Exercises Lie groups

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Exercise 1. Let G be a group, equipped with the structure of a C^∞-manifold. Let $\mu : G \times G \to G,\ (x,y) \mapsto xy$ be the multiplication map. We assume that μ is smooth, i.e., C^∞.

(a) Show that the tangent map $T_{(e,e)}\mu : T_eG \times T_eG \to T_eG$ of μ at (e,e) is given by $(X,Y) \mapsto X+Y$. Hint: use partial derivatives with respect to x and y.

(b) Show that the inversion map $\iota : G \to G,\ x \mapsto x^{-1}$ is smooth in an open neighborhood of e and that the tangent map $T_e\iota : T_eG \to T_eG$ is given by $X \mapsto -X$. Hint: use the implicit function theorem.

(c) Show that G is a Lie group.

Exercise 2. We recall that $\mathrm{SU}(2)$ is the group of unitary 2×2 matrices with determinant 1. Let S^3 denote the unit sphere in $\mathbb{R}^4$, centered at the origin. For $x \in S^3$ we define the complex 2×2 matrix

$$u_x := \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

(a) Show that the map $\varphi : x \mapsto u_x$ is a bijection from S^3 onto $\mathrm{SU}(2)$.

(b) We view S^3 as a smooth submanifold of $\mathbb{R}^4$ and transfer this manifold structure to $\mathrm{SU}(2)$ so that φ becomes a diffeomorphism. Show that $\mathrm{SU}(2)$, equipped with this manifold structure is a Lie group.

(c) Show that $\mathrm{SU}(2)$, equipped with this manifold structure, is a smooth submanifold of $\mathrm{GL}(2,\mathbb{C})$.

Exercise 3. We recall that $\mathrm{O}(n)$ is the group of real $n \times n$ matrices $a \in \mathrm{GL}(n,\mathbb{R})$ such that $aa^t = I$.

(a) Let S be the linear space of symmetric $n \times n$ matrices in $\mathrm{M}(n,\mathbb{R})$. Show that $\varphi : A \mapsto AA^t$ defines a smooth map $\mathrm{M}(n,\mathbb{R}) \to \mathrm{S}$ with tangent map at I given by

$$T_I\varphi : X \mapsto X + X^t, \quad \mathrm{M}(n,\mathbb{R}) \to \mathrm{S}.$$

(b) Show that φ is a submersion at I.

(c) Show that $\mathrm{O}(n, \mathbb{R})$ is a Lie group.

Exercise 4. We identify $\mathrm{M}(n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$ and transfer the standard inner product on $\mathbb{R}^{n^2}$ to an inner product on $\mathrm{M}(n, \mathbb{R})$.

(a) Show that
$$\langle X, Y \rangle = \mathrm{tr}\,(XY^t), \qquad (X, Y \in \mathrm{M}(n, \mathbb{R})).$$

(b) Show that $\mathrm{O}(n)$ is contained in the sphere of center 0 and radius $\sqrt{n}$ for this inner product.

(c) Show that $\mathrm{O}(n)$ is compact.

Exercise 5. We identify $\mathrm{M}(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$ and transfer the standard complex inner product on $\mathbb{C}^{n^2}$ to an inner product on $\mathrm{M}(n, \mathbb{C})$.

(a) Show that
$$\langle X, Y \rangle = \mathrm{tr}\,(XY^*), \qquad (X, Y \in \mathrm{M}(n, \mathbb{C}).$$

(b) Show that $\mathrm{U}(n)$ is contained in the sphere of center 0 and radius $\sqrt{n}$ for this inner product.

(c) Show that $\mathrm{U}(n)$ is compact.

Exercise 6. Let now G be a Lie group. The commutator of two elements $x, y \in G$ is the element $c(x, y) := xyx^{-1}y^{-1}$. Show that for all $X, Y \in T_eG$ we have
$$[X, Y] = \frac{\partial}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} c(\exp sX, \exp tY).$$

Hint: use relations like $x \exp Y x^{-1} = \exp(\mathrm{Ad}\,(x)Y)$.

Exercise 7. If u, v are two smooth vector fields on a smooth manifold M, we recall that their bracket $[u, v]$ is the vector field defined by
$$[u, v] = \frac{d}{dt}\bigg|_{t=0} (\varphi^t)^* v.$$

Here φ^t denotes the flow of u. Moreover, the pull-back of a vector field v by a diffeomorphism φ is given by $\varphi^* v(x) = (T_x \varphi)^{-1} v(\varphi(x))$, for $x \in M$. We recall that the bracket defines a bilinear map $\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$ that is anti-symmetric and satisfies the Jacobi identity, hence turns $\mathcal{V}(M)$ into a Lie algebra.

Let G be a Lie group. The purpose of this exercise is to relate the Lie algebra structures of $\mathcal{V}(G)$ and T_eG. We recall that for $X \in T_eG$ the left G-invariant vector field v_X on G is defined by $v_X(x) = T_e(l_x)X$. Let $\Phi_X : \mathbb{R} \times G \to G$ denote the flow of v_X.

(a) Show that $\Phi_X(t,x) = x\exp(tX)$ for $t \in \mathbb{R}, x \in G$.

(b) Let $Y \in T_eG, t \in \mathbb{R}$. Show that $(\Phi_X^t)^* v_Y$ is a left invariant vector field.

(c) Let Y, t be as above. Show that $(\Phi_X^t)^* v_Y = v_{\mathrm{Ad}\,(\exp tX)Y}$.

(d) Let $X, Y \in T_eG$. Show that $[v_X, v_Y] = v_{[X,Y]}$.

(e) Show that $\mathcal{V}_L(G)$, the space of left invariant vector fields, is a Lie subalgebra of $\mathcal{V}(G)$, and that the map $X \mapsto v_X$ is an isomorphism from the Lie algebra T_eG onto $\mathcal{V}_L(G)$.

Exercise 8. Let V be a finite dimensional real linear space.

(a) Show that $\det : \mathrm{GL}(V) \to \mathbb{R}^*$ is a homomorphism of Lie groups.

(b) Show that $D(\det)(I) = \mathrm{tr}$. Here tr denotes the linear map $\mathrm{End}\,(V) \to \mathbb{R}$, asigning to a linear endomorphism of V its trace.

(c) Show that for all $A \in \mathrm{End}\,(V)$ we have:

$$\det(e^A) = e^{\mathrm{tr}\,A}$$

(hint: avoid computations; use a result from the text instead).

Exercise 9. We recall that a smooth manifold M is connected if and only if it is arcwise connected, i.e. for every two points $p, q \in M$ there exists a continuous curve $c : [0,1] \to M$ with $c(0) = p,\ c(1) = q$. In the following we will use the notation

$$\mathfrak{so}(n) := \{A \in \mathrm{M}(n, \mathbb{R}) \mid A^t = -A\}.$$

(a) Consider the exponential map $\exp : A \mapsto e^A = \sum_{n=0}^{\infty} (n!)^{-1} A^n$. Compute

$$\exp \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}$$

and show that $\exp$ maps $so(2)$ surjectively onto $SO(2)$.

(b) Show that for every $x \in \mathrm{SO}(n)$ there exists a $y \in \mathrm{SO}(n)$ such that $x = yby^{-1}$ with b a matrix consisting of 2×2 matrix blocks $B \in \mathrm{SO}(2)$, and 1×1 matrix blocks $B = (1)$ along the diagonal.

(c) Let $n \geq 2$. Show that the exponential map $\exp$ maps $so(n)$ onto $\mathrm{SO}(n)$, i.e. every $x \in \mathrm{SO}(n)$ may be written as $\exp X$ with $X \in so(n)$.

(d) Show that $\mathrm{SO}(n)$ is connected.

Exercise 10. Show that $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are connected. (Hint: use the method suggested in the previous exercise, but now with diagonal matrices.)

Exercise 11. We recall that the Lie group $\mathrm{SL}(n, \mathbb{R})$ has tangent space at I equal to

$$sl(n, \mathbb{R}) = \{X \in \mathrm{M}_n(\mathbb{R}) \mid \mathrm{tr}\, X = 0\}.$$

(a) Show that the matrix $y = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ belongs to $\mathrm{SL}(2, \mathbb{R})$, but cannot be written as $y = \exp Y$ with $Y \in sl(2, \mathbb{R})$.

(b) Let $n \geq 2$. Show that every $x \in \mathrm{SL}(n, \mathbb{R})$ can be written as $x = \exp X_a \exp X_s$, with X_a an antisymmetric, and X_s a symmetric matrix in $sl(n, \mathbb{R})$. (Hint: consider $x^t x$).

(c) Show that $\mathrm{SL}(n, \mathbb{R})$ is a connected Lie group.

Exercise 12. Let $\varphi, \psi : H \to G$ be two homomorphisms of Lie groups, and assume that H is connected. Show that

$$\varphi = \psi \iff \varphi_* = \psi_*$$

Exercise 13.

(a) Show that $\mathbb{R}^3$ together with the exterior product $(X, Y) \mapsto X \times Y$ is a Lie algebra.

(b) Show that the above Lie algebra is isomorphic to the Lie algebra $\mathfrak{o}(3)$ of the orthogonal group $O(3)$. Hint: Consider the map $\psi : \mathbb{R}^3 \to \mathrm{M}_3(\mathbb{R})$ defined by $v \mapsto \mathrm{mat}(v \times \cdot)$, the matrix with respect to the standard basis of the linear map $v \times \cdot : X \mapsto v \times X, \ \mathbb{R}^3 \to \mathbb{R}^3$.

Exercise 14. Let H_1, H_2 be Lie groups.

(a) If $\varphi : H_1 \to H_2$ is an isomorphism of groups show that φ is C^∞ if and only it is C^∞ on a neighborhood of e. Hint: use left translations. Show that φ is a diffeomorphism if and only if it is local diffeomorphism at e.

We now assume that G is a Lie group and that $i_j : H_j \to G, \ j = 1, 2$, are two injective Lie group homomorphisms with $i_1(H_1) = i_2(H_2)$.

(b) Show that there exists a unique map $\varphi : H_1 \to H_2$ such that $i_2 \circ \varphi = i_1$. Show that φ is an isomorphism of groups.

(c) Show that there exists a unique map $\tau : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that $i_{2*} \circ \tau = i_{1*}$. Show that τ is an isomorphism of Lie algebras.

(d) Show that there exists an open neighborhood Ω of 0 in $\mathfrak{g}$ with the following properties.

- The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism from Ω onto an open subset of G.
- For each $j = 1, 2$, the exponential map $\exp_j : \mathfrak{h}_j \to H_j$ is a diffeomorphism from $\Omega_j := i_{j*}^{-1}(\Omega)$ onto an open subset of H_j.

(e) Show that for every $X \in \Omega_1$ we have $\varphi(\exp_1(X)) = \exp_2(\tau X)$.

(f) Show that φ is an isomorphism of Lie groups and that $\varphi_* = \tau$.

(g) Show that every subgroup H of G carries at most one structure of a Lie subgroup.

Exercise 15. Let $\mathfrak{l}$ be a Lie algebra, and $\mathfrak{a}$ an ideal of $\mathfrak{l}$, that is: $\mathfrak{a} \subset \mathfrak{l}$ is a linear subspace such that $[\mathfrak{l}, \mathfrak{a}] \subset \mathfrak{a}$, i.e. $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{l}$ and $Y \in \mathfrak{a}$.

Show that $\mathfrak{l}/\mathfrak{a}$ has a unique structure of Lie algebra such that the canonical projection $\pi : \mathfrak{l} \to \mathfrak{l}/\mathfrak{a}$ is a Lie algebra homomorphism.

Exercise 16. Recall that a subgroup H of a group G is called normal if $xHx^{-1} \subset H$ for all $x \in G$. We recall that normality of H is equivalent to the existence of a group structure on the coset space G/H for which the canonical map $\pi : G \to G/H$ is a group homomorphism. Note that $H = \ker(\pi)$.

Let $\mathfrak{l}$ be a Lie algebra. By an ideal of $\mathfrak{l}$ we mean a linear subspace $\mathfrak{a} \subset \mathfrak{l}$ with the property that $[\mathfrak{l}, \mathfrak{a}] \subset \mathfrak{a}$, i.e., $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{l},\ Y \in \mathfrak{a}$.

(a) Let $\mathfrak{a} \subset \mathfrak{l}$ be an ideal. Show that the quotient (linear) space $\mathfrak{l}/\mathfrak{a}$ has a unique structure of Lie algebra such that the canonical projection $\pi : \mathfrak{l} \to \mathfrak{l}/\mathfrak{a}$ is a homomorphism of Lie algebras.

(b) Let $\varphi : \mathfrak{l} \to \mathfrak{m}$ be a surjective homomorphism of Lie algebras. Show that $\ker \varphi$ is an ideal in $\mathfrak{l}$ and that the induced map
$$\bar{\varphi} : \mathfrak{l}/\ker \varphi \to \mathfrak{m}$$
is an isomorphism of Lie algebras (this is the analogue of the isomorphism theorem for surjective group homomorphisms).

Exercise 17. Suppose that H is a Lie subgroup of a Lie group G. As usual we denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

(a) Show that if H is normal in G then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Hint: this exercise is more subtle than it may seem at first. The reason is that H need not be a smooth submanifold of G. Use a suitable characterization of $\mathfrak{h}$.

Now assume that G and H are connected.

(b) Show: if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then H is normal in G.

Exercise 18. We define the center $Z(G)$ of a Lie group G by

$$Z(G) = \{ x \in G \mid \forall y \in G : \; xy = yx \}.$$

Now assume that G is connected.

(a) Show that $Z(G) = \ker \mathrm{Ad}$.

(b) Show that $Z(G)$ has Lie algebra equal to $\ker \mathrm{ad}$.

If $\mathfrak{l}$ is a Lie algebra, we define its center by

$$Z(\mathfrak{l}) = \{X \in \mathfrak{l} \mid [X, Y] = 0 \quad \forall Y \in \mathfrak{l}\} = \ker \mathrm{ad}.$$

(c) Show that $Z(\mathfrak{l})$ is an ideal in $\mathfrak{l}$.

(d) Show that the Lie algebra of $Z(G)$ equals $Z(\mathfrak{g})$.

Exercise 19. We consider the Lie algebra $\mathfrak{so}(4)$ of the group $\mathrm{SO}(4)$. It consists of the matrices of the form

$$\begin{pmatrix} A & -C^t \\ C & B \end{pmatrix},$$

with $A, B \in \mathfrak{so}(2)$ and $C \in \mathrm{M}(2, \mathbb{R})$.

Let D_1 be a matrix as above with $A = B \neq 0$ and $C = 0$. Let D_2 be a matrix as above with $A = -B \neq 0$ and $C = 0$. Let $\mathfrak{v} \subset \mathfrak{so}(4)$ be the linear subspace of matrices of the above form with $A = B = 0$.

(a) Show that $\mathfrak{v}$ is invariant under both $\mathrm{ad}\, D_1$ and $\mathrm{ad}\, D_2$, and compute $\mathfrak{v}_2 := \ker \mathrm{ad}\, D_1 \cap \mathfrak{v}$ and $\mathfrak{v}_1 := \ker \mathrm{ad}\, D_2$.

(b) Show that $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ and that $[\mathfrak{v}_1, \mathfrak{v}_2] = 0$, i.e., $[X_1, X_2] = 0$ for all $X_1 \in \mathfrak{v}_1$ and $X_2 \in \mathfrak{v}_2$.

(c) Put $\mathfrak{a}_1 = \mathbb{R}D_1 \oplus \mathfrak{v}_1$ and $\mathfrak{a}_2 = \mathbb{R}D_2 \oplus \mathfrak{v}_2$. Show that $[\mathfrak{a}_1, \mathfrak{a}_2] = 0$.

(d) Show that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are ideals in $\mathfrak{so}(4)$ and show that $\mathfrak{so}(4) = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ as Lie algebras.

(e) Determine a basis R_{j1}, R_{j2}, R_{j3} of $\mathfrak{a}_j$, for $j = 1, 2$, such that the linear map $T_j : \mathfrak{so}(3) \to \mathfrak{a}_j$ determined by $T_j R_i = R_{ji}$ is an isomorphism of Lie algebras.

(f) Given $A \in \mathrm{End}\,(\mathfrak{so}(4))$, let $\mathrm{mat}\, A$ denote its matrix with respect to the basis $R_{11}, R_{12}, \ldots, R_{23}$ of $\mathfrak{so}(4)$. Show that $\varphi = \mathrm{mat} \circ \mathrm{ad}$ determines a Lie algebra embedding from $\mathfrak{so}(4)$ into $\mathrm{M}(6, \mathbb{R})$ with image isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

(g) Show that $\Phi = \mathrm{mat} \circ \mathrm{Ad}$ determines a Lie group homomorphism from $\mathrm{SO}(4)$ into $\mathrm{GL}(6, \mathbb{R})$ with image isomorphic to $\mathrm{SO}(3) \times \mathrm{SO}(3)$. Hint: follow the same reasoning as for $\mathrm{SU}(2) \to \mathrm{SO}(3)$ in the lecture notes.

(h) Determine the kernel of Φ.

Exercise 20.

(a) Show that the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generate the Lie algebra $sl(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$.

(b) Express the commutator brackets $[H,X]$, $[H,Y]$ and $[X,Y]$ as linear combinations of H,X,Y. *Remark*: a triple of elements H,X,Y of an arbitrary Lie algebra satisfying the same commutator relations, is called a standard $sl(2,\mathbb{R})$-triple.

(c) Compute the matrix of $\mathrm{ad}\, A$ with respect to H,X,Y, for $A = H,X,Y$.

Exercise 21. This exercise gives an introduction to the non-commutative field $\mathbb{H}$ of quaternions. We introduce $\mathbb{H}$ as the $\mathbb{R}$-algebra of complex 2×2-matrices of the form

$$m(a,b) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}. \end{pmatrix}$$

The map m is a real linear isomorphism from $\mathbb{R}^4 \simeq \mathbb{C}^2 \to \mathbb{H}$. The images of the standard basis vectors $e_1,\ldots,e_4$ of $\mathbb{R}^4$ are denoted by $1,i,j,k$, respectively. Thus, $1 = I$,

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(a) Show that $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. Determine ik, ki, jk, kj.

We embed $\mathbb{C}$ into $\mathbb{H}$ by $x+yi \mapsto m(x+yi,0)$. Thus $x+yi$ is mapped to the matrix $xI + yi$. In particular, we view $\mathbb{R}$ as a subspace of $\mathbb{H}$ via the map $a \mapsto aI$.

(b) Show that, accordingly, $\mathbb{H}$ is a two dimensional vector space over $\mathbb{C}$ with basis $1,j$. Show that for all $z\in\mathbb{C}$ we have $zj = j\bar{z}$.

We define the conjugation ι on $\mathbb{H}$ by

$$\iota : x_1 + x_2 i + x_3 j + x_4 k \mapsto x_1 - (x_2 i + x_3 j + x_4 k).$$

(c) Considering the elements of $\mathbb{H}$ as matrices, show that $\iota(h)$ equals h^*, the complex conjugate of h. Show that $\iota(\alpha\beta) = \iota(\beta)\iota(\alpha)$ for all $\alpha,\beta\in\mathbb{H}$. Show that $\iota^2 = I_{\mathbb{H}}$.

(d) Via m we transfer the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^4$ to a norm on $\mathbb{H}$. Show that

$$h\,\iota(h) = \iota(h)\,h = \|h\|^2 \qquad (h\in\mathbb{H}).$$

(e) Show that every element $h \in \mathbb{H} \setminus \{0\}$ is invertible, with inverse $h^{-1} = \|h\|^{-2}\iota(h)$ (this extends the similar formula for $\mathbb{C}$). Thus, $\mathbb{H}$ has all properties of a field, except for commutativity of the multiplication.

(f) Show that

$$\|\alpha\beta\| = \|\alpha\|\|\beta\| \qquad (\alpha, \beta \in \mathbb{H}).$$

(g) Show that the unit sphere S in $\mathbb{H}$, is a group for the $\mathbb{H}$-multiplication. Show that $S = \mathrm{SU}(2)$. Show that the Lie algebra of $\mathrm{SU}(2)$ coincides with the space $\mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \simeq \mathbb{R}^3$; this space is called the space of pure quaternions. Show in a direct fashion that the adjoint action of $\mathrm{SU}(2)$ on its Lie algebra is by means of transformations from $\mathrm{SO}(3)$.

Exercise 22. We retain the notation of the previous exercise. We denote by $\mathrm{GL}_{\mathbb{R}}(\mathbb{H})$ the group of invertible real linear transformations of $\mathbb{H}$; thus, $\mathrm{GL}_{\mathbb{R}}(\mathbb{H}) \simeq \mathrm{GL}(4, \mathbb{R})$. We denote by $\mathrm{SO}(\mathbb{H})$ the subgroup corresponding to $\mathrm{SO}(4)$.

(a) Show that for all $A, B \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ the map $\varphi(A, B) : \mathbb{H} \to \mathbb{H}, h \mapsto AhB^{-1}$ belongs to $\mathrm{SO}(\mathbb{H}) \simeq \mathrm{SO}(4)$.

(b) Show that the kernel K of the map φ consists of (I, I) and $(-I, -I)$.

(c) Show that φ factors to a Lie group *isomorphism* $\bar{\varphi} : \mathrm{SU}(2) \times \mathrm{SU}(2)/K \xrightarrow{\simeq} \mathrm{SO}(4)$. Hint: try to use a minimal amount of computation. Be inspired by the lecture notes on $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$. First consider the derivative of $\bar{\varphi}$. What is its kernel?

(d) Let $H = \{I, -I\} \subset \mathrm{SU}(2)$. Let π denote the canonical projection $\mathrm{SU}(2) \times \mathrm{SU}(2)/K \to \mathrm{SU}(2)/H \times \mathrm{SU}(2)/H$. Show that the map $\psi := \pi \circ (\bar{\varphi})^{-1}$ is a surjective Lie group homomorphism from $\mathrm{SO}(4)$ onto $\mathrm{SU}(2)/H \times \mathrm{SU}(2)/H$.

(e) Show that $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and that $\mathrm{SO}(3) \times \mathrm{SO}(3) \simeq \mathrm{SO}(4)/\{\pm I\}$.

Exercise 23. **Grassmannian manifold.** Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. As a set, the Grassmannian manifold $G_{n,k} = G_{n,k}(\mathbb{K})$ consists of all k-dimensional linear subspaces of $\mathbb{K}^n$. We consider the linear space $\mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^n)$ of linear maps $\mathbb{K}^k \to \mathbb{K}^n$. Given a sequence $i = (i_1, \dots, i_k)$ of integers i_j with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and an element $A \in \mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^n)$, we denote by $D_i(A)$ the determinant of the $k \times k$ submatrix of A determined by the rows with numbers $i_1, \dots, i_k$. The set H_i of matrices A with $D_i(A) \neq 0$ is open in $\mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^n)$ (why?). We denote by $\mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$ the collection of $A \in \mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^n)$ with $\ker A = \{0\}$. Then $\mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$ is the union of the sets H_i, hence open in $\mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^n)$.

We define the map $p : \mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n) \to G_{n,k}$ by $p(A) = A(\mathbb{K}^k)$.

(a) Show that p is surjective.

It is known that the set $G_{n,k}$ has a structure of smooth manifold such that p is submersive. Put $G = \mathrm{GL}(n, \mathbb{K})$. We define the map $\alpha : G \times \mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n) \to \mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$by $\alpha(g, A) = g \circ A$. We define the map $\beta : G \times G_{n,k} \to G_{n,k}$ by $\beta(g, V) = g \cdot V := g(V)$.

(b) Show that α and β are actions of G on $\mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$ and $G_{n,k}$ respectively, and that $p(\alpha(g, A)) = \beta(g, p(A))$, for all $A \in \mathrm{Hom}_0(\mathbb{K}^k, \mathbb{K}^n)$ and all $g \in G_{n,k}$.

(c) Show that α is smooth.

(d) Show that β is smooth.

(e) Show that $G_{n,k} \simeq \mathrm{GL}(n, \mathbb{K})/P$, where P is the subgroup of matrices $g \in \mathrm{GL}(n, \mathbb{K})$ of the form

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

with $A \in \mathrm{GL}(k, \mathbb{K})$, $B \in \mathrm{GL}(n-k, \mathbb{K})$ and C a $k \times n$ matrix with entries in $\mathbb{K}$.

(f) Show that $G_{n,k}$ is compact. Hint: treat the fields $\mathbb{R}$ and $\mathbb{C}$ separately.

Exercise 24. **Flag manifold.** Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let $n \geq 2$ and let $d = (d_1, \ldots, d_k)$ be a sequence of positive integers with $\sum_{j=1}^k d_j = n$. We define a *flag* of type d in $\mathbb{K}^n$ to be an ordered sequence $F = (F_0, F_1, \ldots, F_{k-1}, F_k)$ of linear subspaces of $\mathbb{K}^n$ with $0 = F_0 \subset F_1 \subset \cdots \subset F_k = \mathbb{K}^n$ and with $\dim(F_j/F_{j-1}) = d_j$, for all $1 \leq j \leq k$. The collection of all flags of type d, denoted by $\mathcal{F} = \mathcal{F}_d$, is called a flag manifold.

Let $G = \mathrm{GL}(n, \mathbb{K})$ and let $\alpha : G \times \mathcal{F} \to \mathcal{F}$ be defined by $\alpha(g, F) = g \cdot F := (g(F_j) \mid 0 \leq j \leq k)$.

(a) Show that α is a transitive action of G on $\mathcal{F}$.

Let the standard flag E of type d be defined by $E_0 = 0$ and $E_j = \mathrm{span}\,\{e_1, \ldots, e_{d_1 + \cdots + d_j}\}$, for $1 \leq j \leq k$. We define the map $\varphi : G \to \mathcal{F}$ by $\varphi(g) = g \cdot E$.

(b) Determine the stabilizer $P = P_d$ of E in G. Show that P is a closed subgroup of G.

(c) Show that $\varphi : G \to \mathcal{F}$ induces a bijection $\bar{\varphi} : G/P \to \mathcal{F}$. Accordingly, we equip $\mathcal{F}$ with the structure of a smooth manifold such that $\bar{\varphi}$ is a diffeomorphism.

(d) Put $K = \mathrm{O}(n)$ if $\mathbb{K} = \mathbb{R}$ and $K = \mathrm{U}(n)$ if $\mathbb{K} = \mathbb{C}$. In both cases show that $\varphi(K) = \mathcal{F}$. Put $H = K \cap P$ and show that $\mathcal{F}$ is diffeomorphic to K/H. Conclude that $\mathcal{F}$ is compact.

(e) With notation as in (d), show that $m : K \times P \mapsto G$, $(k, p) \mapsto kp$ is a surjective map. Hint: use (d) and (b). Moreover, show that m is a smooth submersion. Hint: use homogeneity.

(f) Determine d such that $\mathcal{F}_d \simeq \mathbb{P}^{n-1}(\mathbb{K})$. More generally, let $1 \leq k < n$. Determine d such that $\mathcal{F}_d \simeq G_{n,k}(\mathbb{K})$.

Exercise 25. Let A and B be commuting endomorphisms of a linear space V over the ground field $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$. Show that for each $\lambda \in \mathbf{k}$ and all $k \in \mathbb{N}$ the endomorphism B leaves the space $\ker(A - \lambda I)^k$ invariant. We define the generalized eigenspace of A for the eigenvalue λ to be the space E_λ of vectors $v \in V$ for which there exists a $k \in \mathbb{N}$ such that $(A - \lambda I)^k v = 0$. Show that B leaves the space E_λ invariant.

Exercise 26. Let G be a commutative Lie group, and let (π, V) be a unitary representation of G.

(a) Show that π is irreducible if and only if $\dim V = 1$. Hint: use Schur's lemma.

(b) Show that there exist mutually orthogonal one dimensional invariant linear subspaces $V_1, \ldots, V_n$ of V, such that $V = V_1 \oplus \cdots \oplus V_n$.

(c) Show that the natural representation of $\mathrm{SO}(2)$ in $\mathbb{C}^2$ is not irreducible.

(d) Show that the natural representation of $\mathrm{SO}(2)$ in $\mathbb{R}^2$ is irreducible.

Exercise 27. Let $n \geq 3$, and let π be the natural representation of $\mathrm{SO}(n)$ in $\mathbb{C}^n$. We consider the centralizer $\mathfrak{c}$ of $\mathrm{SO}(n)$ in $\mathrm{M}(n, \mathbb{C})$.

(a) Let v_1, v_2 be a pair of orthonormal vectors in $\mathbb{R}^n$ and w_1, w_2 a second such pair. Show that there exists an element $g \in \mathrm{SO}(n)$ such that $gv_j = w_j$, for $j = 1, 2$.

(b) Let v_1, v_2 and w_1, w_2 be two pairs as in item (a). Show that for all $T \in \mathfrak{c}$ we have $\langle Tv_1, v_2 \rangle = \langle Tw_1, w_2 \rangle$.

(c) Show that π is irreducible.

Exercise 28. Let (δ, V) be an irreducible finite dimensional representation of the Lie group G.

(a) Let π be the n-fold direct product of δ in $W = V \oplus \cdots \oplus V$ (n summands). Show that $\mathrm{Hom}_G(V, W)$ is a linear space of dimension n.

(b) Show that $\mathrm{End}\,_G(V \oplus V)$ is 4-dimensional.

(c) Show that the space $V^2 = V \oplus V$ has a direct sum decomposition $V^2 = U_1 \oplus U_2$ into G-invariant non-trivial subspaces different from $V \oplus \{0\}$ and $\{0\} \oplus V$. From this we draw the conclusion that V^2 has no canonical decomposition into irreducibles.

Exercise 29. Let $(\delta_1, V_1),\ (\delta_2, V_2)$ be two irreducible finite dimensional representations of the Lie group G. Show that

(a) $\dim \mathrm{Hom}_G(V_1, V_2) = 1 \iff \delta_1 \sim \delta_2$ (hint: use the previous exercise).

(b) $\dim \mathrm{Hom}_G(V_1, V_2) = 0 \iff \delta_1 \not\sim \delta_2$.

In the following exercises, (π, V) and (ρ, W) will be finite dimensional continuous representations of a compact Lie group G. We recall that the character of the representation π is defined to be the function $\chi_\pi : G \to \mathbb{C}$ given by

$$\chi_\pi(x) = \mathrm{tr}\,(\pi(x)), \qquad (x \in G).$$

Exercise 30. Show that:

(a) $\chi_\pi(e) = \dim V$;

(b) $\chi_\pi(xyx^{-1}) = \chi_\pi(y)$, for all $x, y \in G$;

(c) $\chi_\pi(x^{-1}) = \overline{\chi_\pi(x)}$, for $x \in G$.

Exercise 31. Let $G = \mathrm{SU}(2)$. For $\varphi \in \mathbb{R}$ we define $t_\varphi \in G$ to be the diagonal matrix with entries $e^{i\varphi}$ in the upper left corner, and $e^{-i\varphi}$ in the lower right corner.

(a) Show that $T = \{t_\varphi \mid \varphi \in \mathbb{R}\}$ is a commutative compact subgroup of G. Show that every element of G that commutes with T belongs to T. (Such a group is called a maximal torus of G.)

(b) Let χ_n be the character of the irreducible representation π_n of G, for $n \in \mathbb{N}$. Show that

$$\chi_n(t_\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}.$$

(c) Prove the following Clebsch-Gordon formula, for $m, n \in \mathbb{N}$ with $m \leq n$:

$$\pi_n \otimes \pi_m \sim \pi_{n+m} \oplus \pi_{n+m-2} \oplus \cdots \oplus \pi_{n-m}.$$

Hint: establish an identity of characters.

In the following exercises, G will be a compact Lie group, and dx normalized Haar measure on G. For two continuous functions $f, g : G \to \mathbb{C}$ we define the convolution product $f * g : G \to \mathbb{C}$ by

$$f * g(x) = \int_G f(xy^{-1})g(y)\; dy.$$

Exercise 32. Let $\delta_1, \delta_2 \in \widehat{G}$ be inequivalent representations.

(a) Show that $f * g = 0$ for all $f \in C(G)_{\delta_1}$ and $g \in C(G)_{\delta_2}$.

(b) Show that $\chi_{\delta_1} * \chi_{\delta_2} = 0$.

(c) Let $\delta \in \widehat{G}$. Show that $\chi_\delta * \chi_\delta = \frac{1}{\dim\delta}\chi_\delta$

If (π, V) is a continuous finite dimensional representation of G and $f \in C(G)$, we define the endomorphism $\pi(f)$ of V by

$$\pi(f) = \int_G f(x)\pi(x)\, dx.$$

Exercise 33. Let (π, V) be a finite dimensional unitary representation of G.

(a) Show that, for all $f, g \in C(G)$:

$$\pi(f * g) = \pi(f) \circ \pi(g).$$

For $\delta \in \widehat{G}$ we put $P_\delta = \dim(\delta)\, \pi(\bar{\chi}_\delta)$, where the bar indicates that the complex conjugate of the character is taken.

(b) Show that P_δ is G-intertwining.

(c) Show that P_δ is an orthonormal projection, i.e. P_δ is symmetric and $P_\delta^2 = P_\delta$.

(d) Show that $P_{\delta_1} \circ P_{\delta_2} = 0$ if $\delta_1, \delta_2 \in \widehat{G}$ are inequivalent.

(e) Show that the set $S(\pi)$ of $\delta \in \widehat{G}$ with $P_\delta \neq 0$ is finite.

(f) Assume that W is an irreducible G-submodule of V, and let $\pi|_W \sim \delta \in \widehat{G}$. Show that $P_\delta = I$ on W. Hint: first show that P_δ is a scalar on W.

(g) Show that

$$V = \oplus_{\delta \in S(\pi)} P_\delta(V)$$

is an orthogonal direct sum of invariant subspaces of V. Moreover, show that the restriction of π to $P_\delta(V)$ is equivalent to a finite direct sum of copies of δ.

(h) Assume that $V = V_1 \oplus \cdots \oplus V_m$ is a decomposition of V into irreducibles. Show that for every $\delta \in \widehat{G}$ we have

$$P_\delta(V) = \sum_{i : \pi|_{V_i} \sim \delta} V_i$$

Show that the number of terms in the above sum equals the L^2-inner product of χ_π with χ_δ.

Angular momentum operators

The purpose of the following set of exercises is to clarify the connection between the angular momentum operators in quantum mechanics, and the representation theory of the Lie algebra $\mathfrak{su}(2)$.

In the following, e_1, e_2, e_3 will denote the standard basis in $\mathbb{R}^3$. Matrices will be taken with respect to this basis. We will use the notation $a \times b$ for the exterior product of two vectors $a, b \in \mathbb{R}^3$. Recall that this product is determined by the requirement

$$\langle x, a \times b\rangle = \det(x, a, b)$$

for all $x \in \mathbb{R}^3$. By substituting the elements of the standard basis for x one obtains the usual determinant formulas for the components of $a \times b$.

We start with a general exercise on smooth group actions and the associated representations on function spaces.

Exercise 34. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Let $\tau : G \times M \to M, (g, m) \mapsto gm$ be a smooth action of G on a smooth manifold M. There is an associated representation π of G on the space $C^\infty(M)$ of smooth functions on M. It is given by the formula:

$$\pi(g)\varphi\,(m) = \varphi(g^{-1}m),$$

for $\varphi \in C^\infty(M)$, $g \in G$, $m \in M$.

(a) Check that indeed π is a representation of G in $C^\infty(M)$; we assert nothing on continuity here.

Given $X \in \mathfrak{g}$ and $\varphi \in C^\infty(M)$ we define the function $\pi_*(X)\varphi : M \to \mathbb{C}$ by

$$\pi_*(X)\varphi = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp tX)\varphi.$$

(b) Show that $\pi_*(X)$ is a smooth first order linear partial differential operator on M.

(c) Show that $\pi(x)\pi_*(X)\pi(x)^{-1} = \pi_*(\mathrm{Ad}\,(x)X)$ for all $X \in \mathfrak{g}$ and $x \in G$.

(d) Show that $\pi_*([X, Y]) = \pi_*(X)\pi_*(Y) - \pi_*(Y)\pi_*(X)$ for all $X, Y \in \mathfrak{g}$.

Thus, $X \mapsto \pi_*(X)$ is a Lie algebra homomorphism from $\mathfrak{g}$ into the algebra $\mathrm{DO}(M) \subset \mathrm{End}\,(C^\infty(M))$ of smooth linear partial differential operators on M. In particular, π_* is a representation of the Lie algebra $\mathfrak{g}$ in $C^\infty(M)$.

Exercise 35.

(a) Show that for $a \in \mathbb{R}^3$ the matrix R_a of the linear map $x \mapsto a \times x$ belongs to $\mathfrak{so}(3)$, the Lie algebra of $\mathrm{SO}(3)$; thus $\mathfrak{so}(3)$ consists of all real anti-symmetric 3×3 matrices.

(b) Show that for $a, b \in \mathbb{R}^3$ we have $R_{a\times b} = [R_a, R_b]$.

(c) Show that $(\mathbb{R}^3, \times)$ is a Lie algebra isomorphic to $\mathfrak{so}(3)$; here $a \mapsto R_a$ is the isomorphism. Observe that $R_{e_j} = R_j$, the infinitesimal generating rotation around the x_j-axis.

In the following we shall apply both of the above exercices to the natural smooth action of the group $\mathrm{SO}(3)$ on $\mathbb{R}^3$ by the usual matrix multiplication.

Let π denote the associated representation of $\mathrm{SO}(3)$ in $C^\infty(\mathbb{R}^3)$, and let π_* be the representation of $\mathfrak{so}(3)$ in $\mathrm{End}\,(C^\infty(\mathbb{R}^3))$, defined as above. Thus, if $R \in \mathfrak{so}(3)$, then $\pi_*(R)$ is a first order linear partial differential operator on $\mathbb{R}^3$.

Exercise 36. Show that for every $a \in \mathbb{R}^3$, $\varphi \in C^\infty(\mathbb{R}^3)$,

$$\pi_*(R_a)\varphi(x) = -a \cdot (x \times \nabla\varphi(x)) \qquad (x \in \mathbb{R}^3).$$

In the following we shall write $\bar{R}_a$ for $\pi_*(R_a)$. By linearity of the map $a \mapsto \bar{R}_a$, the formula of the above exercise is equivalent to:

$$\begin{aligned} \bar{R}_1 &= -(x_2\partial_3 - x_3\partial_2) \\ \bar{R}_2 &= -(x_3\partial_1 - x_1\partial_3) \\ \bar{R}_3 &= -(x_1\partial_2 - x_2\partial_1) \end{aligned}$$

In classical mechanics, angular momentum of a particle tested with a vector $a \in \mathbb{R}^3$ is given by

$$L_a := a \cdot (r \times p).$$

The quantum mechanical analogue of momentum p is the momentum operator $\frac{\hbar}{i}\nabla$. The analogue of angular momentum is then the operator:

$$L_a := \frac{\hbar}{i} a \cdot (x \times \nabla).$$

Comparing with the above we see that $L_a = i\,\hbar\bar{R}_a$, for every $a \in \mathbb{R}^3$. In the following we shall replace $L/\hbar$ by L, so that

$$L_a = i\bar{R}_a \qquad (a \in \mathbb{R}^3).$$

Since $a \mapsto R_a$ and $R \mapsto \bar{R}$ are Lie algebra homomorphisms, we have that

$$\bar{R}_{a\times b} = [\bar{R}_a, \bar{R}_b].$$

For the angular momentum operators this means that we have the commutation rule:

$$[L_a, L_b] = iL_{a\times b} \qquad (a, b \in \mathbb{R}^3).$$

For the components $L_j = L_{e_j}$ this means

$$\begin{aligned} L_1L_2 - L_2L_1 &= iL_3 \\ L_2L_3 - L_3L_2 &= iL_1 \\ L_3L_1 - L_1L_3 &= iL_2. \end{aligned}$$

In the physics literature this set of equations is often briefly written as

$$L \times L = iL.$$

We thus see that the assertion expressed by the latter formula is equivalent to the assertion that the map $a \mapsto i^{-1}L_a$ is a Lie algebra homomorphism from $\mathbb{R}^3$, equipped with the exterior product, to $\mathrm{End}\,(C^\infty(\mathbb{R}^3))$, equipped with the commutator bracket. One readily checks that the map $a \mapsto L_a$ is injective. Hence the operators $i^{-1}L_1, i^{-1}L_2, i^{-1}L_3$ generate a sub Lie algebra of $\mathrm{End}\,(C^\infty(\mathbb{R}^3))$ isomorphic to $(\mathbb{R}^3, \times)$, hence to $\mathfrak{so}(3)$.

In quantum mechanics one now calls any observable L with the property $L \times L = iL$ an angular momentum operator. In our language we may translate this statement as follows. To avoid technicalities we assume here that an observable corresponds to a *bounded* Hermitian operator in a Hilbert space (which in general is too strong a restriction). Thus, an angular momentum operator is by definition a map L from $\mathbb{R}^3$ to the space of bounded Hermitian operators of $\mathcal{H}$ such that $a \mapsto i^{-1}L_a$ is a non-trivial homomorphism of the Lie algebra $(\mathbb{R}^3, \times)$ into $\mathrm{End}\,(\mathcal{H})$.

Assumption: In the following two exercises we assume such a general angular momentum operator $L : \mathbb{R}^3 \to \mathrm{End}\,(\mathcal{H})$ to be fixed.

Exercise 37.

(a) Show that $\mathfrak{so}(3)$ contains no ideals different from 0 and $\mathfrak{so}(3)$.

(b) Show that the map $a \mapsto L_a$ is injective.

(c) Show (with minimum of computation) that the operators L_1, L_2, L_3 satisfy the same commutation relations as iR_1, iR_2 and iR_3.

(d) Show (with minimum of computation) that the operators L_j satisfy the same commutation relations as $\frac{\sigma_j}{2}$, where σ_j are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(e) Show that the map $\frac{\sigma_j}{2i} \mapsto \frac{1}{i}L_j$ extends to a representation ρ of $\mathfrak{su}(2)$ in $\mathcal{H}$.

The representation ρ extends to a complex linear representation of the complexification $sl(2, \mathbb{C})$ of $\mathfrak{su}(2)$ in $\mathcal{H}$. Let H, X, Y be the standard basis of $sl(2, \mathbb{C})$.

We define the following bounded, but not Hermitian, operators of $\mathcal{H}$.

$$L_\pm := L_1 \pm iL_2.$$

(f) Show that $L_+ = \rho(X)$, $L_- = \rho(Y)$ and $2L_3 = \rho(H)$. Conclude that $2L_3, L_+, L_-$ is a standard sl_2-triple.

(g) Show that $L_+^* = L_-$.

Exercise 38. Let $L_j,\ L_\pm$ be as above. We shall now discuss the raising and lowering procedure one finds in the physics literature. We advise the reader to keep the observation about the standard triple in mind.

We define the bounded operator L^2 of $\mathcal{H}$ by

$$L^2 := L_1^2 + L_2^2 + L_3^2.$$

(a) Show that L^2 is Hermitian and commutes with L_j for every $j = 1, 2, 3$.

(b) Show that

$$L_+L_- = L^2 - L_3(L_3 - I), \qquad L_-L_+ = L^2 - L_3(L_3 + I).$$

(c) Assume that $v \in \mathcal{H}$ is an eigenvector for both L^2 and L_3 with eigenvalues λ and λ_3, respectively. Show that then L_+v is an eigenvector for L^2 and L_3 with the eigenvalues λ and $\lambda_3 + 1$, respectively. Show also that

$$\|L_+v\|^2 = [\lambda - \lambda_3(\lambda_3 + 1)]\|v\|^2$$

(d) Let $v \in \mathcal{H}$ be an eigenvector for both L^2 and L_3. Show that there exists a $k \in \mathbb{N}$ such that $L_+^k v = 0$. Show that there exists a $l \in \mathbb{N}$ such that $L_-^l v = 0$.

(e) Let $v \in \mathcal{H}$ be an eigenvector for both L^2 and L_3. Show that there exists a unique linear ρ-invariant linear subspace $V \subset \mathcal{H}$ containing v and such that $\rho|_V$ is a finite dimensional irreducible representation of $\mathfrak{su}(2)$.

We know that the irreducible representation $\rho|_V$ is completely determined by its dimension $n+1$, where $n \geq 0$. Put $j = \frac{n}{2}$.

(f) Show that the eigenvalues of $L_3|_V$ are $j, j-1, \dots, -j$.

(g) Show that L^2 acts by a scalar on V. Show that this scalar is $j(j+1)$. Hint: select $v \in V\setminus\{0\}$ such that $L_+v = 0$ and compute L^2v by using (b).

Exercise 39. Let H, X, Y be the standard triple for $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}}$, and let α be the root of $\mathfrak{t} = i\mathbb{R}H$ in $\mathfrak{g}_{\mathbb{C}}$ determined by $\alpha(H) = 2$. Let (π, V) be a finite dimensional continuous representation of $\mathrm{SU}(2)$. The collection of weights of $\pi_*|\mathfrak{t}$ is denoted by $\Lambda(\pi_*)$. Show that

(a) $\Lambda(\pi_*) \subset \frac{1}{2}\mathbb{Z}\alpha$.

(b) If $0 \notin \Lambda(\pi_*)$, then $\frac{1}{2}\alpha \in \Lambda(\pi_*)$.

(c) If π is irreducible, then 0 and $\frac{1}{2}\alpha$ do not both belong to $\Lambda(\pi_*)$.

(d) The representation π is the direct sum of $\dim V_0 + \dim V_{\alpha/2}$ irreducibles (among which equivalent ones may occur; they are all counted).

(e) π is irreducible if and only if $\dim V_0 + \dim V_{\alpha/2} = 1$.

Exercise 40. We consider the connected compact Lie group $G = \mathrm{SU}(n),\ n \geq 2$.

(a) Show that the complexification of its Lie algebra equals

$$\mathfrak{g}_{\mathbb{C}} = \{X \in \mathrm{M}_n(\mathbb{C}) :\ \operatorname{tr} X = 0\} = \mathfrak{sl}(n, \mathbb{C}).$$

Show that the linear space $\mathfrak{t}$ consisting of all diagonal matrices in $\mathfrak{g}$ is a maximal torus.

(b) For every $1 \leq k \leq n$ we define $\epsilon_k \in i\mathfrak{t}^*$ to be the real linear map $\mathfrak{t} \to i\mathbb{R}$ that assigns to $X \in \mathfrak{t}$ the k-th diagonal element X_{kk}.
If $1 \leq i, j \leq n$ then we denote by E_{ij} the matrix whose entries are zero, except on the i-th row and the j-th column, where the entry is 1.
Show that the linear subspaces $\mathbb{C}E_{ij}(i \neq j)$ are root spaces of $\mathfrak{g}_{\mathbb{C}}$. Determine the set $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$ of roots in terms of the $\epsilon_k,\ 1 \leq k \leq n$.

(c) Put $E = i\mathfrak{t}^*$. Show that (E, R) is a root system, by verifying the conditions of the definition of a root system. Determine, for every $\alpha \in R$, the reflection $s_\alpha : E \to E$.

(d) Show that the Weyl group W of (E, R) is isomorphic to the permutation group S_n of n-elements. More precisely, define an explicit map $S_n \to W$, and show that this map is an isomorphism of groups.

(e) Determine a fundamental system S for R.

(f) Prove that the reflections s_α $(\alpha \in S)$ already generate W.

(g) Determine explicitly a W-invariant inner product on E.

(h) Determine the Cartan integers associated with S.

(i) Determine the Dynkin diagram of $\mathrm{SU}(n)$.

In the following we assume that $n = 3$.

(j) Let $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = \epsilon_2 - \epsilon_3$. Show that $R = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$. Show that the angle beteen α and β equals $2\pi/3$. Make a picture of R.

Exercise 41. **Induced representation.** Let G be a Lie group, H a closed subgroup, and (ξ, V) a finite dimensional representation of H.

(a) Show that the action of H on $G \times V$ given by $h \cdot (g, v) = (gh, \xi(h)^{-1})$ is proper and free, hence of PFB type.

It follows from the above that the quotient space $\mathcal{V} = G \times_H V := (G \times V)/H$ is a smooth manifold.

(b) Show that the map $p : \mathcal{V} \to G/H$ induced by the projection $G \times V \to G$ is smooth.

(c) Show that for each $g \in G$ the map $\varphi_g : v \mapsto [(g, v)]$ defines a bijection from V onto the fiber $p^{-1}(gH)$.

(d) Show that $p : \mathcal{V} \to G/H$ has a unique structure of vector bundle such that all maps φ_g $(g \in G)$ are linear. This vector bundle is said to be associated with the representation ξ.

(e) Show that the action of G on $G \times V$ given by $g \cdot (x, v) = (gx, v)$ factorizes to a smooth action of G on $\mathcal{V}$. Show that for each $x \in G/H$ the action by $g \in G$ maps the fiber $\mathcal{V}_x$ linearly and bijectively onto the fiber $\mathcal{V}_{gx}$ (a vector bundle over G/H with this property is said to be homogeneous).

(f) Let $C(\mathcal{V})$ denote the space of continuous sections of the vectorbundle $\mathcal{V}$. For $g \in G$ and $s \in C(\mathcal{V})$ we define $\pi(g)s : G/H \to \mathcal{V}$ by $\pi(g)s(x) = g \cdot s(g^{-1}x)$. Show that π defines a representation of G in $C(\mathcal{V})$. This representation is said to be induced from ξ. Notation $\pi = \mathrm{Ind}\,_H^G(\xi)$.

(g) Let $C(G, V, \xi)$ denote the space of continuous functions $\varphi : G \to V$ such that

$$\varphi(gh) = \xi(h)^{-1}\varphi(g) \qquad (g \in G,\ h \in H).$$

For φ in this space we define $\tilde{s}_\varphi : G \to \mathcal{V}$ by $s_\varphi(x) = [(x, \varphi(x))]$. Show that $\tilde{s}_\varphi$ factorizes to a section s_φ of $\mathcal{V}$. Show that $\varphi \mapsto s_\varphi$ is a linear bijection $C(G, V, \xi) \simeq C(\mathcal{V})$. Via this bijection we may realize the representation π on the space $C(G, V, \xi)$. Show that it is then given by the formula:

$$\pi(g)\varphi(x) = \varphi(g^{-1}x) \qquad (\varphi \in C(G, V, \xi),\ g, x \in G).$$

This realization of the induced representation is called ‘the induced picture.’

Extra exercises 2012

Exercise 42. We consider the action of the group $A = (\mathbb{R}, +, 0)$ on $M = \mathbb{R}^2 \setminus \{0\}$ given by

$$t(x_1, x_2) = (e^t x_1, e^{-t} x_2).$$

(a) Show that the action is free.

(b) Show that for every $m \in M$ there exists an open A-invariant neighborhood U of m such that the restriction of A to M is of principal fiber bundle type.

(c) Show that the quotient topology on $A \backslash M$ is not Hausdorff.

(d) Show that the action of A on M is not of principal fiber bundle type.

Exercise 43. In this exercise, we will give an interesting application of the formula for the derivative of the exponential map to the polar decomposition of matrices.

Let $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$ and let $\mathfrak{s}$ denote the space of symmetric matrices in $\mathrm{M}_n(\mathbb{R})$ of trace zero.

(a) Show that the map $\varphi : K \times \mathfrak{s} \to G$ given by $(k, X) \mapsto k \exp X$ is bijective. Hint: use the ideas of another exercise.

(b) Let $k \in K$ and $X_0 \in \mathfrak{s}$. Show that $T_{k,X_0}\varphi$ is bijective if and only if the map $T_{e,X_0}\varphi$ is injective.

(c) Let $\psi = r_{\exp X_0} : G \to G$. Show that the tangent map T of $\psi^{-1} \circ \varphi$ at the point (e, X_0) is the linear map $\mathfrak{k} \times \mathfrak{s} \to \mathfrak{g}$ given by

$$T(Y, X) = Y + F(X_0)X,$$

where

$$F(X_0) = \frac{e^{\mathrm{ad}\, X_0} - I}{\mathrm{ad}\,(X_0)} = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}\,(X_0))^n}{(n+1)!}.$$

(d) Show that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$ and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$.

(e) Show that $\mathrm{ad}\,(X_0) : \mathfrak{g} \to \mathfrak{g}$ is symmetric with respect to the positive definite inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g} \subset \mathrm{M}_n(\mathbb{R})$ given by

$$\langle A, B \rangle = \mathrm{tr}\,(AB^{\mathrm{T}}).$$

(f) Let $\mathfrak{p} : \mathfrak{s} \to \mathfrak{s}$ be the projection with kernel $\mathfrak{k}$. Show that $p \circ F(\mathrm{ad}\,(X_0))|_{\mathfrak{s}}$ is an injective linear map $\mathfrak{s} \to \mathfrak{s}$.

(g) Show that φ is a diffeomorphism.

Exercise 44. Let M be a smooth manifold, and ω a continuous density on M. Let V be finite dimensional complex vector space.

(a) Show that there exists a unique linear map $I_\omega : C_c(M,V) \to V$ such that for all $f \in C_c(M,V)$ and $\xi \in V^*$,

$$\xi(I_\omega(f)) = \int_M (\xi \circ f)\, \omega.$$

We agree to write

$$\int_M f\omega := I_\omega(f).$$

(b) Let W be a second finite dimensional complex vector space. Show that for any linear map $A : V \to W$ and every $f \in C_c(M,V)$,

$$A \int_M f\omega = \int_M (A \circ f)\omega.$$

Exercise 45. Let M be a smooth manifold.

(a) Show that M has a smooth density ω such that ω_a is positive for every $a \in M$. In the rest of this exercise, we assume such a density to be fixed.

(b) Let ω be a smooth density on M, G a compact Lie group, and assume that G acts smoothly on M from the left. For each $a \in M$ we define $\lambda_a \in \mathcal{D}T_aM$ by

$$\lambda_a = \int_G (l_x^*\omega)_a \, dx$$

where dx is a choice of right invariant positive density on G. Show that this definition is rigorous, and that $\lambda : a \mapsto \lambda_a$ defines a continuous density on M.

- Show: λ is G-invariant.
- Show: if ω is smooth, then λ is smooth;
- Show: if ω is positive, then λ is positive.

Conclusion: there exists a smooth positive G-invariant density on M.

Exercise 46. Let M be smooth manifold. Let h be a (smooth) Riemannian metric on M. Thus, for each $m \in M$, the map $h_m : T_mM \times T_mM \to \mathbb{R}$ defines a positive definite inner product on T_mM, and $m \mapsto h_m$ is smooth as a section of $T^*M \otimes T^*M$. In other words, in local coordinates $x^1, \dots, x^n$ on a coordinate patch U,

$$h = \sum_{ij} h_{ij} dx^i \otimes dx^j,$$

with $h_{ij} \in C^\infty(U)$.

Given a diffeomorphism $\varphi : M \to M$ we define the pull-back $\varphi^*(h)$ as usual, by

$$\varphi^*(h)_a(X, Y) = h_{\varphi(a)}(T_a\varphi(X), T_a\varphi(Y)),$$

for $a \in M$ and $X, Y \in T_aM$.

Assume now that M is equipped with a smooth left action by a compact Lie group G, and that $dx \in \Gamma^\infty(\mathcal{D}TG)$ is a right G-invariant and positive smooth density on G.

(a) Show that

$$g = \int_G l_x^* h \; dx$$

defines a smooth Riemannian metric on M.

(b) Show that g is left G-invariant, i.e., $l_y^* g = g$ for all $y \in G$.

(c) Let M be an arbitrary smooth manifold, and $G \times M \to M$ a left action of a compact Lie group G on M. Conclude that M has a smooth Riemannian structure for which G acts by isometries.

Exercise 47. The purpose of this exercise is to extend Proposition 20.17 to the setting of continuous representations in Hilbert space. The proof suggested below depends on application of the principle of uniform boundedness (also known as the Banach-Steinhaus theorem) from Functional Analysis.

Let G be a compact Lie group. We assume that π is a continuous representation of G in a complex Hilbert space $\mathcal{H}$. By this we mean that $\pi : G \to \mathrm{GL}(\mathcal{H})$ is a group homomorphism, and that the associated action map $G \times \mathcal{H} \to \mathcal{H}$ is continuous.

Let $(\,\cdot\,,\,\cdot\,)$ denote the (Hermitian, positive definite) inner product on H and let $\|\cdot\|$ denote the associated norm. In the course of this exercise, you will need to make use of the principle of uniform boundedness or the Banach-Steinhaus theorem from Functional Analysis.

(a) Show that there exists a constant $C > 0$ such that

$$\|\pi(g)v\| \leq C\|v\|, \qquad (\forall v \in \mathcal{H}, g \in G).$$

(b) Show that for each $v, w \in \mathcal{H}$ the integral

$$\langle v, w\rangle = \int_G (\pi(x)v, \pi(x)w) \; dx$$

is well-defined and defines a sesquilinear form on $\mathcal{H}$. Show that this sesquilinear form is positive definite and also that it is continuous.

(c) Let (g_n) be a sequence in G and let (v_n) be a sequence in $\mathcal{H}$ such that $\lim_{n\to\infty} \pi(g_n)v_n = 0$. Show that $\lim_{n\to\infty} v_n = 0$.

(d) Show that there exists a constant $c > 0$ such that $\|\pi(g)v\| \geq c\|v\|$ for all $g \in G$.

(e) Show that the norm $\|\cdot\|_{new}$ defined by $\langle\cdot\,,\,\cdot\rangle$ is equivalent to the original norm $\|\cdot\|$.

(f) Show that $(\mathcal{H}, \langle\cdot\,,\,\cdot\rangle)$ is a Hilbert space and that π is a continuous unitary representation in this Hilbert space.

Exercise 48. The purpose of this exercise is to investigate the appropriate continuity for representations in infinite dimensional Hilbert space. We consider the group $U(1)$, realized as the unit circle in $\mathbb{C}$.

We consider the complex linear space $l^2(\mathbb{N})$ of sequences $c = (c_n)_{n\in\mathbb{N}}$ of complex numbers with $\sum_n |c_n|^2 < \infty$. For each $c \in l^2(\mathbb{N})$ we write

$$\|c\| := \left(\sum_{n=0}^{\infty} |c_n|^2\right)^{1/2}$$

(a) Show that for all $c, d \in \mathbb{N}$ the sum

$$\langle c, d\rangle := \sum_{n\in\mathbb{N}} c_n \overline{d}_n$$

is absolutely convergent and satisfies $|\langle c, d\rangle| \leq \|c\|\|d\|$.

(b) Show that $\langle\cdot\,,\,\cdot\rangle$ defines a positive definite inner product for which $l^2(\mathbb{N})$ becomes a Hilbert space.

(c) For $z \in \mathrm{U}(1)$ we define the operator $\pi(z) : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ by

$$(\pi(z)c)_n = z^n c_n.$$

Show that $\pi(z)$ is a unitary isomorphism of $l^2(\mathbb{N})$, for all $z \in U(1)$.

(d) Show that the map $U(1) \times l^2(\mathbb{N}) \to l^2(\mathbb{N})$ given by $(z, c) \mapsto \pi(z)c$ is continuous.

(e) Show that for each $z \in U(1) \setminus \{1\}$ there exists an $n \in \mathbb{N}$ such that $|z^n - 1| \geq \sqrt{2}$.

(f) Show that for each $z \in U(1) \setminus \{1\}$ there exists a $c \in l^2(\mathbb{N})$ with $\|c\| = 1$ such that $\|\pi(z)c - c\| \geq \sqrt{2}$.

(g) Let $\mathrm{End}\,(l^2(\mathbb{N}))$ denote the space of continuous linear endomorphisms of $l^2(\mathbb{N})$, equipped with the operator norm. Show that the map $\pi : \mathrm{U}(1) \to \mathrm{End}\,(l^2(\mathbb{N}))$ is not continuous at 1.

The purpose of the next two exercises is to go through a proof of the Peter-Weyl theorem for finite groups, which may be viewed as zero-dimensional compact Lie groups.

Exercise 49. In this exercise we assume that G is a finite group, equipped with the trivial topology. Then $C(G)$ equals the complex linear space of all functions $G \to \mathbb{C}$. We first concentrate on the notion of invariant integral.

Given $f \in C(G)$ we define

$$I(f) = \frac{1}{|G|} \sum_{x \in G} f(x).$$

(a) Show that $C(G)$ is a finite dimensional linear space.

(b) Show that $I : G \to \mathbb{C}$ is complex linear, and has the following properties

  (1) If $f \geq 0$ then $I(f) \geq 0$.

  (2) If $f \geq 0$ and $I(f) = 0$, then $f = 0$.

  (3) If $f \in C(G)$ and $y \in G$ then $I(l_y^* f) = I(f)$.

(c) If $J : C(G) \to \mathbb{C}$ is linear and satisfies conditions (1) - (3), then there exists a constant $c > 0$ such that $I = cJ$.

Exercise 50. In this exercise, G is assumed to be a finite group. We equip $C(G)$ with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}.$$

Let $\widehat{G}$ denote the collection of equivalence classes of finite dimensional representations of G. Let S denote the sum of the subspaces $C(G)_\delta$ in $C(G)$ with $\delta \in \widehat{G}$.

(a) Show that $\widehat{G}$ is finite.

(b) Let (π, V_π) be any finite dimensional representation of G and let $v_1, v_2 \in V_\pi$. Show that the function $m : x \mapsto \langle \pi(x)v_1, v_2 \rangle$ belongs to S.

(c) Show that $S^\perp$ is invariant for both L and R, the left and the right regular representations of G, respectively.

(d) Show that $S = C(G)$.