A(t) - B(t)\|,$$

for all $A, B \in \mathcal{B}$. This result is important, as it allows perturbation of $A \in \mathcal{A}^{w1}$ in $\mathcal{A}^1$ without changing the Fredholm index.

Theorem 6.5 *Let $A \in \mathcal{A}^{w1}$. Then $D_A: \mathcal{W} \to \mathcal{H}$ is Fredholm. In particular, if A is constant then D_A is bijective.*

Proof. The proof uses the same ideas as the proof in the finite dimensional case, with an added elliptic regularity result, needed to pass to the transpose operator, see [2]. □

In the next section we will define a map $\mu = \mu_{W,H}: \mathcal{A}(\mathbb{R}, H, W) \to \mathbb{Z}$, which turns out to be an appropriate generalization of the spectral flow. We will then finally show that for $A \in \mathcal{A}^{w1}$, we have

$$\text{index}\,(D_A) = -\mu(A).$$

The proof will make use of reduction to the finite dimensional case.

7 Spectral flow for curves in $\mathcal{S}(H, W)$.

Let now $A \in \mathcal{A}^1(\mathbb{R}, W, H)$. Then using the transversality theory of Fredholm maps between Banach manifolds, we may define the intersection number $\mu(A)$ of A with $\mathcal{S}_1$ in the usual fashion.

This intersection number is a C^1-homotopy invariant and for A transversal to all $\mathcal{S}_k$, $k \geq 1$, it may be computed as follows. The fact that A is transversal to all $\mathcal{S}_k$ means that A is disjoint from $\mathcal{S}_k$ for $k \geq 2$ and intersects $\mathcal{S}_1$ transversally. The set of $t \in \mathbb{R}$ with $A(t) \in \mathcal{S}_1$ is discrete and contained in a compact subset of $\mathbb{R}$ hence finite. Let $t_1 < \cdots < t_n$ be an ordering of its elements. In view of the earlier considerations concerning the orientation of $\mathcal{S}_1$, see Lemma 5.8, the intersection number $\mu(A)$ of A with $\mathcal{S}_1$ is given by

$$\mu(A) = \sum_{j=1}^{n} \text{sign } \nu_{A(t_j)}(A'(t_j)) = \sum_{j=1}^{n} \text{sign}\,\text{trace}\,[P(t_j) \circ A'(t_j) \circ P(t_j)]. \tag{7.8}$$

Here $P(t_j)$ denotes the rank one orthogonal projection onto the kernel of $A(t_j)$.

Lemma 7.1 *In the above setting, let $1 \leq j \leq n$ be fixed. Then there exist constants $\varepsilon, \delta > 0$ and a C^1-function λ: $I_\delta :=]\, t_j - \delta, t_j + \delta\, [\;\longrightarrow\;] - \varepsilon, \varepsilon\, [$, such that for $t \in I_\delta$,*

(a) $\sigma(A(t)) \cap] - \varepsilon, \varepsilon\, [= \{\lambda(t)\}$ *and the multiplicity of $\lambda(t)$ is one;*

(b) $\text{trace}\,[P(t_j)A'(t_j)P(t_j)] = \lambda'(t_j)$.

Proof. The existence of $\delta, \varepsilon, \lambda$ such that (a) holds is a consequence of Proposition 5.4. We turn to the proof of (b). Let $P(t)$ denote the orthogonal projection on $\ker(A(t) - \lambda(t)I)$, then $P(t)$ has rank one, and is a C^1-function of t. Let x be a unit vector in $\ker A(t_j)$, and define $x(t) = \|P(t)x\|^{-1}P(t)x$. Then for δ sufficiently small, the function $x: I_\delta \to W$ is C^1 on I_δ. Moreover, the projection $P(t)$ is given by $v \mapsto \langle v\,,\, x(t)\rangle x(t)$. Accordingly, we have

$$\lambda(t) = \langle A(t)x(t)\,,\, x(t)\rangle.$$

Differentiating this expression with respect to t, and evaluating at $t = t_j$, we obtain

$$\lambda'(t_j) = \langle A'(t_j)x(t_j)\,,\, x(t_j)\rangle + \langle A(t_j)x'(t_j)\,,\, x(t_j)\rangle + \langle A(t_j)x(t_j)\,,\, x'(t_j)\rangle.$$

Using that $A(t_j)x(t_j) = 0$, that x' has values in W, and that $A(t_j) \in L(W, H)$ is symmetric, we see that the last two terms in the above expression equal zero. It follows that

$$\lambda'(t_j) = \langle A'(t_j)x(t_j)\,,\, x(t_j)\rangle = \text{trace}\,[P(t_j)A'(t_j)P(t_j)].$$

This proves the assertion. □

With notation of the above lemma, we see that (7.8) becomes

$$\mu(A) = \sum_{j=1}^{n} \text{sign}\,\lambda'(t_j). \tag{7.9}$$

This means that $\mu(A)$ may be interpreted as the spectral flow of the family $A(t)$ as t goes from $-\infty$ to ∞.

8 The main theorem

We now come to the main theorem. First of all, we note that the intersection number $\mu_{W,H}$ can be extended to a continuous homotopy invariant $\mathcal{A} \to \mathbb{Z}$, so that in particular it becomes well defined on $\mathcal{A}^{w1}$.

Theorem 8.1 *Let $A \in \mathcal{A}^{w1}$. Then* $\text{index}\,(D_A) = -\mu(A)$.

Proof in the finite dimensional case. In this case, $W = H$ are finite dimensional, and $A\colon \mathbb{R} \to \text{End}(H)$ is a C^1-map. It suffices to show that $\dim E^s(A^-) - \dim E^s(A^+)$ equals $-\mu(A)$. Both numbers are invariant under homotopies with fixed endpoints $A^\pm$, so that we may assume that $A(t)$ is transversal to $\mathcal{S}_k(\mathbb{R}, W, W)$, for every $k \geq 1$, as in the beginning of Section 7. In particular, we have the validity of formula (7.9). For $t \in \mathcal{O} = \mathbb{R} \setminus \{t_1, \dots, t_n\}$, we define $P_-(t)$ to be the orthogonal projection onto the sum of the eigenspaces of $A(t)$ for the negative eigenvalues, and we define $P_+(t)$ to be the orthogonal projection onto the sum of the eigenspaces for the positive eigenvalues. Then it follows from the perturbation theory discussed before that P_- and $P_=$ are C^1 functions, with ranks that are locally constant. Moreover, $\text{rk}P_-(t)$ equals $\dim A^s(A^-)$ for $t < t_1$ and equals $\dim A^s(A^+)$ for $t > t_n$. We now claim that at every point t_j, we have

$$\lim_{t\uparrow t_j} \text{rk}\, P_-(t) - \lim_{t\downarrow t_j} \text{rk}\, P_-(t) = \text{sign}\, \lambda'(t_j).$$

Adding these equalities for $j = 1, \dots, n$, we obtain the desired equality. We restrict our attention to a sufficiently small neighborhood I of t_j and put $I^\pm\colon= \{t \in I \mid \pm(t - t_j) > 0\}$. Let $P(t)$ denote the projection onto the one dimensional eigenspace of $A(t)$, of eigenvalue $\lambda(t)$. Then $t \mapsto P(t)$ is a C^1-map. We will discuss the case $\text{sign}\, \lambda'(t_j) = +1$. The other case is handled similarly. In the present case, $\lambda < 0$ on I^- and $\lambda > 0$ on I^+. Put $P^*_\pm(t) = P_\pm(t)(I - P(t))$, then $P^*_\pm$ is readily seen to extend C^1 to the interval I. Moreover, $I = P^*_-(t) + P(t) + P^*_+(t)$, where for each t the images of the projections are mutually orthogonal. Now $P_-(t) = P^*_-(t) + P(t)$ for $t \in I^-$ and $P_-(t) = P^*_-(t)$ for $t \in I^+$. The result follows. □

We proceed to prove the general case by reduction to the finite dimensional case. For this we need the index and spectral flow to behave well with respect to direct sums.

If $A_j \in \mathcal{A}(\mathbb{R}, W_j, H_j)$, for $j = 1, 2$, we define $A_1 \oplus A_2\colon t \mapsto A_1(t) \oplus A_2(t)$. Then $A_1 \oplus A_2 \in \mathcal{A}(\mathbb{R}, W_1 \oplus W_2, H_1 \oplus H_2)$. Moreover, if $A_j \in \mathcal{A}^{w1}$ then $A_1 \oplus A_2 \in \mathcal{A}^{w1}$ and a similar statement holds with $\mathcal{A}^1$ instead of $\mathcal{A}^{w1}$.

Lemma 8.2 *Let $A_j \in \mathcal{A}(\mathbb{R}, W_j, H_j)$, for $j = 1, 2$.*

(a) $\mu(A_1 \oplus A_2) = \mu(A_1) + \mu(A_2)$.

(b) *If $A_j \in \mathcal{A}^{w1}$, then* $\text{index}\,(D_{A_1 \oplus A_2}) = \text{index}\,(D_{A_1}) + \text{index}\,(D_{A_2})$.

Proof. Assertion (b) is an immediate consequence of the definitions. For (a), we note that by standard approximation arguments we may reduce the identity to the case that $A_j \in \mathcal{A}^1(\mathbb{R}, W_j, H_j)$. By obvious homotopy arguments, we may then reduce to the situation that A_1 is constant on $[-1, \infty[$ and that A_2 is constant on $]-\infty, 1]$. By a further homotopy we may further reduce to the case that, in addition, A_j intersects $\mathcal{S}_k(\mathbb{R}, W_j, H_j)$ transversally. Property (b) now follows by a straightforward application of (7.9). □

The key to the reduction to the finite dimensional case is the following result, formulated as Theorem 4.5 in [2].

Theorem 8.3 *Let $A \in \mathcal{A}(\mathbb{R}, W, H)$. Then there exists a finite dimensional space V and a $b \in \mathcal{A}(\mathbb{R}, V, V)$, such that $A \oplus b$ is homotopic to a constant curve inside $\mathcal{A}(\mathbb{R}, W \oplus V, H \oplus V)$.*

Proof. Let $\mathcal{A}^1_0$ denote the set of $A \in \mathcal{A}^1(\mathbb{R}, W, H)$ intersecting all $\mathcal{S}_k$ transversally. For A in this set, we define $m_0(A)$ to be the number of intersection points of A with $\mathcal{S}_1$. Moreover, for $A \in \mathcal{A}(\mathbb{R}, W, H)$ we define $m(A)$ to be the minimum of $m_0(B)$ as B ranges over the elements of $\mathcal{A}^1_0$ that are continuously homotopic to A.

If $m(A) = 0$, then A is homotopic to a curve $B \in \mathcal{A}^1$ that has no intersections with $\cup_{k \geq 1} \mathcal{S}_k$. This means that $B(t)$ is bijective for all t. Now $B_\tau(t) = B(\tan(\tau \arctan t))$ defines a continuous homotopy of B with the constant family $B_0 : t \mapsto B(0)$, showing that A is homotopic to a constant family.

We will now establish the result by induction on $m(A)$. Thus, let $m(A) = m > 0$ and assume the result has been established for A with $m(A) < m$. Then it suffices to establish the existence of a $b \in \mathcal{A}^1(\mathbb{R}, \mathbb{C}, \mathbb{C})$ such that $m(A \oplus b) < m$. This is done as follows.

By homotopy we may as well assume that $A \in \mathcal{A}^1_0$ and that A intersects $\mathcal{S}_1$ in precisely m points. Let $t_1 < t_2 < \ldots < t_m$ be the points of intersection, and use the notation of ..... In particular, let $\lambda(t)$ be the eigenvalue of $A(t)$ for t in a sufficiently small neighborhood I_δ of t_1, such that λ is C^1 and $\lambda(t_1) = 0$.

We choose a C^1-function $b : \mathbb{R} \to \mathbb{R}$ such that $b(t) = -\lambda(t)$ on I_δ, such that b is locally constant outside a compact set, and such that b has $t = t_j$ as its only zero. Then b determines an element of $\mathcal{A}^1(\mathbb{R}, \mathbb{C}, \mathbb{C})$. We will show that $m(A \oplus b) < m$, establishing the induction step.

We select a unit vector $x(t) \in N(t) := \ker(A(t) - \lambda(t)I)$, depending on $t \in I_\delta$ in a C^1-fashion. Moreover, for $t \in I_\delta$ we define two orthogonal unit vectors in $W \oplus \mathbb{C}$ by $e_1(t) = (x(t), 0)$ and $e_2(t) = (0, 1)$. Let $E(t)$ be the span of these vectors. We note that the restriction of $A(t) \oplus b(t)$ to $E(t)$ has matrix with respect to $e_1(t), e_2(t)$ given by

$$M(t) = \begin{pmatrix} \lambda(t) & 0 \\ 0 & -\lambda(t) \end{pmatrix}.$$

We will first define a suitable homotopy M_τ of this matrix. Let $\beta \in C^\infty_0(\mathbb{R})$ be a real valued cut off function with $\beta(0) = 1$ and $\operatorname{supp} \beta \subset I_\delta$. Put

$$M_\tau(t) = \begin{pmatrix} \lambda(t) & \tau\beta(t) \\ \tau\beta(t) & -\lambda(t) \end{pmatrix}.$$

Then M_τ is a symmetric matrix of determinant $-\lambda(t)^2 - \tau\beta(\tau)$ which is strictly positive on I_δ as soon as $\tau \neq 0$. It follows that for $\tau \neq 0$ the matrix $M_\tau(t)$ has two non-zero real eigenvalues, one positive and one negative. We now define the homotopy B_τ of $A \oplus b$ on I_δ by $B_\tau(t) = A(t) \oplus b(t)$ on $N(t)^\perp \oplus \mathbb{C}$ and by $B_\tau(t)|E(t)$ determined by the matrix $M_\tau(t)$. By the choice of the cut off function β it follows that $B_\tau = A \oplus b$ outside a fixed compact subset of I_δ. We extend B_τ outside I_δ by putting $B_\tau = A \oplus b$. If $\tau \neq 0$, then $B_\tau(t)$ has no non-zero eigenvalues for $t \in I_\delta$, whereas $B_\tau = A \oplus b$ outside I_δ. It follows that $m_0(B_\tau) = m - 1$, showing that $m(A \oplus b) \leq m - 1$. □

Proof of Theorem 8.1. By standard approximation arguments (e.g. involving convolution operators), it follows that Theorem 8.3 holds in the $\mathcal{A}^1$ setting.

Let now $A \in \mathcal{A}^1(\mathbb{R}, W, H)$, and let $b \in \mathcal{A}^1(\mathbb{R}, V, V)$ be such that $A \oplus b$ is C^1-homotopic to a constant curve. Then $\operatorname{index}(D_{A\oplus b}) = 0$. Similarly, $\mu(A \oplus b) = 0$. It follows that

$$\operatorname{index}(D_A) + \operatorname{index}(D_b) = \operatorname{index}(D_{A\oplus b}) = -\mu(A \oplus b) = -(\mu(A) + \mu(b)).$$

By the finite dimensional case, $\operatorname{index}(D_b) = -\mu(b)$. This implies the result. $\square$

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