Harmonic Analysis

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1 Introduction

For an introduction to the following material we refer to the first lecture given by the author. One may also confer [Sug], [1] which will be the basic reference for the rest of this course.

2 Haar measure

Let X be a locally compact Hausdorff space.

Definition 2.1 *A positive Radon integral on X is a linear functional $I : C_c(X) \to \mathbb{C}$ such that $f \geq 0 \Rightarrow I(f) \geq 0$.*

Remark. By the Riesz representation theorem I is actually the integral associated with a regular Borel measure μ. We shall refer to this measure as the Radon measure associated with I, and we shall use the notation:

$$I(f) = \int_G f(x)\, d\mu(x).$$

Lemma 2.2 *Let I be a positive Radon measure on X. Then:*

(a) $f \leq g \Rightarrow I(f) \leq I(g) \quad (f, g \in C_c(X))$.

(b) $f \in C_c(X) \Rightarrow |I(f)| \leq I(|f|)$;

(c) *The map $f \mapsto I(f)$ is continuous on $C_c(G)$.*

Proof. For (a) use that $g - f \geq 0$ and the definition of I.

(b): First assume that f is real-valued, and define continuous functions by $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Then $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Hence:

$$|I(f)| = |I(f_+) - I(f_-)| \leq I(f_+) + I(f_-) = I(|f|).$$

Now assume that f is complex valued, and that $I(f) \in \mathbb{R}$. Then $I(\mathrm{Im}\, f) = 0$. Hence

$$|I(f)| = |I(\mathrm{Re}\, f)| \leq I(|\mathrm{Re}\, f|) \leq I(|f|),$$

[1][Sug]: M. Sugiura, Unitary Representations and Harmonic Analysis, 2nd ed. North-Holland/Koddansha, Amsterdam 1990.

by what we proved above. Let f now be arbitrary. Then there exists a $z \in \mathbb{C}$ with $|z| = 1$ and $zI(f) \geq 0$. Then $I(zf) \in \mathbb{R}$, hence $|I(f)| = |I(zf)| \leq I(|zf|) = I(|f|)$.

It remains to establish (c). Let $K \subset X$ be a compact subset. Then we must show that there exists a $C > 0$ such that for all $f \in C_c(K)$ we have $|I(f)| \leq C\|f\|_\infty$. There exists a non-negative function $g \in C_c(X)$ such that $g = 1$ on K. Hence for all $f \in C_c(K)$ we have $|f| = |f|g \leq \|f\|_\infty g$, and it follows that

$$|I(f)| \leq I(|f|) \leq \|f\|_\infty I(g),$$

which establishes the desired estimate. Note that by taking the infimum over g one obtains the estimate $|I(f)| \leq \|f\|_\infty \operatorname{volume}(K)$. □

By a G–space we shall from now on mean a locally compact Hausdorff space X together with a continuous action of G on X.

We agree to write L for the induced action of G on $C(X)$. Thus, if $\varphi \in C(X)$, then $L_g\varphi(x) = \varphi(g^{-1}x)$.

The G–space X is called homogeneous if there is only one G–orbit, i.e. if $x_0 \in X$ is a point, then $G \cdot x_0 = X$. One readily verifies that in this setting the stabilizer

$$G_{x_0} = \{g \in G\,; \quad gx_0 = x_0\}$$

is a closed subgroup of G. Moreover, the map $G \to X, g \mapsto gx$ induces a homeomorphism $G/G_x \to X$. Thus any homogeneous G–space is of the form G/H with H a closed subgroup of G. Conversely, if H is a closed subgroup, then G/H is a homogeneous space in a natural way.

Example. The unit sphere S^2 in $\mathbb{R}^3$ is a homogeneous space for the special orthogonal group $\mathrm{SO}(3)$. Let $e_1 = (1, 0, 0)$. Then the stabilizer of e_1 in $\mathrm{SO}(3)$ is the group of rotations with axis $\mathbb{R}e_1$, which is isomorphic to $\mathrm{SO}(2)$. The sphere is thus identified with the quotient $\mathrm{SO}(3)/\mathrm{SO}(2)$. This is the starting point for the spherical harmonics to be discussed later.

Lemma 2.3 *Let X be a G–space and let $\varphi \in C_c(X)$. Then $g \mapsto L_g\varphi$ is a continuous map $G \to C_c(X)$.*

Proof. Let $g_0 \in G$. Then we must show that $L_g\varphi - L_{g_0}\varphi$ is close to 0 in $C_c(X)$ for g close to g_0. Now $L_g\varphi - L_{g_0}\varphi = L_{gg_0^{-1}}\psi - \psi$, where $\psi = L_{g_0}\varphi$, and we see that without loss of generality we may as well assume that $g_0 = e$. Let C be a compact neighbourhood of e in G. Then the set $K = C\operatorname{supp}\varphi$ is compact in X. Moreover, for $g \in C$ the function $L_g\varphi - \varphi$ has support contained in K. Fix $\varepsilon > 0$. Then we must show that there exists an open neighbourhood U of e in C such that

$$||L_g\varphi - \varphi||_K = \sup_{x \in K} |L_g\varphi(x) - \varphi(x)| < \varepsilon \tag{1}$$

for all $x \in U$. The map $(g, x) \mapsto \varphi(g^{-1}x)$ is continuous, hence for every $x \in K$ there exists an open neighbourhood $N_x \ni x$ in X and an open neighbourhood $U_x \ni e$ in $C \subset G$, such that $|\varphi(g^{-1}y) - \varphi(y)| < \varepsilon$ for all $y \in N_x, g \in U_x$. By compactness of K there exists a finite collection $x_1, \ldots, x_n$ of points in K such that the sets N_{x_j} cover K. Let U be the intersection of the sets U_j. Then U is an open neighbourhood of e in G, and one readily checks that (1) holds for $g \in U$. □

Corollary 2.4 *Let X be a G–space, and let I be a positive Radon integral on G. Then for $\varphi \in C_c(X)$ we have that $g \mapsto I(L_g\varphi)$ defines a continuous function $G \to \mathbb{C}$.*

A positive Radon integral I on a G–space X is called invariant if for all $\varphi \in C_c(X)$ and $g \in G$ we have

$$I(L_g\varphi) = I(\varphi),$$

or equivalently, if μ denotes the measure associated with I,

$$\int_X \varphi(g^{-1}x)d\mu(x) = \int_X \varphi(x)d\mu(x).$$

Lemma 2.5 *Let I be a G–invariant positive non-trivial Radon measure on a G–homogeneous space X. Let $f \in C_c(X)$, and suppose that $f \geq 0$ everywhere. Then $I(f) = 0 \Rightarrow f = 0$.*

Proof. Suppose that f is as above, and $I(f) = 0$, but $f \neq 0$. Then there exists a point $x_0 \in X$ such that $f(x_0) \neq 0$. Hence there exists a compact neighbourhood $U \ni x_0$ and a positve constant $\varepsilon > 0$ such that $f \geq \varepsilon$ on U. Let now $g \in C_c(X), g \geq 0$. Let $C = \mathrm{supp} g$ be the compact support of g. Then every point of C is contained in Gx_0 and by compactness there exists a finite subset $S \subset G$, such that the sets sU, $s \in S$ cover C. For every $s \in S$ there exists a $\lambda_s > 0$ such that on sU we have: $g \leq \lambda_s\varepsilon \leq \lambda_s L_s f$. Hence $g \leq \sum_{s\in S} \lambda_s L_s(f)$, and it follows that

$$0 \leq I(g) \leq \sum_{s\in S} \lambda_s I(L_s f) = 0,$$

the latter equality being a consequnece of the invariance of the integral. It follows from this that $I = 0$, contradiction. □

Corollary 2.6 *Let the hypothesis of the above lemma be fulfilled. Then*

$$\langle f|g\rangle = I(f\bar{g}) = \int_X f(x)\overline{g(x)}\, dx$$

defines a positive definite Hermitian inner product on $C_c(X)$.

We denote the completion of $C_c(X)$ with respect to the above inner product by $L^2(X)$. Let L denote the left represenation of G on X. Then by invariance of the measure the map L_x is unitary for the inner product on $C_c(x)$ and therefore has a unique extension to a unitary map $L_x : L^2(X) \to L^2(X)$.

The group G may be viewed as a homogeneous space for its left (right) action. A left–(right–) invariant positive non-trivial Radon integral on X is called a left (right) Haar integral on G.

Theorem 2.7 *Let G be a locally compact group. Then there exists a left (right) Haar integral I on G. If I' is a second left (right) Haar integral on G, then $I' = cI$ for a positive constant $c > 0$.*

Proof. For the existence part of this theorem we refer the reader to [BI 7].[2] We will prove uniqueness. If $f \in C_c(X)$ then we define the function $f^\vee \in C_c(X)$ by $f^\vee(x) = f(x^{-1})$. If J is a

[2] [BI 7]: Bourbaki, Intégration, Chapter 7

right Haar integral then we define the left Haar integral $J^\vee$ by $J^\vee(f) = J(f^\vee)$. Thus we have a bijection between left and right Haar integrals.

Let now I and J be a left and a right Haar integral respectively. Then it suffices to show that $J^\vee = cI$ for some $c > 0$. Let μ and ν denote the measures associated with I and J respectively, and let $f \in C_c(G)$ be such that $I(f) \neq 0$. Define

$$D_f : x \mapsto I(f)^{-1} \int_G f(y^{-1}x)\, d\nu(y). \tag{2}$$

Then D_f is a continuous function on G, by Lemma 2.3.

Let $g \in C_c(G)$. Then

$$J(g) = \int_G g(xy) d\nu(x)$$

for every $y \in G$. Multiplying the above identity with $f(y)$ and integrating with respect to $d\mu(y)$ we obtain:

$$I(f)J(g) = \int_G \int_G f(y)g(xy)\, d\nu(x)d\mu(y) = \int_G \int_G f(y)g(xy)\, d\mu(y)d\nu(x)$$

where the change of order of integration is allowed by Fubini's therorem. Note that the integrand is compactly supported as a function on $G \times G$. Using left invariance of μ, and after that again changing the order of integration we may rewrite the integral as:

$$\int_G \left[\int_G f(x^{-1}y) d\nu(x) \right] g(y) d\mu(y) = \mu(\mu(f) D_f g).$$

Hence $J(g) = I(D_f g)$ for every $g \in C_c(G.)$ Let $f_1, f_2 \in C_c(G)$ be two functions with $I(f_j) \neq 0$, then it follows from the above that $I((D_{F_1} - D_{f_2})g) = 0$ for all g, hence $D_{f_1} = D_{f_2}$. Thus it follows that $D = D_f$ is a continuous function which is actually independent of the particular choice of f. Substituting $x = e$ in the definition (2) of D_f we now infer that

$$D(e)I(f) = J^\vee(f) \tag{3}$$

for all f in the set S of functions $\varphi \in C_c(G)$ with $I(\varphi) \neq 0$. The set S is readily seen to be dense in $C_c(G)$, so that (3) is actually valid for all $f \in C_c(G)$. Hence $J^\vee = D(e)I$. □

Remark 2.8 The existence (and the uniqueness) of Haar measure is much easier to establish when G is a Lie group, i.e. G possesses the structure of C^∞-manifold, compatible with its topology, and such that the map $G \times G \to G$, $(x, y) \mapsto xy^{-1}$ is smooth (i.e. C^∞). Examples of Lie groups are: $\mathrm{SU}(n), \mathrm{SO}(n), \mathrm{SL}(n, \mathbb{C}), \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R})$. For simplicity we assume that G is connected. Let $n = \dim G$, and fix a non-trivial n-form $\omega_e \in \wedge^n T_e^* G$. Here $T_x^* G$ denotes the cotangent-space of G at an element $x \in G$. Let $l_x : G \to G, g \mapsto xg$, and define, for $x \in G$, the n-form $\omega_x \in \wedge^n T_x^* G$ by

$$\omega_x = (T_e l_x)^{-1^*} \omega_e.$$

Then $\omega : x \mapsto \omega_x$ is a nowhere vanishing smooth differential n–form on G, hence defines an orientation on G. We define $I : C_c(G) \to \mathbb{C}$ by the oriented integral:

$$I(f) = \int_G f\, \omega \qquad (f \in C_c(G)).$$

One readily verifies that I is a positive Radon integral which is non-trivial. Moreover, one also readily checks that $(l_g)^*\omega = \omega$ for all $g \in G$ from which it follows that I is a left Haar integral. For further details we refer to [BD] [3] or [Wa]. [4]

Let I be a left Haar integral on G. Then for every $x \in G$ we define the Radon integral $R_x I$ by $R_x I(\varphi) = I(R_x^{-1}\varphi)$. Then one readily verfies that $R_x I$ is a left Haar integral (use that R_x and L_y commute). Hence $R_x I = \Delta(x) I$ for a uniquely determined $\Delta(x) > 0$. The function $\Delta : G \to]0, \infty[$ which is readily seen to be independent of tyhe particular choice of I, is called: the modular function of G.

Exercise 2.9 Show that Δ is a continuous group homomorphism.

A group G with $\Delta \equiv 1$ is called unimodular. Thus, a group is unimodular if its left Haar measure is a right Haar measure. For obvious reasons the Haar measure is said to be bi-invariant in this case.

Exercise 2.10 Show that a compact group G is unimodular. Hint: use that the image $\Delta(G)$ is a compact group.

3 Representations

From now on G will always be a locally compact topological group.

Exercise 3.1 Let π be a representation of G in a Banach space V. Show that the following conditions are equivalent:

(a) $\pi : G \times V \to V$ is continuous.

(b) For every $x \in G$ the map $\pi(x)$ is coninuous, and for every $v \in V$ the map $G \to V, x \mapsto \pi(x)v$ is continuous.

Hint: to show that (a) follows from (b), use the Banach-Steinhaus theorem.

Lemma 3.2 *Let X be a G–homogeneous space, and let I be a non-trivial invariant positive Radon integral on X. Then the representation L of G in $L^2(X)$ defined in the previous section is continuous.*

Proof. In view of the above exercise it suffices to show that for every $\varphi \in L^2(X)$ the map $\Phi : x \mapsto L_x\varphi, G \to L^2(X)$ is continuous. Using that L is a representation, we readily see that it suffices to establish the continuity of Φ at e. Thus we must estimate the L^2-norm of the function $L_x\varphi - \varphi$ as $x \to e$. Let $\varepsilon > 0$. Then there exists a $\psi \in C_c(X)$ such that $\|\varphi - \psi\| < \frac{1}{3}\varepsilon$. Let

[3] [BD]: T. Bröcker, T. tom Dieck, Representations of Compact Lie Groups, Graduate Texts in Math. **98**, Springer-Verlag 1985.

[4] [Wa]: F. Warner, Foundations of differentiable manifolds and Lie groups, Scott, Foresman and Co., Glenview Illinois 1971.

$g \in C_c(G)$ be a non-negative function such that $g = 1$ on an open neigbourhood of $\mathrm{supp}\psi$. Then for x sufficiently close to e we have $g = 1$ on $\mathrm{supp}L_x\psi$. Thus for such x we have:

$$\begin{aligned} \|L_x\varphi - \varphi\| &\leq \frac{2}{3}\varepsilon + \|L_x\psi - \psi\|_2 \\ &= \frac{2}{3}\varepsilon + \|(L_x\psi - \psi)g\|_2 \\ &\leq \frac{2}{3}\varepsilon + \|L_x\psi - \psi\|_\infty \|g\|_2. \end{aligned}$$

Using Lemma 2.3 one sees that the last term becomes smaller than $\frac{\varepsilon}{3}$ as $x \to e$. □

Let π be a representation of G in a (complex) linear space V. By an *invariant subspace* we mean a linear subspace $W \subset V$ such that $\pi(x)W \subset W$ for every $x \in G$.

A continuous representation π of G in a Banach space V is called irreducible, if 0 and V are the only closed invariant subspaces of V.

Remark. Note that for a finite dimensional representation an invariant subspace is automatically closed.

By a unitary representation of G we will always mean a continuous representation π of G in a Hilbert space $\mathcal{H}$, such that $\pi(x)$ is unitary for every $x \in G$.

Proposition 3.3 *Let G be compact, and suppose that (π, V) is a continuous finite dimensional representation of G. Then there exists a positive definite Hermitian inner product $\langle\cdot|\cdot\rangle$ on V for which the representation π is unitary.*

Proof. Let dx denote Haar measure on G, and fix any positive definite Hermitian inner product $\langle\cdot|\cdot\rangle_1$ on V. Then we define a new Hermitian pairing on V by

$$\langle v|w\rangle = \int_G \langle \pi(x)v|\pi(x)w\rangle_1 \, dx \qquad (v, w \in V).$$

Notice that the integrand $\iota_{v,w}(x) = \langle\pi(x)v|\pi(x)w\rangle_1$ in the above equation is a continuous function of x. We claim that the pairing thus defined is positive definite. Indeed, if $v \in V$ then the function $\iota_{v,v}$ is continuous and positive on G. Hence $\langle v|v\rangle = \int_G \iota_{v,v}(x)\,dx \geq 0$ by positivity of the measure. Also, if $\langle v|v\rangle = 0$, then $\iota_{v,v} \equiv 0$ by Lemma 2.2, and hence $\langle v|v\rangle = \iota_{v,v}(e) = 0$, and positive definiteness follows.

Finally we claim that π is unitary for the inner product thus defined. Indeed this follows from the invariance of the measure: If $y \in G$, and $v, w \in V$, then

$$\langle\pi(y)v|\pi(y)w\rangle = \int_G \iota_{v,w}(xy)dx = \int_G \iota_{v,w}(x)dx = \langle v|w\rangle.$$

□

Lemma 3.4 *Let $(\pi, \mathcal{H})$ be a continuous representation of G. If $\mathcal{H}_1$ is an invariant subspace for π, then its orthocomplement $\mathcal{H}_2 = \mathcal{H}_1^\perp$ is a closed invariant subspace for π. If $\mathcal{H}_1$ is closed, then we have the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of closed invariant subspaces.*

Proof. Left to the reader □

Corollary 3.5 *Assume that G is compact, and let (π, V) be a continuous finite dimensional representation of G. Then π decomposes as a finite direct sum of irreducibles, i.e. there exists a direct sum decomposition $V = \oplus_{1 \le j \le n} V_j$ of V into invariant subspaces such that for every j the representation π_j defined by $\pi_j(x) = \pi(x)|V_j$ is irreducible.*

Proof. Fix an inner product for which π is unitary, and apply the above lemma repeatedly. □

Definition 3.6 *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then by a matrix coefficient of π we mean any function $m : G \to \mathbb{C}$ of the form*

$$m(x) = m_{v,w}(x) := \langle \pi(x)v|w\rangle$$

with $v, w \in \mathcal{H}$.

Notice that the matrix coefficent m definied above is a continuous function on G.

Let now (π, V) be a finite dimensional unityary representation of G, and fix an orthonormal basis $u_1, \ldots, u_n$ of V. Then for every $x \in G$ we define the matrix $M(x) = M_{\underline{u}}(x)$ by

$$M(x)_{ij} = m_{u_i,u_j}(x).$$

This is just the matrix of $\pi(x)$ with respect to the basis $\underline{u}$. Note that it is unitary. Note also that $M(xy) = M(x)M(y)$. Thus M is a continuous group homomorphism from G to the group $U(n)$ of unitary $n \times n$ matrices.

If (π_j, V_j) $(j = 1, 2)$ are continuous representations of G in topological linear spaces, then a continuous linear map $T : V_1 \to V_2$ is said to be equivariant, or called interwining if the following diagram commutes for all $x \in G$:

$$\begin{array}{ccccc} & V_1 & \xrightarrow{T} & V_2 & \\ \pi_1(x) & \uparrow & & \uparrow & \pi_2(x) \\ & V_1 & \xrightarrow{T} & V_2 & \end{array}$$

The representations π_1 and π_2 are said to be equivalent if there exists a topological linear isomorphism T from V_1 onto V_2 which is equivariant.

Exercise 3.7 Let (π_j, V_j) be two finite dimensional representations of G. Show that π_1 and π_2 are equivalent if and only if there exists choices of bases for V_1 and V_2, such that for the assocxiated matrices one has:

$$\mathrm{mat}\, \pi_1(x) = \mathrm{mat}\, \pi_2(x).$$

We will now discuss an important example of representations. Let SU(2) denote the group of 2 by 2 unitary matrices with determinant one. Thus SU(2) is the group of matrices of the form

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \alpha \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Then SU(2) acts on $\mathbb{C}^2$ in a natural way, and we have the associated representation π on the space $P(\mathbb{C})$ of polynomial functions $p : \mathbb{C}^2 \to \mathbb{C}$. It is given by the formula

$$\pi(g)p(z) = p(g^{-1}z) = p(\alpha z_1 + \bar{\beta} z_2, -\beta z_1 + \bar{\alpha} z_2)$$

The subspace $P_n = P_n(\mathbb{C}^2)$ of homogeneous polynomials of degree n is an invariant subspace for π. We write π_n for the restriction of π to P_n.

We will now discuss a result that will allow us to show that the representations π_n of the above example are irreducible. We first need the following (fundamental) lemma from linear algebra. If V is a complex linear space, we write $\mathrm{End}(V)$ for the space of linear maps from V to itself, and $\mathrm{GL}(V)$ for the group of invertible elements in $\mathrm{End}(V)$. If π is a representation of G in V, then we may define a representation $\tilde{\pi}$ of G in $\mathrm{End}(V)$ by

$$\tilde{\pi}(g)A = \pi(g)A\pi(g)^{-1}.$$

Note that if π is finite dimensional and continuous, then so is $\tilde{\pi}$. Note also that the space

$$\mathrm{End}(V)^G = \{A \in V \; ; \quad \tilde{\pi}(g)A = A\}$$

of G-invariants in V is just the space of G–equivariant linear maps $V \to V$.

Lemma 3.8 *Let V be a finite dimensional complex linear space, and let $A, B \in \mathrm{End}(V)$ be such that $AB = BA$. Then A leaves $\ker B, \mathrm{im} B$ and all the eigenspaces of B invariant.*

Proof. Elementary, and left to the reader. □

From now on we assume that G is a locally compact group. All representations are assumed to be continuous.

Lemma 3.9 (Schur's lemma) *Let (π, V) be a representation of G in a finite dimensional complex linear space V. Then the following holds.*

(a) *If π is irreducible then* $\mathrm{End}(V)^G = \mathbb{C}\mathrm{I}_V$.

(b) *If G is compact and* $\mathrm{End}(V)^G = \mathbb{C}\mathrm{I}_V$, *then π is irreducible.*

Proof. "(a)" Suppose that π is irreducible, and let $A \in \mathrm{End}(V)^G$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A, and let $E_\lambda = \ker(A - \lambda \mathrm{I})$ be the associated eigenspace. Note that for the existence of this eigenspace we need that V is complex. For every $x \in G$ we have that $\pi(x)$ commutes with A, hence leaves E_λ invariant. In view of the irreducibility of π it now follows that $E_\lambda = V$, hence $A = \lambda \mathrm{I}$.

"(b)" By compactness of G there exists a Hermitean inner product $\langle \cdot | \cdot \rangle$ for which π is unitary.

Let $0 \neq W \subset V$ be a G–invariant subspace. For the proof that π is irreducible it suffices to show that we must have $W = V$. Let P be the orthogonal projection $V \to W$. Since W and $W^\perp$ are both G–invariant, we have, for $g \in G$, that $\pi(g)P = \pi(g) = P\pi(g)$ on W, and $\pi(g)P = 0 = P\pi(g)$ on $W^\perp$. Hence $P \in \mathrm{End}(V)^G$, and it follows that $P = \lambda \mathrm{I}$ for some $\lambda \in \mathbb{C}$. Now $P \neq 0$, hence $\lambda \neq 0$. Also, $P^2 = P$, hence $\lambda^2 = \lambda$, and we see that $\lambda = 1$. Therefore $P = I$, and $W = V$. □

We will now apply the above lemma to prove the following.

Proposition 3.10 *The representations $(\pi_n, P_n(\mathbb{C}^2))$ $(n \geq 0)$ of* $\mathrm{SU}(2)$ *are irreducible.*

For the proof we will need compactness of $\mathrm{SU}(2)$. In fact we have the following more general result.

Exercise 3.11 For $n \geq 1$, let $\mathrm{M}_n(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{C})$ denote the linear spaces of $n \times n$ matrices with entries in $\mathbb{R}$ and $\mathbb{C}$ respectively. Show that $\mathrm{SU}(n)$ is a closed and bounded subset of $\mathrm{M}_n(\mathbb{C})$. Show that $\mathrm{SO}(n) = \mathrm{SU}(n) \cap \mathrm{M}_n(\mathbb{R})$, and finally show that $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ are compact topological groups.

Proof of Prop. 3.10 Let $n \geq 0$ be fixed, and put $\pi = \pi_n$ and $V = P_n(\mathbb{C}^2)$. Suppose that $A \in \mathrm{End}(V)$ is equivariant. Then in view of Lemma 3.9 (b) it suffices to show that A is a scalar.

For $0 \leq k \leq n$ we define the polynomial $p_k \in V$ by

$$p_k(z) = z_1^{n-k} z_2^k.$$

Then $\{p_k\,;\quad 0 \leq k \leq n\}$ is a basis for V. For $\varphi \in \mathbb{R}$ we put

$$t_\varphi = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \qquad r_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

Then

$$T = \{t_\varphi\,;\quad \varphi \in \mathbb{R}\} \quad \text{and} \quad R = \{r_\varphi\,;\quad \varphi \in \mathbb{R}\}$$

are (closed) subgroups of $\mathrm{SU}(2)$. One readily verifies that for $0 \leq k \leq n$ and $\varphi \in \mathbb{R}$ we have:

$$\pi(t_\varphi)p_k = e^{i(2k-n)\varphi} p_k.$$

Thus every p_k is a joint eigenvector for T. Fix a φ such that the numbers $e^{i(2k-n)\varphi}$ are mutually different. Then for every $0 \leq k \leq n$ the space $\mathbb{C}p_k$ is eigenspace for $\pi(t_\varphi)$ with eigenvalue $e^{i(2k-n)\varphi}$. Since A and $\pi(t_\varphi)$ commute it follows that A leaves all the spaces $\mathbb{C}p_k$ invariant. Hence there exist $\lambda_k \in \mathbb{C}$ such that

$$Ap_k = \lambda_k p_k, \quad 0 \leq k \leq n.$$

Let E_1 be the eigenspace of A with eigenvalue λ_1. We will show that $E_1 = V$, thereby completing the proof. The space E_1 is $\mathrm{SU}(2)$–invariant, and contains p_1. Hence it contains $\pi(r_\varphi)p_1$ for every $\varphi \in \mathbb{R}$. By a straightforward computation one sees that

$$\pi(r_\varphi)p_1(z_1, z_2) = (\cos\varphi\, z_1 + \sin\varphi\, z_2)^n = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\varphi\, \sin^k\varphi\; p_k.$$

From this one sees that the $\pi(r_\varphi)p_1$ (which are contained in E_1) span V. Hence $E_1 = V$. □

We end this section with two useful consequences of Schur's lemma.

Lemma 3.12 *Let (π, V), $\pi', V')$ be two irreducible finite dimensional representations of the locally compact group G. If π and π' are not equivalent, then every intertwining linear map $T : V \to V'$ is trivial.*

Proof. Let T be intertwining, and non-trivial. Then $\ker T \subset V$ is a proper G–invariant subspace. Hence $\ker T = 0$, and it follows that T is injective. Therefore its image $\mathrm{im} T$ is a non-trivial G–invariant subspace of V'. It follows that $\mathrm{im} T = V'$, hence T is a bijection, contradicting the inequivalence. □

Let V be a complex linear space. Then by a sesquilinear form on V we mean a map $\beta : V \times V \to \mathbb{C}$ which is linear in the first variable, additive in the second, and which satisfies $\beta(v, \lambda w) = \bar{\lambda}\beta(v, w)$, $v, w \in V$, $\lambda \in \mathbb{C}$. Thus a Hermitean inner product is a sesquilinear form. If (π, V) is a representation for a group G, then a sesquilinear form β on V is called equivariant if $\beta(\pi(g)v, \pi(g)w) = \beta(v, w)$ for all $v, w \in V$, $g \in G$.

Lemma 3.13 *Let (π, V) be a finite dimensional unitary representation of a locally compact group G. Then the equivariant sesquilinear forms on V are precisely the maps $\beta : V \times V \to \mathbb{C}$ of the form $\beta = \lambda\langle\cdot|\cdot\rangle$, $\lambda \in \mathbb{C}$. Here $\langle\cdot|\cdot\rangle$ denotes the (equivariant) inner product of the Hilbert space V.*

Proof. Let $\beta : V \times V \to \mathbb{C}$ be sesquilinear. Then for every $w \in V$ the map $v \mapsto \beta(v, w)$ is a linear functional on V. Hence there exists a unique $A(w) \in V$ such that $\beta(v, w) = \langle v, A(w)|.\rangle$ One readily verifies that $A : V \to V$ is a linear map. Moreover, the equivariance of β and $\langle\cdot|\cdot\rangle$ implies that A is equivariant. Hence $A = \lambda \mathrm{I}$ for some $\lambda \in \mathbb{C}$, and the result follows. $\square$

4 Schur orthogonality

In this section G will be a compact topological group, unless stated otherwise. Let I be a left Haar integral on G. Then I is determined up to multiplication by a positive constant. Thus I is uniquely determined by the additional requirement that $I(1) = 1$. We denote the measure associated with I by dx. Then $\int_G dx = 1$.

If π is a finite dimensional irreducible unitary representation of G we write $C(G)_\pi$ for the linear span of the space of matrix coefficients of π. Notice that the space $C(G)_\pi$ does not depend on the chosen (unitary) inner product on V_π. Thus by Proposition 3.3 we can define $C(G)_\pi$ for any irreducible finite dimensional representation π of G.

There is a nice way to express sums of matrix coefficients of a finite dimensional unitary representation (π, V) of a locally compact group G. Let $v, w \in V$. Then we shall write $L_{v,w}$ for the linear map $V \to V$ given by

$$L_{v,w}(u) = \langle u|w\rangle v.$$

One readily sees that

$$\mathrm{tr}\,(L_{v,w}) = \langle v|w\rangle, \qquad v, w \in V. \tag{4}$$

Indeed both sides of the above equation are sesquilinear forms in (v, w), so it suffices to check the equation for v, w members of an orthonormal basis, which is easily done.

It follows from the above equation that

$$m_{v,w}(x) = \mathrm{tr}\,(\pi(x)L_{v,w}).$$

Hence every sum m of matrix coefficients is of the form $m(g) = \mathrm{tr}\,(\pi(g)A)$, with $A \in \mathrm{End}(V)$. Conversely if $\{e_k\ ;\quad 1 \leq k \leq n\}$ is an orthonormal basis for V, then one readily sees that

$$A = \sum_{1 \leq i,j \leq n} \langle Ae_j|e_i\rangle L_{e_i,e_j}.$$

Using this one may express every function of the form $g \mapsto \mathrm{tr}\,(\pi(g)A)$ as a sum of matrix coefficients. We now define the linear map $T_\pi : \mathrm{End}(V) \to C(G)$ by

$$T_\pi(A)(x) = \mathrm{tr}\,(\pi(x)A), \qquad x \in G,$$

for every $A \in \mathrm{End}(V)$. Let π be irreducible, then it follows from the above discussion that T_π maps V onto $C(G)_\pi$. Define the representation $\pi \otimes \pi^*$ of $G \times G$ on $\mathrm{End}(V)$ by

$$(\pi \otimes \pi^*)(x, y)A = \pi(x)A\pi(y)^{-1},$$

for $A \in \mathrm{End}(V), x, y \in V$.

We define the representation $R \times L$ of $G \times G$ on $C(G)$ by

$$(R \times L)(x, y) = L_y \circ R_x.$$

Then we have the following:

Lemma 4.1 *Let (π, V) be a finite dimensional irreducible representation of G. Then $C(G)_\pi$ is invariant under $R \times L$, and the map $T_\pi : V \to C(G)_\pi$ is surjective, and intertwines the representations $\pi \otimes \pi^*$ and $R \times L$ of $G \times G$.*

Proof. We first prove the equivariance of $T_\pi : \mathrm{End}(V) \to C(G)$. Let $A \in \mathrm{End}(V)$, and $x, y \in G$, then

$$T_\pi(\pi \otimes \pi^*(x, y)A)(g) = \mathrm{tr}\,(\pi(g)\pi(x)A\pi(y^{-1})) = \mathrm{tr}\,(\pi(y^{-1}gx)A) = (R_x L_y T_\pi(A))(x).$$

Note that it follows from this equivariance that the image of T_π is $R \times L$–invariant. In a previous dicussion we saw already that $\mathrm{im}(T_\pi) = C(G)_\pi$. $\square$

Corollary 4.2 *If π and π' are equivalent finite dimensional irreducible representations of G, then $C(G)_\pi = C(G)_{\pi'}$.*

Proof. Let V, V' be the associated representation spaces. Then by equivalence there exists a linear isomorphism $T : V \to V'$ such that $T \circ \pi(x) = \pi'(x) \circ T$ for all $x \in G$. Hence for $A \in \mathrm{End}(V)$, $x \in G$ we have:

$$T_{\pi'}(TAT^{-1})(x) = \mathrm{tr}\,(\pi'(x)TAT^{-1}) = \mathrm{tr}\,(T^{-1}\pi'(x)TA) = \mathrm{tr}\,(\pi(x)A) = T_\pi(A)(x).$$

Now use Lemma 4.1. $\square$

We now have the following.

Theorem 4.3 (Schur orthogonality). *Let (π, V) and (π', V') be two irreducible finite dimensional representations of G. Then we have the following.*

(a) *If π and π' are not equivalent, then $C(G)_\pi \perp C(G)_{\pi'}$ (with respect to the Hilbert structure of $L^2(G)$).*

(b) *Let V be equipped with an inner product for which π is unitary. If $v, w, v', w' \in V$, then the L^2-inner product of the matrix coefficients $m_{v,w}$ and $m_{v',w'}$ is given by:*

$$\int_G m_{v,w}(x)\, \overline{m_{v',w'}(x)}\, dx = \dim(\pi)\, \langle v|v'\rangle \overline{\langle w|w'\rangle} \tag{5}$$

Remark. The relations (5) are known as the Schur orthogonality relations.

Proof. For $w \in V$ and $w' \in V'$ we define the linear map $L_{w',w} : V \to V'$ by $L_{w',w}u = \langle u|w\rangle w'$. Consider the map:

$$I_{w',w} = \int_G \pi'(x)^{-1} \circ L_{w',w} \circ \pi(x)\, dx.$$

Then one readily verifies that

$$\langle I_{w',w}v|v'\rangle = \langle m_{v,w}|m_{v',w'}\rangle_2. \tag{6}$$

Moreover, using invariance of the measure one readily sees that $I_{w',w}$ is intertwining.

(a): If π and π' are inequivalent then the interwtwining map $I_{w',w}$ is trivial by Lemma 3.13. Now apply (6) to prove (a).

(b): Now assume $V = V'$. Then for all $w, w' \in V$ we have $I_{w',w} \in \mathrm{End}(V)^G$, hence $I_{w',w}$ is a scalar. It follows that there exists a sesquilinear form β on V such that

$$I_{w',w} = \beta(w', w)\mathrm{I}.$$

Applying the trace to both sides of the above equation we find that $d_\pi\beta(w', w) = \mathrm{tr}\,(I_{w',w})$. Here we have abbreviated $d_\pi = \dim(\pi)$. On the other hand, since tr is linear, we have that

$$\mathrm{tr}\,(I_{w',w}) = \int_G \mathrm{tr}\,(\pi(x)^{-1}L_{w',w}\pi(x))\, dx = \int_G \mathrm{tr}\,(L_{w',w})\, dx = \mathrm{tr}\,(L_{w',w}) = \langle w'|w\rangle.$$

Hence

$$I_{w',w} = d_\pi^{-1}\langle w'|w\rangle.$$

Now apply (6) to prove (b). □

Another way to formulate the orthogonality relations is the following (V is assumed to be equipped with an inner product for which π is unitary). If $A \in \mathrm{End}(V)$, let A^* denote the Hermitean adjoint of A. Then one readily verifies that

$$(A, B) \mapsto \langle A|B\rangle := \mathrm{tr}\, B^*A$$

defines a Hermitean inner product on End$V$. Moreover, the representation $\pi \otimes \pi^*$ is readily seen to be unitary for this inner product.

Corollary 4.4 *The map* $\sqrt{d}_\pi T_\pi$ *is a unitary isomorphism from* $\mathrm{End}(V)$ *onto* $C(G)_\pi$.

Notice that it follows from the above result that $C(G)_\pi$ has a unique conjugation invariant function χ_π whose L^2–norm is one. In fact one has

$$\chi_\pi(x) = T_\pi(\mathrm{I}_V)(x) = \mathrm{tr}\,\pi(x).$$

The function χ_π is called the character of π.

5 Characters

For the moment assume that G is a locally compact group. If π is any finite dimensional representation of G, we define its character $\chi_\pi \in C(G)$ by $\chi_\pi(x) = \operatorname{tr} \pi(x)$.

Exercise 5.1 Let (π, V) be a finite dimensional representation of G. Show that:

(a) $\chi_\pi(xyx^{-1}) = \chi_\pi(y)$;

(b) $\overline{\chi_\pi(x)} = \chi_\pi(x^{-1})$.

For $x \in G$, define $\pi^*(x) \in \operatorname{End}(V^*)$ by $\pi^*(x) = \pi(x^{-1})^t$.

(c) Show that π^* is a (continuous) representation of G in V^*. (This representation is called the contragredient of π.)

(d) Show that $\chi_{\pi^*} = \overline{\chi_\pi}$.

Exercise 5.2 Let π, ρ be finite dimensional representations of G. Show that:

$$\pi \sim \rho \Rightarrow \chi_\pi = \chi_\rho.$$

If (π_1, V_1) and (π_2, V_2) are two continuous representations of G, then we define the direct sum representation $\pi = \pi_1 \oplus \pi_2$ in the direct sum $V = V_1 \oplus V_2$ of topological linear spaces by

$$\pi(x)(v_1, v_2) = (\pi_1(x)v_1, \pi_2(x)v_2) \qquad (v_1 \in V_1,\ v_2 \in V_2,\ x \in G).$$

Exercise 5.3 Let π_1, π_2 be finite dimensional representations of G. Show that $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$.

Exercise 5.4 Recall the definition of the irreducible representations π_n, $n \in \mathbb{N}$ of $\mathrm{SU}(2)$. Show that the character χ_n of π_n is completely determined by its restriction to $T = \{t_\varphi\ ; \quad \varphi \in \mathbb{R}\}$. Hint: use that every matrix in $\mathrm{SU}(2)$ is conjugate to a matrix of T.

Show that:

$$\chi_n(t_\varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi}.$$

From now on we assume that G is a compact group.

Lemma 5.5 *Let π, ρ be finite dimensional irreducible representations of G.*

(a) *If $\pi \sim \rho$ then $\langle \chi_\pi | \chi_\rho \rangle = 1$.*

(b) *If $\pi \not\sim \rho$ then $\langle \chi_\pi | \chi_\rho \rangle = 0$.*

Proof. This follows easily from Theorem 4.3. □

Let π be a finite dimensional representation of the compact group G. Then π is unitarizable, and therefore is equivalent to a direct sum $\oplus_{i=1}^n \pi_i$ of irreducible representations. It follows that $\chi_\pi = \sum_{i=1}^n \chi_{\pi_i}$. Using the lemma above we see that for every irreducible representation δ of G we have

$$\#\{i\,; \quad \pi_i \sim \delta\} = \langle \chi_\pi | \chi_\delta \rangle. \tag{7}$$

In particular this number is independent of the particular decomposition of π into irreducibles. For obvious reasons the number (7) is called the multiplicity of δ in π. We shall also denote it by $m(\delta, \pi)$.

Let $\widehat{G}$ denote the set of equivalence classes of finite dimensional irreducible representations of G. Then by abuse of language we shall write $\delta \in \widehat{G}$ to indicate that δ is a representative for an element of $\widehat{G}$. (A better notation would perhaps be $[\delta] \in \widehat{G}$.) If $\delta \in \widehat{G}$ and $m \in \mathbb{N}$, then we write $m\delta$ for (the equivalence class of) the direct sum of m copies of δ.

We have proved the folllowing lemma.

Lemma 5.6 *Let π be a finite dimensional representation of the compact group G. Then*

$$\pi \sim \bigoplus_{\delta \in \widehat{G}} m(\delta, \pi)\delta.$$

Moreover, any decomposition of π into irreducibles is equivalent to the above one.

Exercise 5.7 This exercise is meant to illustrate that a decomposition of a representation into irreducibles is not unique. Let π_1, π_2 be irreducible representations in V_1, V_2 respectively. Assume that π_1, π_2 are equivalent, and let $T : V_1 \to V_2$ be an intertwining isomorphism.

Equip $V = V_1 \oplus V_2$ with the direct sum representation π, and show that $W_1 = \{(v, Tv)\,; \quad v \in V_1\}$ is an invariant subspace of V. Show that the restriction of π to W_1 is irreducible, and equivalent to π_1. Find a complementary invariant subspace W_2 and show that the restriction of π to this space is also equivalent to π_1.

The following result expresses that the character is a powerful invariant.

Corollary 5.8 *Let π, ρ be two finite dimensional representations of G. Then*

$$\pi \sim \rho \iff \chi_\pi = \chi_\rho.$$

Proof. The 'only if' part is obvious. To prove the 'if' part, assume that $\chi_\pi = \chi_\rho$. Then for every $\delta \in \widehat{G}$ we have $m(\delta, \pi) = \langle \chi_\pi | \delta \rangle = \langle \chi_\rho | \delta \rangle = m(\delta, \rho)$. Now use the above lemma. $\square$

6 Integration with values in a locally convex space

Let V be a complex linear space. Then by a seminorm on V we mean a map $p : V \to [0, \mathbb{R}[$ satisfying

(a) $p(v + w) \leq p(v) + p(w) \qquad (v, w \in V)$;

(b) $p(\lambda v) = |\lambda| p(v) \qquad (v \in V, \lambda \in \mathbb{C})$.

If p is a seminorm on V, $v \in V$ and $\varepsilon > 0$, we write $B(v, p, \varepsilon) = \{w \in V\ ;\quad p(v - w) < \varepsilon\}$.

A set $\mathcal{P}$ of seminorms on V will be called fundamental if for every $p_1, p_2 \in \mathcal{P}$ there exists a $p \in \mathcal{P}$ such that $p_1, p_2 \leq p$, and moreover, if $p(v) = 0\ (\forall\, p \in \mathcal{P}) \Rightarrow v = 0$. To a set $\mathcal{P}$ of seminorms one may associate the set $\mathcal{F}_{\mathcal{P}} = \{B(0, p, \varepsilon)\ ;\quad p \in \mathcal{P},\ \varepsilon > 0\}$. If $\mathcal{P}$ is fundamental, then there exists a unique structure of topological linear space on V such that the set $\mathcal{F}_{\mathcal{P}}$ is a fundamental system of neighbourhoods, i.e. for every open neighbourhood $\Omega \ni 0$ in V there exist $p \in \mathcal{P}$ and $\varepsilon > 0$ such that $B(0, p, \varepsilon) \subset \Omega$. Note that $\{0\}$ is closed for this topology, so the topology is Hausdorff. By a locally convex space we will always mean a (complex) topological linear space whose topology is given by a fundamental system of seminorms in the fashion described above. The name convex refers to the fact that the fundamental neigbourhoods $B(0, p, \varepsilon)$ are convex in V viewed as a real vector space. One may also use convexity properties of neighbourhoods to characterize locally convex spaces[5].

Note that a normed linear space $(V, \|\cdot\|)$ is locally convex, because $\mathcal{P} = \{\|\cdot\|\}$ is fundamental.

We recall that a filter in a set S is any collection $\mathcal{F}$ of subsets of S satisfying the following conditions

(a) $A, B \in \mathcal{F} \Rightarrow A \cap B \subset \mathcal{F}$;

(b) $A \in \mathcal{F}$ *and* $A \subset B \subset S \Rightarrow B \in \mathcal{F}$.;

(c) $\emptyset \notin \mathcal{F}$.

A filter $\mathcal{F}$ in a topological linear space V is said to be convergent with limit $v \in V$ if for every neighbourhood $\Omega \ni v$ there exists a $A \in \mathcal{F}$ such that $A \subset \Omega$. If V is Hausdorff then every filter in V has at most one limit (because of conditions (b) and (c)). The notion of convergence of a filter extends the notion of convergence of a sequence. Indeed, let v_k be a sequence. To this sequence one may associate the filter $\mathcal{G}$ consisting of all of sets $A \subset V$ such that $\{v_k\ ;\quad k \in \mathbb{N}\} \setminus A$ is finite. Then $\{v_k\}$ is convergent with limit v if and only if $\mathcal{G}$ is convergent with limit v.

A filter $\mathcal{F}$ in a topological linear space V is called a Cauchy filter if for every neighbourhood Ω of 0 in V there exists a $A \in \mathcal{F}$ such that $A + (-A) \subset \Omega$. This notion generalizes the notion of Cauchy sequence in the same fashion as above.

A topological linear space is said to be complete if every Cauchy filter is convergent. This definition obviously extends the old definition of completeness for a normed space $(V, \|\cdot\|)$.

In the following we shall need the following property of complete topological linear Hausdorff spaces.

Lemma 6.1 *Let V_0 be a dense linear subspace of a topological linear space V (equipped with the restriction toplogy). Moreover, let A be a continuous linear map of V_0 to a complete topological linear Hausdorff space. Then A has a unique continuous linear extension to a map $V \to W$.*

Proof. Uniqueness is obvious. To establish existence, let $v \in V$, let $\mathcal{N}$ be the filter of all neighbourhoods of v in V, and let $\mathcal{F}_0$ be the filter of sets $B \cap V_0$, $B \in \mathcal{F}$. Let $\mathcal{G}$ be the filter of all sets $S \subset W$ such that $A^{-1}S \in \mathcal{F}$. Then from the continuity and the linearity of A it follows that $\mathcal{G}$ is Cauchy, hence has a (unique) limit $\tilde{A}v$. One readily checks that $\tilde{A}$ is the desired extension. $\square$

[5] see e.g. Bourbaki, Espaces Vectoriels Topologiques, Chap. 2, par. 4.

A locally convex space whose topology is determined by a countable fundamental set of seminorms $\{p_j \; ; \quad j \in \mathbb{N}\}$ is complete if and only if a sequence which is Cauchy with respect to every p_j is convergent. Such a space is called a Fréchet space.

Example. Let X be a locally compact Hausdorff space, and V a locally convex space, with a fundamental set $\mathcal{P}$ of seminorms. Then $C(X, V)$, the space of continuous functions $X \to V$ may be equipped with a locally convex topology in the following natural way. For every compact subset $K \subset X$ and every $p \in \mathcal{P}$ we define the seminorm

$$s_{K,p} : f \mapsto \sup_{x \in K} p(f(x)).$$

The set of these seminorms is easily seen to be fundamental.

Exercise 6.2 Show that the locally convex space $C(X, V)$ defined above is complete if V is complete. Show that if X is σ-compact, i.e. X is a countable union of compact sets, and V is Fréchet, then $C(X, V)$ is a Fréchet space.

In the following we assume that X is a locally compact Hausdorff space, and V a complete locally convex space. We also assume that I is a positive Radon integral on X. Our purpose is to extend the definition of I to the space $C_c(X, V)$ of compactly supported continuous functions with values in V. We will first do this under the assumption that X is compact, which we assume to be fulfilled until asserted otherwise. Let V^* denote the space of continuous linear functionals on V.

Proposition 6.3 *There exists a unique continuous linear map $I : C(X, V) \to V$ such that for every $\xi \in V^*$ we have:*

$$\xi(I(f)) = I(\xi \circ f) \qquad f \in C(X, V). \tag{8}$$

Moreover, if p is a continuous seminorm on V then for all $f \in C(X, V)$ we have:

$$p(I(f)) \leq I(p(f)). \tag{9}$$

We will prove this proposition in a number of steps. If V is finite dimensional, then one readily verifies the existence of I. For the estimate we need the following.

Lemma 6.4 *Let W be a finite dimensional real linear space, and p a seminorm on W. Then for every $w \in W \setminus \{0\}$ and every $\varepsilon > 0$ there exists a linear hyperplane $H \subset W$ such that for all $h \in H$ we have $p(w + h) > p(w) - \varepsilon$.*

Proof. If $p(w) - \varepsilon < 0$, there is nothing to prove. Thus assume that $p(w) - \varepsilon \geq 0$. Let L be the linear subspace of W consisting of $v \in W$ with $p(v) = 0$. Then the image $\bar{w}$ of w in $\bar{W} = W/L$ is non-zero, and p induces a norm $\bar{p}$ on $\bar{W}$. It suffices to prove the assertion for $\bar{W}, \bar{p}, \bar{w}$. Thus without loss of generality we may assume from the start that p is a norm. Then the set

$$S = \{v \in W \; ; \quad p(v) \leq p(w) - \varepsilon\}$$

is compact and convex, and does not contain w. Hence there exists a linear hyperplane $H \subset W$ such that $(w + H) \cap S = \emptyset$. This implies the desired estimate. $\square$

Proof of Prop. 6.3 when V is finite dimensional. It remains to prove the estimate. Let p be a seminorm, let $f \in C(X, V)$, and put $w = I(f)$. Let $\varepsilon > 0$ be arbitrary. Then by the above lemma there exists a real linear subspace $H \subset V$ of real codimension 1 such that for all $h \in H$ we have $p(w+h) > p(w) - \varepsilon$. Obviously this must imply that $V = \mathbb{R}w \oplus H$ as a real linear space. Write $f = f_1 w + f_2$, where $f_1 \in C(X, \mathbb{R})$ and $f_2 \in C(X, H)$. Then $I(f) = w$ implies $I(f_1) = 1$ and $I(f_2) = 0$. Now $I(|f_1|) \geq |I(f_1)| = 1$, hence

$$p(I(f)) = p(w) \leq I(|f_1|)p(w) = I(p(w)|f_1|) = I(p(f_1 w)) \leq I(p(f) + \varepsilon\, 1_X),$$

where 1_X denotes the constant function with value 1. Hence

$$p(I(f)) \leq I(p(f)) + \varepsilon I(1_X)$$

for arbitrary $\varepsilon > 0$, from which the desired estimate follows. □

In general, if $W \subset V$ is a finite dimensional linear subspace, then we may view $C(X, W)$ as a subspace of $C(X, V)$ and we obtain a continuous linear functional $I : C(X, W) \to W$ satisfying (8) with W instead of V. One now readily sees that I extends to the algebraic direct sum of such subspaces $C(X, W)$ with W finite dimensional; we denote this direct sum by $C(X) \otimes V$. If p is a continuous seminorm on V, then for every $f \in C(X) \otimes V$ we have $p(I(f)) \leq I(p(f))$. Hence I on $C(X) \otimes V$ is continuous for the topology on $C(X, V)$. Thus in view of Lemma 6.5 the proof of Proposition 6.3 is completed by the following lemma.

Lemma 6.5 *The space $C(X) \otimes V$ is dense in $C(X, V)$.*

Proof. Let $f \in C(X, V)$, let p be a continuous seminorm on V, and let $\varepsilon > 0$. Then it suffices to show that there exists a $g \in C(X) \otimes V$ such that

$$p(f(y) - g(y)) < \varepsilon \tag{10}$$

for all $y \in X$. For every $x \in X$ there exists a neighbourhood N_x such that for all $y \in N_x$ we have $p(f(x) - f(y)) < \varepsilon$. By compactness there exists a finite set of $x_j \in X$ such that the sets $N_j = N_{x_j}$ cover X. Let φ_j be a partition of unity subordinate to the covering $\{N_j\}$. Then we claim that $g = \sum_j \varphi f(x_j)$ satisfies our requirements. First of all it is clear that g has its values in the linear span of the $f(x_j)$, hence belongs to $C(X) \otimes V$. Secondly, let $y \in X$, and let J_y denote the set of j such that $y \in N_j$. Then $j \in J_y$ implies $p(f(y) - f(x_j)) < \varepsilon$, hence

$$p(\varphi_j(y)f(y) - \varphi_j(y)f(x_j)) < \varphi_j(y)\varepsilon.$$

If $j \notin J_y$, then $\varphi_j(y) = 0$. Thus summing the above inequality over the j, we obtain the desired inequality (10). □

We will now describe the extension of the Radon integral to $C_c(X, V)$ when X is locally compact. Let $Y \subset X$ be a compact subset, then it suffices to define I on the space $C_Y(X, V)$ of continuous functions $X \to V$ with support contained in Y. Let Z be a compact neighbourhood of Y, and select a function $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on Y, and such that Z is a neighbourhood of suppφ. If $f \in C(Z)$, then φf may be viewed as an element of $C_c(X)$, with support contained in Z. Hence we may define the positive Radon integral $I_\varphi : C(Z) \to \mathbb{C}$ by

$f \mapsto I(\varphi f)$. We now define I to be restriction to $C_Y(X,V)$ of I_φ's extension to $C(Z,V)$. One readily verifies that this definition of I does not depend on the choices of Z, φ involved.

We end this section with a useful result, assertying that continuous linear maps commute with integration.

Lemma 6.6 *Let X be a locally compact Hausdorff space, and I a positive Radon integral on X. Let $A : V \to W$ be a continous linear map between complete locally convex spaces. Then for all $f \in C_c(X,V)$ we have $A \circ f \in C_c(X,W)$. Moreover:*

$$A(I(f)) = I(A \circ f).$$

Proof. As in the previous discussion we may reduce to the situation that X is compact. Let η be any continuous linear functional on W. Then $A^t\eta = \eta \circ A$ is a continous linear functional on V. Hence

$$\eta \circ A(I(f)) = [A^t\eta](I(f)) = I([A^t\eta] \circ f) = I(\eta \circ A \circ f) = \eta(I(A \circ f)).$$

Since η was arbitrary this completes the proof. $\square$

We shall now apply the material of this section to representation theory. Let G be a locally compact group, and let dx be a choice of left Haar measure on G. Let (π, V) be a (continous) representation of G in a complete locally convex space. If $f \in C_c(G)$, then we define the linear operator $\pi(f) : V \to V$ by

$$\pi(f)v = \int_G f(x)\pi(x)v\, dx.$$

Lemma 6.7 *If $f \in C_c(G)$, then:*

(a) *the linear operator $\pi(f)$ is continuous;*

(b) *for all $y \in G$ we have*

$$\pi(L_y f) = \pi(y) \circ \pi(f). \tag{11}$$

Proof. Exercise for the reader. To prove the last equality one needs the previous lemma. $\square$

Remark. Note that if dx is a right Haar measure then the above lemma is also valid, but with eqn. (11) replaced by $\pi(R_y f) = \pi(f) \circ \pi(y)^{-1}$.

The following lemma asserts that the identity operator I_V can be approximated by $\pi(f)$.

Lemma 6.8 *For every $v \in V$, every continuous seminorm p on V and every $\varepsilon > 0$ there exists an open neighbourhood U of e in G such that for every $\varphi \in C_c(U)$ with $\varphi \geq 0$ and $\int_G \varphi(x)dx = 1$ we have:*

$$p(\pi(\varphi)v - v) < \varepsilon.$$

Proof. There exists an open neighbourhood $U \ni e$ such that $x \in U \Rightarrow p(\pi(x)v - v) < \frac{1}{2}\varepsilon$. Let $\varphi \in C_c(U)$ satisfy the above hypotheses. Then:

$$\begin{aligned} p(\pi(\varphi)v - v) &= p(\int_G [\varphi(x)\pi(x)v - \varphi(x)v]\, dx) \\ &\leq \int_G \varphi(x) p(\pi(x)v - v)\, dx \\ &\leq \int_G \tfrac{1}{2}\varepsilon\varphi(x)dx = \tfrac{1}{2}\varepsilon < \varepsilon. \end{aligned}$$

□

7 The algebra of representative functions

Let (π, V) be a continuous representation of a locally compact group G. Then a vector $v \in V$ is called G–finite if the linear span $\langle \pi(x)v\,; \quad x \in G\rangle$ is finite dimensional.

Let $\mathcal{R}(G)$ denote the space of left and right G–finite functions in $C(G)$. This space is called the algebra of representative functions.

Exercise 7.1 Show that $\mathcal{R}(G)$ is indeed a subalgebra of $C(G)$.

Proposition 7.2 *Let G be compact. Then*

$$\mathcal{R}(G) = \bigoplus_{\delta \in \widehat{G}} C(G)_\delta. \tag{12}$$

Proof. For every $\delta \in \widehat{G}$ the space $C(G)_\delta$ is finite dimensional and left and right G-invariant, hence contained in $\mathcal{R}(G)$. Therefore it remains to be shown that every element of $\mathcal{R}(G)$ is a finite linear combination of matrix coefficients of finite dimensional representations. Let $\varphi \in \mathcal{R}(G)$, and let V_1 be the span of all translates of the form $L_x R_y \varphi$, $x, y \in G$. Moreover, let $V = V_1 + \overline{V_1}$, the bar denoting complex conjugation. Then V is a finite dimensional left and right G–invariant subspace of $\mathcal{R}(G)$, hence decomposes as a finite direct sum of irreducible $G \times G$-modules. We must show that every direct summand V_0 in this decomposition is contained in the right hand side of 12. Let $f \in V_0 \setminus \{0\}$. Then

$$W = \{L(\psi)f\,; \quad \psi \in V\}$$

is a left and right G-invariant subspace of V_0. Moreover, since $L(\bar{f})f(e) = \langle f|f\rangle \neq 0$, we see that W is non-trivial. Therefore $W = V_0$. In particular there exists a $\psi \in V$ such that $L(\psi)f = f$. But this means that:

$$R(\psi)f(x) = \int_G \psi(y)f(xy)\, dy = \langle L_x\psi|\bar{f}\rangle.$$

Hence f is a matrix coefficient of the left regular representation restricted to the finite dimensional space V. □

Let $\mathcal{H}_\alpha$ be a collection of Hilbert spaces, indexed by a set α. Then the algebraic direct sum $\oplus \mathcal{H}_\alpha$ is a pre-Hilbert space when equipped with the direct sum inner product: $\langle \sum_\alpha v_g a | \sum_\alpha w_\alpha \rangle = \sum_\alpha \langle v_\alpha | w_\alpha \rangle$. Its completion is called the Hilbert direct sum of the spaces $\mathcal{H}_\alpha$, and denoted by

$$\widehat{\bigoplus_{\alpha \in \mathcal{A}}} \mathcal{H}_\alpha. \tag{13}$$

If π_α is a unitary representation of G in $\mathcal{H}_\alpha$, for every $\alpha \in \mathcal{A}$, then the direct sum of the π_α extends to a unitary representation of G in (13). We call this representation the Hilbert sum of the π_α.

Theorem 7.3 (The Peter-Weyl Theorem). *The space $L^2(G)$ decomposes as the Hilbert sum*

$$L^2(G) = \widehat{\bigoplus_{\delta \in \widehat{G}}} \; C(G)_\delta,$$

each of the summands being an irreducible invariant subspace for the representation $R \times L$ of $G \times G$.

It remains to establish density of $\mathcal{R}(G)$ in $L^2(G)$. This will be achieved in the next section (cf. Lemma 9.1), after some necessary preparations.

Exercise 7.4 Fix, for every (equivalence class of an) ireducible unitary representation (δ, V_δ) an orthonormal basis $e_{\delta,1}, \ldots, e_{\delta,\dim(\delta)}$. Denote the matrix coefficient associated to $e_{\delta,i}$ and $e_{\delta,j}$ by $m_{\delta,ij}$. Use Schur orthogonality and the Peter-Weyl theorem to show that the functions

$$\sqrt{\dim(\delta)}\, m_{\delta,ij} \qquad \delta \in \widehat{G}, \;\; 1 \leq i, j \leq \dim(\delta)$$

constitute a complete orthonormal system for $L^2(G)$.

8 Compact operators

Let X, Y be locally compact Hausdorff spaces. If $\varphi \in C(X)$, and $\psi \in C(Y)$, then we write $\varphi \otimes \psi$ for the continuous function on $X \times Y$ defined by:

$$\varphi \otimes \psi : (x, y) \mapsto \varphi(x)\psi(y).$$

The linear span of such functions in $C(X \times Y)$ is denoted by $C(X) \otimes C(Y)$. If $\varphi \in C_c(X)$ and $\psi \in C_c(Y)$ then $\varphi \otimes \psi$ is compactly supported. Hence the span $C_c(X) \otimes C_c(Y)$ of such functions is a subspace of $C_c(X \times Y)$.

Lemma 8.1 *Let X, Y be locally compact Hausdorff spaces. Then the space $C_c(X) \otimes C_c(Y)$ is dense in $C_c(X \times Y)$.*

Proof. Fix $\Phi \in C_c(X \times Y)$, and let $K = \operatorname{supp} \Phi$. Then $K \subset K_X \times K_X$ for compact subsets $K_X \subset X, K_Y \subset Y$. Fix an open neighbourhood $U_X \supset K_X$ with a compact closure. Let $\varepsilon > 0$.

Then by compactness there exists a finite open covering $\{V_j\}$ of K_X such that for every j and all $x_1, x_2 \in V_j, y \in K_Y$ one has

$$\Phi(x_1, y) - \Phi(x_2, y) < \varepsilon.$$

Without loss of generality we may assume that $V_j \subset U_X$ for all j. Select a partition of unity $\{\varphi_j\}$ which is subordinate to the covering $\{V_j\}$, and fix for every j a point $\xi_j \in V_j$. Let $x \in K_X, y \in K_Y$. If j is such that $x \in V_j$, then $|\Phi(x_j, y) - \Phi(x, y)| < \varepsilon$. It follows from this that

$$\begin{aligned} |[\sum_j \varphi_j(x)\Phi(x_j, y)] - \Phi(x, y)| &= |\sum_j[\varphi_j(x)\Phi(x_j, y) - \varphi_j(x)\Phi(x, y)]| \\ &\leq \sum_j \varphi_j(x)|\Phi(x_j, y) - \Phi(x, y)| \\ &\leq \sum_j \varepsilon\varphi_j(x) = \varepsilon. \end{aligned}$$

Hence if we put $\psi_j(y) = \Phi(x_j, y)$, then

$$\|\sum_j \varphi_j \otimes \psi_j - \Phi\|_\infty \leq \varepsilon.$$

Moreover, $\mathrm{supp}\varphi_j \otimes \psi_j \subset U_X \times K_Y$, and we see that we have support control which is uniform in ε. □

Let now G be a locally compact group. Let J be a left Haar integral on $G \times G$. Then one readily verifies that for a fixed $\psi \in C_c(G)$ with $\psi \geq 0$ one has that $\varphi \mapsto J(\varphi \otimes \psi)$ is a left Haar integral on G, hence equal to a constant times the Haar integral I on G. Applying the same reasoning with interchanged variables, one sees that there exists a constant $c > 0$ such that $J(\varphi \otimes \psi) = cI(\varphi)I(\psi)$. Without loss of generality we may assume that $c = 1$. Of course in the sense of measure theory this means that the Haar measure of $G \times G$ is the square of the Haar measure of G. More precisely we have:

Lemma 8.2 *Let $f \in C_c(G \times G)$. Then*

$$J(f) = \int_G (\int_G f(x, y)dx)\, dy.$$

Proof. By what we said above the identity is valid for $f \in C_c(G) \otimes C_c(G)$. Now use a density argument. □

If $K \in C_c(G \times G)$, then we define the linear operator $T_K : C_c(G) \to C_c(G)$ by

$$T_K(\varphi)(x) = \int_G K(x, y)\varphi(y)dy.$$

For obvious reasons this is called an integral operator with kernel K.

Lemma 8.3 *Let $K \in C_c(G \otimes G)$. Then the operator T_K extends uniquely to a bounded linear endomorphism of $L^2(G)$ with operator norm $\|T_K\| \leq \|K\|_2$. Moreover, this extension is compact.*

Proof. Let $\varphi \in C_c(G)$. Then

$$\langle T_K(\varphi)|\psi\rangle = \langle K|\varphi \otimes \bar{\psi}\rangle \leq \|K\|_2 \|\varphi \otimes \bar{\psi}\|_2 = \|K\|_2 \|\varphi\|_2 \|\psi\|_2.$$

Hence $\|T_K\varphi\|_2 \leq \|K\|_2 \|\varphi\|_2$. This implies the first assertion, since $C_c(G)$ is dense in $L^2(G)$.

For the second assertion, note that there exists a sequence K_j in $C_c(G) \otimes C_c(G)$ which converges to K. It follows that

$$\|T_{K_j} - T_K\| \leq \|K_j - K\|_2 \to 0.$$

Every operator T_{K_j} has a finite dimensional image hence is compact. The subspace of compact operators is closed for the operator norm, hence T_K is compact. □

Corollary 8.4 *Assume that G is compact, and let $f \in C(G)$. Then the operator $R(f) : L^2(G) \to L^2(G)$ is compact.*

Proof. If $\varphi \in C(G)$, then

$$R(f)\varphi(x) = \int_G f(y)\varphi(xy)\, dy = \int_G f(x^{-1}y)\varphi(y)\, dy.$$

Hence $R(f) = T_K$, with $K(x, y) = f(x^{-1}y)$. (Note that for this argument it is crucial that G be compact. For if not, and $f \in C_c(G)$ then the associated K need not be compactly supported.
□

Exercise 8.5 Let $(\pi, \mathcal{H})$ be a unitary representation of G. Let $f \in C(G)$, and put $f^*(x) = \overline{f(x^{-1})}$. Show that

$$\pi(f)^* = \pi(f^*).$$

Show also: if f is conjugation invariant, i.e. $f(xyx^{-1}) = f(y)$ for all $x, y \in G$, then $\pi(f)$ is intertwining.

Corollary 8.6 *Assume that G is compact, and let $f \in C(G)$ be such that $f^* = f$. Then $R(f)$ (and $L(f)$ as well) is a compact self-adjoint operator.*

We now recall the important spectral theorem for compact self-adjoint operators in Hilbert space.

Theorem 8.7 *Let T be a compact self-adjoint operator in the (complex) Hilbert space $\mathcal{H}$. Then there exists a discrete subset $\Lambda \subset \mathbb{R} \setminus \{0\}$ such that the following holds.*

(a) *For every $\lambda \in \Lambda$ the associated eigenspace $\mathcal{H}_\lambda$ of T in $\mathcal{H}$ is finite dimensional;*

(b) *If $\lambda, \mu \in \Lambda, \lambda \neq \mu$ then $\mathcal{H}_\lambda \perp \mathcal{H}_\mu$.*

(c) *For every $\lambda \in \Lambda$, let P_λ denote the orthogonal projection $\mathcal{H} \to \mathcal{H}_\lambda$. Then*

$$T = \sum_{\lambda \in \Lambda} \lambda\, P_\lambda,$$

the convergence being absolute with respect to the operator norm.

9 Proof of the Peter-Weyl Theorem

In this section we assume that G is a compact group. We will establish the Peter-Weyl theorem by proving the following lemma.

Lemma 9.1 *Let G be compact. Then the space $\mathcal{R}(G)$ is dense in $L^2(G)$.*

We first need some preparation.

Lemma 9.2 *Let U be a neighbourhood of e in G. Then there exists a $\varphi \in C_c(G)$ such that:*

(a) $\varphi \geq 0$ *and* $\int_G \varphi(x)\, dx = 1$;

(b) $\varphi^* = \varphi$;

(c) φ *is conjugation invariant.*

Proof. From the continuity of the map $x \mapsto x^{-1}$ one sees that there exists a compact neighbourhood V of e such that $V \subset U$ and $V^{-1} \subset U$. For every $x \in G$ there exist an open neighbourhood N_x of x and a compact neighbourhood V_x of e in V such that $zyz^{-1} \in V$ for all $z \in N_x, y \in V_x$. By compactness of G finitely many of the N_x cover G. Let Ω be the intersection of the corresponding V_x. Then Ω is a compact neighbourhood of e and for all $x \in G$ and $y \in \Omega$ we have $xyx^{-1} \in V$.

Now select $\psi_0 \in C_c(\Omega)$ such that $\psi_0 \geq 0$ and $\int_G \psi_0(x)\, dx = 1$. Define

$$\psi(x) = \int_G \psi_0(yxy^{-1})\, dy.$$

Using the result of the exercise below we see that ψ is a continuous function. Clearly $\psi \geq 0$. Moreover, by application of Fubini's theorem and bi-invariance of the Haar measure it follows that $\int_G \varphi(x)\, dx = 1$. If $\psi(x) \neq 0$, then $yxy^{-1} \in \mathrm{supp}\psi_0$ for some $y \in G$, hence $x \in \cup_{y\in G} y^{-1}\Omega y \subset V$. It follows that $\mathrm{supp}\psi \subset V$. One now readily verifies that the function $\varphi = \frac{1}{2}(\psi + \psi^*)$ satisfies all our requirements. □

Exercise 9.3 Let G be a compact group, I the normalized Haar integral, and assume that $f : G \times G \to \mathbb{C}$ is a continuous function. Define the function $F : G \to C(G)$ by $F(y)(x) = f(x, y)$.

(a) Show that F is a continuous function with values in the Fréchet space $C(G)$.

(b) Show that for all $x \in G$ we have: $I(F)(x) = \int_G f(x, y)\, dy$.

Corollary 9.4 *Let $f \in L^2(G)$, $f \neq 0$. Then there exists a left and right G–equivariant bounded linear operator $T : L^2(G) \to L^2(G)$ with:*

(a) $Tf \neq 0$.

(b) T *is self-adjoint and compact;*

(c) *T maps every right G–invariant closed subspace of $L^2(G)$ into itself.*

Proof. Let $\varepsilon = \frac{1}{2}\|f\|_2$, and fix an open neighbourhood U of e in G that satisfies the assertion of Lemma 6.8 with $V = L^2(G)$, $v = f$. Let $\varphi \in C_c(U)$ be as in Lemma 9.2, and define $T = R(\varphi)$. Then T satisfies (a) and (c). T is left G–equivariant, since L and R commute. It is right G–equivariant because φ is conjugation invariant, cf. Exercise 8.5. Finally (b) follows from Corollary 8.6. □

Proof of Lemma 9.1. The space $\mathcal{R}(G)$ is left and right G–invariant, and by unitarity so is its orthocomplement V. Suppose that V contains a non-trivial element f. Let T be as in the above lemma. Then $T|V : V \to V$ is a non-trivial compact self-adjoint operator which is both left and right G–equivariant. By the spectral theorem for compact self-adjoint operators, there exists a $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that the eigenspace $V_\lambda = \ker(T - \lambda \mathrm{I}_V)$ is non-trivial. By compactness of T the eigenspace V_λ is finite dimensional, and by equivariance of T it is both left and right G–invariant. Hence $V_\lambda \in \mathcal{R}(G)$, contradiction. Therefore, V must be trivial. □

10 Density of G–finite vectors

For the moment we assume that G is a locally compact group, and that dx is a choice of left Haar measure on G. If $f, g \in C_c(G)$, we define the convolution product of f and g to be the function $f * g : G \to \mathbb{C}$ given by

$$f * g\,(x) = \int_G f(y)\, g(y^{-1}x)\, dy.$$

If $x \in G$, then $\mathrm{ev}_x : f \mapsto f(x)$, is a continuous linear functional on $C_c(G)$; thus applying Proposition 6.3 we see that $f * g(x) = \mathrm{ev}_x L(f) g$. Hence $f * g \in C_c(G)$. One readily checks that $\mathrm{supp}(f * g) \subset \mathrm{supp} f \cdot \mathrm{supp} g$.

Exercise 10.1 Show that $C_c(G)$ equipped with addition and the convolution product is an associative algebra. Show that in general this algebra has no unit element.

An important motivation for the definition of the convolution product is the following.

Exercise 10.2 Let π be a continuous representation of G in a complete locally convex space V. Show that for all $f, g \in C_c(G)$ one has

$$\pi(f * g) = \pi(f) \circ \pi(g).$$

The following fact will be needed in the sequel.

Exercise 10.3 Let $f, g \in C_c(G)$ and let $x \in G$. Then

$$L_x(f * g) = L_x f * g \qquad \text{and} \qquad R_x(f * g) = f * R_x g.$$

We will also need the following.

Lemma 10.4 *Let $f, g \in C_c(G)$. Then*

$$\|f * g\|_\infty \leq \|f\|_2 \|g\|_2.$$

Proof. Define $\check{g} : G \to \mathbb{C}$ by $\check{g}(x) = g(x^{-1})$. Then from the definition of the convolution product it follows that for $x \in G$ we have:

$$f * g(x) = \langle f | L_x \check{g} \rangle_2.$$

By the Cauchy-Schwartz inequality this implies that $|f * g(x)| \leq \|f\|_2 \|L_x \check{g}\|_2$. From the bi-invariance of the Haar integral it follows that $\|L_x \check{g}\|_2 = \|g\|_2$. Hence $|f * g(x)| \leq \|f\|_2 \|g\|_2$ for every $x \in G$. □

From now on we assume that the group G is compact, and that dx is normalized Haar measure. Moreover, we assume that π is a continuous representation of G in a complete locally convex space V.

Let V_G denote the space of G–finite vectors in V. Then V_G is a G–invariant linear subspace of V. A vector $v \in V_G$ is called isotypical if there exists an irreducible finite dimensional representation δ of G such that the linear span $\langle \pi(G)v \rangle$ is equivalent to a (necessarily finite) multiple of δ. The representation δ (or rather its equivalence class) is then called the type of the isotypical vector v. If $\delta \in \widehat{G}$ then by V_δ we denote the set of isotypical G–finite vectors of type δ in V. Obviously V_δ is a G–invariant linear subspace of V. Moreover, in view of Corollary 3.5 the space V_G is the algebraic direct sum of the spaces V_δ :

$$V_G = \bigoplus_{\delta \in \widehat{G}} V_\delta. \tag{14}$$

Notice that the above decomposition of V_G is canonical. The decomposition of each V_δ into irreducibles is not canonical in general, cf. Exercise 5.7.

The notation introduced above is consistent with the notation $C(G)_\delta$ introduced before:

Lemma 10.5 *Let $\delta \in \widehat{G}$. Then $C(G)_\delta$ equals the space of isotypical right G–finite functions of type δ.*

Proof. Let V_δ denote the space of right G–finite functions of type δ in $C(G)$. Then obviously $C(G)_\delta \subset V_\delta$. It remains to prove the reversed inclusion.

If $\rho \in \widehat{G}$, let $P_\rho : L^2(G) \to C(G)_\rho$ denote the orthogonal projection. By the invariance of the subspace $C(G)_\rho$ it follows that P_ρ is equivariant. The restriction T of P_ρ to V_δ is an intertwining operator $V_\delta \to C(G)_\rho$ for the right regular representation. If $\rho \not\sim \delta$ then using Lemma 3.12 we see that $T = 0$. By the Peter-Weyl theorem this implies that V_δ is contained in the closure of $C(G)_\delta$. This closure equals $C(G)_\delta$, by finite dimensionality. □

Exercise 10.6 Let $\delta \in \widehat{G}$. Show that $C(G)_\delta$ equals the space of isotypical left G–finite vectors of type δ^*.

Lemma 10.7 *Let $f \in C(G)_{\delta^*}$, $\delta \in \widehat{G}$. Then $\pi(f)$ maps V into the space V_δ.*

Proof. Let $v \in V$, and consider the map $T : C(G) \to V$, $\varphi \mapsto \pi(\varphi)v$. Then it follows from (11) that T intertwines L with π. Therefore it maps L–isotypical vectors of type δ to π–isotypical vectors of type δ. Now use Exercise 10.6. □

Proposition 10.8 *The space $\mathcal{R}(G)$ is dense in $C(G)$ with respect to the sup-norm.*

Proof. Fix $f \in C(G)$, and let $\varepsilon > 0$. Then by Lemma 6.8 there exists a $\varphi \in C(G)$ such that $\|f - \varphi * f\|_\infty < \varepsilon$. By Lemma 9.1 there exists a $\psi \in \mathcal{R}(G)$ such that $\|\varphi - \psi\|_2 < \varepsilon$, hence $\|\varphi * f - \psi * f\|_\infty < \varepsilon \, \|f\|_2$, by Lemma 10.4. Combining these estimates we infer that

$$\|f - \psi * f\|_\infty < (\|f\|_2 + 1)\varepsilon.$$

Using Exercise 10.6 we see that $\psi * f = L(\psi)f \in \mathcal{R}(G)$. Since ε was arbitrary this establishes density of $\mathcal{R}(G)$ in $C(G)$. □

Corollary 10.9 *The space V_G of G–finite vectors is dense in V.*

Proof. Let $v \in V$ and let p be a continuous seminorm on V. Let $\varepsilon > 0$. Then by Lemma 6.8 there exists a $\varphi \in C(G)$ such that $p(v - \pi(\varphi)v) < \varepsilon$. By Lemma 9.1 there exists a left G–finite $\psi \in C(G)$ such that $\|\varphi - \psi\|_\infty < \varepsilon$. One now readily verifies that $p(\pi(\varphi)v - \pi(\psi)v) < \varepsilon p(v)$, so that $p(v - \pi(\psi)v) < \varepsilon(p(v) + 1)$. But $\pi(\psi)v$ is G-finite by Lemma 10.7, and we see that $p(v, V_G) = 0$. This establishes density. □

Corollary 10.10 *If π is irreducible, then V is finite dimensional.*

Proof. By the previous corollary we may select a non-trivial element $v \in V_G$. The linear span W of $\pi(G)v$ is a non-trivial invariant subspace, whcih is finite dimensional hence closed. Since π is irreducible we must have $W = V$. □

Corollary 10.11 $V = \mathrm{cl}\left(\oplus_{\delta \in \widehat{G}} V_\delta\right)$.

Proof. This follows from combining (14) with Corollary 10.9. □

We will end this section by characterizing the projections onto components of the above decomposition. If $\delta \in \widehat{G}$, we denote its representation space by H_δ.

Lemma 10.12 *Let $\delta, \rho \in \widehat{G}$. Then the following holds.*

(a) $\delta \not\sim \rho \Rightarrow \rho(d_\delta \check{\chi}_\delta) = 0$.

(b) $\delta \sim \rho \Rightarrow \rho(d_\delta \check{\chi}_\delta) = I_{\mathrm{H}_\rho}$.

Proof. Let H_ρ be the representation space of ρ, and put $T = \rho(\check{\chi}_\delta)$. Then from the conjugation invariance of χ_δ and the bi-invariance of dx it follows that $T : \mathrm{H}_\rho \to \mathrm{H}_\rho$ is equivariant. Hence $T = \lambda I_{\mathrm{H}_\rho}$ for some $\lambda \in \mathbb{C}$, by Schur's lemma. From this we find that

$$d_\rho \, \lambda = \mathrm{tr}\,(T) = \int_G \chi_\delta(x^{-1}) \mathrm{tr}\,(\rho(x))\, dx = \langle \chi_\rho | \chi_\delta \rangle.$$

Now apply Lemma 5.5. □

Exercise 10.13 Let $\delta, \rho \in \widehat{G}$. Show that the following holds.

(a) If $\delta \not\sim \rho$, then $d_\delta \chi_\delta * \chi_\rho = 0$.

(b) If $\delta \sim \rho$, then $d_\delta\, \chi_\delta * \chi_\rho = \chi_\rho$.

Theorem 10.14 *Let $\delta \in \widehat{G}$. Then V_δ is a closed subspace of V. Moreover,*

$$V = V_\delta \oplus \mathrm{cl}\,(\oplus_{\rho \in \widehat{G}\setminus\{0\}} V_\rho)$$

and $P_\delta := d_\delta \pi(\check{\chi}_\delta)$ is the associated projection operator $V \to V_\delta$. Finally, P_δ is equivariant.

Proof. From the above lemma it follows that P_δ is the identity on V_δ, and annihilates every V_ρ, $\rho \not\sim \delta$. By continuity of P_δ it follows that $P_\delta = I$ on $\mathrm{cl}\, V_\delta$. On the other hand P_δ maps into V_δ by Lemma 10.7. It follows that $\mathrm{cl}\, V_\delta = V_\delta$, hence V_δ is closed. Moreover, P_δ is a projection operator with image V_δ. Hence $V = W \oplus V_\delta$ with $W = \ker P_\delta$. Since χ_δ is conjugation invariant, it follows that P_δ is equivariant and therefore W is a G–invariant complete locally convex space. It follows from Corollary 10.9 that W_G is dense in W. On the other hand, obviously $W_\delta \subset W \cap V_\delta = 0$. Hence W equals the closure of $\oplus_{\rho \in \widehat{G}\setminus\{\delta\}} V_\rho$. □

We can now completely describe the structure of every unitary representation of the compact group G in terms of irreducibles.

Corollary 10.15 *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then for every $\delta \in \widehat{G}$ the space $\mathcal{H}_\delta$ is closed, and $\pi(d_\delta \check{\chi}_\delta)$ is the orthogonal projection $\mathcal{H} \to \mathcal{H}_\delta$. Moreover we have the following Hilbert decomposition:*

$$\mathcal{H} = \widehat{\bigoplus_{\delta \in \widehat{G}}}\; \mathcal{H}_\delta.$$

Proof. The only thing that remains to be verified is the orthogonality of the summands. If $\delta \in \widehat{G}$, let P_δ denote the orthogonal projection $\mathcal{H} \to \mathcal{H}_\delta$. Then P_δ is equivariant since π is unitary, and $\mathcal{H}_\delta$ is invariant. Let $\delta, \rho \in \widehat{G}$, $\delta \not\sim \rho$. Then the restriction of P_δ to $\mathcal{H}_\rho$ is an intertwining operator $\mathcal{H}_\rho \to \mathcal{H}_\delta$. Using Lemma 3.12 we see that $P_\delta|\mathcal{H}_\rho = 0$, whence $\mathcal{H}_\delta \perp \mathcal{H}_\rho$. □

Exercise 10.16 Let G be a compact group, and dx the normalized Haar measure on G. Show that for every $f \in L^2(G)$ we have

$$f = \sum_{\delta \in \widehat{G}} d_\delta\, \chi_\delta * f,$$

with convergence in the L^2–norm.

We complete the description of unitary representations by describing the isotypical ones. Let $\delta \in \widehat{G}$, V^δ its representation space, and let S be a set provided with the counting measure. Then we define the representation $1_{L^2(S)} \otimes \delta$ of G on $L^2(S, V^\delta)$ by

$$[1_{L^2(S)} \otimes \delta](x) f(s) = \delta(x) \circ f(s), \qquad (s \in S,\, x \in G)$$

for $f \in L^2(S, V^\delta)$.

Proposition 10.17 *Let* $(\pi, \mathcal{H})$ *be an isotypical unitary representation of* G*, i.e. there exists a* $\delta \in \widehat{G}$ *such that* $\mathcal{H}_\delta = \mathcal{H}$*. Then there exists a set* S*, such that*

$$\pi \sim 1_{L^2(S)} \otimes \delta.$$

Proof. If $v \in \mathcal{H}$ we put $\mathcal{H}(v)$ for the linear span of the $\pi(x)v$, $x \in G$. Note that $\mathcal{H}(v)$ is finite dimensional by the assumption, and the first assumption of Theorem 10.14. Let $\mathcal{S}$ be the collection of subsets $S \subset \mathcal{H} \setminus \{0\}$ satisfying:

(a) if $v \in S$, then $\mathcal{H}(v)$ is irreducible;

(b) if $v, w \in S, v \neq w$ then $\mathcal{H}(v) \perp \mathcal{H}(w)$.

We order $\mathcal{S}$ by inclusion. Let $\mathcal{C} \subset \mathcal{S}$ be a completely ordered subset. Then $\cup\mathcal{C}$ belongs to $\mathcal{S}$ and dominates $\mathcal{C}$. By Zorn's lemma the set $\mathcal{S}$ has a maximal element S. For every $s \in S$ we may fix an intertwining operator $T_s : V^\delta \to \mathcal{H}$ with $\mathrm{im} T_s = \mathcal{H}(s)$. We now define $T : L^2(S, V^\delta) \to \mathcal{H}$ by

$$Tf = \sum_{s \in S} T_s f(s).$$

Then one readily verifies that T is an equivariant isometry. It remains to prove that T is surjective. Assume not. Then $\mathrm{im}(T)^\perp$ is a non-trivial G–invariant subspace of $\mathcal{H}$, hence contains an element s_1 such that $\mathcal{H}(s_1)$ is irreducible. Obviously $\mathcal{H}(s_1) \perp \mathrm{im}T$, so $\mathcal{H}(s_1) \perp \mathcal{H}(s)$ for every $s \in S$. This contradicts the maximality of S. □

11 Class functions

By a class function on a locally compact group G we mean a function $f : G \to \mathbb{C}$ which is conjugation invariant, i.e. $L_x R_x f = f$ for all $x \in G$. The space $C(G, \mathrm{class})$ of continuous class functions is a closed subspace of $C(G)$.

Now assume that G is compact. Then the projections $P_\delta : C(G) \to C(G)_\delta$ $(\delta \in \widehat{G})$ map class functions to class functions, by equivariance. Hence

$$P_\delta C(G, \mathrm{class}) \subset C(G)_\delta \cap C(G, \mathrm{class}) = \mathbb{C}\chi_\delta.$$

It follows from this that the space $\mathcal{R}(G, \mathrm{class}) = C(G, \mathrm{class}) \cap \mathcal{R}(G)$ of bi-G-finite class functions is the linear span of the characters χ_δ, $\delta \in \widehat{G}$.

Lemma 11.1 *Let* $f \in C(G, \mathrm{class})$*. Then*

$$f = \sum_{\delta \in \widehat{G}} \langle f | \chi_\delta \rangle \, \chi_\delta,$$

with convergence in the L^2*–norm. Moreover, the space* $\mathcal{R}(G, \mathrm{class})$ *is dense in* $C(G, \mathrm{class})$*, equipped with the sup-norm.*

Proof. By the Peter-Weyl theorem we have $f = \sum_{\delta \in \widehat{G}} P_\delta f$ in L^2–sense. Now $P_\delta f \in \mathbb{C}\chi_\delta$, and hence:

$$P_\delta f = \langle P_\delta f | \chi_\delta \rangle \, \chi_\delta = \langle f | \chi_\delta \rangle \, \chi_\delta.$$

The second result is proved as follows. Let $f \in \mathcal{R}(G, \text{class})$, and $\varepsilon > 0$. Then there exists a function $g \in \mathcal{R}(G)$ such that $\|f - g\|_\infty < \varepsilon$. The function g is contained in a finite direct sum W of spaces of the form $C(G)_\delta$. The same is true for the function $\tilde{g} = \int_G L_y R_y g \, dy$. By invariance of the Haar measure, $\tilde{g}$ is a continuous class function; thus $g \in \mathcal{R}(G, \text{class})$. Finally, if $x \in G$, then

$$\begin{aligned} |f(x) - \tilde{g}(x)| &= |\int_G [f(x) - g(y^{-1}xy)] \, dy| \\ &= |\int_G [f(yxy^{-1}) - g(y^{-1}xy)] \, dy| \\ &\leq \int_G \|f - g\|_\infty \, dy < \varepsilon. \end{aligned}$$

This shows that $\|f - \tilde{g}\| \leq \varepsilon$, and establishes density. □

12 Abelian groups

We now consider the case that the compact group G is abelian (i.e. $xy = yx$ for all $x, y \in G$). By a multiplicative character of G we mean a continuous group homomorphism $\xi : G \to \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is equipped with complex multiplication. Notice that every compact subgroup of $\mathbb{C}^*$ must be contained in the unit circle $|z| = 1$. Therefore, if ξ is a multiplicative character, then $|\xi(x)| = 1$, $x \in G$.

Lemma 12.1 *Let G be a compact abelian group. If (δ, V^δ) is a finite dimensional irreducible representation of G, then $\dim V^\delta = 1$. Moreover, $\delta(x) = \chi_\delta(x) I_{V^\delta}$. The map $\delta \mapsto \chi_\delta$ induces a bijection from $\widehat{G}$ onto the set of multiplicative characters of G.*

Proof. If $x \in G$, then $\delta(y)\delta(x) = \delta(yx) = \delta(xy) = \delta(x)\delta(y)$ for all $x, y \in G$, hence $\delta(x)$ is equivariant, and it follows that

$$\delta(x) = \xi(x) I, \tag{15}$$

for some $\xi(x) \in \mathbb{C}$, by Schur's lemma. It follows from this that every linear subspace of V^δ is invariant. Therefore, the dimension of V^δ must be one. From the fact that δ is a representation it follows immediately that $x \mapsto \xi(x)$ is a character. Applying the trace to (15) we see that $\xi = \chi_\delta$, the character of δ. Thus $\delta \mapsto \chi_\delta$ induces a map from the space $\widehat{G}$ of equivalence classes (!) of finite dimensional irreducible representations to the set of multiplicative characters of G. If ξ is a multiplicative character then (15) defines an irreducible representation δ of G in $\mathbb{C}$, and $\xi = \chi_\delta$. Therefore the map $\delta \to \chi_\delta$ is onto the multiplicative characters. □

Corollary 12.2 *Assume that G is compact and abelian. Then the set of multiplicative characters χ_δ, $\delta \in \widehat{G}$ is a complete orthonormal system for $L^2(G)$.*

Proof. This follows immediately from the previous lemma combined with the theorem of Peter and Weyl (Theorem 7.3). □

Corollary 12.3 *Assume that G is compact and abelian. Then the linear span of the set of multiplicative characters is dense in $C(G)$ (for the sup-norm).*

Proof. This follows from Proposition 10.8. $\square$

In the present setting we define the Fourier transform $\hat{f} : \widehat{G} \to \mathbb{C}$ of a function $f \in L^2(G)$ by

$$\hat{f}(\delta) = \langle f | \chi_\delta \rangle.$$

Let $\widehat{G}$ be equipped with the counting measure. Then the associated L^2-space is $l^2(\widehat{G})$, the space of functions $\varphi : \widehat{G} \to \mathbb{C}$ such that $\sum_{\delta \in \widehat{G}} |\varphi(\delta)|^2 < \infty$, equipped with the inner product:

$$\langle \varphi | \psi \rangle := \sum_{\delta \in \widehat{G}} \varphi(\delta) \, \overline{\psi(\delta)}.$$

Corollary 12.4 (The Plancherel theorem). *Let G be a compact abelian group. Then the Fourier transform $f \mapsto \hat{f}$ is an isometry from $L^2(G)$ onto $l^2(\widehat{G})$. Moreover, if $f \in L^2(G)$, then*

$$f = \sum_{\delta \in \widehat{G}} \hat{f}(\delta) \, \chi_\delta.$$

Proof. Exercise for the reader. $\square$

The purpose of the following exercise is to view the classical theory of Fourier series as a special case of the Peter-Weyl theory.

Exercise 12.5 Let $G = \mathbb{R}^n / 2\pi \mathbb{Z}^n$. If $m \in \mathbb{Z}^n$, show that

$$\chi_m : x \mapsto e^{i(m \cdot x)}$$

defines a multiplicative character of G. (Here $m \cdot x = m_1 x_1 + \cdots + m_n x_n$.) Show that every multiplicative character is of this form. Thus $\widehat{G} \simeq \mathbb{Z}^n$. Accordingly for $f \in L^2(G)$ we view the Fourier transform $\hat{f}$ as a map $\mathbb{Z}^n \to \mathbb{C}$.

Show that the normalized Haar integral of G is given by

$$I(f) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n.$$

Show that for $f \in L^2(G)$, $m \in \mathbb{Z}^n$ we have:

$$\hat{f}(m) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} f(x_1, \ldots, x_n) \, e^{-i(m_1 x_1 + \cdots + m_n x_n)} \, dx_1 \ldots dx_n.$$

Moreover, show that we have the inversion formula

$$f(x) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) \, e^{i(m \cdot x)} \qquad (x \in \mathbb{R}^n / 2\pi \mathbb{Z}^n)$$

in the L^2–sense.

13 The group $\mathrm{SU}(2)$

Recall the definition of the representation π_n of $\mathrm{SU}(2)$ in the space $V_n = P_n(\mathbb{C}^2)$ of homogeneous polynomials of degree n from Section 3. In Proposition 3.10 it was shown that π_n is irreducible. Moreover, the associated character is determined by the formula:

$$\chi_n(t_\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi} \qquad (\varphi \in \mathbb{R}) \tag{16}$$

(see Exercise 5.4). The purpose of this section is to prove the following result:

Proposition 13.1 *Every finite dimensional irreducible representation of* $\mathrm{SU}(2)$ *is equivalent to one of the* π_n.

We recall that every element of $\mathrm{SU}(2)$ is conjugate to an element of T. Therefore a class function on $\mathrm{SU}(2)$ is completely determined by its restriction to T. This restriction to T is obviously invariant under the substitution $t \mapsto t^{-1}$. Thus, if $C(T)_{\mathrm{ev}}$ denotes the space of continuous functions $f : T \to \mathbb{C}$ satisfying $f(t^{-1}) = f(t)$ for all $t \in T$, then restriction to T defines an injective linear map $r : C(G, \mathrm{class}) \to C(T)_{\mathrm{ev}}$.

Lemma 13.2 *The map* $r : C(G, \mathrm{class}) \to C(T)_{\mathrm{ev}}$ *is a bijective isometry (for the sup–norms).*

Proof. That r is isometric follows from the observation that the set of values of a function $f \in C(G, \mathrm{class})$ is equal to the set of values of its restriction $r(f)$. Thus it remains to establish the surjectivity of r. Let $g \in C(T)_{\mathrm{ev}}$. An element $x \in \mathrm{SU}(2)$ has two eigenvalues $e^{i\varphi(x)}$ and $e^{-i\varphi(x)}$ which may locally be chosen such that $x \mapsto \varphi(x)$ depends continuously on x. Define $f(x) = g(t_{\varphi(x)})$. Then f is well defined, and independent of the particular choice of $\varphi(x)$. Moreover, $f \in C(G, \mathrm{class})$ and $r(f) = g$. $\square$

Corollary 13.3 *The linear span of the characters* χ_n, $n \in \mathbb{N}$ *is dense in* $C(G, \mathrm{class})$.

Proof. By Lemma 13.2 it suffices to show that the linear span S of the functions $\chi_n|T$ is dense in $C(T)_{\mathrm{ev}}$. From formula (16) we see that $\chi_n(t_\varphi) = \sum_{k=0}^n e^{i(n-2k)\varphi}$. Hence S equals the linear span of the functions $\gamma_n : t_\varphi \mapsto e^{in\varphi} + e^{-in\varphi}$ $(n \in \mathbb{N})$. The latter span is dense in $C(T)_{\mathrm{ev}}$, by classical Fourier theory (see Section 12). $\square$

Corollary 13.4 *Every finite dimensional irreducible representation of* $\mathrm{SU}(2)$ *is equivalent to one of the* π_n, $n \in \mathbb{N}$.

Proof. The representations π_n are irreducible, and mutually inequivalent. Therefore their characters χ_n constitute an orthonormal system in $L^2(G, \mathrm{class})$. If $f \in C(G, \mathrm{class})$, and $\langle f|\chi_n\rangle = 0$ for all $n \in \mathbb{N}$, then $\langle f|g\rangle = 0$ for all $g \in C(G, \mathrm{class})$. Let $\varphi \in C(G)$. Then by the fact that dy is normalized and f is a class function, one obtains:

$$\langle f|\varphi\rangle_2 = \int_G f(x)\,\overline{\varphi(x)}\,dx = \int_G \int_G f(yxy^{-1})\,\overline{\varphi(x)}\,dydx.$$

Applying Fubini's theorem to interchange the order of integrations, and bi-invariance of the measure dx, one sees that the above integral equals:

$$\int_G f(x)\, \overline{\varphi(y^{-1}xy)}\, dxdy = \int_G f(x)\, \overline{\varphi(y^{-1}xy)}\, dydx = \langle f|\psi\rangle_2,$$

where $\psi = \int_G L_y R_y \varphi\, dy$ is a continuous class function. Here Fubini's theorem has been applied once more. The inner product at the extreme right of the latter equation vanishes, and therefore $f \perp C(G)$. Hence $f = 0$. It follows that the orthonormal system $\{\chi_n\ ;\quad n \in \mathbb{N}\}$ is complete. But this must imply that the π_n exhaust the irreducible representations of $\widehat{G}$, by the Peter-Weyl theorem. □

From the fact that every element of SU(2) is conjugate to an element of T it follows that there should exist a Jacobian $J : T \to [0, \infty[$ such that for every continuous class function f on SU(2) we have

$$\int_{\mathrm{SU}(2)} f(x)\, dx = \int_0^{2\pi} f(t_\varphi)\, J(t_\varphi)\, d\varphi.$$

It is possible to compute this Jacobian by a substitution of variables. We shall obtain it by other means:

Lemma 13.5 *For every continuous class function $f : \mathrm{SU}(2) \to \mathbb{C}$ we have:*

$$\int_{\mathrm{SU}(2)} f(x)\, dx = \int_0^{2\pi} f(t_\varphi)\, \frac{\sin^2 \varphi}{\pi}\, d\varphi. \tag{17}$$

Proof. Consider the linear map L which assigns to $f \in C(G, \text{class})$ the expression on the left hand side minus the expression opn the right hand side of the above equation. Then we must show that L is zero.

Obviously L is continuous linear, so that it suffices to show that $L\chi_n = 0$ for every $n \in \mathbb{N}$. The function χ_0 is identically one; therefore left and right hand side of (17) both equal 1 if one substitutes $f = \chi_0$. Hence $L\chi_0 = 0$. On the other hand, if $n \geq 1$, and $f = \chi_n$, then the left hand side of (17) equals $\langle \chi_n | \chi_0 \rangle = 0$. The right hand side of (17) also equals 0, hence $L\chi_n = 0$ for all n. □

14 Manifolds

Notations and preliminaries

Let V, V' be finite dimensional real linear spaces, and let Ω be an open subset of V. We recall that a map $\varphi : \Omega \to V'$ is called differentiable at a point $a \in \Omega$ in the direction $v \in V$ if

$$\partial_v \varphi(a) = \frac{d}{dt}[\varphi(a + tv)]_{t=0}$$

exists.

The map f is called differentiable at $a \in \Omega$ if there exists a linear map $Df(a) : V \to V'$ such that

$$f(a + h) - f(a) = Df(a)h + \mathfrak{o}(h) \qquad (h \to 0).$$

The linear map $Df(a)$, which is unique when it exists, is called the derivative of f at a. If f is differentiable in a, then $\partial_v f(a)$ exists for every $v \in V$ and we have

$$\partial_v f(a) = Df(a)v.$$

When $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$, then the above formula may be used to express the matrix of $Df(a)$ in terms of the partial derivatives $\partial_j f_i = \partial_{e_j} f_i$ (the Jacobi matrix).

If f is differentiable in (any point of) Ω, then Df is a map from Ω to the space $\mathrm{Hom}(V, V')$ of linear maps $V \to V'$. If this map is differentiable, then f is called twice differentiable. The derivative of Df is denoted by $D^2 f$. It is now clear how to define the notion of an p-times differentiable function and its p–th derivative $D^p f$. A function f is called p–times continuously differentiable, or briefly C^p, if it is p–times differentiable and $D^p f$ is continuous. We recall that f is C^p on Ω if all mixed partial derivatives of f order at most p exist and are continuous on Ω. Let $C^p(\Omega, V')$ denote the linear space of C^p–maps $\Omega \to V'$. Then the effect of any sequence of at most p–partial derivatives applied to $C^p(\Omega, V')$ is independent of the order of the sequence.

A map $f : \Omega \to V'$ is called smooth (or C^∞) if it is C^p for every $p \geq 0$. We put

$$C^\infty(\Omega, V') = \cup_{p \geq 0} C^p(\Omega, V)$$

for the space of smooth maps $\Omega \to V'$.

Let $e_1, \ldots, e_n$ be a basis of V, and abbreviate $\partial_j = \partial_{e_j}$. Then ∂_j is a linear operator on the space $C^\infty(\Omega, V')$. By the above mentioned result on the order of mixed partial derivatives we have that ∂_i and ∂_j commute ($1 \leq i, j \leq n$). Hence, as an endomorphism of $C^\infty(\Omega, V')$ every mixed partial derivatives of order at most p is of the form:

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n},$$

with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq p$.

We briefly write $C^p(\Omega)$ for $C^p(\Omega, \mathbb{C})$ ($0 \leq p \leq \infty$). By a linear partial differential operator with C^∞–coefficients on Ω we mean a linear endomorpism P of $C^\infty(\Omega)$ of the form:

$$P = \sum_\alpha c_\alpha \, \partial^\alpha$$

with finitely many non-trivial functions $c_\alpha \in C^\infty(\Omega)$. The number $k = \max\{|\alpha| \; ; \quad c_\alpha \neq 0\}$ is called the order of P.

Manifolds

Let $\varphi : \Omega \to \Omega'$ be a bijection between open subsets of finite dimensional real linear spaces. Then φ is called a C^p–diffeomorphism ($0 \leq p \leq \infty$) if φ and φ^{-1} are C^p. Note that by the inverse function theorem this is equivalent to the requirement that $\varphi \in C^p$ and $D\varphi(a)$ is bijective for every $a \in \Omega$.

We shall now develop the theory of C^∞–manifolds (we leave it to reader to keep track of what can be done in a C^p–context, for further reading we suggest the references [La][6], [Wa][7]

[6] [La]: S. Lang, Differential Manifolds, Addison Wesley, Reading Massachusetts 1972

[7] [Wa]: F. Warner, Foundations of differentiable manifolds and Lie groups, Scott, Foresman and Co., Glenview Illinois 1971.

Let X be a Hausdorff topological space. A pair (U, χ), consisting of an open subset $U \subset X$ and a homeomorphism χ from U onto an open subset of $\mathbb{R}^n$ is called an n–dimensional chart of X. If (U', χ') is a second n–dimensional chart of X, such that $U \cap U' \neq \emptyset$, then the map $\chi' \circ \chi^{-1}$ is a homeomorphism from $\chi(U \cap U')$ onto $\chi'(U \cap U')$. This homeomorphism is called the transition map from the chart χ to the chart χ'.

A set $\{(U_\alpha, \chi_\alpha)\,;\ \alpha \in \mathcal{A}\}$ of n-dimensional charts is called a C^∞ (or smooth) n-dimensional atlas of X, if

(a) $\cup_{\alpha \in \mathcal{A}} U_\alpha = X$;

(b) all transition maps $\tau_{\beta\alpha} = \chi_\beta \circ \chi_\alpha^{-1}$ are smooth (i.e. C^∞).

Remark. Note that since $\tau_{\beta\alpha}$ is the inverse of $\tau_{\alpha\beta}$, it actually follows that all transition maps are diffeomorphisms.

An n–dimensional smooth (or C^∞) manifold is a Hausdorff topological space X equipped with a smooth n-dimensional atlas $\{(U_\alpha, \chi_\alpha)\,;\ \alpha \in \mathcal{A}\}$. An n–dimensional chart (U, χ) of the manifold X is called smooth if all the transition maps $\chi_\alpha \circ \chi^{-1}$ are diffeomorphisms. The components $\chi_1, \ldots, \chi_n$ will then be called a system of local coordinates of X. The collection of all smooth charts of X is an atlas by its own right, called the maximal atlas of the smooth manifold X.

Remark. Any open subset of a finite dimensional linear space is a smooth manifold in a natural way, its dimension being the dimension of the linear space. More generally any open subset of an n–dimensional smooth manifold X is a smooth manifold of dimension n in a natural way.

A map $f : X \to Y$ of smooth manifolds (of possibly different dimensions) is called C^p at a point $x \in X$ if there exist smooth charts (U, χ) and (V, ψ) of X and Y respectively, such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \chi^{-1}$ is a C^p map from $\chi(U)$ to $\psi(V)$. (Similarly one may define the concept of a k times differentiable map between smooth manifolds.)

One readily checks that the composition of C^p maps between smooth manifolds is C^p, etc.

The map $f : X \to Y$ is called a (C^∞) diffeomorphism if it is bijective, and if f and its inverse f^{-1} are smooth (i.e C^∞). Note that diffeomorphic manifolds have the same dimension. The present notion generalizes that of a diffeomorphism of open subsets of finite dimensional real linear spaces.

Our next objective is to generalize the notion of derivative of a differentiable smooth map between manifolds. The key to this is the concept of a tangent vector. Since our manifold is not contained in an ambient linear space, it may seem strange that tangent vectors can be defined at all. The basic idea is that it makes sense to say that two curves are tangent at a point. A tangent vector of a manifold is then defined as an equivalence class of tangential curves. More precisely, let X be smooth manifold of dimension n. Then by a differentiable curve in X, we mean a differentiable map $c : I \to X$, where $I \subset \mathbb{R}$ is some open interval containing 0. The point $c(0)$ is called the initial point of c. Let $x \in X$ be a fixed point. Then two differentiable curves c, d with initial point x are said to be tangent at x if there exists a smooth chart (U, χ) containing x, such that

$$\frac{d}{dt}\,\chi \circ c(t)|_{t=0} = \frac{d}{dt}\,\chi \circ d(t)|_{t=0}. \tag{18}$$

Suppose now that (V, ψ) is another smooth chart, and let $\tau = \psi \circ \chi^{-1}$ is the associated transition map. Then by the chain rule we have:

$$\frac{d}{dt}\, \psi \circ c(t)|_{t=0} = D(\tau)(\chi(x))\, \frac{d}{dt}\, \chi \circ c(t)|_{t=0}. \tag{19}$$

From this we see that if (18) holds in one chart containing x, then it holds in any other chart containing x. Let $\mathcal{C}_x$ denote the set of all differentiable curves in X with initial point x. Define the equivalence relation $\sim$ on $\mathcal{C}_x$ by $c \sim d$ if and only if c and d are tangential at x. We define

$$T_x X := \mathcal{C}_x / \sim .$$

The class of an element $c \in \mathcal{C}_x$ is denoted by $\dot{c}(0)$. The elements of $T_x X$ are called the tangent vectors of X at x.

Let (U, χ) be a chart containing x. Then for every $c \in \mathcal{C}_x$, the vector $d/dt\, [\chi \circ c](0)$ only depends on the equivalence class $\dot{c}(0)$. We denote it by $T_x \chi\ \dot{c}(0)$ (this notation will be justified at a later stage).

Lemma 14.1 *The map $T_x\chi : T_x X \to \mathbb{R}^n$ is bijective.*

Proof. The injectivity of $T_x\chi$ is an immediate consequence of the definitions. To establish its surjectivity, let $v \in \mathbb{R}^n$ and fix any differentiable curve $\underline{c}$ in $\chi(U)$, with initial point $\chi(x)$, and with $d/dt[\underline{c}](0) = v$. Let $c = \chi^{-1}\underline{c}$. Then $c \in \mathcal{C}_x$, and by definition we have $T_x\chi\ \dot{c}(0) = v$. □

Let now (V, ψ) be another chart containing x. Then by (19) we have that

$$T_x\psi = D[\psi \circ \chi^{-1}]\,(\chi(x)) \circ T_x\chi \qquad \text{on} \qquad T_x X.$$

This implies the following.

Corollary 14.2 *The set $T_x X$ has a unique structure of real linear space such that for every chart (U, χ) containing x the map $T_x\chi : T_x X \to \mathbb{R}^n$ is linear.*

The set $T_x X$, equipped with the structure of linear space described in the above corollary, is called the tangent space of X at x.

The tangent map

We can now generalize the concept of derivative to manifolds. Let $f : X \to Y$ be a map between smooth manifolds, and suppose that f is differentiable at the point $x \in X$. If $c, d \in \mathcal{C}_x$, then $f \circ c, f \circ d \in \mathcal{C}_{f(x)}$. Let (U, χ), and (V, ψ) be charts of X and Y such that $x \in U$, $f(U) \subset V$. Then the map $F = \psi \circ f \circ \chi^{-1}$ is differentiable. Moreover, if $\dot{c}(0) = \dot{d}(0)$, then by definition we have

$$\frac{d}{dt}[\chi \circ c](0) = \frac{d}{dt}[\chi \circ d](0).$$

If we apply $DF(\chi(x))$ to this expression we obtain

$$\frac{d}{dt}[\psi \circ f \circ c](0) = \frac{d}{dt}[\psi \circ f \circ d](0),$$

by the chain rule. It follows from this that $f\circ c$ and $f\circ d$ are equivalent elements of $\mathcal{C}_{f(x)}$. This shows that the map $\mathcal{C}_x \to \mathcal{C}_{f(x)}, c \mapsto f\circ c$ induces a map $T_xX \to T_{f(x)}Y$, which we denote by T_xf.

Note that it is immediate from the above discussion that the following diagram commutes:

$$\begin{array}{ccc} T_xX & \xrightarrow{T_xf} & T_{f(x)}Y \\ {\scriptstyle T_x\chi}\downarrow & & \downarrow{\scriptstyle T_{f(x)}\psi} \\ \mathbb{R}^n & \xrightarrow{D(\psi\circ f\circ\chi^{-1})(\chi(x))} & \mathbb{R}^m. \end{array} \tag{20}$$

Hence it follows from Lemma 14.1 and Cor. 14.2 that the map $T_xf : T_x \to T_{f(x)}$ is linear; it is called the tangent map of f at x.

Theorem 14.3 (The chain rule). *Let $f : X \to Y$ and $g : Y \to Z$ be maps, such that f is differntaible at $x \in X$ and g is differentiable at $f(x)$. Then $g\circ f$ is differentiable at x, and*

$$T_x(g\circ f) = T_{f(x)}g\circ T_xf.$$

Proof. This follows from the ordinary chain rule by using the commutative diagram (20) three times, once for $f : X \to Y$ at x, once for $g : Y \to Z$ at $f(x)$, and once for $g\circ f : X \to Z$ at x. $\square$

Let U be an open subset of X. Then if $x \in U$ one readily checks that the tangent map T_xi of the inclusion map $i : U \to X$ is an isomorphism $T_xU \to T_xX$. Via this isomorphism we shall identify $T_xU \simeq T_xX$. In particular, if U is an open subset of $\mathbb{R}^n$, then $T_xU \simeq T_x\mathbb{R}^n$. The latter space is identified with $\mathbb{R}^n$ as follows. For $v \in \mathbb{R}^n$, define the curve $c_v : [0,1] \to \mathbb{R}^n$ by $c_v(t) = x + tv$. Then we identify $\mathbb{R}^n \simeq T_x\mathbb{R}^n$ via the map $v \mapsto \dot{c}_v(0)$. We leave it to the reader to check the following. If $f : U \to V$ is a map between open subsets $V \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ then via the identifications discussed above, the tangent map $T_xf : T_xU \to T_{f(x)}V$ $(x \in U)$ corresponds to the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$. Also, if (U,χ) is a chart of a smooth manifold X containing the point $x \in X$, then the map $T_x\chi : T_xX \to \mathbb{R}^n$ of Lemma 14.1 corresponds to the map $T_x\chi : T_xX \to T_x\mathbb{R}^n$. Finally, observe that when $c \in \mathcal{C}_x$, then the element $\dot{c}(0)$, defined as the $\sim$ class of c, equals $T_0c(1)$.

Remark. In the literature one also finds the notations $df(x)$ and $Df(x)$ for T_xf.

Submanifolds

Let X be a n–dimensional smooth manifold. A subset $Y \subset X$ is called a smooth submanifold of dimension k if for every $y \in Y$ there exists a chart (U,χ) containing y, such that $\chi(U\cap Y) = \chi(U)\cap\mathbb{R}^k$. Here we agree to identify $\mathbb{R}^k$ with the subspace $\{x \in \mathbb{R}^n;\ x_j = 0\ \ (j \geq k)\}$.

Suppose that Y is a submanifold of X, and let $i_Y : Y \to X$ denote the inclusion map. Then one readily verifies that for every $y \in Y$ the map T_yi_Y is an injective linear map. Via this map we shall identify T_yY with a linear subspace of T_yX.

Notice that the notion of submanifold as defined above generalizes the notion of smooth submanifold of $\mathbb{R}^n$. Notice also that a subset $Y \subset X$ is a submanifold of X if it looks like a submanifold of $\mathbb{R}^n$ in any set of local coordinates. More precisely, Y is a submanifold if for every smooth chart (U,χ) of X the set $\chi(U\cap Y)$ is a smooth submanifold of $\mathbb{R}^n$ (which may be empty).

Let X, Y be smooth manifolds. A map $f : X \to Y$ is called an immersion at a point $x \in X$ if the tangent map $T_x f : T_x X \to T_{f(x)} Y$ is injective. It is called a submersion at $x \in X$ if for all the tangent map $T_x f$ is surjective.

One now has the following useful result, which is a consequence of the implicit function theorem for $\mathbb{R}^n$.

Lemma 14.4 (Immersion Lemma). *Let $f : X \to Y$ be a smooth map, and let $x \in X$. Then f is an immersion at x if and only if $\dim X - \dim Y = p \geq 0$ and there exist open neigbourhoods $X \supset U \ni x$ and $Y \supset V \ni f(x)$ and a diffeomorphism φ of V onto a product $U \times \Omega$ with $\Omega \ni 0$ an open subset of $\mathbb{R}^p$, such that the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ I_U \downarrow & & \downarrow \varphi \\ U & \xrightarrow{i_1} & U \times \Omega \end{array}$$

Here i_1 denotes the inclusion $x \mapsto (x, 0)$.

Lemma 14.5 (Submersion Lemma). *Let $f : X \to Y$ be a smooth map between smooth manifolds. Then f is submersive at $x \in X$ if and only if $\dim X - \dim Y = p \geq 0$, and there exist open neigbourhoods $X \supset U \ni x$ and $Y \supset V \ni f(x)$ and a diffeomorphism $\varphi : U \to V \times \Omega$ with Ω an open subset of $\mathbb{R}^p$, such that the following diagram commutes*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi \downarrow & & \downarrow I_V \\ V \times \Omega & \xrightarrow{\mathrm{pr}_1} & V \end{array}$$

Here pr_1 denotes the projection on the first component.

In particular it follows from the above lemmas that an immersion is locally injective, and that a submersion always has an open image. From the definitions given before combined with the two lemmas above we now obtain:

Theorem 14.6 *Let X be a smooth manifold, and let $Y \subset X$ be a subset. Then the following conditions are equivalent.*

(a) *Y is a smooth submanifold;*

(b) *Y is locally closed in X, and for every $y \in Y$ there exists an open neighbourhood $U \supset y$ such that $U \cap Y$ is the image of an injective immersion;*

(c) *for every $y \in Y$ there exists an open neigbourhood $X \supset U \ni y$ and a submersion φ of U onto a smooth manifold Z such that $Y \cap U = \varphi^{-1} z$ for some $z \in Z$.*

15 Vector fields

If X is a smooth manifold, we write TX for the disjoint union of the tangent spaces T_x, $x \in X$. The set TX is called the tangent bundle of X.[8] We define the map $\pi : TX \to X$ by the requirement that $\pi(T_xX) \subset \{x\}$ for every $x \in X$.

A map $v : X \to TX$ with $v(x) \in T_xX$ for every $x \in X$ is called a vector field on X. The set of vector fields on X is denoted by $\Gamma(TX)$. By defining addition and scalar multiplication of vector fields pointwise, we turn $\Gamma(TX)$ into a linear space.

Recall that if $U \subset \mathbb{R}^n$ is open, and $x \in U$, then we have a natural identification $\nu_x : T_xU \xrightarrow{\simeq} \mathbb{R}^n$. We identify TU with $U \times \mathbb{R}^n$ via the map ν given by $\nu(\xi) = (x, \nu_x(\xi))$, for $x \in U$, $\xi \in T_xU$. Via this identification a vector field on U may be viewed as a map $v : U \to U \times \mathbb{R}^n$, with $v(x) \in \{x\} \times \mathbb{R}^n$ for all $x \in U$. Thus $v(x) = (x, f(x))$ for a uniquely defined function $U \to \mathbb{R}^n$. In this way we may identify $\Gamma(TU)$ with the linear space of maps $U \to \mathbb{R}^n$. We now agree to call a vector field $v \in \Gamma(U)$ of class C^p, if it is C^p as a map $U \to \mathbb{R}^n$.

If $f : X \to Y$ is a differentiable map then we define the map $Tf : TX \to TY$ by $Tf = T_xf$ on T_xX.

A smooth map $f : X \to Y$ is a diffeomorphism if and only if $Tf : TX \to TY$ is bijective. Moreover, if this is the case then we have an induced bijective map $f_* : \Gamma(TX) \to \Gamma(TY)$, defined by the formula:

$$[f_*v](f(x)) = T_xf\, v(x).$$

After these preliminaries we can introduce the notion of a C^p vector field. On open subsets of $\mathbb{R}^n$ this has been done already. If X is a smooth manifold, then a vector field $v \in \Gamma(TX)$ is said to be C^p if for every $x \in X$ there exists a chart (U, χ) containing x so that $\chi_*(v|U)$ is smooth. By what we said above the latter assertion can also be rephrased as: the map

$$\chi_*(v|_U) \circ \chi : x \mapsto T_x\chi\, [v(x)] \tag{21}$$

is C^p. By the chain rule it follows that if v is a C^p vector field on X, then for *any* smooth chart (U, χ) the map (21) is C^p. The set $\Gamma^p(TX)$ of C^p vector fields on X is obviously a linear subspace of $\Gamma(TX)$.

From now on we assume that $1 \leq p \leq \infty$, that X is a smooth manifold, and that $v \in \Gamma^p(X)$. Let $x \in X$. Then by an integral curve for v with initial point x we mean a differentiable map $c : I \to X$, with I an open interval containing 0, such that

$$\begin{aligned} c(0) &= x \\ \dot{c}(t) &= v(c(t)) \qquad (t \in I). \end{aligned}$$

Here we have written $\dot{c}(t)$ for $\frac{d}{dt}c(t) = T_tc \cdot 1$. We now come to a nice reformulation of the existence and uniqueness theorem for systems of first order ordinary differential equations (use local coordinates to see this).

Theorem 15.1 *Let $v \in \Gamma^p(TX)$, $x \in X$. Then there exists an open interval $I \ni 0$ such that:*

(a) *there exists an integral curve $c : I \to X$ for v with initial point x;*

(b) *if $d : J \to X$ is a second integral curve for v with initial point x, then $d = c$ on $I \cap J$.*

[8] usually one reserves this name for TX equipped with a canonical structure of vector bundle

Lemma 15.2 *Let $c : I \to X$ be an integral curve for v with initial point x. Fix $t_1 \in I_x$, let $I_1 = I - t_1$ be the translated interval, and let $c_1 : I_1 \to X$ be defined by $c_1(t) = c(t + t_1)$. Then c_1 is an integral curve for v with initial point x_1.*

Proof. By an easy application of the chain rule it follows that

$$\dot{c}_1(t) = \dot{c}(t + t_1) = v(c(t + t_1)) = v(c_1(t)).$$

Moreover, $c_1(0) = x_1$ by definition. $\square$

Corollary 15.3 *Let $c, d : I \to X$ be integral curves for v with initial point x. Then $c = d$.*

Proof. Let J be the set of $t \in I$ for which $c(t) = d(t)$. Then J is a closed subset of I by continuity of c and d. On the other hand, if $t_1 \in J$, then $c(t + t_1) = d(t + t_1)$ for t in a neighbourhood of 0, in view of Lemma 15.2 and Theorem 15.1. This implies that J is open in I as well. Hence J is an open and closed subset of I containing 0, and we see that $J = I$. $\square$

From this corollary it follows that there exists a maximal open interval $I_x \ni 0$ for which there exists an integral curve $c : I_x \to X$ for v with initial point x. Indeed I_x is the union of all the intervals which are domain for an integral curve with initial point x.

The associated unique integral curve $I_x \to X$ is called the maximal integral curve with initial point x.

Exercise 15.4 Let v be a C^1 vector field on a compact manifold X, and let $x \in X$. Show that $I_x = \mathbb{R}$. Hint: assume that I_x is bounded from above, and let s be its sup. Let $\alpha : I_x \to X$ be the maximal integral curve. Show that there exists a sequence $s_n \in I_x$ with $s_n \to s$ so that $\alpha(s_n) \to x_1$. Now apply the existence and uniqueness theorem to v and the starting point x_1.

The following results will be of crucial importance in the theory of Lie groups.

Corollary 15.5 *Let v be a C^p vector field on a smooth manifold X. Let $x \in X$, and let $\alpha : I_x \to X$ be the associated maximal integral curve. Let $t_1 \in I_x$, and let α_1 be the maximal integral curve with initial point $x_1 = \alpha(t_1)$. Then I_{x_1} equals the translated interval $I_x - t_1$. Moreover, for $t \in I_x$ we have:*

$$\alpha_1(t - t_1) = \alpha(t).$$

Proof. It follows from Lemma 15.2 that $c : I_x - t_1 \to X,\ t \mapsto \alpha(t_1 + t)$ defines an integral curve with initial point x_1. Hence $I_x - t_1 \subset I_{x_1}$. Moreover, $c = \alpha_1$ on $I_x - t_1$. In particular it follows that $-t_1 \in I_{x_1}$. Applying the same argument to α_1 and $x = \alpha(-t_1)$ we see that $I_{x_1} + t_1 \subset I_x$. Hence $I_{x_1} = I_x - t_1$. The desired equality now follows from $c = \alpha_1$ on $I_x - t_1$. $\square$

The following result, which is stated without proof, expresses that the integral curve depends smoothly on the initial value. Let Ω be the union of the subsets $I_x \times \{x\}$ of $\mathbb{R} \times X$. For $x \in X$, let $\alpha_x : I_x \to X$ be the maximal integral curve of v with initial point x. Then we define the flow of the vector field v to be the map $\Phi : \Omega \to X$ given by $\Phi(t, x) = \Phi_t(x) = \alpha_x(t)$.

Theorem 15.6 *Let v be a C^p vectorfield on X. Then Ω is an open subset of $\mathbb{R} \times X$, and the flow $\Phi : \Omega \to X$ is a C^p map.*

16 Lie groups

A Lie group is a smooth manifold G equipped with a group structure so that the maps $m : (x, y) \mapsto xy$, $G \times G \to G$ and $\iota : x \mapsto x^{-1}$, $G \to G$ are smooth.

Example. An important example of a Lie group is the group $\mathrm{GL}(V)$ of invertible linear endomorphisms of a finite dimensional real linear space V. Indeed $\mathrm{GL}(V)$ is the subset of elements in $\mathrm{End}(V)$ with determinant non-zero, hence an open subset of a linear space, and therefore a smooth manifold. Moreover, the group operations are obviously smooth for this manifold structure.

If $x \in G$, then the map $l_x : y \mapsto xy$, $G \to G$ is a smooth bijective map, whose inverse $l_{x^{-1}}$ is also smooth. Therefore, l_x is a diffeomorphism from G onto itself. Likewise the right multiplication map $r_x : y \mapsto yx$ is a diffeomorphism of G onto itself, and therefore so is the conjugation map $Ad(x) = l_x r_x^{-1} : y \mapsto xyx^{-1}$. The latter map fixes the neutral element e; therefore its tangent map

$$\mathrm{Ad}(x) := T_e Ad(x)$$

is a linear automorphism of T_eG. From the fact that $x, y \mapsto xyx^{-1}$ is smooth, it follows that $x \mapsto \mathrm{Ad}(x)$ is a smooth map from G to $\mathrm{GL}(T_eG)$. From $Ad(e) = I_G$ it follows that $\mathrm{Ad}(e) = I_{T_eG}$. Moreover, differentiating the relation $Ad(xy) = Ad(x)Ad(y)$ at e, we find: $\mathrm{Ad}(xy) = \mathrm{Ad}(x)\mathrm{Ad}(y)$ for all $x, y \in G$. A smooth map from a Lie group G_1 to a Lie group G_2 which is a homomorphism of groups is called a Lie group homomorphism. Thus we have proved:

Lemma 16.1 $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ *is a Lie group homomorphism.*

Example. We return to the example of $\mathrm{GL}(V)$, with V a finite dimensional real linear space. Since $\mathrm{GL}(V)$ is an open subset of the linear space $\mathrm{End}(V)$ we may identify its tangent space at I with $\mathrm{End}(V)$. If $x \in \mathrm{GL}(V)$, then $Ad(x)$ is the restriction of the linear map $C_x : A \mapsto xAx^{-1}$, $\mathrm{End}(V) \to \mathrm{End}(V)$. Hence $\mathrm{Ad}(x) = C_x$ is conjugation by x.

A vector field $v \in \Gamma(TG)$ is called left invariant, if $(l_x)_* v = v$ for all $x \in G$, or, equivalently if

$$v(xy) = T_y(l_x)\, v(y) \qquad (x, y \in G). \tag{22}$$

From the above equation with $y = e$ we see that a left invariant vector field is completely determined by its value $v(e) \in T_eG$ at e. Conversely, if $X \in T_eG$, then we may define a vector field v_X on G by $v_X(x) = T_e(l_x)X$, for $x \in G$. Differentiating the relation $l_{xy} = l_x \circ l_y$ and applying the chain rule we see that $T_e(l_{xy}) = T_y(l_x)T_e(l_y)$. Applying this to the definition of v_X we see that v_X satisfies (22), hence is left invariant. Thus we see that $X \to v_X$ is a bijection from T_eG onto the left invariant vector fields on G. Notice that v_X is smooth by smoothness of the group structure.

If $X \in T_eG$, we define α_X to be the maximal integral curve of v_X with initial point e.

Lemma 16.2 *Let $X \in T_eG$. Then the integral curve α_X has domain $\mathbb{R}$. Moreover, we have $\alpha_X(s + t) = \alpha_X(s)\alpha_X(t)$ for all $s, t \in \mathbb{R}$. Finally the map $(t, X) \mapsto \alpha_X(t)$, $\mathbb{R} \to T_eG$ is smooth.*

Proof. Let α be any integral curve for v_X, let $y \in G$, and put $\alpha_1(t) = y\alpha(t)$. Differentiating this relation with respect to t we obtain:

$$\frac{d}{dt}\alpha_1(t) = T_{\alpha(t)} l_y \frac{d}{dt}\alpha_X(t) = T_{\alpha(t)} l_y v_X(\alpha_X(t)) = v_X(\alpha_1(t)),$$

by left invariance of v_X. Hence α_1 is an integral curve for v_X as well.

Let now I be the domain of α_X, fix $t_1 \in I$, and put $x_1 = \alpha_X(t_1)$. Then $\alpha_1(t) = x_1\alpha_X(t)$ is an integral curve for v_X with starting point x_1, hence has domain contained in $I - t_1$, in view of Corollary 15.5. On the other hand from its definition we see that the domain of α_1 is I, so that $I \subset I - t_1$. Since this holds for any $t_1 \in I$ this implies that $s + t \in I$ for all $s, t \in I$. Hence $I = \mathbb{R}$.

Fix $s \in \mathbb{R}$, then by what we saw above $c : t \mapsto \alpha_X(s)\alpha_X(t)$ is the maximal integral curve for v_X with initial pont $\alpha_X(s)$. On the other hand, the same holds for $d : t \mapsto \alpha_X(s + t)$, cf. Corollary 15.5. Hence $c = d$.

The final assertion follows from Theorem 15.6. $\square$

We now define the exponential map $\exp : T_eG \to G$ by

$$\exp(X) = \alpha_X(1).$$

Lemma 16.3 *For all $s, t \in \mathbb{R}$, $X \in T_eG$ we have*

(a) $\exp(sX) = \alpha_X(s)$.

(b) $\exp(s + t)X = \exp sX \, \exp tX$.

Moreover, the map $\exp : T_eG \to G$ is a local diffeomorphism at 0. Its tangent map at the origin is given by $T_0 \exp = I_{T_eG}$.

Proof. Consider the curve $c(t) = \alpha_X(st)$. Then $c(0) = e$, and

$$\frac{d}{dt}c(t) = s\dot{\alpha}_X(st) = s\, v_X(\alpha_X(st)) = v_{sX}(c(t)).$$

Hence c is the maximal integral curve of v_{sX} with initial point e, and we conclude that $c(t) = \alpha_{sX}(t)$. Now evaluate at $t = 1$ to obtain the equality.

Formula (b) is an immediate consequence of (a) and Lemma 16.2 Finally we note that

$$T_0(\exp)X = \frac{d}{dt}\exp(tX)|_{t=0} = \dot{\alpha}_X(0) = v_X(e) = X.$$

Hence $T_0(\exp) = I_{T_eX}$, and from the inverse function theorem it follows that exp is a local diffeomorphism at 0, i.e. there exists an open neighbourhood U of 0 in T_eG such that exp maps U diffeomorphically onto an open neighbourhood of e in G. $\square$

Example. We return to the example of the group $\mathrm{GL}(V)$, with V a finite dimensional real linear space. If $x \in \mathrm{GL}(V)$, then l_x is the restriction of the linear map $L_x : A \mapsto xA$, $\mathrm{End}(V) \to \mathrm{End}(V)$, to $\mathrm{GL}(V)$, hence $T_e(l_x) = L_x$, and we see that for $X \in \mathrm{End}(V)$ the invariant vectorfield v_X is given by $v_X(x) = xX$. Hence the integral curve α_X satifies the equation:

$$\frac{d}{dt}\alpha(t) = \alpha(t)X.$$

Since $t \mapsto e^{tX}$ is a solution to this equation with the same initial value, we must have that $\alpha_X(t) = e^{tX}$. Thus in this case exp is the ordinary exponential map $X \mapsto e^X$, $\mathrm{End}(V) \to \mathrm{GL}(V)$.

A smooth group homomorphism $\alpha : (\mathbb{R}, +) \to G$ is called a one parameter subgroup of G.

Lemma 16.4 *If $X \in T_eG$, then $t \mapsto \exp tX$ is a one parameter subgroup of G. Moreover, all one parameter subgroups are obtained in this way. More precisely, let α be a one parameter subgroup in G, and put $X = \dot{\alpha}(0)$. Then $\alpha(t) = \exp(tX)$ $(t \in \mathbb{R})$.*

The first assertion follows from Lemma 16.2. Let $\alpha : \mathbb{R} \to G$ be a one parameter subgroup. Then $\alpha(0) = e$, and

$$\frac{d}{dt}\alpha(t) = \frac{d}{ds}\alpha(t+s)|_{s=0} = \frac{d}{ds}\alpha(t)\alpha(s)|_{s=0} = T_e(l_{\alpha(t)})\dot{\alpha}(0) = v_X(\alpha(t)),$$

hence α is an integral curve for v_X with initial point e. Hence $\alpha = \alpha_X$ by Corollary 15.3. Now apply Lemma 16.3. $\square$

We now come to a very important corollary.

Corollary 16.5 *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \exp_G \uparrow & & \uparrow \exp_H \\ T_eG & \xrightarrow{T_e\varphi} & T_eH \end{array}$$

Proof. Let $X \in T_eG$. Then $\alpha(t) = \varphi(\exp_G(tX))$ is a one parameter subgroup of H. Differentiating at $t = 0$ we obtain $\dot{\alpha}(0) = T_e(\varphi)T_0(\exp_G)X = T_e(\varphi)X$. Now apply the above lemma to conclude that $\alpha(t) = \exp_H(T_e(\varphi)X)$. $\square$

Exercise 16.6 Let V be a finite dimensional real linear space. Show that $\det : \mathrm{GL}(V) \to \mathbb{R}^*$ is a Lie group homomorphism. Show that $T_I(\det) = \mathrm{tr}$. Show that for all $A \in \mathrm{End}(V)$ we have:

$$\det(e^A) = e^{\mathrm{tr}\, A}.$$

Applying the above corollary to the Lie group homomorphism $Ad(x) : G \to G$ for $x \in G$, we obtain the following:

Corollary 16.7 *Let $x \in G$, then for every $X \in T_eG$ we have*

$$x \exp X\, x^{-1} = \exp(\mathrm{Ad}\,(x)\, X).$$

Differentiating Ad at e we obtain a linear map

$$\mathrm{ad} = T_e\mathrm{Ad} : T_eG \to \mathrm{End}(T_eG).$$

Moreover, applying Corollary 16.5 to the Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$, we obtain:

Corollary 16.8 *For all $X \in T_eG$ we have:*

$$\mathrm{Ad}\,(\exp X) = e^{\mathrm{ad}\, X}.$$

Example. Let V be finite dimensional real linear space. Then for $x \in \mathrm{GL}(V)$ the linear map $\mathrm{Ad}\,(x) : \mathrm{End}(V) \to \mathrm{End}(V)$ is given by $\mathrm{Ad}\,(x)Y = xYx^{-1}$.] Substituting $x = e^{tX}$ and differentiating this with respect to t at $t = 0$ we obtain:

$$(\mathrm{ad}\,X)Y = \frac{d}{dt}[e^{tX}Ye^{-tX}]_{t=0} = XY - YX.$$

Hence in this case $(\mathrm{ad}\,X)Y$ is the commutator bracket of X and Y.

In general we put

$$[X,Y] := (\mathrm{ad}\,X)Y \qquad \text{for} \qquad X, Y \in T_eG.$$

Then $(X,Y) \mapsto [X,Y]$ defines a bilinear map $T_eG \times T_eG \to T_eG$. This map is anti-symmetric:

Lemma 16.9 *For all $X, Y \in T_eG$ we have $[X,Y] = -[Y,X]$.*

Proof. Let $Z \in T_eG$. Then for all $s, t \in \mathbb{R}$ we have

$$\exp(tZ) = \exp(sZ)\exp(tZ)\exp(-sZ) = \exp(t\mathrm{Ad}\,(\exp(sZ))\,Z),$$

by Lemma 16.3 and Corollary 16.7. Differentiating this relation with respect to t at $t = 0$ we obtain:

$$Z = \mathrm{Ad}\,(\exp(sZ))\,Z \qquad (s \in \mathbb{R}).$$

Differentiating this with respect to s at $s = 0$ we obtain:

$$0 = \mathrm{ad}\,(Z)Z = [Z,Z].$$

Now substitute $Z = X + Y$ and use the bilinarity to arrive at the desired conclusion. $\square$

Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then one readily verifies that $\varphi \circ Ad_G(x) = Ad_H(\varphi(x)) \circ \varphi$. Taking the tangent map of both sides of this equation at e, we obtain that the following diagram commutes:

$$\begin{array}{ccc} T_eG & \xrightarrow{T_e\varphi} & T_eH \\ \mathrm{Ad}_G(x) \uparrow & & \uparrow \mathrm{Ad}_H(\varphi(x)) \\ T_eG & \xrightarrow{T_e\varphi} & T_eH \end{array}$$

Differentiating once more at $x = e$, in the direction of $X \in T_eG$, we obtain that the following diagram commutes:

$$\begin{array}{ccc} T_eG & \xrightarrow{T_e\varphi} & T_eH \\ \mathrm{ad}_G(X) \uparrow & & \uparrow \mathrm{ad}_H(T_e\varphi X) \\ T_eG & \xrightarrow{T_e\varphi} & T_eH \end{array}$$

We now agree to write $[X,Y] = \mathrm{ad}\,(X)Y$. Then by applying $T_e\varphi \circ \mathrm{ad}_G X$ to $Y \in T_eG$ the commutativity of the above diagram yields

$$T_e\varphi\,[X,Y]_G = [T_e\varphi\,X, T_e\varphi\,Y]_H. \tag{23}$$

Applying the above relation to $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ we obtain

$$\mathrm{ad}\,([X,Y]) = \mathrm{ad}\,X\,\mathrm{ad}\,Y - \mathrm{ad}\,Y\,\mathrm{ad}\,X.$$

Applying the latter relation to $Z \in T_eG$, we obtain

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] \tag{24}$$

for all $X, Y, Z \in T_eG$.

A real Lie algebra is a real linear space $\mathfrak{a}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$, such that for all $X, Y, Z \in \mathfrak{a}$ we have:

(a) $[X, Y] = -[Y, X]$ (anti-symmetry);

(b) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

Remark. Note that condition (a) may be replaced by the equivalent condition (a'): $[X, X] = 0$ for all $X \in \mathfrak{a}$.

Lemma 16.10 *Let G be a Lie group. Then T_eG equipped with the bilinear map $(X, Y) \mapsto [X, Y] := (\mathrm{ad}\, X)Y$ is a Lie algebra.*

Proof. The anti-linearity was established before. The Jacobi identity follows from (24) combined with the anti-linearity. $\square$

A homomorphism of Lie algebras $\varphi : \mathfrak{a} \to \mathfrak{b}$ is a linear map $\mathfrak{a} \to \mathfrak{b}$ such that $\varphi([X, Y]_{\mathfrak{a}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{b}}$, for all $X, Y \in \mathfrak{a}$.

Exercise 16.11 Let A be an associative real algebra. Show that $[x, y] = xy - yx$ $(x, y \in A)$ defines a Lie algebra structure on A.

From now on we will adopt the convention that Roman capitals denote Lie groups. The corresponding Gothic lower case letters will denote the associated Lie algebras. If $\varphi : G \to H$ is a Lie group homomorphism then the associated tangent map $T_e\varphi$ will be denoted by φ_*. We now have the following.

Lemma 16.12 *Let $\varphi : H \to G$ be a homomorphism of Lie groups. Then the associated tangent map $\varphi_* : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \exp_G \uparrow & & \uparrow \exp_H \\ \mathfrak{g} & \xrightarrow{\varphi_*} & \mathfrak{h} \end{array}$$

Proof. The first assertion follows from (23), the second is Corollary 16.5. $\square$

17 Intermezzo: partial differentiations

Let X, Y be smooth manifolds of dimensions m and n respectively. Then $X \times Y$ is a manifold in a natural way (for the charts one may take products of $X-$ and $Y-$charts). Fix $x_0 \in X, y_0 \in Y$, and consider the maps $\iota_X : X \to X \times Y,\ x \mapsto (x, y_0)$ and $\iota_Y : Y \to X \times Y,\ y \mapsto (x_0, y)$. Then the associated tangent maps are injective linear maps:

$$T_{x_0}\iota_X : T_{x_0} \to T_{(x_0,y_0)}(X \times Y) \qquad \text{and} \qquad T_{y_0}\iota_Y : T_{y_0} \to T_{(x_0,y_0)}(X \times Y).$$

Using local coordinates one sees that their images have zero intersection. Via the above tangent maps we identify $T_{x_0}X$ and $T_{y_0}Y$ with linear subspaces of $T_{(x_0,y_0)}(X \times Y)$: thus we see that

$$T_{(x_0,y_0)}(X \times Y) = T_{x_0}X \oplus T_{y_0}Y.$$

If $f : X \times Y \to Z$ is a differentiable map to a smooth manifold Z we write $T^X_{(x_0,y_0)}f$ for the tangent map of $x \mapsto f(x, y_0)$ at $x = x_0$ and $T^Y_{(x_0,y_0)}f$ for the tangent map of $y \mapsto f(x_0, y)$ at $y = y_0$.

Lemma 17.1 *Let $\xi \in T_{x_0}, \eta \in T_{y_0}$. Then*

$$T_{(x_0,y_0)}f\ (\xi, \eta) = T^X_{(x_0,y_0)}f\ \xi + T^Y_{(x_0,y_0)}f\ \eta.$$

Proof. It is immediate from the definitions that $T^X_{(x_0,y_0)}f = T_{x_0}[f \circ \iota_X]$. Applying the chain rule we see that $T^X_{(x_0,y_0)}f = T_{(x_0,y_0)}f \circ T_{x_0}\iota_X$. The identification of $T_{x_0}X$ with a subspace of $T_{(x_0,y_0)}(X \times Y)$ means that:

$$T^X_{(x_0,y_0)}f\ \xi = T_{(x_0,y_0)}f\ (\xi, 0).$$

Now apply a similar reasoning, but with interchanged roles of X and Y, to finish the proof. $\square$

Corollary 17.2 *Let $m : G \times G \to G$ be the multiplication map $(x, y) \mapsto xy$. Then for all $X, Y \in T_eG$ we have:*

$$T_{(e,e)}m\ (X, Y) = X + Y.$$

Another useful application is the following.

Corollary 17.3 *Let $f : \mathbb{R}^2 \to X$ be differentiable at $(0, 0)$. Then*

$$\frac{\partial}{\partial t} f(t,t)|_{t=0} = \frac{\partial}{\partial t} f(t,0)|_{t=0} + \frac{\partial}{\partial t} f(0,t)|_{t=0}.$$

Proof. Define $\varphi(t) = (t, t)$ and apply the chain rule to $f(\varphi(t))$. Then one sees that

$$\frac{\partial}{\partial t} f(t,t)|_{t=0} = T_{(0,0)}f \circ \frac{\partial}{\partial t}\varphi(0) = T_{(0,0)}f\,(1,1) = T_{(0,0)}f\,(1,0) + T_{(0,0)}f\,(0,1).$$

The right side of this equation equals the right side of the desired equation. $\square$

Note that in the above proof we have made use of the following lemma in a special case:

Lemma 17.4 *Let $f = (f_1, f_2) : Z \to X \times Y$ be differentiable at the point $z_0 \in Z$, and put $(x_0, y_0) = f(z_0)$. Then*

$$T_{z_0} f = (T_{z_0} f_1, T_{z_0} f_2).$$

Proof. Left to the reader. $\square$

Exercise 17.5 Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Show that for $X, Y \in \mathfrak{g}$ we have:

$$[X,Y] = \frac{\partial}{\partial s}\frac{\partial}{\partial t} \exp(sX)\exp(tY)\exp(sX)^{-1}\exp(tY)^{-1}|_{s=t=0}.$$

18 Commutativity of a Lie group

A Lie algebra $\mathfrak{l}$ is said to be abelian (or commutative) if $[X, Y] = 0$ for all $X, Y \in \mathfrak{l}$. Obviously the Lie algebra of an abelian Lie group is abelian. The converse is true if the group is connected. We will finish this section by establishing this result.

Lemma 18.1 *Let G be a Lie group, and assume that $X, Y \in \mathfrak{g}$. Then*

$$[X,Y] = 0 \quad \Rightarrow \quad \exp X \ \exp Y = \exp(X+Y).$$

Remark. If $X, Y \in \mathfrak{g}$ satify the hypothesis of the above lemma, we say that X and Y commute.

Proof. Assume that $X, Y \in \mathfrak{g}$ commute. It follows from the above assertion that $\exp X$ and $\exp Y$ commute. We will begin by proving this result, and then we will deduce the lemma from it.

By Corollaries 16.7 and 16.8 we have:

$$\exp X \ \exp Y\,(\exp X)^{-1} = \exp[\,\mathrm{Ad}\,(\exp X)Y\,] = \exp[\,e^{\mathrm{ad}\,X}Y\,] = \exp Y.$$

Hence $\exp X$ and $\exp Y$ commute. Applying the same reasoning to tX and tY we see that $\exp tX$ and $\exp tY$ commute for all $t \in \mathbb{R}$. Hence $\alpha(t) = \exp tX \ \exp tY$ is a 1 parameter subgroup of G. It follows that $\alpha(t) = \exp tZ$, where

$$Z = \frac{d}{dt}[\exp tX \ \exp tY]_{t=0} = \frac{d}{dt}[\exp tX]_{t=0} + \frac{d}{dt}[\exp tY]_{t=0} = X + Y$$

(see also Corollary 17.3). $\square$

If G is a Lie group, then the connected component of G containing e is denoted by G_e. This component is called the identity component of G. Obviously $e \in G_e$. Moreover, if $x \in G_e$ then xG_e is connected and open and closed in G, and it contains $x = xe$. Hence $xG_e = G_e$. We see that G_e is a subgroup of G.

Exercise 18.2 Show that G_e is a normal subgroup, i.e. $xG_ex^{-1} = G_e$ for all $x \in G$.

Lemma 18.3 *The identity component G_e of a Lie group G equals the subgroup generated by the elements* $\exp X$, $X \in \mathfrak{g}$.

Remark. Thus G_e is the subgroup consisting of elements of the form

$$\exp X_1 \exp X_2 \cdots \exp X_k \qquad (X_1, \ldots, X_k \in \mathfrak{g}).$$

Proof. Let H be the subgroup of G generated by the elements $\exp X$, $X \in \mathfrak{g}$. Since obviously $\exp \mathbb{R}X \subset G_e$ for all $X \in \mathfrak{g}$, we see that $H \subset G_e$. To prove that this inclusion is an equality, it suffices to show that H is open and closed in G. Let $\Omega \subset \mathfrak{g}$ be an open neighbourhood of 0 in $\mathfrak{g}$ such that exp is a diffeomorphism of Ω onto an open subset $U \subset G$. Let $h \in H$. Then $h \exp \Omega$ is an open neighbourhood of h contained in H. Hence H is an open subgroup of G. Now use the lemma below to conclude that H is closed as well. $\square$

Lemma 18.4 *Let G be a topological group. Then any open subgroup of G is closed as well.*

Proof. Let H be an open subgroup of G. Select, for every coset $x \in G/H$ a representative $g_x \in G$. Then G is the disjoint union of the sets g_xH, $x \in G/H$. Left translation being a homeomorphism, these cosets are all open. The complement of H in G equals the union of the cosets g_xH, $x \in G/H$, $x \neq eH$. Thus we see that this complement is open so that H is closed. $\square$

Proposition 18.5 *Let G be a connected Lie group. Then G is commutative if and only if its Lie algebra $\mathfrak{g}$ is commutative.*

Proof. If G is commutative then $Ad\,(x) = I_G$ for all $x \in G$. Differentiation yields $\mathrm{Ad}\,(x) = I_{\mathfrak{g}}$ for all $x \in G$, and differentiating this at $x = e$, we see that $\mathrm{ad} = 0$. Hence $\mathfrak{g}$ is commutative.

Conversely, assume that $\mathfrak{g}$ is commutative. Then $\exp X$ and $\exp Y$ commute, for all $X, Y \in \mathfrak{g}$. Hence the subgroup G_e generated by $\exp \mathfrak{g}$ is commutative. But this subgroup equals G, since G is connected. $\square$

19 Subgroups

Let G be a Lie group. Then a subgroup H which is a submanifold is called a Lie subgroup of G. Obviously a Lie subgroup H is a Lie group in a natural way, and the inclusion map $\iota : H \to G$ is a homomorphism of Lie groups. The associated tangent map $\iota_* : \mathfrak{h} \to \mathfrak{g}$ is an injective homomorphism of Lie algebras. Via this map we view $\mathfrak{h}$ as a subalgebra $\mathfrak{g}$. The commutative diagram of Lemma 16.12 now means that the exponential map of H is the restriction of the exponential map of G to $\mathfrak{h}$.

Remark. If H is a Lie subgroup of $\mathrm{GL}(V)$, then $\mathfrak{h}$ is a linear subspace of $\mathrm{End}(V)$ such that $X, Y \in \mathfrak{h} \Rightarrow XY - YX \in \mathfrak{h}$, and for all $X \in \mathfrak{h}$ we have $\exp X = e^X$. Many Lie groups may be realized as Lie subgroups of a general linear group: such Lie groups are called linear.

Lemma 19.1 *Let H be a Lie subgroup of a Lie group G. Then its Lie algebra $\mathfrak{h}$ is given by*

$$\mathfrak{h} = \{ X \in \mathfrak{g} \mid \exp \mathbb{R}X \subset H \}.$$

Proof. This is immediate from the above remarks. $\square$

Lemma 19.2 *Let H be a subgroup of a Lie group G. Then H is a Lie subgroup if and only if H is closed.*

Proof. If H is a Lie subgroup, then it is a submanifold, hence locally closed. Hence there exists a compact neighbourhood U of e in G such that $U \cap H$ is closed. Select an open neighbourhood V of e in G such that $VV^{-1} \subset U$.

Let h_n be a sequence in H converging to an element $g \in G$. Then there exists a $N > 0$ such that $p \geq N \Rightarrow h_p g^{-1} \in V$. Hence $p, q \geq N \Rightarrow h_p h_q^{-1} \in VV^{-1} \subset U$. In particular this implies that $h_N h_n^{-1} \in U \cap H$ for all $n \geq N$ and taking the limit for $n \to \infty$ we see that $h_N g^{-1} \in U \cap H$. This implies that $g \in H$. Hence H is closed.

The proof of the converse implitcation is deeper. We do not give its proof here, but refer to [BD], Theorem 3.11[9] instead. $\square$

Corollary 19.3 *Let H_1, H_2 be Lie subgroups of a Lie group G. Then $H = H_1 \cap H_2$ is a Lie subgroup of G with Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \cap \mathfrak{h}_2$.*

Proof. This is an easy consequence of the previous results. $\square$

We will now illustrate how to use the above tools to determine Lie groups and there Lie algebras. Let V be a complex linear space. Then V viewed as a real linear space is denoted by $V_{\mathbb{R}}$. Let $J \in \mathrm{End}(V_{\mathbb{R}})$ be the linear map $v \mapsto iv$ (multiplication by i). Then $\mathrm{End}(V)$, the space of complex linear maps $V \to V$, may be viewed as the space of $A \in \mathrm{End}(V_{\mathbb{R}})$ such that $AJ = JA$. Similarly $\mathrm{GL}(V)$ is the group of $g \in \mathrm{GL}(V_{\mathbb{R}})$ such that $gJ = Jg$. From this it is obvious that $\mathrm{GL}(V)$ is a closed subgroup of $\mathrm{GL}(V_{\mathbb{R}})$, hence a Lie subgroup. We claim that its Lie algebra equals $\mathrm{End}(V)$. To see this, let $X \in \mathrm{End}(V)$. Then $J \exp X J^{-1} = J e^X J^{-1} = e^{JXJ^{-1}} = e^X$. Hence exp maps $\mathrm{End}(V)$ into $\mathrm{GL}(V)$. On the other hand, if $X \in \mathrm{End}(V_{\mathbb{R}})$ and $\exp \mathbb{R}X \subset \mathrm{GL}(V)$, then

$$\exp tX = J \exp tX J^{-1} = e^{t\,JXJ^{-1}}.$$

Differentiating this expression at $t = 0$, we find that X commutes with J, hence belongs to $\mathrm{End}(V)$. In view of Lemma 19.1 the claim has now been established. Use similar techniques to make the following exercises.

[9][BD]: T. Bröcker, T. tom Dieck, Representations of Compact Lie Groups, Graduate Texts in Math. **98**, Springer-Verlag 1985.

Exercise 19.4 Let V be a real linear space, and define $\mathrm{SL}(V)$ to be the group of $g \in \mathrm{GL}(V)$ with determinant 1. Show that $\mathrm{SL}(V)$ is a Lie subgroup of $\mathrm{GL}(V)$, with Lie algebra

$$sl(V) = \{X \in \mathrm{End}(V) \,|\, \mathrm{tr}\, X = 0\}.$$

Let V be equipped with an inner product, and for $X \in \mathrm{End}(V)$ let X^* denote the adjoint of X with respect to the given inner product. Let $\mathrm{O}(V)$ be the associated group of orthogonal maps, i.e.

$$\mathrm{O}(V) = \{x \in \mathrm{GL}(V) \,|\, x^* = x^{-1}\}.$$

Show that $\mathrm{O}(V)$ is a Lie subgroup of $\mathrm{GL}(V)$ with Lie algebra

$$o(V) = \{X \in \mathrm{End}(V) \,|\, X^* = -X\}.$$

Show that $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}(V)$ is a Lie subgroup with Lie algebra $so(V) = o(V)$.

Exercise 19.5 Let V be a complex linear space, and define $\mathrm{SL}(V)$ to be the group of $g \in \mathrm{GL}(V)$ with determinant 1. Show that $\mathrm{SL}(V)$ is a Lie subgroup of $\mathrm{GL}(V)$, with Lie algebra

$$sl(V) = \{X \in \mathrm{End}(V) \,|\, \mathrm{tr}\, X = 0\}.$$

Let V be equipped with a Hermitean inner product, and for $X \in \mathrm{End}(V)$ let X^* denote the Hermiten adjoint of X with respect to the given inner product. Let $\mathrm{U}(V)$ be the associated group of unitary maps, i.e.

$$\mathrm{U}(V) = \{x \in \mathrm{GL}(V) \,|\, x^* = x^{-1}\}.$$

Show that $\mathrm{U}(V)$ is a Lie subgroup of $\mathrm{GL}(V)$ with Lie algebra

$$u(V) = \{X \in \mathrm{End}(V) \,|\, X^* = -X\}.$$

Show that $\mathrm{SU}(V) = \mathrm{U}(V) \cap \mathrm{SL}(V)$ is a Lie subgroup with Lie algebra

$$su(V) = \{X \in \mathrm{End}(V) \,|\, X^* = -X \quad \text{and} \quad \mathrm{tr}\, X = 0\}.$$

20 The groups SU(2) and SO(3)

In the previous section we saw that $\mathrm{SU}(2)$ is a Lie subgroup of $\mathrm{GL}(2, \mathbb{C})$. The Lie algebra of the latter group is the algebra $\mathrm{M}(2, \mathbb{C})$ of complex 2×2 matrices. The Lie algebra $su(2)$ is the algebra of $X \in \mathrm{M}(2, \mathbb{C})$ with

$$X^* = -X, \qquad \mathrm{tr}\, X = 0.$$

From this one sees that as a real linear space $su(2)$ is generated by the elements

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

One readily verifies that $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \sigma_3$, and $\sigma_2\sigma_3 = -\sigma_3\sigma_2 = \sigma_1$. Hence the commutator brackets are given by

$$[\sigma_1, \sigma_2] = 2\sigma_3, \quad [\sigma_2, \sigma_3] = 2\sigma_1, \quad [\sigma_3, \sigma_1] = 2\sigma_2.$$

From this it follows that the endomorphisms $\mathrm{ad}\,\sigma_j \in \mathrm{End}(su(2))$ have the following matrices with respect to the basis $\sigma_1, \sigma_2, \sigma_3$:

$$\mathrm{mat\,ad}\,\sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \mathrm{mat\,ad}\,\sigma_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad \mathrm{mat\,ad}\,\sigma_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The above elements belong to

$$so(3) = \{X \in \mathrm{M}(3, \mathbb{R}) \mid X^* = -X\},$$

the Lie algebra of the group SO(3).

If $a \in \mathbb{R}^3$, then the exterior product map $X \mapsto a \times X$, $\mathbb{R}^3 \to \mathbb{R}^3$ has matrix

$$R_a = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

with respect to the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$. Clearly $R_a \in so(3)$.

Lemma 20.1 *Let $t \in \mathbb{R}$. Then $\exp tR_a$ is the rotation with axis a and angle $t|a|$.*

Proof. Let $r \in \mathrm{SO}(3)$. Then one readily verifies that $R_a = r \circ R_{r^{-1}a} \circ r^{-1}$, and hence

$$\exp tR_a = r \circ \exp[tR_{r^{-1}a}] \circ r^{-1}.$$

Selecting r such that $r^{-1}a = |a|e_1$, we see that we may reduce to the case that $a = |a|e_1$. In that case one readily computes that:

$$\exp tR_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t|a| & -\sin t|a| \\ 0 & \sin t|a| & \cos t|a| \end{pmatrix}.$$

□

Write $R_j = R_{e_j}$, for $j = 1, 2, 3$. Then by the above formulas for $\mathrm{mat\,ad}\,(\sigma_j)$ we have

$$\mathrm{mat\,ad}\,(\sigma_j) = 2R_j \qquad (j = 1, 2, 3). \tag{25}$$

We now define the map $\varphi : \mathrm{SU}(2) \to \mathrm{GL}(3, \mathbb{R})$ by $\varphi(x) = \mathrm{mat\,Ad}\,(x)$, the matrix being taken with respect to the basis $\sigma_1, \sigma_2, \sigma_3$. Then φ is a homomorphism of Lie groups. Moreover, from

$$\varphi(\exp X) = \mathrm{mat}\, e^{\mathrm{ad}\, X} = e^{\mathrm{mat\,ad}\, X}$$

we see that φ maps $\mathrm{SU}(2)_e$ into SO(3). Since SU(2) is obviously connected, we have $\mathrm{SU}(2) = \mathrm{SU}(2)_e$, so that φ is a Lie group homomorphism from SU(2) to SO(3). The tangent map of

φ is given by $\varphi_* : X \mapsto \mathrm{mat}\,\mathrm{ad}\,X$. It maps the basis $\{\sigma_j\}$ of $su(2)$ onto the basis $\{2R_j\}$ of $so(3)$, hence is a linear isomorphism. It follows that φ is a local diffeomorphism at I, hence its image $\mathrm{im}\varphi$ contains an open neighbourhood of I in $\mathrm{SO}(3)$. Hence $\mathrm{im}\varphi$ is an open connected subgroup of $\mathrm{SO}(3)$, and we see that $\mathrm{im}\varphi = \mathrm{SO}(3)_e$. The latter group equals $\mathrm{SO}(3)$, by the lemma below. From this we conclude that $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a surjective group homomorphism. Hence $\mathrm{SO}(3) = \mathrm{SU}(2)/\ker\varphi$. The kernel of φ may be computed as follows. If $x \in \ker\varphi$, then $\mathrm{Ad}\,(x) = I$. Hence $x\sigma_j = \sigma_j x$ for $j = 1, 2, 3$. From this one sees that $x \in \{-I, I\}$. Hence $\ker\varphi = \{-I, I\}$.

Proposition 20.2 *The map $\varphi : \mathrm{SU}(2) \mapsto \mathrm{GL}(3)$, $x \mapsto \mathrm{mat}\,\mathrm{Ad}\,(x)$ is a surjective group homomorphism onto $\mathrm{SO}(3)$, and induces an isomorphism:*

$$\mathrm{SU}(2)/\{\pm I\} \simeq \mathrm{SO}(3).$$

For $k \in \mathbb{N}$ the representation π_{2k} of $\mathrm{SU}(2)$ factorizes to a representation $\bar{\pi}_{2k}$ of $\mathrm{SO}(3)$. The representations π_{2k} are mutually inequivalent and exaust $\widehat{\mathrm{SO}(3)}$.

Proof. By the preceding discussion it suffices to prove the assertions about the representations. One readily verifies that $\pi_{2k}(x) = I$ for $x \in \{\pm I\}$. Hence π_{2k} factorizes to a representation $\bar{\pi}_{2k}$ of $\mathrm{SO}(3)$. Every invariant subspace of the representation space V_{2k} of π_{2k} is $\pi_{2k}(\mathrm{SU}(2))$ invariant if and only if it is $\bar{\pi}_{2k}(\mathrm{SO}(3))$ invariant. A non-trivial $\mathrm{SO}(3)$–equivariant map $V_{2k} \to V_{2l}$ would also be $\mathrm{SU}(2)$–equivariant. Hence the $\bar{\pi}_{2k}$ are mutually inequivalent. Finally, to see that they exhaust $\widehat{\mathrm{SO}(3)}$, assume that (π, V) is an irreducible representation of $\mathrm{SO}(3)$. Then $\varphi^*\pi := \pi \circ \varphi$ is an irreducible representation of $\mathrm{SU}(2)$, hence equivalent to some π_n, $n \in \mathbb{N}$. From $\varphi^*\pi = I$ on $\ker\varphi$ it follows that $\pi_n = I$ on $\{\pm I\}$, hence n is even. $\square$

Lemma 20.3 *The group $\mathrm{SO}(3)$ is connected.*

Proof. Let $x \in \mathrm{SO}(3)$. There exists an orthogonal matrix m such that $mxm^{-1} = \exp tR_1$. The curve $c(t) = m^{-1}\exp tR_1 m$ $(0 \le t \le 1)$ lies in $\mathrm{SO}(3)$ and connects I with x. $\square$

Exercise 20.4 Let $n \ge 2$. Show that $\mathrm{SO}(n) = \mathrm{O}(n)_e$.

21 Invariant differential operators

Let X be a smooth manifold and v a smooth vector field on X. If $f \in C^\infty(G)$, then we define the function $vf \in C^\infty(G)$ by:

$$vf(x) = T_x f v(x).$$

If c is an integral curve for v, then using the chain rule one sees that

$$vf(c(t)) = \frac{d}{dt} f(c(t)).$$

If (U, x) is a (n–dimensional) chart for X, then the component functions x_j are called local coordinates for X. We define the vector fields $\partial/\partial x_j$ on U by:

$$(\frac{\partial}{\partial x_j})_p = (T_p\chi)^{-1} e_j.$$

Here e_j denotes the j–th standard basis vector of f. We now see that

$$\frac{\partial}{\partial x_j} f(p) = \partial_j (f \circ \chi^{-1})(\chi(p));$$

thus the left side of the above equality corresponds indeed to a partial derivative of f in the local coordinates x_j. From the fact that partial differentiations in $\mathbb{R}^n$ commute when applied to smooth functions we now conclude that for $1 \leq i, j \leq n$ the endomorphisms $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ of $C^\infty(U)$ commute.

For a multi-index $\alpha \in \mathbb{N}^n$ we define the following endomorphism of $C^\infty(U)$:

$$\frac{\partial^\alpha}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

A (linear) differential operator (with C^∞–coefficients) of order at most $m \in \mathbb{N}$ is a linear map $P : C^\infty(X) \to C^\infty(X)$, such that for every chart (U, x) of X there exist finitely many $c_\alpha \in C^\infty(U)$, $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$ such that

$$Pf|_U = \sum_\alpha c_\alpha \frac{\partial^\alpha}{\partial x^\alpha}(f|_U)$$

for all $f \in C^\infty(X)$.

In particular it follows that if $v \in \Gamma^\infty(TX)$ then $\partial_v : f \mapsto vf$ is a first order differential operator on X which annihilates the constant functions. Using local coordinates one sees that the linear map $v \mapsto \partial_v$ is injective. Using local coordinates one also sees that any first order differential operator on X which annihilates the constant functions arises from a unique vector field in this way.

Using local coordinates one sees that for $v, w \in \Gamma^\infty(TX)$ the differential operator $[\partial_v, \partial_w] = \partial_v\partial_w - \partial_w\partial_v$ is first order and annihilates constants. Hence $[\partial_v, \partial_w] = \partial_u$ for a unique smooth vector field u. This vector field, denoted by $[v, w]$, is called the Lie bracket of the vector fields v and w. From the definitions one readily sees that $\Gamma^\infty(TX)$ equipped with the Lie bracket is (an infinite dimensional) Lie algebra.

Let G be a Lie group. Then one readily sees that a vector field $v \in \Gamma^\infty(TG)$ is left invariant iff

$$v(L_x f) = L_x(vf) \quad \text{for all} \quad f \in C^\infty(G),\ x \in G,$$

i.e. iff ∂_v commutes with every left translation L_x, $x \in G$. From this one sees that the bracket of left invariant vector fields is left invariant again. Thus the space $\mathcal{L}$ of left invariant vector fields is a Lie subalgebra of $\Gamma^\infty(TG)$.

Lemma 21.1 *The map $X \to v_X$ is a Lie algebra isomorphism from $\mathfrak{g}$ onto $\mathcal{L}$.*

Proof. Let $X, Y \in \mathfrak{g}$. Then $[v_X, v_Y]$ is left invariant hence equals v_Z for some $Z \in \mathfrak{g}$. Thus, if $f \in C^\infty(G)$, then

$$T_e f\ Z = [v_X, v_Y]f(e) = v_X v_Y f(e) - v_Y v_X f(e).$$

Now

$$
\begin{aligned}
v_X v_Y f(e) &= \frac{\partial}{\partial s} v_Y f(\exp sX)|_{s=0} \\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial t} f(\exp sX \exp tY)|_{s=t=0} \\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial t} f(\exp(se^{t\,\mathrm{ad}\,X}Y)\exp(tX))|_{s=t=0} \\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial t}[f(\exp(se^{t\,\mathrm{ad}\,X}Y)) + f(\exp(sY)\exp(tX))]|_{s=t=0} \\
&= \frac{\partial}{\partial t} v_{e^{t\,\mathrm{ad}\,X}Y} f(e)|_{t=0} + \frac{\partial}{\partial t} v_X f(\exp sY)|_{s=0} \\
&= \frac{\partial}{\partial t} T_e f\,[e^{t\,\mathrm{ad}\,X}Y]|_{t=0} + v_Y v_X f(e) \\
&= T_e f[X,Y] + v_Y v_X f(e).
\end{aligned}
$$

Hence $T_e f\, Z = T_e f\,[X,Y]$ for every $f \in C^\infty(G)$, and it follows that $Z = [X,Y]$. $\square$

22 The algebra of invariant differential operators

A differential operator P on a Lie group G is called left invariant if for all $x \in G$ we have:

$$L_x \circ P = P \circ L_x \qquad \text{on} \qquad C^\infty(G).$$

Here L denotes the left regular action of G on $C^\infty(G) : L_x f(y) = f(x^{-1}y)$. The space $U(G)$ of left invariant differential operators is an algebra. From now on we shall use the notation ∂_X for ∂_{v_X}, $X \in \mathfrak{g}$. Then the map $X \mapsto \partial_X$ is a linear embedding of $\mathfrak{g}$ into $U(G)$. We have seen that for all $X, Y \in \mathfrak{g}$ we have:

$$\partial_X \partial_Y - \partial_Y \partial_X = \partial_{[X,Y]}. \tag{26}$$

From now on, let $X_1, \ldots, X_n$ be a fixed linear basis for $\mathfrak{g}$. If $\alpha \in \mathbb{N}^n$, we shall use the multi-index notation

$$\partial^\alpha = \partial_{X_1}^{\alpha_1} \cdots \partial_{X_n}^{\alpha_n}.$$

Here it should be noted that the order of the operators ∂_{X_j} is important, since they do not commute.

Lemma 22.1 *Let $\alpha \in \mathbb{N}^n$. Then for all $f \in C^\infty(G)$ we have:*

$$\partial^\alpha f(x) = \left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\alpha_n} f(x \exp t_1 X_1 \cdots \exp t_n X_n)|_{t_1 = \ldots = t_n = 0}.$$

Proof. Let $f \in C^\infty(G)$ and $X \in \mathfrak{g}$. Then we recall that for every $x \in G$ we have

$$\frac{\partial}{\partial s}[f(x \exp sX)]_{s=0} = \partial_X f(x).$$

Hence

$$\begin{aligned}
(\frac{\partial}{\partial t})^k [f(x \exp tX)]_{t=0} &= (\frac{\partial}{\partial t})^{k-1} [\frac{\partial}{\partial s} f(x \exp(t+s)X)|_{s=0}]_{t=0} \\
&= (\frac{\partial}{\partial t})^{k-1} [\partial_X f\,(x \exp tX)]_{t=0}.
\end{aligned}$$

By induction we thus see that

$$(\frac{\partial}{\partial t})^k [f(x \exp tX)]_{t=0} = (\partial_X)^k f\,(x). \tag{27}$$

The lemma follows from repeatedly applying the latter equation with (X, k) equal to $(X_n, \alpha_n), \ldots, (X_1, \alpha_1)$, respectively. $\square$

Proposition 22.2 *The elements ∂^α, $\alpha \in \mathbb{N}^n$ constitute a basis for the complex linear space $U(G)$.*

Proof. The smooth map

$$\varphi : (t_1, \ldots, t_n) \mapsto \exp t_1 X_1 \ldots \exp t_n X_n, \quad \mathbb{R}^n \to G$$

has tangent map $T_0\varphi : \mathbb{R}^n \to \mathfrak{g}$ given by:

$$(\tau_1, \ldots, \tau_n) \mapsto \tau_1 X_1 + \cdots + \tau_n X_n.$$

Hence there exists an open neighbourhood Ω of 0 in $\mathbb{R}^n$ such that φ maps Ω diffeomorphically onto an open neighbourhood U of e in G. Any m-th order differential operator P on U is given by:

$$Pf(\varphi(t)) = Q(f \circ \varphi)(t) \quad (f \in C^\infty(U)),$$

with $Q = \sum_{|\alpha| \le m} c_\alpha(t)(\frac{\partial}{\partial t})^\alpha$ a differential operator on Ω. Write $p_\alpha = c_\alpha(0)$. Then in particular we see that

$$\begin{aligned} Pf(e) &= \sum_{|\alpha| \le m} p_\alpha (\frac{\partial}{\partial t})^\alpha f(\exp t_1 X_1 \ldots \exp t_n X_n) \\ &= \sum_{|\alpha| \le m} p_\alpha \left[\partial^\alpha f\right](e), \end{aligned}$$

by repeated application of (27). If we assume, in addition, that P is left invariant, then

$$\begin{aligned} Pf(x) &= [L_{x^{-1}} Pf](e) = P(L_{x^{-1}} f)(e) \\ &= \sum_{|\alpha| \le m} p_\alpha \left[\partial^\alpha L_{x^{-1}} f\right](e) = \sum_{|\alpha| \le m} p_\alpha \left[\partial^\alpha f\right](x). \end{aligned}$$

Hence $P = \sum_{|\alpha| \le m} p_\alpha \partial^\alpha$ and we see that the ∂^α span $U(G)$. To see that they are linearly independent, let $m > 0$, and assume that $p_\alpha \in \mathbb{C}$ are such that

$$\sum_{|\alpha| \le m} p_\alpha \partial^\alpha = 0.$$

Then in particular it follows that for all $f \in C^\infty(G)$ we have

$$\sum_{|\alpha| \le m} p_\alpha \left(\frac{\partial}{\partial t}\right)^\alpha [f \circ \varphi](t_1, \ldots, t_n)] = \sum_{|\alpha| \le m} p_\alpha \partial^\alpha f(e) = 0.$$

From the fact that φ is a diffeomorphism it follows that f can be found such that $f \circ \varphi$ has any prescribed Taylor expansion at 0. Therefore all $p_\alpha, |\alpha| \le m$ must be zero. $\square$

From now on it will be convenient to view $\mathfrak{g}$ as a subspace of the algebra $U(G)$, via the injective map $\partial : X \mapsto \partial_X$. Thus, if $X, Y \in \mathfrak{g}$ then we have $[X, Y] = XY - YX$ in the algebra $U(G)$. Moreover, we have:

$$U(G) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathbb{C} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \tag{28}$$

as a linear space. Since the $X_1, \ldots, X_n$ do not commute in general, the product in $U(G)$ does not well behave with respect to the above decompostion. Write $[X_k, X_l] = \sum_j c_{kl}^j X_j$. Then we have for instance

$$\begin{aligned} (X_1 X_2)^2 &= X_1(X_2 X_1) X_2 = X_1(X_1 X_2 - \sum_{j=1}^n c_{12}^j X_j) X_2 \\ &= X_1^2 X_2^2 - c_{12}^1 X_1^2 X_2 - c_{12}^2 X_1 X_2^2 - \sum_{j \le 3} c_{12}^3 X_1 X_j X_2. \end{aligned}$$

The terms of the last sum can be rewritten as:

$$X_1 X_j X_2 = X_1 X_2 X_j + X_1 [X_j, X_2] = X_1 X_2 X_j + \sum_{k=1}^{n} c_{j2}^k \, X_1 X_k.$$

This indicates how to rewrite a general product $\partial^\alpha \partial^\beta$ as a linear combination of the ∂^γ, by using commutation relations. The decomposition (28) and the commutation relations $XY = YX + [X,Y]$ completely determine the structure of the algebra $U(G)$. This is best formulated by means of the following universal property.

Proposition 22.3 *Let G be a Lie group, and A an associative algebra (over $\mathbb{C}$ and with unit element). Assume that $\varphi : \mathfrak{g} \to A$ is a linear map such that*

$$\varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) \qquad \text{for all} \quad X, Y \in \mathfrak{g}.$$

Then there exists a unique algebra homomorphism $\Phi : U(G) \to A$ such that $\Phi = \varphi$ on $\mathfrak{g}$.

Uniqueness follows from the fact that $\mathfrak{g}$ generates $U(G)$ as an algebra, by (28). In this proof it will be convenient to use the notation

$$X^\alpha = \partial^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

We may define a linear extension of Φ of φ by setting

$$\Phi(X^\alpha) = \varphi(X_1)^{\alpha_1} \cdots \varphi(X_n)^{\alpha_n},$$

for all $\alpha \in \mathbb{N}^n$. The problem is then to show that Φ is a homomorphism of algebras. This can be done by a double induction argument, involving commutator relations. The first induction occurs in the proof of the following lemma. If $m \in \mathbb{N}$, let $U_m(G)$ denote the (finite dimensional) subspace of operators of order at most m in $U(G)$.

Lemma 22.4 *Let $X \in \mathfrak{g}$, $P \in U_m(G)$. Then $\Phi(XP) = \varphi(X)\,\Phi(P)$.*

Proof. We prove this by induction on m. For $m = 0$ the result is obvious. So, let $m > 0$ and assume the result has been proved already for smaller values of m. By linearity of Φ it suffices to prove the result for $X = X_j$, with $1 \leq j \leq n$. By linearity of Φ and the induction hypothesis we may as well assume that $P = X^\alpha$, where $\alpha \in \mathbb{N}^n$, $|\alpha| = m$. Let k be the smallest index for which $\alpha_k \neq 0$. Then $XP = X_j X_k^{\alpha_k} \cdots X_n^{\alpha_n}$. If $j \leq k$, then the result follows from the definition of Φ. So assume that $j > k$. Let α' be the multi-index $\alpha - e_k$, where e_k is the n-tuple with zeros in all coordinates except for the k-th, which is 1. Then $X^\alpha = X_k X^{\alpha'}$. Hence:

$$XP = X_j X_k X^{\alpha'} = X_k X_j X^{\alpha'} + [X_j, X_k] X^{\alpha'}.$$

From the definition and the linearity of Φ, and using the induction hypothesis, we now find:

$$\begin{aligned} \Phi(XP) &= \varphi(X_k)\,\Phi(X_j X^{\alpha'}) + \varphi([X_j, X_k])\,\Phi(X^{\alpha'}) \\ &= \varphi(X_k)\,\varphi(X_j)\Phi(X^{\alpha'}) + \varphi([X_j, X_k])\Phi(X^{\alpha'}). \end{aligned}$$

Using the rule $\varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$ we may rewrite the above as:

$$\Phi(XP) = \varphi(X_j)\varphi(X_k)\Phi(X^{\alpha'}).$$

The expression on the right side of this equation equals $\varphi(X_j)\Phi(X^\alpha)$, in view of the definition of Φ. $\square$

To complete the proof of Proposition 22.3 it suffices to show:

$$\Phi(PQ) = \Phi(P)\Phi(Q), \qquad P \in U_m(G), Q \in U(G). \tag{29}$$

We prove (29) by induction on m. For $m = 0$ the result is obvious from the definition of Φ. Thus assume that $m > 0$ and that the result has been proved already for smaller values of m. By linearity of Φ and the induction hypothesis we may as well assume that $P = P_1 X$, with $P_1 \in U_m(G)$, $X \in \mathfrak{g}$. Then by the induction hypothesis we have:

$$\Phi(PQ) = \Phi(P_1)\Phi(XQ).$$

From the previous lemma we have $\Phi(XQ) = \varphi(X)\Phi(Q) = \Phi(X)\Phi(Q)$, and using the induction hypothesis once more we now find:

$$\Phi(PQ) = \Phi(P_1)\Phi(X)\Phi(Q) = \Phi(P_1X)\Phi(Q) = \Phi(P)\Phi(Q).$$

□

The universal property described in the above proposition determines the algebra $U(G)$ up to isomorphism. Moreover, if G_1 and G_2 are Lie groups with isomorphic Lie algebras, then $U(G_1)$ and $U(G_2)$ are isomorphic. In fact we have:

Lemma 22.5 *Let G_1, G_2 be Lie groups, and assume that $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of their Lie algebras. Then φ has a unique extension to an algebra homomorphism $\Phi : U(G_1) \to U(G_2)$. If φ is an isomorphism of Lie algebras, then Φ is an isomorphism of associative algebras.*

Proof. As a map $\mathfrak{g}_1 \to U(G_2)$, the map φ satisfies $\varphi([X,Y]) = [\varphi(X), \varphi(Y)] = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$. By the universal property φ has a unique extension to an algebra homomorphism $\Phi : U(G_1) \to U(G_2)$. This proves the first assertion. Now assume that φ is an isomorphism, and let $\psi : \mathfrak{g}_2 \to \mathfrak{g}_1$ be its inverse. Then by the first part of the lemma ψ has a unique extension to an algebra homomorphism $\Psi : U(G_2) \to U(G_1)$. The composition $\Psi \circ \Phi$ extends $\psi \circ \varphi = I_{\mathfrak{g}_1}$, hence must equal the identity of $U(G_1)$, by uniqueness. Likewise $\Phi \circ \Psi = I$, and we see that Φ is an isomorphism with inverse Ψ. □

Because of the universal property of Proposition 22.3 the algebra $U(G)$ is called a universal enveloping algebra for $\mathfrak{g}$. In view of the above remark its isomorphism class only depends on the isomorphism class of the Lie algebra $\mathfrak{g}$, and therefore it is natural to construct the universal enveloping algebra in terms of $\mathfrak{g}$, without reference to a group. The construction is as follows.

Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification of $\mathfrak{g}$ and

$$T\mathfrak{g}_{\mathbf{C}} = \oplus_{k \in \mathbb{N}} \otimes^k \mathfrak{g}_{\mathbf{C}}$$

its tensor algebra. Let I be the two-sided ideal in $T\mathfrak{g}_{\mathbf{C}}$ generated by the elements $X \otimes Y - Y \otimes X - [X,Y]$, $X, Y \in \mathfrak{g}$. Then $U(\mathfrak{g}_{\mathbf{C}})$ is defined as the quotient algebra of $T\mathfrak{g}_{\mathbf{C}}$ by I. It is not hard to show that the map $\mathfrak{g} \to U(\mathfrak{g}_{\mathbf{C}})$ fulfills the universal property of Proposition 22.3. For details, see e.g. [Hum][10], Section 17.2. In particular if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then we map $\mathfrak{g} \to U(\mathfrak{g}_{\mathbf{C}})$ induces a natural isomorphism $U(G) \simeq U(\mathfrak{g}_{\mathbf{C}})$.

[10][Hum]: J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, New York Heidelberg Berlin, 1972

23 Bi-invariant differential operators

Let $x \in G$. Then the map $\mathrm{Ad}\,(x) : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra isomorphism, hence by Lemma 22.5 extends to an algebra automorphism of $U(G)$, which we also denote by $\mathrm{Ad}\,(x)$. By the uniqueness part of that lemma it follows that Ad defines a linear representation of G in $U(G)$.

Lemma 23.1 *For $x \in G$, let R_x denote the right translation $C^\infty(G) \to C^\infty(G)$ defined by $R_x f(y) = f(yx)$ for all $f \in C^\infty(G)$, $y \in G$. Then for all $D \in U(G)$ we have:*

$$\mathrm{Ad}\,(x)D = R_x \circ D \circ R_{x^{-1}}.$$

Proof. Since $\mathfrak{g}$ generates the algebra $U(G)$ it suffices to prove this for $D = X \in \mathfrak{g}$. Let $f \in C^\infty(G)$, $y \in G$. Then

$$\begin{aligned} R_x[X(R_{x^{-1}}f)](y) &= \partial_X[R_{x^{-1}}f](yx) \\ &= \frac{\partial}{\partial t} R_{x^{-1}} f(yx \exp tX)|_{t=0} \\ &= \frac{\partial}{\partial t} f(yx(\exp tX)x^{-1})|_{t=0} \\ &= \frac{\partial}{\partial t} f(y \exp t\mathrm{Ad}\,(x)X)|_{t=0} \\ &= \partial_{\mathrm{Ad}\,(x)X} f(y) = [\mathrm{Ad}\,(x)X]f(y). \end{aligned}$$

□

An operator $P \in U(G)$ is called bi-G-invariant if for all $x \in G$ we have $R_x \circ P \circ R_x^{-1} = P$. The algebra of such operators is denoted by $\mathbb{D}(G)$. By the above lemma it equals the algebra $U(G)^G$ of $\mathrm{Ad}\,(G)$-invariants in $U(G)$.

If $X \in \mathfrak{g}$, then the map $u \mapsto Xu - uX$ extends the endomorphism $\mathrm{ad}\,X$ of $\mathfrak{g}$. Therefore we write $\mathrm{ad}\,X$ for this linear endomorphism of $U(G)$.

Lemma 23.2 *Let $X \in \mathfrak{g}$. Then $\mathrm{ad}\,X$ is a derivation of $U(G)$, i.e. it is a linear endomorphism such that*

$$\mathrm{ad}\,X\,(uv) = (\mathrm{ad}\,X\,u)\,v + u\,(\mathrm{ad}\,X\,v) \qquad \text{for all} \quad u, v \in U(G).$$

Proof. This is immediate from the definitions. □

Remark. Notice that any derivation of $U(G)$ is completely determined by what it does on the generating space $\mathfrak{g}$.

Lemma 23.3 *The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(U(G))$ is a Lie algebra homomorphism.*

Proof. We must show that $\mathrm{ad}\,[X, Y] = \mathrm{ad}\,X\mathrm{ad}\,Y - \mathrm{ad}\,Y\mathrm{ad}\,X$. Now this is an immediate consequence of the definition. □

In the following sense the map ad is the tangent map of Ad. Let $m \in \mathbb{N}$. Then obviously for all $x \in G$ the map $\mathrm{Ad}\,(x)$ leaves the finite dimensional subspace $U_m(G)$ of $U(G)$ invariant. Let $\mathrm{Ad}_m(x) := \mathrm{Ad}\,(x)|U_m(G)$. Then $\mathrm{Ad}_m : G \to \mathrm{GL}(U_m(G))$ is readily seen to be a homomorphism of Lie groups.

Lemma 23.4 *The tangent map of* $\mathrm{Ad}_m : G \to \mathrm{GL}(U_m(G))$ *at* e *is given by* $X \mapsto \mathrm{ad}\,(X)|U_m(G)$.

Proof. Let $Y_1, \ldots, Y_k \in \mathfrak{g}$, $0 \le k \le m$. Then it suffices to show that for all $X \in \mathfrak{g}$ we have:

$$\mathrm{Ad}_{m*}(X)Y_1 \cdots Y_k = \mathrm{ad}\,(X)Y_1 \cdots Y_k.$$

Now this is seen as follows:

$$\begin{aligned}
\mathrm{Ad}_{m*}(X)(Y_1 \cdots Y_k) &= \frac{d}{dt}\mathrm{Ad}\,(\exp tX)Y_1 \cdots Y_k\,|_{t=0} \\
&= \frac{d}{dt}(\mathrm{Ad}\,(\exp tX)Y_1) \cdots \mathrm{Ad}\,(\exp tX)Y_k)\,|_{t=0} \\
&= \sum_{j=1}^{k} \frac{d}{dt} Y_1 \cdots Y_{j-1}\,[\,\mathrm{Ad}\,(\exp tX)Y_j\,]\,Y_{j+1} \cdots Y_k|_{t=0} \\
&= \sum_{j=1}^{l} Y_1 \cdots Y_{j-1}\,[\,\mathrm{ad}\,X(Y_j)\,]\,Y_{j+1} \cdots Y_k.
\end{aligned}$$

Since $\mathrm{ad}\,X$ is a derivation, the last member of this equation equals $\mathrm{ad}\,X(Y_1 \cdots Y_k)$. □

Lemma 23.5 *Let* $u \in U(G)$. *Then the following statements are equivalent.*

(a) *u is $\mathrm{Ad}\,(G_e)$–invariant;*

(b) *u is $\mathrm{ad}\,\mathfrak{g}$–invariant;*

(c) *u belongs to the center of the algebra $U(G)$.*

Proof. (a) ⇒ (b): suppose that $u \in U_m(G)$ is $\mathrm{Ad}\,(G_e)$ invariant. Then it follows that $\mathrm{Ad}_{m*}(X)u = 0$ for all $X \in \mathfrak{g}$. Now use that $\mathrm{ad}\,(X)u = \mathrm{Ad}_{m*}(X)u$.

(b) ⇒ (c): suppose that u is $\mathrm{ad}\,X$–invariant. Then $Xu - uX = \mathrm{ad}\,Xu = 0$ for all $X \in \mathfrak{g}$. Hence u commutes with a set of generators of the algebra $U(G)$ so that it must belong to the center.

(c) ⇒ (a): Suppose that $u \in U(G)$ is a central element. Fix $m \in \mathbb{N}$ such that $u \in U_m(G)$. Then $\mathrm{Ad}_{m*}(X) = \mathrm{ad}\,X\,u = 0$ for all $X \in \mathfrak{g}$. Hence $\mathrm{Ad}\,(\exp X)u = e^{\mathrm{Ad}_{m*}X}u = u$ for $X \in \mathfrak{g}$ (here we have used Lemma 16.12). It now follows that $\mathrm{Ad}\,(x)u = u$ for all $x \in G_e$. □

We shall now use the above to determine a particular bi-invariant differential operator of SU(2).

Lemma 23.6 *The element*

$$\Sigma = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

belongs to $\mathbb{D}(\mathrm{SU}(2))$.

Proof. Since $\mathrm{SU}(2)$ is connected, it suffices to show that $\mathrm{ad}\,(su(2))$ annihilates Σ. For this it suffices to show that Σ commutes with $\sigma_1, \sigma_2, \sigma_3$.

Since $\mathrm{ad}\,\sigma_1$ is a derivation, it follows that

$$[\sigma_1, \Sigma] = \sum_{j=1}^{3}([\sigma_1, \sigma_j]\sigma_j + \sigma_j[\sigma_1, \sigma_j]) = 0 + (\sigma_3\sigma_2 + \sigma_2\sigma_3) + (-\sigma_3\sigma_2 - \sigma_2\sigma_3) = 0.$$

Likewise one sees that Σ commutes with σ_2 and σ_3. □

The element

$$\mathcal{C} = -\frac{1}{4}\Sigma = -\frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \in \mathbb{D}(\mathrm{SU}(2))$$

is called the Casimir operator of $\mathrm{SU}(2)$.

The algebra isomorphism $\varphi = \mathrm{mat} \circ \mathrm{ad}$ from $su(2)$ onto $so(3)$ induces an algebra isomorphism $U(\mathrm{SU}(2)) \to U(\mathrm{SO}(3))$ which we also denote by φ. The image of the Casimir operator under this isomorphism is also denoted by $\mathcal{C}$. From (25) it follows that in $U(\mathrm{SO}(3))$ we have:

$$\mathcal{C} = -(R_1^2 + R_2^2 + R_3^2).$$

This element belongs to the center of $U(\mathrm{SO}(3))$. In view of Lemma 23.5 it therefore is a bi-invariant differential operator on $\mathrm{SO}(3)$.

24 Smooth functions with values in a locally convex space

Let $\Omega \subset \mathbb{R}^n$ be an open subset and V a complete locally convex space. A map $f : \Omega \to V$ is said to be differentiable at a point $a \in \Omega$ if there exists a linear map $Df(a) : \mathbb{R}^n \to V$ such that

$$f(a + h) - f(a) - Df(a)h = o(h) \quad (h \to 0).$$

The linear map $Df(a)$, which is uniquely determined by the above property, is called the derivative of f at a. Here the 'small oh' notation is defined as follows. If φ is a V–valued function defined on an open neighbourhood of 0 in $\mathbb{R}^n$ then

$$\varphi(h) = o(h) \quad (h \to 0)$$

means that $\lim_{h\to 0} \frac{\varphi(h)}{\|h\|} = 0$. Equivalently, this means that for every continuous seminorm p on V we have $\lim_{h\to 0} \|h\|^{-1} p(\varphi(h)) = 0$.

The function $f : \Omega \to V$ is said to have a directional derivative at $a \in \Omega$ in the direction $v \in \mathbb{R}^n \setminus \{0\}$ if

$$\partial_v f = \lim_{\substack{t\to 0 \\ t\in\mathbb{R}}} \frac{f(a + tv) - f(a)}{t}$$

exists. The function is said to be partially differentiable in a with respect to the j-th variable if $\partial_{e_j} f(a)$ exists (here e_j is the j–th standard basis vector of $\mathbb{R}^n$). We write $\partial_j f$ for ∂_{e_j}.

In a natural way $\mathrm{Hom}(\mathbb{R}^n, V)$, the space of linear maps $\mathbb{R}^n \to V$, may be equipped with the structure of a locally convex space. If p is a continuous seminorm of V, then we have an associated seminorm $\|\cdot\|_p$ on $\mathrm{Hom}(\mathbb{R}^n, V)$ defined by

$$\|T\|_p = \sup_{\|x\|=1} p(Tx) \qquad (T \in \mathrm{Hom}(\mathbb{R}^n, V)).$$

Thus $\|Tx\|_p \leq \|T\|_p \|x\|$ for all $x \in \mathbb{R}^n$. The space $\mathrm{Hom}(\mathbb{R}^n, V)$, equipped with the seminorms $\|\cdot\|_p$ is a complete locally convex space of its own right.

A differentiable map $f : \Omega \to V$ is said to be C^1 if $Df : \Omega \to \mathrm{Hom}(\mathbb{R}^n, V)$ is continuous. It is said to be twice differentiable if Df is differentiable. Using recursion we define f to be C^p if its derivative Df is C^{p-1}. We write $C^p(\Omega, V)$ for the space of C^p maps $\Omega \to V$, and put

$$C^\infty(\Omega, V) = \bigcap_{p \in \mathbb{N}} C^p(\Omega, V).$$

Lemma 24.1 *Let $I \subset \mathbb{R}$ be an open interval and $f \in C^1(I, V)$. Then for all $a, h \in \mathbb{R}$ with $a, a+h \in I$ we have:*

$$p(f(a+h) - f(a) - f'(a)h) \leq |h| \sup_{0 \leq t \leq 1} p(f'(a+th) - f'(a)).$$

Proof. We claim that

$$f(a+h) - f(a) - f'(a)h = h \int_0^1 [f'(a+th) - f'(a)] dt. \tag{30}$$

It suffices to prove the equation which arises if one applies a continuous linear functional $\eta \in V^*$ to the above equation. The functional η passes through the differentiation and the integration (use Proposition 6.3). Therefore the claim follows from elementary calculus.

The proof is completed by applying the estimate (9) to (30). □

Using the above estimate one can prove the following result in the same fasion as in the finte dimensional case:

Lemma 24.2 *Let $p \in \mathbb{N} \cup \{\infty\}$. Then the function $f : \Omega \to V$ is of class C^p if and only if for all sequences $j_1, \ldots, j_k \in \{1, \ldots, n\}$ of length $k \leq p$ the partial derivative:*

$$\partial_{j_1} \cdots \partial_{j_k} f$$

exists and defines a continuous function $\Omega \to V$.

It is now straightforward to define the notion of a differentiable map and of a C^p map $X \to V$, for X a smooth (finite dimensional) manifold (use local coordinates). Moreover, if $f : X \to V$ is differentiable at a point $a \in X$ then we define an associated tangent map $T_a f : T_a X \to V$ by

$$T_a f \dot{c}(0) = \frac{d}{dt} f(c(t))|_{t=0}$$

for c a differentiable curve in X with initial point a. Notice that this tangent map is linear and uniquely characterized by the property that

$$\eta \circ T_a f = T_a(\eta \circ f)$$

for every continuous linear functional η on V. By using local coordinates one may now prove the following versions of the chain rule:

Lemma 24.3 *Let X, Y be smooth manifolds, and V, W complete locally convex spaces. Assume that the map $f : X \to Y$ is differentiable at $a \in X$ and that the map $g : Y \to V$ is differentiable at $b = f(a)$. Then the composition $g \circ f$ is differentiable at a, and its tangent map is given by:*

$$T_a(g \circ f) = T_b g \circ T_a f.$$

Assume that $f : X \to V$ is differentiable at $a \in X$ and that $A : V \to W$ is a continuous linear map. Then $A \circ f$ is differentiable at $a \in X$ and

$$T_a(A \circ f) = A \circ T_a f.$$

Proof. Left to the reader $\square$

Let $\mathrm{PDO}(X)$ denote the algebra of all smooth differential operators on X. The elements of $\mathrm{PDO}(X)$ are linear endomorphisms of $C^\infty(X)$. Let $P \in \mathrm{PDO}(X)$. Then by using local coordinates we see that we may define an endomorphism $\tilde{P}$ of $C^\infty(X, V)$ by

$$\eta\,(\tilde{P} f) = P\,(\eta \circ f)$$

for all $f \in C^\infty(X, V)$ and every continuous linear functional η on V. From now on we shall suppress the tilde in the notations.

25 Smooth vectors in a representation

In this section we assume that G is a Lie group and that π is a continuous representation of G in a complete locally convex space V. A vector $v \in V$ is called smooth if $x \mapsto \pi(x)v$ is a smooth (i.e. C^∞) map from G to V. The space V^∞ of smooth vectors is a linear subspace of V which is readily seen to be G–invariant.

We shall first discuss some examples of smooth vectors.

Example. Let X be a smooth manifold, equipped with a (continuous) action of the Lie group G. We assume the action to be smooth, i.e. the map $(g, x) \mapsto gx$, $G \times X \to X$ is C^∞. This action naturally induces the following representation L in the space $C^\infty(G)$. If $\varphi \in C^\infty(X)$, $g \in G$, then $L_g\varphi(x) = \varphi(g^{-1}x)$.

The space $C^\infty(X)$ can be equipped with the structure of a locally convex space. If $K \subset X$ is a compact subset and $\mathcal{P} \subset \mathrm{PDO}(X)$ a finite set of differential operators, then we define the seminorm $\nu_{\mathcal{P},K}$ by

$$\nu_{\mathcal{P},K}(\varphi) = \max_{P \in \mathcal{P}} \sup_{x \in K} |P\varphi(x)|.$$

The set of seminorms defined in this way is complete, and thus $C^\infty(X)$ becomes a locally convex space.

Let us consider the particular case that X is an open subset of $\mathbb{R}^n$. For $K \subset X$ compact and $m \in \mathbb{N}$ we define the seminorm $\nu_{K,m}$ on $C^\infty(X)$ by

$$\nu_{K,m} = \max_{|\alpha| \leq m} \sup_{x \in K} | \left(\frac{\partial}{\partial x} \right)^\alpha f(x)|.$$

The set of seminorms $\nu_{K,m}$ thus obtained is already complete, so that it determines the topology of $C^\infty(X)$. It is well known (and not hard to prove) that the space $C^\infty(X)$ is complete.

One may limit the set of seminorms to a countable set, by restricting K to a sequence K_j of compact subsets with $K_j \subset K_{j+1}$ and $\cup_j K_j = X$. Thus one sees that $C^\infty(X)$ is a Fréchet space.

We now return to the case of an arbitrary smooth manifold X. Then by using local coordinates one can show that the locally convex space $C^\infty(X)$ is complete. Moreover, if X is σ–compact, i.e. X allows a countable covering by compact sets, then one can show that $C^\infty(X)$ is Fréchet.

Lemma 25.1 *Let G act smoothly on the smooth manifold X. Then the natural representation L of G in the complete locally convex space $C^\infty(X)$ is continuous. Moreover, every function of the space $C^\infty(G)$ is a smooth vector for L.*

Proof. See appendix. □

Another example of smooth vectors in a representation is provided by the following. Let dx be left Haar measure on G, and $L^1(G)$ the associated Banach space of integrable functions. The left regular representation L of G in $L^1(G)$ is continuous.

Lemma 25.2 *Let $\varphi \in C_c^\infty(G)$. Then φ is a smooth vector for the left regular representation $(L, L^1(G))$.*

Proof. See appendix. □

The following result garantees the existence of many smooth vectors in a representation.

Lemma 25.3 *Let $f \in C_c^\infty(G)$. Then $\pi(f)$ maps V to V^∞.*

Proof. Fix $v \in V$ and consider the continuous linear map: $A : \varphi \mapsto \pi(\varphi)v,\ L^1(G) \to V$. Let $f \in C_c^\infty(G)$. Then $F : f \mapsto L_x f$ is a smooth map $G \to L^1(G)$, by the previous lemma. The composition $A \circ F : G \to V$ is smooth by Lemma 24.3. But:

$$A \circ F(x) = A(L_x f) = \pi(L_x f)v = \pi(x)\pi(f)v$$

by Lemma 6.7, and it follows that $x \mapsto \pi(x)\pi(f)v$ is smooth. Hence $\pi(f)v \in V^\infty$. □

Remark. A smooth vector of the form $\pi(f)v$, with $f \in C_c^\infty(G)$, $v \in V$, is called a Gårding vector in V. A famous result of Dixmier and Malliavin, cf. [DM] [11], asserts that if V is a Fréchet space, then every smooth vector is a Gårding vector.

[11][DM]: J. Dixmier, P. Malliavin: Factorisations de fonctions et de vecteurs indéfiniment différentiables. Bull. Sci. Math. **102** (1978), 307 – 330.

Corollary 25.4 *The subspace V^∞ of smooth vectors is dense in V.*

Proof. This follows from combining the previous lemma with Lemma 6.8. □

Corollary 25.5 *If π is a finite dimensional representation then all of its vectors are smooth: $V = V^\infty$. Moreover, π is a Lie group homomorphism as a map $G \to \mathrm{GL}(V)$.*

Proof. The first assertion is an immediate consequence of the previous corollary. As for the second one, if $v \in V$, $\eta \in V^*$ then $x \mapsto \eta(\pi(x)v$ is a smooth function $G \to \mathbb{C}$. This implies that the coefficients of $\pi(x)$ with respect to a fixed basis of V depend smoothly on $x \in G$. Hence π is smooth as a map $G \to \mathrm{End}(V)$. Since $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$, this implies that π is smooth as a map $G \to \mathrm{GL}(V)$. □

The algebra $U(G)$ acts in a natural way on the space of smooth V–valued functions on G. This allows us to define, for $P \in U(G)$ and $v \in V^\infty$:

$$\pi(P)v := P(x \mapsto \pi(x))|_{x=e}.$$

One readily verifies that $\pi(P)$ belongs to $\mathrm{End}(V^\infty)$, the space of linear endomorphisms of V^∞. Moreover, the map $P \mapsto \pi(P)$ is a homomorphism of algebras $U(G) \to \mathrm{End}(V^\infty)$. The space V^∞ is a $U(G)$–module in this way.

If $X \in \mathfrak{g}$, $v \in C^\infty$, then retracing the definitions we see that

$$\pi(X)v = \frac{d}{dt}\pi(\exp tX)v|_{t=0}. \tag{31}$$

From (26) and the fact that $\pi : U(\mathfrak{g}) \to \mathrm{End}(V^\infty)$ is an algebra homomorphism it follows that for all $X, Y \in \mathfrak{g}$ we have:

$$\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X) \qquad \text{on} \qquad V^\infty.$$

Example. Let R be the right regular representation of G on $C^\infty(G)$. All vectors of this representation are smooth, so we have an associated algebra homomorphism $R : U(G) \to \mathrm{End}(C^\infty(G))$. Let $f \in C^\infty(G), X \in \mathfrak{g}$, then from (31) we see that

$$R(X)f(x) = \frac{d}{dt}f(x \exp tX)|_{t=0} = \partial_X f(x) = Xf(x).$$

Thus we see that $R = I$ on $\mathfrak{g}$. Since $\mathfrak{g}$ generates the algebra $U(\mathfrak{g})$ this implies that for all $D \in U(G)$ we have:

$$R(D) = D \qquad \text{on} \qquad C^\infty(G). \tag{32}$$

26 Lie algebra representations

Let W be a complex linear space. Then by a representation of $\mathfrak{g}$ in W we mean a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(W)$, i.e. ρ is a linear map such that for all $X, Y \in \mathfrak{g}$ we have: $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$. By the universal property ρ extends to an algebra homomorphism $U(\mathfrak{g}_\mathbf{c}) \to \mathrm{End}(W)$, turning W into a $U(\mathfrak{g}_\mathbf{c})$–module. Conversely, if W is a $U(\mathfrak{g}_\mathbf{c})$–module, then we may define a representation δ of $\mathfrak{g}$ in W by $\delta(X) : w \mapsto Xw$, $W \to W$ for $X \in \mathfrak{g}$. Thus we see that $\mathfrak{g}$ representations are in one to one correspondence with $U(\mathfrak{g}_\mathbf{c})$–modules. In the literature it also customary to speak of $\mathfrak{g}$–modules instead of $U(\mathfrak{g})$–modules. Likewise, if (π, V) is a (continuous) representation of G, then the topological linear space V together with the map $G \times V \to V$, $(x,v) \mapsto \pi(x)v$ is called a G–module.

Finite dimensional representations

If π is a finite dimensional representation, then by Corollary 25.5 π is a Lie group homomorphism $G \to \mathrm{GL}(V)$. Let $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ be the associated homomorphism of Lie algebras. Then from (31) we readily deduce that $\pi(X) = \pi_*(X)$. Hence in a setting like this we will from now on suppress the star in the notations.

Thus we see that a finite dimensional G-module V is automatically a $\mathfrak{g}$–module. Let π denote the associated representations of G and $\mathfrak{g}$ in V. Since $\pi : \mathfrak{g} \to \mathrm{End}(V)$ is the tangent map of the Lie homomorphism $\pi : G \to \mathrm{GL}(V)$, it follows from Lemma 16.12 that for all $X \in \mathfrak{g}$ we have:

$$\pi(\exp X) = e^{\pi(X)}. \tag{33}$$

When G is connected this equation allows us to compare the G– and the $\mathfrak{g}$–module structures on V.

Lemma 26.1 *Assume that G is connected, and let V, V' be two finite dimensional G–modules.*

(a) *Let W be a linear subspace of V. Then W is G–invariant if and only if W is $\mathfrak{g}$–invariant.*

(b) *The G–module V is irreducible if and only V is irreducible as a $\mathfrak{g}$–module.*

(c) *Let $T : V \to V'$ be a linear map. Then T is G–equivariant if and only if T is $\mathfrak{g}$–equivariant.*

Proof. Write π and π' for the representations of G in V and V' respectively.

(a): If W is $\mathfrak{g}$–invariant, then it follows from (33) that W is invariant under the group G_e which is generated by $\exp \mathfrak{g}$. But $G_e = G$, since G is connected. The converse implication is proved by differentiating $\pi(\exp(tX))$ at $t = 0$.

(b): This is now an immediate consequence of (a).

(c): Suppose that T is $\mathfrak{g}$–equivariant. Then for all $X \in \mathfrak{g}$ we have: $\pi'(X) \circ T = \pi(X) \circ T$, hence $\pi'(X)^n \circ T = \pi(X)^n \circ T$ for all $n \in \mathbb{N}$, and it follows that

$$e^{\pi'(X)} \circ T = T \circ e^{\pi(X)}.$$

From this it follows that $\pi'(x) \circ T = T \circ \pi(x)$ for all $x \in \exp \mathfrak{g}$, and hence for $x \in G_e = G$. The reverse implication follows by a straightforward differentiation argument as in part (a) of this proof. $\square$

Lemma 26.2 *Let G be a connected compact Lie group, and let π be a representation of G in a finite dimensional Hilbert space V. Then π is unitary if and only if*

$$\pi(X)^* = -\pi(X) \tag{34}$$

for all $X \in \mathfrak{g}$.

Proof. We recall that $\pi : G \to \mathrm{GL}(V)$ is a Lie group homomorphism. Hence for all $X \in \mathfrak{g}$, $t \in \mathbb{R}$ we have:

$$\pi(\exp tX) = e^{t\pi(X)}.$$

If π is unitary, then $\pi(\exp tX)^* = \pi(\exp(-tX))$, hence

$$e^{t\pi(X)^*} = e^{-t\pi(X)}. \tag{35}$$

Differentiating this relation at $t = 0$ we find (34). Conversely, if (34) holds, then (35) holds for all X, t and it follows that $\pi(x)$ is unitary for $x \in \exp \mathfrak{g}$. This implies that $\pi(x)$ is unitary for $x \in G_e = G$. $\square$

27 The action of bi-invariant differential operators

We now come to an important aspect of harmonic analysis on a Lie group G, involving the action of $\mathbb{D}(G)$ on the space V^∞ of π-smooth vectors in the complete locally convex space V.

Lemma 27.1 *Let $D \in U(G)$. Then for all $x \in G$ we have:*

$$\pi(x)\pi(D)\pi(x^{-1}) = \pi(\mathrm{Ad}\,(x)D) \qquad \text{on} \qquad V^\infty.$$

In particular, if $D \in \mathbb{D}(G)$, then $\pi(D)$ commutes with the action of G on V^∞.

Proof. It suffices to prove the first assertion for the generating elements $D = X \in \mathfrak{g}$. On V^∞ we have:

$$\begin{aligned} \pi(x)\circ\pi(X)\circ\pi(x^{-1}) &= \pi(x)\circ[\frac{d}{dt}\pi(\exp tX)|_{t=0}]\circ\pi(x^{-1}) \\ &= [\frac{d}{dt}\pi(x)\circ\pi(\exp tX)\circ\pi(x^{-1})]_{t=0} \\ &= \frac{d}{dt}\pi(\exp(t\mathrm{Ad}\,(x)X))|_{t=0} = \pi(\mathrm{Ad}\,(x)X). \end{aligned}$$

$\square$

Corollary 27.2 *Let (π, V) be finite dimensional and irreducible. Then the algebra $\mathbb{D}(G)$ acts by scalars on V.*

Proof. This follows from the previous lemma in combination with Schur's lemma (Lemma 3.9). $\square$

Let π be finite dimensional and irreducible. Then it follows from the above lemma that there exists an algebra homomorphism $\chi_\pi : G \to \mathbb{C}$ such that $\pi(D) = \chi_\pi(D)\, I_V$. The homomorphism χ_π is called the infinitesimal character of π.

Relation to the Peter-Weyl theorem

For the rest of this section we assume that G is a compact Lie group. Our goal is to relate the action of $\mathbb{D}(G)$ on $C^\infty(G)$ to the Peter-Weyl decomposition

$$L^2(G) = \widehat{\oplus}_{\delta\in\widehat{G}}\, C(G)_\delta.$$

If $\delta \in \widehat{G}$, then the elements of $C(G)_\delta$ are finite linear combinations of matrix coefficents of δ, hence smooth functions (cf. Corollary 25.5). The $C(G)_\delta$ are finite dimensional and right invariant; hence if $\varphi \in C(G)_\delta$, then $\Phi : x \mapsto R(x)\varphi$ is a smooth map $G \to C(G)_\delta$ (use Corollary 25.5 once more). Since $C(G)_\delta$ is finite dimensional this implies that Φ is smooth as a map $G \to L^2(G)$. Thus we see that the spaces $C(G)_\delta$ consist of smooth vectors for $(R, L^2(G))$.

Let $D \in U(G)$. Then from the discussion in the final example of Section 25 we have that $R(D) = D$ on $C(G)_\delta$. In particular we see that $C(G)_\delta$ is invariant under the left invariant differential operators.

Proposition 27.3 *Let $\delta \in \widehat{G}$. Then $C(G)_\delta$ is a finite dimensional subspace of $C^\infty(G)$, consisting of smooth vectors for $(R, L^2(G))$. Moreover, $C(G)_\delta$ is invariant under the action of all left invariant differential operators. Finally if $D \in \mathbb{D}(G)$, then D acts by the scalar $\chi_\delta(D)$ on $C(G)_\delta$.*

Proof. By the above discussion it remains to establish the last assertion. The right regular representation restricted to $C(G)_\delta$ splits as a finite direct sum of copies of δ. From this it follows that $R(D)$ acts by the scalar $\chi_\delta(D)$ on $C(G)_\delta$. Now use that $R(D) = D$ on $C^\infty(G)$, by (32). $\square$

Thus we see that the algebra $\mathbb{D}(G)$ admits a simultaneous diagonalization over the Peter-Weyl decomposition. The eigenvalues are given by the infinitesimal characters $\chi_\delta, \delta \in \widehat{G}$.

28 Representations of $su(2)$

Let $\mathcal{C}$ be the Casimir operator of SU(2). Then we shall compute the scalar $\chi_n(\mathcal{C}) := \chi_{\pi_n}(\mathcal{C})$ by which $\mathcal{C}$ acts on the representation space of the irreducible representation π_n of SU(2) introduced in Section 3.

We recall that the representation space of π_n is the space $P_n(\mathbb{C}^2)$ of polynomial functions $\mathbb{C}^2 \to \mathbb{C}$ which are homogeneous of degree n. The action of SU(2) on this space is given by

$$\pi_n(x)p(z) = p(x^{-1}z).$$

Let $X = (X_{ij})$ be a matrix in $su(2)$. Then the associated infinitesimal representation π_n of $su(2)$ is given by

$$\pi_n(X)p(z) = \frac{d}{dt}p(e^{-tX}z)|_{t=0}.$$

Applying the chain rule to the right side of the above equation we obtain:

$$\begin{aligned} \pi_n(X)p(z) &= \left[\frac{d}{dt}e^{-tX}|_{t=0}z\right]_1 \frac{\partial p}{\partial z_1}(z) + \left[\frac{d}{dt}e^{-tX}|_{t=0}z\right]_2 \frac{\partial p}{\partial z_2}(z) \\ &= -[X_{11}z_1 + X_{12}z_2]\frac{\partial p}{\partial z_1}(z) - [X_{21}z_1 + X_{22}z_2]\frac{\partial p}{\partial z_2}(z). \end{aligned} \tag{36}$$

This equation allows us to compute the action of the generators $\sigma_1, \sigma_2, \sigma_3$ of the algebra $U(\mathrm{SU}(2))$. However, for reasons that will become apparent in a moment, it will be more convenient to use the generators σ_1 and $X^\pm = -\frac{1}{2}(\pm\sigma_2 + i\sigma_3)$, i.e.

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Recall that the polynomials

$$p_k(z_1, z_2) = z_1^{n-k}\, z_2^k \qquad (0 \leq k \leq n)$$

constitute a basis for $P_n(\mathbb{C}^2)$. Using (36) one sees that the action of σ_1 on this basis is given by:

$$\sigma_1 p_k = i(n-2k)p_k \tag{37}$$

Here we have suppressed the π_n in the notation; we will continu to do so.

The action of the other two basis elements is described by

$$X^+ p = -z_2 \frac{\partial p}{\partial z_1}, \qquad X^- p = -z_1 \frac{\partial p}{\partial z_2}.$$

Hence

$$X^+ p_k = (k-n)\, p_{k+1}, \qquad X^- p_k = -k\, p_{k-1}. \tag{38}$$

These equations hold for all $0 \leq k \leq n$ if we agree to write $p_k = 0$ for $k < 0$ and $k > n$.

We are now ready to compute the scalar $\chi_n(\mathcal{C})$.

Lemma 28.1 *Let $n \in \mathbb{N}$. Then*

$$\chi_n(\mathcal{C}) = \frac{1}{4} n(n+2). \tag{39}$$

Proof. It suffices to compute the action of $\mathcal{C}$ on p_0. One readily verifies that in $U(\mathrm{SU}(2))$ we have:

$$\sigma_2^2 + \sigma_3^2 = -2(X^- X^+ + X^+ X^-).$$

In view of (38) the above element maps p_0 to $-2np_0$. Using this in combination with (37) we see that $\Sigma = \sum_j \sigma_j^2$ acts by the scalar $-n^2 - 2n = -n(n+2)$ on p_0. Hence $\mathcal{C} = -1/4\, \Sigma$ acts by the scalar (39). $\square$

The invariant inner product

We end this section with the computation of a SU(2)-invariant inner product on $P_n(\mathbb{C}^2)$. In view of Lemma 3.13 this inner product is determined up to a positive factor. Thus it is uniquely determined if we require that $\|p_0\| = 1$. Applying Lemma 26.2 to the representation π_n of SU(2), we see that the action of σ_1 is anti-Hermitian with respect to the inner product. This implies that its eigenspaces are mutually orthogonal. Hence the p_k constitute an orthogonal basis for $P_n(\mathbb{C}^2)$, and it remains to compute their lengths. Put $c_k = \|p_k\|^2$. Then $c_0 = 1$ by assumption. The element $\sigma_2 = X^- - X^+$ acts anti-symmetrically, hence for all $k \geq 0$ we have:

$$\langle (X^- - X^+)p_k\, ,\, p_{k+1}\rangle = \langle p_k\, ,\, (X^+ - X^-)p_{k+1}\rangle.$$

Using that $(X^- - X^+)p_k = -kp_{k-1} + (n-k)p_{k+1}$ and $(X^+ - X^-)p_{k+1} = (k+1-n)p_{k+2} + (k+1)p_k$, we see that this leads to:

$$(n-k)c_{k+1} = (k+1)c_k \qquad (0 \leq k \leq n).$$

This enables us to solve the c_k recursively from $c_0 = 1$, and we find:

$$c_k = \binom{n}{k}^{-1}.$$

We have proved:

Lemma 28.2 *Let $P_n(\mathbb{C}^2)$ be equipped with an inner product for which the representation π_n of $\mathrm{SU}(2)$ is unitary. Then the polynomials*

$$p_k(z) = z_1^{n-k} z_2^k \qquad (0 \le k \le n)$$

are mutually orthogonal, and

$$\frac{\|p_0\|}{\|p_k\|} = \binom{n}{k}^{1/2} \qquad (0 \le k \le n).$$

The following corollary will be used at a later stage.

Corollary 28.3 *The polynomials $(X^-)^k p_n$ constitute an orthogonal basis for $P_n(\mathbb{C}^2)$. Moreover,*

$$\frac{\|(X^-)^k p_n\|}{\|p_n\|} = k! \binom{n}{k}^{1/2}.$$

Proof. From repeated application of (38) it follows that

$$(X^-)^k p_n = (-1)^k \frac{n!}{(n-k)!}\, p_{n-k} = (-1)^k\, k! \binom{n}{k} p_{n-k}.$$

Now use the above lemma to complete the proof. □

29 Harmonic analysis on a homogeneous space

In this section we assume that G is a compact topological group, and that H is a closed subgroup. Then G/H is a compact topological space. Let $p : G \to G/H$ be the canonical projection. Then the pull-back map $p^* : f \mapsto f \circ p$ maps $C(G/H)$ injectively into $C(G)$. The image of this map equals the space $C(G)^H$ of right H-invariant continuous functions on G. Via the topological linear isomorphism p^* we shall identify $C(G/H)$ with $C(G)^H$.

Let I_G be the normalized Haar integral on G. Then one readily verifies that the restriction I of I_G to $C(G/H)$ is a positive Radon integral on G/H which is invariant for the left regular representation L of G on $C(G/H)$. Moreover, $I(1) = 1$.

Let $L^2(G/H)$ be the completion of $C(G/H)$ with respect to the L^2–inner product associated with I. Then the left regular representation extends to a unitary representation of G on $L^2(G/H)$. Thus the theory of Section 10 may be applied, and we see that we have the left invariant Hilbert-decomposition

$$L^2(G/H) = \widehat{\bigoplus}_{\delta \in \widehat{G}}\, L^2(G/H)_\delta. \tag{40}$$

Here $L^2(G/H)_\delta$ denotes the space of isotypical left G–finite vectors of type δ in $L^2(G/H)$.

Notice that the L^2 structure is defined in such a way that the map p^* extends to an isometric and equivariant embedding of $L^2(G/H)$ into $L^2(G)$, the image being the subspace of right H-invariant functions. Let $L^2_\delta(G)$ denote the space of isotypical left G–finite vectors of type δ in $L^2(G)$. Then $L^2_\delta(G) = C(G)_{\delta^*}$, cf. Exercise 10.6. By equivariance we have that

$$L^2(G/H)_\delta = L^2_\delta(G) \cap L^2(G/H). \tag{41}$$

If $\delta \in \widehat{G}$, define the map $S_\delta : \mathrm{End}(\mathrm{H}_\delta) \to C(G)$ by

$$S_\delta(A)(x) = \mathrm{tr}\,(\delta(x)^{-1}A).$$

Then one readily verifies that S_δ intertwines the representations $\delta \otimes \delta^*$ and $L \times R$ of $G \times G$. In analogy with Corollary 4.4 we now have:

Lemma 29.1 *The map $\sqrt{d(\delta)}S_\delta$ is an equivariant isometry from* $\mathrm{End}(\mathrm{H}_\delta)$ *onto* $L^2_\delta(G)$.

Proof. It remains to verify that $\sqrt{d(\delta)}S_\delta$ is an isometry. For this we compare with the map T_δ of Corollary 4.4. The map $\nu : A \mapsto A^t$ is an isometry from $\mathrm{End}(\mathrm{H}_\delta)$ onto $\mathrm{End}(\mathrm{H}_\delta^*)$. Moreover, one readily verifies that $S_\delta = T_\delta \circ \nu$. The result now follows from Corollary 4.4. $\square$

By equivariance S_δ^{-1} maps the space (41) bijectively onto the space V of endomorphisms $A \in \mathrm{End}(\mathrm{H}_\delta)$ with $A \circ \delta(x)^{-1} = A$ for all $x \in H$. By unitarity of δ this condition on A is equivalent to the condition that $A = 0$ on $(\mathrm{H}_\delta^H)^\perp$. Therefore, restriction to H_δ^H induces a bijective linear map $V \simeq \mathrm{Hom}(\mathrm{H}_\delta^H, \mathrm{H}_\delta)$. We thus view the latter space as a subspace of $\mathrm{End}(\mathrm{H}_\delta)$. It follows that S_δ restricts to a linear isomorphism

$$S_\delta : \mathrm{Hom}(\mathrm{H}_\delta^H, \mathrm{H}_\delta) \xrightarrow{\simeq} L^2(G/H)_\delta.$$

Let $\delta \otimes 1$ be the representation of G on $\mathrm{Hom}(\mathrm{H}_\delta^H, \mathrm{H}_\delta)$ defined by $\delta \otimes 1(x)A = \delta(x)A$. Then the above map S_δ intertwines $\delta \otimes 1$ and the left action L of G on $L^2(G/H)_\delta$.

In particular it follows that $L^2(G/H)_\delta$ is non-trivial if and only if δ has non-trivial H-fixed vectors. Hence the decomposition (40) ranges over the set $\widehat{G}_H$ of $\delta \in \widehat{G}$ for which $\mathrm{H}_\delta^H \neq 0$.

For $\delta \in \widehat{G}_H$ let n_δ denote the dimension of the space H_δ^H. Then $n_\delta \geq 1$. Let $u_1, \dots, u_{n_\delta}$ be a basis of H_δ^H. Then from the natural (equivariant) isomorphisms

$$\mathrm{Hom}(\mathrm{H}_\delta^H, \mathrm{H}_\delta) \simeq \oplus_{j=1}^{n_\delta} \mathrm{Hom}(\mathbb{C}u_j, \mathrm{H}_\delta) \simeq \oplus_{j=1}^{n_\delta} \mathrm{H}_\delta$$

one sees that $\delta \otimes 1$ is a multiple of δ : in fact $\delta \otimes 1 \sim n_\delta \delta$. Thus $n_\delta = \dim \mathrm{H}_\delta^H$ may be interpreted as the multiplicity of the representation δ in $L^2(G/H)$.

Of particular interest is the case that $\dim \mathrm{H}_\delta^H = 1$ for all $\delta \in \widehat{G}_H$. We say that the decomposition (40) is multiplicity free in this case. For every $\delta \in \widehat{G}_H$ fix a non-trivial element $u_\delta \in \mathrm{H}_\delta^H$ of unit length and define the map $\mathbf{s}_\delta : \mathrm{H}_\delta \to L^2(G/H)$ by $\mathbf{s}_\delta(v)(x) = \sqrt{d(\delta)}\,\langle v | \delta(x) u_\delta \rangle$. Then it follows from the above discussion that $\mathbf{s}_\delta$ is an equivariant isometry from H_δ onto $L^2(G/H)_\delta$.

We now consider the particular case that G is a compact Lie group. Then by Lemma 19.2 the group H is a Lie subgroup. One can show that G/H carries the structure of a smooth manifold such that the action map $G \times G/H \to G/H$ is smooth, and such that p^* maps $C^\infty(G/H)$

bijectively onto $C^\infty(G)^H$ (for details we refer the reader to [BD][12], Section I.4. Thus, by the above discussion and the results of Section 27 we have that

$$L^2(G/H)_\delta \subset C^\infty(G/H) \subset C^\infty(G).$$

Moreover, the algebra $\mathfrak{g}$ acts from the left by first order differential operators on $L^2(G/H)_\delta$. This action lifts to an action L of $U(G)$. If $D \in \mathbb{D}(G)$, then $L(G)$ is a left invariant differential operator on $C^\infty(G/H)$, which diagonalizes over the decomposition (40). On $L^2(G/H)_\delta$ the action is given by the scalar $\chi_\delta(D)$.

30 Harmonic analysis on the sphere

In this section we study harmonic analysis on the two dimensional sphere from the viewpoint of group theory. We shall compare our results with the classical theory of spherical harmonics.

The relation to group theory is rooted in the fact that S^2, the unit sphere in $\mathbb{R}^3$, is an orbit for the natural action of SO(3) on $\mathbb{R}^3$. Let e_1 be the first standard basis vector in $\mathbb{R}^3$. Then the stabilizer of e_1 in SO(3) is the image of SO(2) under the injective group homomorphism

$$\iota : \mathrm{SO}(2) \to \mathrm{SO}(3), \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Via this homomorphism we view SO(2) as a closed subgroup of SO(3). Thus we see that the map $a : \mathrm{SO}(3) \to S^2$, $x \mapsto x\, e_1$ factorizes to an isomorphism

$$\mathbf{a} : \mathrm{SO}(3)/\mathrm{SO}(2) \xrightarrow{\simeq} S^2.$$

Equip S^2 with the rotation invariant Radon measure of total measure 1, and let $L^2(S^2)$ be the corresponding L^2–space. Then via the left action L the group SO(3) acts unitarily on $L^2(S^2)$. Moreover, the map $\mathbf{a}^* : f \mapsto f \circ \mathbf{a}$ is an equivariant isometry from $L^2(S^2)$ onto $L^2(\mathrm{SO}(3)/\mathrm{SO}(2))$. Applying the theory of Section 29 we thus obtain the decomposition:

$$L^2(S^2) = \widehat{\oplus}_{m \in \mathbb{N}} \; L^2(S^2)_m, \tag{42}$$

where $L^2(S^2)_m$ denotes the space of functions of left type $\bar{\pi}_{2m}$. The latter space consists of smooth functions, and is equivariantly isomorphic to $\mathrm{Hom}(P_{2m}(\mathbb{C}^2)^{\mathrm{SO}(2)}, P_{2m}(\mathbb{C}^2))$. Note that $\mathrm{SO}(2) = \exp \mathbb{R} R_1$. Hence the preimage of SO(2) under the epimorphism $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ of Proposition 20.2 equals T. It follows that

$$P_{2m}(\mathbb{C}^2)^{\mathrm{SO}(2)} = P_{2m}(\mathbb{C}^2)^T. \tag{43}$$

We recall that the action of T diagonalizes over the polynomials $p_k = z_1^{2m-k} z_2^k$, which constitute a basis of $P_{2m}(\mathbb{C}^2)$. The action of T on p_k is trivial if and only if $k = m$. Hence the dimensions of the spaces in (43) equal 1. Thus we see that $L^2(S^2)_m$ is isomorphic to the SO(3)–module $P_{2m}(\mathbb{C}^2)$ (for every $m \in \mathbb{N}$). In particular it follows that the decomposition (42) is multiplicity free, and that its m–th componentis $2m+1$–dimensional.

The left action induces an algebra homomorphism from $U(\mathrm{SO}(3))$ to the differential operators on S^2. The image $L(\mathcal{C})$ of the Casimir under this map is a second order differential operator which diagonalizes over the decomposition (42); in particular it acts by the scalar $\chi_{2m}(\mathcal{C}) = m(m+1)$ on the component $L^2(S^2)_m$.

[12][BD]: T. Bröcker & T. tom Dieck, Representations of compact Lie groups. Springer-Verlag, 1985

31 Spherical harmonics

In this section we compare the results of the previous section with classical spherical harmonics. The natural action of SO(3) on $\mathbb{R}^3$ induces a representation L of $U(SO(3))$ in $C^\infty(R^3)$ by means of differential operators. Let $a \in \mathbb{R}^3$, and let $R \in so(3)$ be the matrix of the map $x \mapsto a \times x$. Then the first order differential operator $L(R)$ is given by:

$$\begin{aligned} L(R)f(x) &= \frac{d}{dt} f(e^{-tR}x)|_{t=0} = Df(x)(-Rx) \\ &= \operatorname{grad} f(x) \cdot (x \times a) = \det(\operatorname{grad} f(x), x, a). \end{aligned}$$

Substituting $a = e_1, e_2, e_3$ respectively one thus finds:

$$\begin{aligned} L(R_1) &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \\ L(R_2) &= x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \\ L(R_3) &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}. \end{aligned} \tag{44}$$

We now recall that the Casimir is given by $\mathcal{C} = -(R_1^2 + R_2^2 + R_3^2)$. Hence

$$L(\mathcal{C}) = -\sum_{j=1}^{3} L(R_j)^2 = -x^2\Delta + E(E+1), \tag{45}$$

where Δ is the ordinary Laplacian, where x^2 denotes multiplication by $x^2 = \sum_j x_j^2$, and where E denotes the Euler operator:

$$E = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

For m a non-negative integer let $P_m(\mathbb{R}^3)$ denote the (finite dimensional) space of homogeneous polynomial functions $\mathbb{R}^3 \to \mathbb{C}$ of degree m. Then E acts by the scalar m on $P_m(\mathbb{R}^3)$. Let

$$\mathfrak{H}_m = \{p \in P_m(\mathbb{R}^3)\,; \quad \Delta p = 0\},$$

the space of harmonic polynomials in $P_m(\mathbb{R}^3)$. This space is non-trivial, since it obviously contains the function: $\mathbf{Y}(x) = (x_2 + ix_3)^m$.

Moreover, the space $\mathfrak{H}_m$ is invariant under the action L of SO(3). In particular $L(\mathcal{C})$ leaves this space invariant, and by using (45) we see that $L(\mathcal{C})$ acts by the scalar $m(m+1)$ on it. From general theory we know that the finite dimensional SO(3)–module $\mathfrak{H}_m$ splits as a direct sum of irreducibles. If $\bar{\pi}_{2k} \in \mathrm{SO}(3)^\wedge$ occurs in $\mathfrak{H}_m$ then we must have $\chi_{2k}(\mathcal{C}) = m(m+1)$. This implies that $k = m$. Thus $\mathfrak{H}_m$ is equivalent to a multiple of the irreducible module $P_{2m}(\mathbb{C}^2)$. In fact we have:

Lemma 31.1 *Restriction to S^2 induces an equivariant isomorphism of $\mathfrak{H}_m$ onto $L^2(S^2)_m$. In particular the SO(3)–module $\mathfrak{H}_m$ is equivalent to the irreducible module $P_{2m}(\mathbb{C}^2)$.*

Proof. Consider the restriction map

$$\rho : \mathfrak{H}_m \to C^\infty(S^2),\ f \mapsto f|S^2.$$

It maps $\mathfrak{H}_m$ equivariantly and injectively into $C^\infty(S^2)$. By the above discussion its image is a non-trivial subspace of $L^2(S^2)_m$, and since the latter module is irreducible, it follows that the image of ρ equals $L^2(S^2)_m$. $\square$

We shall now use our insight in the structure of the representations $\bar{\pi}_{2m}$ to compute the elements of $L^2(S^2)_m$ explicitly in terms of spherical coordinates φ, θ on S^2. These coordinates are determined by the relations:

$$x = (\cos\theta, \sin\theta\, e^{i\varphi}) \qquad (0 \leq \varphi \leq 2\pi,\, 0 \leq \theta \leq \pi).$$

Here we identify the x_2, x_3–plane with the complex plane via $(x_2, x_3) \mapsto x_2 + ix_3$.[13]

In spherical coordinates the function $Y = \mathbf{Y}|S^2$ is given by:

$$Y = (\sin\theta)^m e^{im\varphi}. \tag{46}$$

It follows that the rotation $\exp(\psi R_1)$ acts by the scalar $e^{-im\psi}$ on Y. Via the epimorphism $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ defined in Proposition 20.2 we may view $C^\infty(\mathbb{R}^3)$ as a SU(2)–module. Recall that $\varphi(t_\psi) = \exp(2\psi R_1)$. Hence t_ψ acts by the scalar $e^{-2im\psi}$ on Y.

By the results of Section 30 there exists an equivariant isomorphism

$$T : L^2(S^2)_m \xrightarrow{\simeq} P_{2m}(\mathbb{C}^2) \tag{47}$$

of SU(2)–modules. The image $T(Y)$ of Y is a non-trivial element of $P_{2m}(\mathbb{C}^2)$ on which $t_\psi \in T$ acts by the scalar $e^{-i2m\psi}$. This implies that $T(Y)$ is a non-trivial multiple of $p_{2m} = z_2^{2m}$. Each polynomial $\pi_{2m}(iX^-)^k p_{2m}$ $(0 \leq k \leq 2m)$ is a non-trivial multiple of p_{2m-k}. These polynomials constitute a basis for $P_{2m}(\mathbb{C}^2)$. The corresponding functions

$$Y_{m-k} = (iX^-)^k Y \qquad (0 \leq k \leq 2m)$$

therefore constitute a basis for $L^2(S^2)_m$.

We recall that t_ψ acts by the scalar $e^{i(2k-2m)\psi}$ on p_{2m-k}. Hence the rotation $\exp(\psi R_1)$ acts by the scalar $e^{i(k-m)\psi}$ on Y_{m-k}. This implies that in spherical coordinates Y_k must be of the form:

$$Y_k = e^{ik\varphi} f_k(\theta) \qquad (-m \leq k \leq m). \tag{48}$$

Notice that the function Y_0 is invariant under rotations around the x_1–axis. Its level curves thus divide the sphere into spherical zones. The function Y_0 is therefore called a zonal spherical function.

The φ–image of $X^- = \frac{1}{2}(\sigma_2 - i\sigma_3)$ in $so(3)$ equals $R_2 - iR_3$. Hence we obtain:

$$Y_{m-k} = (iR_2 + R_3)^k Y \qquad (0 \leq k \leq 2m). \tag{49}$$

Using (44) and a substitution of variables, one finds that in terms of spherical coordinates the operator $(iR_2 + R_3)$ is given by

$$iR_2 + R_3 = -e^{-i\varphi}\Big(\frac{\partial}{\partial\theta} - i\cot\theta\,\frac{\partial}{\partial\varphi}\Big).$$

[13] Usually a cyclic permutation of these coordinates is used, but the present choice is more convenient for our purposes

Combining this with (46), (48) and (49) we see that

$$f_m = (\sin\theta)^m \tag{50}$$

and the f_k are related by the formula

$$-\frac{df_k}{d\theta} - k\cot\theta\, f_k = f_{k-1} \qquad (|k| \le m).$$

We now substitute $s = \cos\theta$. This substitution is allowed, since $\theta \mapsto \cos\theta$ is a monotonically decreasing function on $[0, \pi]$. If we write $p_k(s)$ for $f_k(\theta)$ viewed as a function of s, the above relation becomes:

$$(1-s^2)^{1/2}\left(\frac{dp_k}{ds} - \frac{ks}{1-s^2}\, p_k\right) = p_{k-1}.$$

This relation may be simplified by putting $u_k(s) = (1-s^2)^{k/2}\, p_k(s)$. It then becomes:

$$\frac{du_k}{ds} = u_{k-1}.$$

From (50) we see that

$$u_m(s) = (1-s^2)^{m/2} p_m(s) = (1-s^2)^m.$$

Hence

$$u_k(s) = \frac{d^{m-k}}{ds^{m-k}}(1-s^2)^m \qquad (|k| \le m),$$

and it follows that

$$p_k(s) = (1-s^2)^{-k/2}\frac{d^{m-k}}{ds^{m-k}}(1-s^2)^m \qquad (|k| \le m).$$

One normally uses a different normalization for the p_k. Put

$$P_m^k(s) = \frac{(1-s^2)^{-k/2}}{2^m\, m!}\frac{d^{m-k}}{ds^{m-k}}(s^2-1)^m \qquad (|k| \le m).$$

Then $P_m = P_m^0$ is called the m–th Legendre polynomial and the P_m^k $(|k| \le m)$ are called its associated Legendre functions. Notice that the chosen normalization has the effect that $P_m(1) = 1$.

It follows from the above discussion that the functions

$$\tilde{Y}_m^k = e^{ik\varphi}\, P_m^k(\cos\theta)$$

constitute a basis for $L^2(S^2)_m$. This basis is not yet orthonormal. Put

$$Y_m^k = (-1)^k\left[\frac{(2m+1)(m+k)!}{4\pi(m-k)!}\right]^{1/2}\tilde{Y}_m^k. \tag{51}$$

Proposition 31.2 *The functions Y_m^k, $m \in \mathbb{N}, |k| \le m$ constitute a complete orthonormal system for $L^2(S^2)$. Here S^2 is equipped with the rotation invariant measure for which S^2 has surface 4π.*

Proof. The spaces $L^2(S^2)$ are orthogonal. Thus it suffices to show that for a fixed m the Y_m^k constitute an orthonormal basis for $L^2(S^2)_m$.

From the above discussion it follows that $\tilde{Y}_m^k = (iX^-)^{m-k}\tilde{Y}_m^m$. We recall that the isomorphism T of (47) maps $\tilde{Y}_m^m$ onto a multiple of p_{2m}. After composing with a suitable scalar multiplication we may as well assume that $T(\tilde{Y}_m^m) = p_{2m}$. Then $T(\tilde{Y}_m^k) = (iX^-)^{m-k}p_{2m}$, by equivariance.

The image of the inner product on $L^2(S^2)_m$ under T is an inner product on $P_{2m}(\mathbb{C}^2)$ for which π_{2m} is unitary. Thus by Corollary 28.3 we see that the $\tilde{Y}_m^k$ are mutually orthogonal, and their lengths are given by:

$$\frac{\|\tilde{Y}_m^k\|^2}{\|\tilde{Y}_m^m\|^2} = \frac{\|(X^-)^{m-k}p_{2m}\|^2}{\|p_{2m}\|^2} = [(m-k)!]^2 \binom{2m}{m-k}. \tag{52}$$

The function $Y = (\sin\theta)^m e^{im\varphi}$ has L^2–norm

$$\|Y\|_2 = \int_0^{2\pi}\int_0^{\pi} |Y(\theta,\varphi)|^2 \sin\theta \; d\theta d\varphi = 2\pi\int_0^{\pi}(\sin\theta)^{2m+1}\; d\theta = 4\pi\frac{(2^m m!)^2}{(2m+1)!}.$$

Hence

$$\|\tilde{Y}_m^m\|_2 = \frac{4\pi}{(2m+1)!}.$$

By combining the last equality with (52) it now follows that the Y_m^k have length 1.

Final comments

The Casimir of $SO(3)$ is closely related to the spherical Laplacian Δ^*. If $f \in C^\infty(S^2)$, let $\tilde{f}$ denote its extension to $\mathbb{R}^3$ defined by setting: $\tilde{f}(tx) = f(x)$, $x \in S^2$, $t > 0$. Then the spherical Laplacian is defined by

$$\Delta^* f = (\Delta\tilde{f})|S^2.$$

It is a second order differential operator on S^2 which annihilates the constants and is formally self-adjoint. In addition, it is rotation invariant. (By these properties it is in fact characterized up to a constant factor.) From (45) we immediately see that $L(\mathcal{C}) = -\Delta^*$. A function $f \in C^\infty(S^2)$ which is an eigenfunction for the spherical Laplacian is called a surface spherical harmonic. According to the theory developed above, the Hilbert space $L^2(S^2)$ has a complete orthonormal system of surface spherical harmonics. Explicitly such a system is given by (51). The eigenvalues of the spherical Laplacian are $\lambda_m = -m(m+1)$, and the eigenvalue λ_m occurs with multiplicity $2m+1$. For more details, background material and information about applications we refer the reader to [BD][14], Section II.10, [T] [15], Chapter II and [HC].[16]

[14][BD]: T. Bröcker & T. tom Dieck, Representations of compact Lie groups. Springer-Verlag, 1985

[15][T]: A. Terras, Harmonic Analysis on symmetric spaces and applications I. Springer-Verlag, 1985

[16][CH]: R. Courant & D. Hilbert, methoden der Mathematischen Physik I. Edition 3. Springer-Verlag, 1968.

32 Appendix

In this appendix we give the proofs of Lemmas 25.1 and 25.2. We start with the following lemma.

Lemma 32.1 *Let X, Y be smooth manifolds, and let $f \in C^\infty(X \times Y)$. Then the map $F : X \to C^\infty(Y)$ defined by $F(x)(y) = f(x, y)$ is smooth.*

Proof. The assertion is local in the x–variable. Hence we may as well assume that X is an open susbset of $\mathbb{R}^m$.

By using smooth partitions of unity over compact subsets of Y we see that it suffices to prove the assertion for functions f with $\mathrm{supp} f \subset X \times U$, where U is a chart in Y. Thus we may as well assume that Y is an open subset of $\mathbb{R}^n$. A complete set of seminorms for $C^\infty(Y)$ is then given by

$$\nu_{K,k}(\varphi) = \max_{|\beta| \leq k} \|(\frac{\partial}{\partial y})^\beta \varphi\|_K.$$

Here $k \in \mathbb{N}$, K is a compact subset of Y, and $\|\cdot\|_K$ denotes the sup-norm over K.

We will first show that F is continuous. Fix $a \in X$. By uniform continuity of the partial derivatives $(\frac{\partial}{\partial y})^\beta f$ over compact subsets of $X \times Y$ we have

$$\lim_{x \to a} \|(\frac{\partial}{\partial y})^\beta F(x) - (\frac{\partial}{\partial y})^\beta F(a)\|_K = 0,$$

for $K \subset Y$ compact. From this it follows that $\lim_{x \to a} \nu_{K,k}(F(x) - F(a)) = 0$, whence the continuity of F in a.

Next we show that F has first order partial derivatives. Let $a \in X$ and $v \in \mathbb{R}^m$. Then by the mean value theorem we have

$$t^{-1}[(\frac{\partial}{\partial y})^\beta f(a + tv, y) - (\frac{\partial}{\partial y})^\beta f(a)] = \partial_v^X (\frac{\partial}{\partial y})^\beta f(a + \tau(t, y)v, y)$$

with $\tau(t, y)$ between 0 and t. If y is restricted to a compact subset of Y then the above expression tends uniformly to $(\frac{\partial}{\partial y})^\beta \partial_v^X f(a, y)$ as $t \to 0$. From this it follows that

$$\lim_{t \to 0} \frac{F(a + tv) - F(a)}{t} = \partial_v^X f(a, \cdot)$$

in $C^\infty(Y)$. Hence F has the directional derivative $\partial_v F(a) : y \mapsto \partial_v^X f(x, y)$.

The directional derivatives $\partial_v F$ are continuous by the first part of the proof. Hence F is C^1. By using an obvious induction one may now complete the proof. □

Proof of Lemma 25.1. Let $\varphi \in C^\infty(X)$. Then $f(g, x) = L_g\varphi(x) = \varphi(g^{-1}x)$ is a smooth function on $G \times X$. Applying the above lemma we obtain that $g \mapsto F(g) = L_g\varphi$ depends smoothly on $g \in G$. □

It remains to prove Lemma 25.2.

Proof of Lemma 25.1. Let $\varphi \in C_c^\infty(G)$, and define the map $F : G \to L^1(G)$ by $F(x) = L_x\varphi$. Replacing φ by a left translate if necessary we see that it suffices to establish the smoothness of F in a neighbourhood of e. This can be done as follows. Fix a function $\psi \in C_c^\infty(G)$ such that $\psi = 1$ on an open neighbourhood of $\mathrm{supp}\varphi$. Select an open neighbourhood U of e such that $\psi = 1$ on $\mathrm{supp} L_x\varphi$ for every $x \in U$.

The map $T : C^\infty(G) \to L^1(G)$, $\varphi \mapsto \psi\varphi$ is continuous linear. Using Lemma 25.1 and Lemma 25.3 we now see that $T \circ F$ is a smooth map $G \to L^1(G)$. In particular this map is smooth on U. But for $x \in U$ we have: $T \circ F(x) = T(L_x\varphi) = L_x\varphi F(x)$. □

33 Erratum A

In Section 2, p.2 it is asserted that every homogeneous G-space is a coset space, i.e. of the form G/H with H a closed subgroup of G. This is true if G is compact, but in general one needs an additional requirement on the action.

All the topological spaces occurring in this section are assumed to be Hausdorff.

A continuous map $f : X \to Y$ between topological spaces is called proper if the preimage $f^{-1}(C)$ of every compact subset $C \subset Y$ is compact.

Lemma 33.1 *Let $f : X \to Y$ be proper, and assume that Y is locally compact. Then f is closed, i.e. the image of any closed subset of X is a closed subset of Y.*

Proof. Let $S \subset X$ be closed. Let t be a point in the closure of $f(S)$. Then there exists a compact neighbourhood C of t in Y. Its preimage $f^{-1}C$ is closed in X. Hence $f^{-1}C \cap S$ is compact in X. The image of this set under f is compact in Y. But this image equals $C \cap f(S)$. Hence $C \cap f(S)$ is closed. This implies that $t \in f(S)$, hence $f(S)$ is closed. □

Let X be a topological space and G a locally compact topological group acting on X. The action is called proper if the action map $G \times X \to X$ is proper.

Lemma 33.2 *Suppose that the action of G on X is proper and transitive. Let $x \in X$, and let G_x be the stabilizer of x in G. Then the natural bijection $G/G_x \to X$ is a homeomorphism.*

Proof. Let $p : G \to G/G_x$ denote the canonical projection. The map $a : g \mapsto gx, G \to X$ is surjective, by transitivity of the action. Hence X is locally compact. The induced map $b : G/G_x \to X$ is a bijection. If $U \subset X$ is open then $a^{-1}(U)$ is open and right G_x–invariant, hence $b^{-1}(U) = p(a^{-1}(U))$ is open. Thus we see that b is continuous. To prove that it is a homeomorphism, it suffices to show that b is closed. For this it suffices to show that a is closed. Now this is a consequence of the properness of a.

Remark. If a compact group G acts continuously on a topological space, then the action is automatically proper. Hence if the action is transitive, then X is homeomorphic to a quotient of G by a compact subgroup.

34 Erratum B

The final argument in the proof of Lemma 16.2 is not complete. The purpose of this section is to complete it.

Let X, Y be smooth manifolds, and v_y a smooth vectorfield on X, depending smoothly on a parameter $y \in Y$. We shall investigate how the flow of v_y depends on y.

We first explain what smoothness of the parameter dependence means. For every $x \in X, y \in Y$ we have $v_y(x) \in T_xX$. Recall that we have a canonical identification $T_{(x,y)}(X \times Y) \simeq T_xX \oplus T_yY$. Via this identification we view $w(x, y) = (v_y(x), 0)$ as an element of $T_{(x,y)}(X \times Y)$. Thus $w : X \times Y \to T(X \times Y)$ is a vectorfield on $X \times Y$. The smoothness requirement on v now means that w is a smooth vector field.

Let $\gamma = (\gamma_1, \gamma_2) : I \to X \times Y$ be an integral curve of w with initial point (x, y). Then from

$$\dot{\gamma}(t) = (\, v_{\gamma_2(t)}(\gamma_1(t)) \, , \, 0)$$

we see that $\dot{\gamma}_2(t) = 0$, hence $\gamma_2(t) = y$ for all $t \in I$. Moreover, we now see that γ_1 is an integral curve of v_y with initial point x.

Let $\Omega \subset \mathbb{R} \times X \times Y$ be the (open) domain of the flow Φ of w. Then it follows that for every $y \in Y$ the flow Φ_y of v_y is given by $\Phi_y(t, x) = \Phi(t, x, y)$. Moreover, its domain Ω_y is given by $\Omega_y \times \{y\} = \Omega \cap \mathbb{R} \times X \times \{y\}$. Thus we see that Φ_y depends smoothly on $y \in Y$.

We now consider the particular case of a Lie group G. Then the left invariant vector field v_X on G depends linearly, hence smoothly, on $X \in \mathfrak{g}$. Its flow is given by $(t, x) \mapsto x\alpha_X(t), \mathbb{R} \times G \to G$. From the above discussion we see that the map $(t, x, X) \mapsto x\alpha_X(t)$, $\mathbb{R} \times G \times \mathfrak{g} \to G$ is smooth.