Uniform temperedness of Whittaker integrals for a real reductive group

Erik P. van den Ban

April 20, 2023

In memory of my son Mark

Contents

1 Whittaker vectors and matrix coefficients 6

2 Moderate estimates for Whittaker coefficients 11

3 The Whittaker Schwartz space 16

4 Sharp estimates for Whittaker coefficients 19

5 Proof of Lemma 4.6: improvement of estimates 22

6 Parabolically induced representations 31

7 Generalized vectors for induced representations 39

8 Whittaker vectors for induced representations 44

9 The Whittaker integral 59

10 Finite dimensional spherical representations 68

11 Projection along infinitesimal characters 71

12 The functional equation 79

13 Proof of Lemma 12.14 87

14 Holomorphy and uniformly moderate estimates 95

15 Uniformly tempered estimates **102**

16 Uniform temperedness of the Whittaker integral **121**

17 Appendix: factorization of polynomial functions **125**

18 Appendix: a Hartog type continuation result **127**

Introduction

In 1982, Harish-Chandra announced the Whittaker Plancherel theorem for real reductive groups in an invited lecture at the AMS summer conference in Toronto. Because of his failing health, the lecture, with the title 'On the theory of the Whittaker integral', was delivered on his behalf by V.S. Varadarajan. As a consequence of Harish-Chandra's untimely death in 1983, the details of the proof remained unpublished until they finally appeared in the posthumous 5th volume of his collected papers [12, pp. 141-307]. That volume also contains the text of the 1982 announcement, see [12, §1.2],

The proof of the Whittaker Plancherel theorem given in [12] seems to be incomplete, mainly since it does not develop a complete theory of Fourier transform for the Whittaker Schwartz space. In particular the required uniformly tempered estimates for the Whittaker integral are not addressed. In the present paper we give a proof of these estimates.

Independently, N. Wallach developed a completely different approach to the Whittaker Plancherel theory in his book [21]. However, the treatment was flawed because of an erroneous estimate, pointed out in [2, Remark 7.5]. Wallach has made several attempts to circumvent the error, see [22], but the final status of his results seems unclear at this point.

Clearly, the present paper has been inspired by both [12] and [21]. My desire to investigate the details of all arguments has led to a somewhat different and rather self-contained treatment of the theory needed for the derivation of a new functional equation for Whitttaker vectors in the generalized principal series, which lies at the basis of the mentioned uniform tempered estimates.

Now that the results of the present paper are available, it is natural to develop a theory of the constant term for tempered families of Whittaker coefficients, as well as a theory of wave packets of Whittaker integrals, in analogy with the Plancherel theory for groups or symmetric spaces. This will be addressed in a follow up article.

We will now describe the contents of our paper in some detail. Throughout the paper, we assume that G is a real reductive Lie group of the Harish-Chandra class, that $G = KAN_0$ is an Iwasawa decomposition and that χ is a unitary character of N_0. The character χ is assumed to satisfy the regularity condition that for each simple root α of $\mathfrak{a}$ in $\mathfrak{n}_0$, the derivative $\chi_* := d\chi(e)$ has a non-zero restriction to the root space $\mathfrak{g}_\alpha$.

Let $C(G/N_0 : \chi)$ be the space of continuous functions $f : G \to \mathbb{C}$ transforming according to the rule

$$f(xn) = \chi(n)^{-1} f(x), \qquad (x \in G, n \in N_0),$$

and let $C_c(G/N_0 : \chi)$ be the subspace of such functions with compact support modulo N_0. The latter space has a natural left G-invariant pre-Hilbert structure, for which the completion is denoted by $L^2(G/N_0 : \chi)$. The representation of G in this completion induced by the left regular action is the unitarily induced representation

$$\mathrm{Ind}_{N_0}^G(\chi). \tag{0.1}$$

The Whittaker Plancherel formula concerns the unitary direct integral decomposition of (0.1). It should be built from pairs (π, λ) with π an irreducible unitary representation in a Hilbert space, and λ a continuous linear functional on the associated space of smooth vectors H_π^∞, transforming according the rule

$$\lambda \circ \pi(n) = \chi(n)\lambda, \qquad (n \in N_0).$$

The functionals of this type are called Whittaker functionals of π, and the space of these is denoted by $\mathrm{Wh}_\chi(H_\pi^\infty)$. An element λ in the latter space determines a G-equivariant (Whittaker) matrix coefficient map $\mathrm{wh}_\lambda : H_\pi^\infty \to C^\infty(G/N_0 : \chi)$ given by

$$\mathrm{wh}_\lambda(v)(x) = \lambda(\pi(x)^{-1}v), \quad (v \in H_\pi^\infty, x \in G).$$

In Sections 1 and 2, these Whittaker coefficients are discussed in more detail. They have moderate growth behavior towards infinity. In Lemma 2.3 we formulate a technique which shows the importance of the regularity condition on χ. As a consequence each Whittaker coefficient of the above type has faster than exponential decay towards infinity in any closed cone disjoint from $\overline{A^+} \setminus \{0\}$, see Corollary 2.4. In Section 2 several related estimates are proven that are needed in the later sections.

Section 3 concerns aspects of the Whittaker Schwartz space $C(G/N_0 : \chi)$ as introduced by [12] and [21].

In Section 4 we discuss sharp estimates for a Whittaker coefficient $\mathrm{wh}_\lambda : H_\pi^\infty \to C^\infty(G/N_0 : \chi)$ in terms of a functional $\Lambda_V \in \mathfrak{a}^*$ attached to π. More precisely, Λ_V is defined in terms of the $\mathfrak{a}$-weights of $V/\mathfrak{n}_0 V$, with V the Harish-Chandra module of K-finite vectors of π, see 4.1. In [21] these estimates were obtained on the positive chamber A^+. In view of the results of Section 2 the estimates turn out to be valid on the entire group A. In [21] the estimates on the positive chamber are obtained by using the method of estimate improvement along maximal parabolic subgroups. We use the same method, cast in the form of Lemma 4.6. This prepares for the lengthy argumentation in Section 15, where the uniformly tempered estimates are obtained. The proof of Lemma 4.6 is deferred to Section 5. We end Section 4 with Cor. 4.8 which is due to both [12] (on the K-finite level) and [21]. It asserts that if G has compact center and π belongs to the discrete series of G, then for every Whittaker vector $\lambda \in \mathrm{Wh}_\chi(H_\pi^\infty)$ the associated

Whittaker coefficient wh_λ is a continuous linear map from H_π^∞ into the Schwartz space $\mathcal{C}(G/N_0 : \chi)$.

In Section 6 we discuss the space of smooth vectors for parabolically (normally) induced representations of the form $\mathrm{Ind}_P^G(\xi)$, with $P = M_P A_P N_P$ a parabolic subgroup of G and ξ a continuous representation of P in a Hilbert space H_ξ. For technical reasons we need to deal with this in the generality of a representation of the form $\xi = \sigma \otimes \pi$, with σ an irreducible unitary representation of M_P, extended to P by triviality on $A_P N_P$ and with (π, F) a continuous representation of P in a finite dimensional Hilbert space. The main result is the characterization of Theorem 6.7, which asserts that the space of smooth vectors of $\mathrm{Ind}_P^G(\xi)$ equals

$$C^\infty(G/P : \xi) := \{f \in C^\infty(G, H_\xi) \mid f(xman) = a^{-\rho_P} \xi(man)^{-1} f(x)\}. \tag{0.2}$$

The left regular representation of G in this space is denoted by $\pi_{P,\xi}^\infty$.

In the subsequent Section 7, the space of generalized vectors for $\mathrm{Ind}_P^G(\xi)$ is defined as a conjugate continuous linear dual by

$$C^{-\infty}(G/P : \xi) := \overline{C^\infty(G/P : \xi^*)'}. \tag{0.3}$$

Here ξ^* is the continuous representation of P in H_ξ defined by $\xi^*(p) := \xi(p^{-1})^*$. This definition has the advantage that (0.2) can be viewed as a subspace of (0.3), via a G-equivariant sesquilinear pairing defined by the usual integration over $K/K \cap M_P$.

In Section 8 we turn to the induced representations $\mathrm{Ind}_{\bar P}^G(\xi)$, with P a standard parabolic subgroup, and with $\xi = \sigma \otimes (-\bar\nu) \otimes 1$, where σ is an irreducible unitary representation of M_P and $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. The Whittaker functionals for the space of smooth vectors $C^\infty(G/\bar P : \sigma : -\bar\nu)$ can then be identified with (conjugates of) elements of the space of Whittaker vectors

$$C^{-\infty}(G/\bar P : \sigma : \nu)_\chi := \{j \in C^{-\infty}(G/\bar P : \sigma : \nu) \mid L_n j = \chi(n) j, \ (n \in N_0)\}. \tag{0.4}$$

The N_0-equivariance of an element j of this space makes that on the open set $N_P \bar P$ it can be represented by a continuous function with values in $H_\sigma^{-\infty} := \overline{H_\sigma^\infty}'$. Subsequent evaluation of this function in the identity element e defines a linear map

$$\mathrm{ev}_e : C^{-\infty}(G/\bar P : \sigma : \nu)_\chi \to (H_\sigma^{-\infty})_{\chi_P}, \tag{0.5}$$

where $\chi_P := \chi|_{M_P \cap N_0}$. At this point we invoke the fundamental result [12, Thm. 1], on which Harish-Chandra's entire treatment of the Whittaker theory is founded, see Theorem 8.1. It allows us to conclude that the map (0.5) is injective, see Corollary 8.11.

Conversely, if σ is a representation of the discrete series of M_P, and if $\mathrm{Re}\,\nu$ is P-dominant, we define for each $\eta \in (H_\sigma^{-\infty})_{\chi_P}$ a continuous $H_\sigma^{-\infty}$-valued function on $N_P \bar P$ which represents an element $j(\bar P, \sigma, \nu, \eta)$ of (0.4) with $\mathrm{ev}_e j(\bar P, \sigma, \nu, \eta) = \eta$, see Propositions 8.12 and 8.14. The element $j(\bar P, \sigma, \nu, \eta)$ depends holomorphically on ν in the region where $\mathrm{Re}\,\nu$ is P-dominant.

In Section 9 we discuss the close relation between $j(P, \sigma, \nu, \eta)$ and the Jacquet integral introduced in [21], see (9.3). Moreover, we discuss the definition of Harish-Chandra's Whittaker integral $\mathrm{Wh}(P, \psi, \nu)$ which is the analogue of the Eisenstein integral for a group or a symmetric space. We show that the Whittaker integral is expressible as a sum of Whittaker matrix coefficients involving Whittaker vectors $j(\nu) = j(\bar{P}, \sigma, \nu, \eta)$, see Corollary 9.10.

In strong analogy with the theory of symmetric spaces one needs to extend the map $\nu \mapsto j(\nu)$ meromorphically in order to reach imaginary ν, which correspond to the unitary principal series. In addition one needs to establish uniformly tempered estimates in regions of the form $|\mathrm{Re}\, \nu| < \varepsilon$, with $\varepsilon > 0$ a suitable constant. In the theory of symmetric spaces, tools for this program where initially developed in [1] for minimal σ-parabolic subgroups and then extended to arbitrary σ-parabolic subgroups in [7]. It turns out that these tools from the theory of symmetric spaces are ideally suited for the Whittaker setting. This unfolds in the final Sections 10 - 16.

In Section 10 we prepare by reviewing the characterization of irreducible finite dimensional spherical representations of G with an M_P-fixed highest weight vector. Then, in Section 11 we consider the action of the center $\mathfrak{Z}$ of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ on a tensor product of the form

$$\mathrm{Ind}_{\bar{P}}^{G}(\sigma \otimes \nu \otimes 1) \otimes \pi_{\mu}, \tag{0.6}$$

with π_μ an irreducible finite dimensional spherical representation of strictly P-dominant highest weight μ, with M_P acting trivially on the highest weight space. Let Λ denote the infinitesimal character of σ, and let $p_{\Lambda+\nu+\mu}$ denote the projection in the space of (0.6) onto the generalized weight space for the infinitesimal character $\Lambda + \nu + \mu$. Then the main result of the section is that there exists a non-zero polynomial function q on $\mathfrak{a}_{P\mathbb{C}}^*$ such that $q(\nu)p_{\Lambda_\sigma+\nu+\mu}$ can be realized by the action of an element $\underline{Z}_\mu(\nu) \in \mathfrak{Z}$ wich depends polynomially on ν, see Corollary 11.13.

In Section 12 the element $\underline{Z}_\mu(\nu) \in \mathfrak{Z}$ is used to define a suitable differential operator $D_\mu(\sigma, \nu) : C^{-\infty}(\bar{P} : \sigma : \nu) \to C^{-\infty}(\bar{P} : \sigma : \nu + \mu)$ such that one has a Bernstein-Sato type functional equation for the Whittaker vector,

$$j(\bar{P}, \sigma, \nu) = D_\mu(\sigma, \nu) \circ j(\bar{P}, \sigma, \nu + \mu) \circ R_\mu(\sigma, \nu),$$

see (12.7). Here $R_\mu(\sigma)$ is a rational function on $\mathfrak{a}_{P\mathbb{C}}^*$ with values in $\mathrm{End}(H_{\sigma,\chi_P}^{-\infty})$. The main problem is to show that $R_\mu(\sigma, \cdot)$ is generically invertible. This is done in the rest of Section 12 and the next.

In Section 14 the functional equation is used to obtain the meromorphic continuation of $\nu \mapsto j(\nu) = j(\bar{P}, \sigma, \nu)$ with estimates in terms of continuous seminorms on $C^{-\infty}(P, \sigma, \nu)$, see Theorem 14.1. The functional equation implies the existence of a non-zero polynomial p_R such that $\nu \mapsto p_R(\nu)j(\nu)$ is holomorphic in the range $\langle \mathrm{Re}\, \nu\,,\, \alpha \rangle > R$ for all $\alpha \in \Delta$. Any singularity of j in this range is contained in the zero set $M = p_R^{-1}(0)$. If it is contained in the regular part of M then by a local analysis transversal to M it can be shown that the singularity produces a non-zero element of

$C^{-\infty}(G/\bar{P}, \sigma, \nu)_\chi$ which must be zero at e hence zero. Therefore, singularities can only occur at singular points of $p^{-1}(0)$. The appearence of singularities would thus violate a form of Hartog's principle formulated and proven in the appendix in Section 18. Therefore, j cannot have singularities. This gives a new proof of Wallach's result [21, Thm. 15.4.1] on the holomorphy of the Jacquet integral, but with strong estimates, see Theorems 14.4 and 14.8. This in turn leads to uniformly moderate estimates for the associated family of Whittaker coefficients in Theorem 14.9.

In Section 15 it is shown that the uniformly moderate estimates for the family of Whittaker coefficients (wh_ν) produced by $j_\nu = j(\bar{P}, \sigma, \nu, \eta)$ can be improved to the so-called uniformly tempered estimates, see Theorem 15.5. This is done by using the differential equations satisfied by (wh_ν) and the method of estimate improvement along maximal parabolic subgroups, as in Section 4, but now with suitable uniformity in the parameter ν. See Lemma 15.9 for the crucial stepwise improvement.

In the final section, 16, the results obtained in the previous sections are applied to the Whittaker integral $\mathrm{Wh}(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi)$. Here '$\tau$:' indicates that left τ-spherical functions are considered, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and ψ is an element of the finite dimensional space $\mathcal{A}_{2,P}$ of $\mathfrak{Z}(\mathfrak{m}_P)$-finite functions in $C(\tau_P : M_P/M_P \cap N_0 : \chi_P)$. The uniformly tempered estimates of Theorem 16.2 thus obtained allow us to define a Fourier transform $\mathcal{F}_P$ in terms of the Whittaker integral, and to show that $\mathcal{F}_P$ defines a continuous linear map from the Whittaker Schwartz space $C(\tau : G/N_0 : \chi)$ to the Euclidean Schwartz space $\mathcal{S}(i\mathfrak{a}^*_P, \mathcal{A}_{2,P})$, see Theorem 16.6.

1 Whittaker vectors and matrix coefficients

We consider a real reductive group G of the Harish-Chandra class and fix an Iwasawa decomposition $G = KAN_0$. We denote by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ the root system of $\mathfrak{a}$ in $\mathfrak{g}$ and by Σ^+ the positive system consisting of the roots $\alpha \in \Sigma$ with $\mathfrak{g}_\alpha$ contained in the Lie algebra $\mathfrak{n}_0$ of N_0. The associated collection of simple roots in Σ^+ is denoted by Δ.

Thus, if $M = Z_K(\mathfrak{a})$, then $P_0 := MAN_0$ is the standard minimal parabolic subgroup associated with Σ^+. Here and in the rest of the paper we adopt the convention to denote Lie groups by Roman capitals and the associated Lie algebras by the corresponding fraktur lower cases.

In this article χ will always be a unitary character of N_0. Following [12, p. 142], we say that χ is *regular* if its derivative $\chi_* := d\chi(e) \in i\mathfrak{n}_0^*$ is non-zero on each of the simple root spaces $\mathfrak{g}_\alpha$, for $\alpha \in \Delta$. Unless otherwise specified, it will always be assumed that χ is regular. Note that the notion of regularity as defined here coincides with the notion of genericity in [21, p. 371].

We consider the function space

$$C(G/N_0 : \chi) := \{f \in C(G) \mid f(xn) = \chi(n)^{-1} f(x), \quad (x \in G, n \in N_0)\}. \qquad (1.1)$$

The subspace of functions with compact support modulo N_0 is denoted $C_c(G/N_0 : \chi)$ and the subspace of smooth functions by $C^\infty(G/N_0 : \chi)$. Finally, the intersection of the latter two is denoted $C_c^\infty(G/N_0\!:\!\chi)$.

We fix a choice of positive invariant Radon measure $d\dot{x}$ on G/N_0 and define a pre-Hilbert structure on $C_c(G/N_0\!:\!\chi)$ by the formula

$$\langle f\,,\,g\rangle := \int_{G/N_0} f(x)\overline{g(x)}\,d\dot{x}, \quad (f,g\in C_c(G/N_0\!:\!\chi)).$$

The associated completion is denoted by $L^2(G/N_0:\chi)$. The Whittaker Plancherel formula concerns the unitary decomposition for the left regular representation of G in the latter space. Here we note that $L^2(G/N_0\!:\!\chi)$ is the space for the unitarily induced representation $\mathrm{Ind}_{N_0}^G(\chi)$. Our notation is slightly different Harish-Chandra's, who uses the notation $L^2(G/N_0\!:\!\chi)$ for the space of the induced representation $\mathrm{Ind}_{N_0}^G(\chi^\vee)$, with $\chi^\vee: n\mapsto \chi(n)^{-1}$, see [12, p. 143]. Finally, note that Wallach [21, p. 365] works with another realization of this representation space, namely $L^2(\chi:N_0\backslash G)$ equipped with the right regular representation.

In the following we will need a bit of background from representation theory that we will now explain.

If V is a locally convex (Hausdorff) space then a representation π of a Lie group L in V is called smooth if it is continuous and if $V=V^\infty$.

If V is Fréchet space, its strong dual V' is a complete locally convex space. Suppose that π is a continuous representation of L in V, then the homomorphism $\pi^\vee: L\to \mathrm{GL}(V')$ defined by

$$\pi^\vee(x)\xi = \xi\circ\pi(x)^{-1}, \qquad (x\in L,\ \xi\in V'). \tag{1.2}$$

need not be a continuous representation. However, if π is smooth, then $\pi^\vee$ is a smooth representation of L in V, called the contragredient of π. For details we refer to [23], Proposition 4.4.1.9 and the definition of $V^\vee$ preceding Proposition 4.1.2.1.

To prepare for the treatment of conjugate representations, we first briefly discuss the notion of conjugate space. Let V and W be complex linear spaces. We denote by $\bar{V}$ the real linear space V equipped with the conjugate complex multiplication $\mathbb{C}\times V\to V$, $(z,v)\mapsto \bar{z}v$. A map $T: V\to W$ is said to be conjugate linear if it is real linear and satisfies $T(\lambda v)=\bar{\lambda}T(v)$ for $v\in V$ and $\lambda\in\mathbb{C}$. The complex linear space of conjugate linear maps $V\to W$ (equipped with the pointwise operations of scalar multiplication and addition) equals the complex linear space $\mathrm{Hom}_{\mathbb{C}}(\bar{V},W)$. Given $T\in\mathrm{Hom}(V,W)$ we denote by $\bar{T}$ the map $V\to W$ viewed as an element of $\mathrm{Hom}(\bar{V},\bar{W})$. We note that the map

$$T\mapsto\bar{T}, \quad \mathrm{Hom}(V,W)\to\mathrm{Hom}(\bar{V},\bar{W})$$

is not complex linear, but conjugate linear. Hence, $T\mapsto\bar{T}$ is a complex linear isomorphism $\overline{\mathrm{Hom}(V,W)}\to\mathrm{Hom}(\bar{V},\bar{W})$. Since the map $T\mapsto\bar{T}$ is the identity on the set $\mathrm{Hom}(V,W)$ we have the following identity of complex linear spaces

$$\overline{\mathrm{Hom}(V,W)} = \mathrm{Hom}(\bar{V},\bar{W}). \tag{1.3}$$

In the sequel we will encounter the conjugate dual space $\overline{V^*} = \overline{\mathrm{Hom}(V, \mathbb{C})}$ and the dual conjugate space $(\bar{V})^* = \mathrm{Hom}(\bar{V}, \mathbb{C})$. These spaces are not equal, but complex linearly isomorphic under the map $\lambda \mapsto \overline{{}^c\lambda}$ given by

$$ {}^c\lambda = \mathbf{c} \circ \lambda, \tag{1.4} $$

where $\mathbf{c} : \mathbb{C} \to \mathbb{C}$ denotes the conjugation map $z \mapsto \bar{z}$.

Indeed, $\mathbf{c} \in \mathrm{Hom}(\bar{\mathbb{C}}, \mathbb{C})$ so in view of the equality (1.3) it follows that ${}^c\lambda \in \mathrm{Hom}(\bar{V}, \mathbb{C})$ $(\lambda \in \mathrm{Hom}(V, \mathbb{C}))$. It is now readily verified that $\lambda \mapsto {}^c\lambda$ is a conjugate linear map $\mathrm{Hom}(V, \mathbb{C}) \to \mathrm{Hom}(\bar{V}, \mathbb{C})$ hence a complex linear isomorphism

$$ \overline{\mathrm{Hom}(V, \mathbb{C})} \xrightarrow{\simeq} \mathrm{Hom}(\bar{V}, \mathbb{C}). $$

Given a representation π of L in a locally convex space V, we denote by $(\bar{\pi}, \bar{V})$ the conjugate of π. Here the conjugate complex linear space $\bar{V}$ is equipped with the locally convex topology of V. Furthermore, for $x \in L$, $\bar{\pi}(x)$ equals the complex linear map $\overline{\pi(x)} : \bar{V} \to \bar{V}$. It is clear that $(\bar{\pi}, \bar{V})$ is a representation of L in a locally convex space again, which is continuous if and only if π is continuous.

We note that the spaces $\overline{V'}$ and $\bar{V}'$ are topologically complex linear isomorphic under the map $\lambda \mapsto {}^c\lambda$ given by (1.4).

If π is a smooth Fréchet representation, then $\bar{\pi}^\vee$ is a smooth continuous representation, and therefore, so is the equivalent representation $\overline{\pi^\vee}$.

We use the notation $U(\mathfrak{l})$ for the universal enveloping algebra of the complexification $\mathfrak{l}_\mathbb{C}$ of $\mathfrak{l}$. The canonical anti-automorphism of $U(\mathfrak{l})$ is denoted by $u \mapsto u^\vee$. It is readily verified that for the associated infinitesimal representations $\pi : U(\mathfrak{l}) \to \mathrm{End}(V)$ and $\pi^\vee : U(\mathfrak{l}) \to \mathrm{End}(V')$ we have

$$ \pi^\vee(u)\xi = \xi \circ \pi(u^\vee), \quad (u \in U(\mathfrak{l}), \xi \in V'). $$

Given a continuous Hilbert space representation (π, H) of L it is known that the contragredient $(\pi^\vee, H')$ is continuous, see [23, Cor. 4.1.2.3]. Therefore, so are $\bar{\pi}^\vee$ and $\overline{\pi^\vee}$. Both duals $(\bar{H})'$ and $(H')^\vee$ come into play through the Hermitian inner product b viewed as a bilinear map $H \times \bar{H} \to \mathbb{C}$. Let $b_1 : H \to (\bar{H})'$ be the linear map defined by $b_1(v) = b(v, \cdot\,)$. Let $b_2 : \bar{H} \to H'$ be the linear map defined by $b_2(v) = b(\,\cdot\,, v)$. Then b_2 can be viewed as a linear map $H \to \overline{H'}$. As such b_1 and b_2 are topological linear isomorphisms from H onto $(\bar{H})'$ and $\overline{H'}$ respectively. Thus, here the isomorphism $\overline{H'} \to \bar{H}'$ is given by $\beta = b_1 \circ b_2^{-1}$. Using the conjugate symmetry of b, it readily follows that β coincides with the isomorphism ${}^c(\,\cdot\,)$ defined by (1.4). Since $\lambda \mapsto {}^c\lambda$ intertwines the representations $\overline{\pi^\vee}$ and $\bar{\pi}^\vee$, it follows that b_2 and b_1, respectively, intertwine these representations with the same continuous representation π^* of L in H, given by

$$ \pi^*(x) = \pi(x^{-1})^* \qquad (x \in L), $$

where the star indicates that the Hilbert adjoint is taken.

For such a continuous Hilbert representation we denote the associated Fréchet representation in the space of smooth vectors by (π^∞, H^∞). As we mentioned above, the continuous linear dual $(H^\infty)'$ of H^∞, equipped with the strong dual topology, is complete. The associated contragredient of π^∞ in $(H^\infty)'$, denoted by $(\pi^{\infty\vee}, (H^\infty)')$ is a smooth representation of L. In this setting, with G in place of L, it is of interest to consider the space of Whittaker functionals

$$\mathrm{Wh}_\chi(H^\infty) := \{\lambda \in (H^\infty)' \mid \forall n \in N_0 : \quad \lambda \circ \pi^\infty(n) = \chi(n)\lambda\}, \tag{1.5}$$

see [21, 15.3.4, p. 378]. Equivalently, $\mathrm{Wh}_\chi(H^\infty)$ consists of the functionals $\lambda \in (H^\infty)'$ such that

$$\pi^{\infty\vee}(n)\lambda = \chi(n)^{-1}\lambda, \qquad (n \in N_0).$$

For a given Whittaker functional $\lambda \in \mathrm{Wh}_\chi(H^\infty)$, the matrix coefficient map $\mathrm{wh}_\lambda : H^\infty \to C^\infty(G/N_0 : \chi)$, given by

$$\mathrm{wh}_\lambda(v)(x) = \lambda(\pi(x)^{-1}v),$$

is readily seen to be continuous and G-equivariant. Moreover, we have the following easy lemma.

Lemma 1.1 *The matrix coefficient map $\lambda \mapsto \mathrm{wh}_\lambda$ is a bijection*

$$\mathrm{Wh}_\chi(H^\infty) \xrightarrow{\simeq} \mathrm{Hom}_G(H^\infty,\ C^\infty(G/N_0 : \chi)), \tag{1.6}$$

where Hom_G indicates the space of intertwining continuous linear maps. The inverse of (1.6) is given by $T \mapsto \mathrm{ev}_e \circ T$, where $\mathrm{ev}_e : C^\infty(G/N_0 : \chi) \to \mathbb{C}$ denotes evaluation at the identity.

The following result, valid for any continuous character χ of N_0, is stated and proven in [21, Cor 15.4.4].

Lemma 1.2 *If (π, H) is admissible and of finite length, then*

$$\dim \mathrm{Wh}_\chi(H^\infty) < \infty.$$

Given a continuous representation ρ of a Lie group in a complete locally convex space V, we use the notation $\bar{\rho}$ for ρ viewed as a representation in the conjugate space $\bar{V}$. Clearly, $\bar{\rho}$ is continuous again and the identifications $\overline{\rho^\infty} = \bar{\rho}^\infty$ and $\overline{V^\infty} = \bar{V}^\infty$ are obvious.

In this paper it will be desirable to view the matrix coefficient map wh_λ, for $\lambda \in \mathrm{Wh}_\chi(H^\infty)$ as a matrix coefficient with a suitable generalized vector. This is possible in the following setting of duality.

Let (π_j, H_j) be two continuous Hilbert representations of a Lie group L, for $j = 1, 2$. By a perfect sesquilinear pairing of π_1 and π_2 we mean an equivariant continuous sesquilinear pairing

$$H_1 \times H_2 \to \mathbb{C} \tag{1.7}$$

such that the induced maps $\alpha_1 : H_1 \to (\bar{H}_2)'$ and $\alpha_2 : H_2 \to \overline{H_1'}$ are unitary isomorphisms. Note that these maps are intertwining. In particular, the restriction map

$$r_1 : \xi \mapsto \xi|_{H_1^\infty}, \qquad \overline{H_1'} \to \overline{H_1^{\infty\prime}}$$

is a continuous linear injection, intertwining $\overline{\pi_1^\vee}$ with $\overline{\pi_1^{\infty\vee}}$.

We put

$$H_2^{-\infty} := \overline{H_1^{\infty\prime}}, \tag{1.8}$$

and accordingly denote by $\pi_2^{-\infty} := \overline{\pi_1^{\infty\vee}}$ the natural continuous representation on (1.8). We consider the canonically associated sesquilinear pairing

$$H_1^\infty \times H_2^{-\infty} \to \mathbb{C}, \quad (v, j) \mapsto \langle v\,,\, j\rangle. \tag{1.9}$$

This pairing is equivariant for π_1^∞ and $\pi_2^{-\infty}$ and induces the inverse of the continuous linear isomorphism (1.12). Put $\iota_2 := r_1 \circ \alpha_2$. Then the map

$$\iota_2 : H_1 \hookrightarrow H_2^{-\infty} \tag{1.10}$$

is a continuous linear injection, intertwining π_1 with $\pi_1^{-\infty}$. We will use it to identify the first of these spaces as an invariant subspace of the second. This allows us to view the elements of (1.8) as generalized vectors for π_2. We now note that for $(v, w) \in H_1^\infty \times H_2$, we have

$$\langle v\,,\, \iota_2(w)\rangle = [r_1(\alpha_2(w))](v) = [\alpha_2(w)](v) = \langle v\,,\, w\rangle,$$

where the last mentioned pairing is (1.7). We thus see that the sesquilinear pairing (1.9) is an extension of the pairing $H_1^\infty \times H_2 \to \mathbb{C}$ given by restricting (1.7).

In the present context it is sometimes convenient to also use the sesquilinear pairing

$$H_2^{-\infty} \times H_1^\infty \to \mathbb{C}, \quad (j, v) \mapsto \langle j\,,\, v\rangle := \overline{\langle v\,,\, j\rangle}. \tag{1.11}$$

Finally, we note that the above definitions imply that the pairing (1.9) induces a topological linear isomorphism

$$\overline{H_2^{-\infty}} \xrightarrow{\simeq} (H_1^\infty)'. \tag{1.12}$$

This isomorphism intertwines $\overline{\pi_2^{-\infty}}$ with $\pi_1^{\infty\vee}$. In the Whittaker setting, with $L = G$ and π_1 and π_2 of finite length, this equivariance implies that (1.12) restricts to a bijective conjugate linear map

$$H_{2,\chi}^{-\infty} \xrightarrow{\simeq} \mathrm{Wh}_\chi(H_1^\infty), \quad j \mapsto {}^\vee\! j, \tag{1.13}$$

where

$$H_{2,\chi}^{-\infty} := \{j \in H_2^{-\infty} \mid \pi_2^{-\infty}(n)j = \chi(n)j \ (n \in N_0)\}. \tag{1.14}$$

For obvious reasons, we agree to call (1.14) the space of Whittaker vectors for π_1. Given $j \in (H_2^{-\infty})_\chi$ there is the associated Whittaker coefficient map $\mathrm{wh}_j = \mathrm{wh}_{({}^\vee j)} : H_1^\infty \to C^\infty(G/N_0 : \chi)$, given by

$$\mathrm{wh}_j(v)(x) = {}^\vee\! j(\pi_1(x)^{-1}v) = \langle \pi_1(x)^{-1}v\,,\, j\rangle, \quad (v \in H_1^\infty,\ x \in G).$$

Remark 1.3 If (π, H) is a continuous Hilbert representation of a Lie group L we define the representation π^* of L in H by $\pi^*(x) = \pi(x^{-1})^*$ for $x \in L$. It is readily verified that the isometry $\mathrm{i} : H \to \bar{H}'$ induced by the Hilbert inner product intertwines π^* with $\bar{\pi}^\vee$. Thus, π^* is continuous since $\bar{\pi}^\vee$ is continuous. Let now H_π and H_{π^*} denote H equipped with the representations π and π^*, respectively. Then the inner product of H gives an equivariant perfect sesquilinear pairing

$$H_\pi \times H_{\pi^*} \to \mathbb{C}. \tag{1.15}$$

Conversely, any equivariant perfect pairing of the form (1.7) can be transfered to a pairing as (1.15) by putting $H = H_1, \pi = \pi_1$ and using the equivariant unitary isomorphism $\mathrm{i}_2^{-1} \circ \alpha_2 : H_2 \to H_1$. The pairing (1.15) gives rise to an intertwining injective linear map $H_{\pi^*} \to H_{\pi^*}^{-\infty} = \overline{H_\pi^{\infty\prime}}$. The representation π is unitary if and only if $\pi = \pi^*$. In that case we obtain the equality $H_\pi^{-\infty} = \overline{H_\pi^{\infty\prime}}$ which is compatible with an existing convention in the literature.

Remark 1.4 The point of view explained above Remark 1.3 will be of particular importance in the setting of parabolically induced representations of the form $\mathrm{Ind}_P^G(\xi)$, with ξ a continuous Hilbert representation of a parabolic subgroup P of G. Let $L^2(G/P : \xi)$ be the Hilbert space in which $\mathrm{Ind}_P^G(\xi)$ is realized by the left regular action. Then, with the similar notation for ξ^*, there exists a natural G-equivariant perfect sesquilinear pairing

$$L^2(G/P : \xi) \times L^2(G/P : \xi^*) \to \mathbb{C}.$$

Applying the formalism introduced above, one obtains a compatible equivariant sesquilinear pairing

$$L^2(G/P : \xi)^\infty \times L^2(G/P : \xi^*)^{-\infty} \to \mathbb{C}$$

which induces an equivariant continuous linear isomorphism

$$L^2(G/P : \xi^*)^{-\infty} \xrightarrow{\simeq} \overline{(L^2(G/P : \xi)^{\infty\prime}}$$

The associated space of Whittaker vectors, $(L^2(G/P : \xi^*)^{-\infty})_\chi$, can thus be viewed as a space of generalized sections of a Hilbert bundle.

2 Moderate estimates for Whittaker coefficients

We fix a non-degenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \tag{2.1}$$

which is negative definite on $\mathfrak{k}$, positive definite on $\mathfrak{p}$ and which restricts to the Killing form on the semisimple part $[\mathfrak{g}, \mathfrak{g}]$. Let θ denote the Cartan involution on $\mathfrak{g}$ associated with K. We define the positive definite $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot\,, \,\cdot \rangle$ on $\mathfrak{g}$ by

$$\langle X\,, Y \rangle := -B(X, \theta Y), \qquad (X, Y \in \mathfrak{g}). \tag{2.2}$$

The restriction of this inner product to $\mathfrak{a}$ induces a dual inner product on $\mathfrak{a}^*$, The latter's extension to a complex biliinear form on $\mathfrak{a}^*_{\mathbb{C}}$ is also denoted $\langle \cdot\, , \,\cdot \rangle$. The associated norms on $\mathfrak{a}$ and $\mathfrak{a}^*$ are denoted by $|\cdot|$. Finally we extend the norm on $\mathfrak{a}^*$ to the norm $|\cdot|$ on $\mathfrak{a}^*_{\mathbb{C}}$ associated with the Hermitian extension of the inner product on $\mathfrak{a}^*$. Accordingly,

$$|\nu|^2 = |\mathrm{Re}\,\nu|^2 + |\mathrm{Im}\,\nu|^2, \qquad (\nu \in \mathfrak{a}^*_{\mathbb{C}}). \tag{2.3}$$

If $\mathfrak{v} \subset \mathfrak{a}$ is a linear subspace, we will use the inner product on $\mathfrak{a}$ to identify the real linear dual $\mathfrak{v}^*$ with a subspace of $\mathfrak{a}^*$, unless otherwise specified.

We define ${}^\circ G$ to be the intersection of the kernels $\ker\xi$ where ξ ranges over the characters $G \to \mathbb{R}_{>0}$. Let $\mathfrak{a}_\Delta = \cap_{\alpha\in\Delta}\ker\alpha$ and put $A_\Delta = \exp(\mathfrak{a}_\Delta)$. Then multiplication induces an isomorphism of Lie groups

$$G \simeq {}^\circ G \times A_\Delta.$$

It follows that $G = {}^\circ G$ if and only if G has compact center.

We define ${}^\circ\|\cdot\| : {}^\circ G \to]0,\infty[$ by ${}^\circ\|x\| = \|\mathrm{Ad}(x)\|_{\mathrm{op}}$ $(x \in G)$, where the subscript 'op' indicates that the operator norm with respect to the inner product (1.2) has been taken.

We put ${}^*A_\Delta := G^0 \cap A$. Then via the direct sum $\mathfrak{a} = {}^*\mathfrak{a}_\Delta \oplus \mathfrak{a}_\Delta$ we identify the elements of the real duals ${}^*\mathfrak{a}^*_\Delta$ and $\mathfrak{a}^*_\Delta$ with elements of $\mathfrak{a}^*$. We select a basis $\mathcal{B}$ of $\mathfrak{a}^*_\Delta$ and define $\|\cdot\|_\Delta : A_\Delta \to [1,\infty[$ by

$$\|b\|_\Delta = \max_{\beta\in\pm\mathcal{B}} b^\beta, \qquad (b \in A_\Delta).$$

Finally we define $\|\cdot\| : G \to [1,\infty[$ by

$$\|xb\| := \max({}^\circ\|x\|, \|b\|_\Delta), \qquad (x \in {}^\circ G,\ b \in A_\Delta).$$

Put $\Sigma_e := \Sigma \cup \mathcal{B} \cup (-\mathcal{B})$. Then it is easily verified that for $k_1, k_2 \in K$ and $a \in A$,

$$\|k_1 a k_2\| = \max_{\alpha\in\Sigma_e} a^\alpha. \tag{2.4}$$

See also [3, Lemma 2.1], where the definition is given for G with compact center.

From the above definitions and (2.4) it readily follows that $\|\cdot\|$ is a norm on G in the sense of [20, Lemma 2.A.2.1].

Lemma 2.1 *If (π, H) is a (continuous) representation of G in a Hilbert space, then there exist constants $r(\pi) \geq 0$ and $C > 0$ such that*

$$\|\pi(g)\|_{\mathrm{op}} \leq C\|g\|^{r(\pi)}, \qquad (g \in G), \tag{2.5}$$

where $\|\cdot\|_{\mathrm{op}}$ indicates the operator norm.

Proof. See [20, Lemma 2.A.2.2]. □

Lemma 2.2 *Let (π, H) be of finite length, and $\lambda \in \mathrm{Wh}_\chi(H^\infty)$. There exists a constant $r > 0$ and a continuous seminorm n on H^∞ such that the Whittaker coefficient wh_λ satisfies*

$$|\mathrm{wh}_\lambda(v)(a)| \leq e^{r|\log a|}\mathrm{n}(v), \tag{2.6}$$

for all $v \in H^\infty$ and $a \in A$.

Proof. Let $r(\pi)$ and C be as in (2.5). Put $m = \max_{\alpha \in \Sigma_e} |\alpha|$. Then it follows that for $a \in A$ we have

$$\|a\|^{r(\pi)} \leq e^{mr(\pi)|\log a|}.$$

By continuity of λ, there exist a finite subset $S \subset U(\mathfrak{g})$ such that $|\lambda(v)| \leq \sum_{u \in S} \|\pi^\infty(u)v\|$, for all $v \in V^\infty$. By decomposing each element of S as a sum of weight vectors for $\mathrm{ad}(\mathfrak{a})$ it is readily seen that we may assume S to consist of weight vectors from the start. Let ξ_u denote the weight by which $\mathrm{ad}(\mathfrak{a})$ acts on u. Then it follows that, for all $v \in H^\infty$ and $a \in A$,

$$\begin{aligned} |\lambda(\pi(a)v)| &\leq \sum_{u \in S} \|\pi^\infty(u)\pi(a)v\| \\ &\leq C\,\|a\|^{r(\pi)} \sum_{u \in S} a^{-\xi_u} \|\pi^\infty(u)v\| \\ &\leq C\,e^{(mr(\pi)+s)|\log a|} \sum_{u \in S} \|\pi^\infty(u)v\|, \end{aligned}$$

where $s = \max_{u \in S} |\xi_u|$. The result follows with $r = mr(\pi) + s$. □

The above proof does not use the assumption that $\chi \in \widehat{N}_0$ is regular. If χ is regular, then the above estimate gives rise to remarkable new exponential estimates. The argumentation for this is suggested by the following lemma.

Lemma 2.3 *Let $u \in U(\mathfrak{n}_0)$ have weight $\eta \in \mathfrak{a}^*$ for the adjoint action of $\mathfrak{a}$. Then for all $f \in C^\infty(G/N_0 : \chi)$ and $a \in A$,*

$$L_u f(a) = a^{-\eta} \chi_*(u) f(a).$$

Proof. Note that

$$L_u f(a) = [R_{\mathrm{Ad}(a)^{-1}(u^\vee)} f](a) = a^{-\eta} R_{u^\vee} f(a) = a^{-\eta} \chi_*(u) f(a).$$ □

The regularity of the character implies that Whittaker coefficients have fast decay outside the closed positive Weyl chamber $\mathrm{cl}(A^+)$.

Corollary 2.4 *Let (π, H) be an admissible Hilbert representation of G of finite length and let $\lambda \in \mathrm{Wh}_\chi(H^\infty)$. Let Γ be a closed cone in $\mathfrak{a}$ which is disjoint from $\mathrm{cl}(\mathfrak{a}^+) \setminus \{0\}$.*

Then for every $s > 0$ there exists a continuous seminorm n of H^∞ such that, for all $v \in H^\infty$, $k \in K$ and $a \in \exp(\Gamma)$,

$$|\mathrm{wh}_\lambda(v)(ka)| \leq e^{-s|\log a|} n(v). \tag{2.7}$$

Proof. Let $r > 0$ and n be as in (2.6). Let S be the unit sphere in $\mathfrak{a}$. Then by compactness of $S \cap \Gamma$ it suffices to show that for every $H_0 \in S \cap \Gamma$ there exists a closed neighborhood $\omega \ni H_0$ in S such that (2.7) holds with suitable s' and n' in place of s and n, for all $v \in H^\infty$, all $k \in K$ and all $a \in \exp(\mathbb{R}_{\geq 0}\omega)$.

Let $H_0 \in S \cap \Gamma$ be given. Then $H_0 \notin \mathrm{cl}(\mathfrak{a}^+)$ and it follows that there exists a simple root $\alpha \in \Delta$ such that $\alpha(H_0) < 0$. We may fix $p \in \mathbb{N}$ such that for $H = H_0$ we have

$$r|H| + p\alpha(H) < -s|H|. \tag{2.8}$$

We may now fix a closed neighborhood ω in S such that this estimate holds for $H \in \omega$. By positive homogeneity (2.8) holds for $H \in \mathbb{R}_+\omega$. Let now $X \in \mathfrak{g}_\alpha$ be such that $\chi_*(X) = 1$. Put $u = X^k \in U(\mathfrak{g})$. Then by application of Lemma 2.3 it follows that, for all $v \in H^\infty$, $k \in K$ and $a \in \exp(\mathbb{R}_{\geq 0}\omega)$,

$$\begin{aligned} |\mathrm{wh}_\lambda(v)(ka)| &= a^{k\alpha}|L_u(\mathrm{wh}_\lambda(\pi(k)^{-1}v))(a)| \\ &= e^{k\alpha(\log a)}|\mathrm{wh}_\lambda(\pi(u)\pi(k)^{-1}v)(a)| \\ &\leq e^{k\alpha(\log a)+r|\log a|}n(\pi(u)\pi(k)^{-1}v) \leq e^{-s|\log a|}n'(v), \end{aligned}$$

where $n'(v) = \sup_{k\in K} n(\pi(u)\pi(k)^{-1}v)$. □

Lemma 2.5 *Let $\vartheta : \mathfrak{a} \to \mathbb{R}$ be either linear, or of the form $\vartheta = r|\cdot|$, with $r > 0$. Let $\xi \in \mathfrak{a}^*$ and assume that $\xi \geq \vartheta$ on $\mathfrak{a}^+$. Then there exists a finite subset $\Theta \subset U(\mathfrak{n})$ such that for all $f \in C^\infty(G/N_0 : \chi)$ and all $a \in A$ we have the estimate*

$$a^{-\xi}|f(a)| \leq e^{-\vartheta(\log a)} \max_{u\in\Theta} |L_u f(a)|. \tag{2.9}$$

Before we start with the proof, we need to introduce suitable notation. As usual, for $\Phi \subset \Delta$ we define $\mathfrak{a}_\Phi$ to be the intersection of the spaces $\ker\alpha$, for $\alpha \in \Phi$. In particular, $\mathfrak{a}_\Delta$ equals the centralizer of $\mathfrak{g}$ in $\mathfrak{a}$. We agree to write ${}^*\mathfrak{a}_\Phi$ for the orthocomplement of $\mathfrak{a}_\Phi$ in $\mathfrak{a}$. Then $\mathfrak{a} = {}^*\mathfrak{a}_\Phi \oplus \mathfrak{a}_\Phi$.

The collection of restrictions $\alpha|_{{}^*\mathfrak{a}_\Delta}$, for $\alpha \in \Delta$, is a basis of ${}^*\mathfrak{a}_\Delta^*$. The associated dual basis of ${}^*\mathfrak{a}_\Delta$ is denoted by $\{h_\alpha \mid \alpha \in \Delta\}$. We define ${}^\backprime\mathfrak{a}_\Phi := \mathrm{span}_\mathbb{R}\{h_\alpha \mid \alpha \in \Phi\}$. Then we have the following direct sum decomposition

$$\mathfrak{a} = {}^\backprime\mathfrak{a}_\Phi \oplus \mathfrak{a}_\Phi. \tag{2.10}$$

This decomposition will be important in the proof of Lemma 2.5.

We denote by ${}^\backprime\mathfrak{a}_\Phi^-$ the interior in ${}^\backprime\mathfrak{a}_\Phi$ of the closed cone spanned by the elements $-h_\alpha$ for $\alpha \in \Phi$. Then

$${}^\backprime\mathfrak{a}_\Phi^- = \{H \in {}^\backprime\mathfrak{a}_\Phi \mid (\forall\alpha \in \Phi) : \alpha(H) < 0\}.$$

In addition, we define $\mathfrak{a}(\Phi) := {}^\backprime\mathfrak{a}_\Phi^- + \mathrm{cl}(\mathfrak{a}_\Phi^+)$.

Lemma 2.6 *The set $\mathfrak{a}$ is the disjoint union of the sets $\mathfrak{a}(\Phi)$, for $\Phi \subset \Delta$.*

Proof. We write $\mathbb{R}^\Delta$ for the real linear space of functions $\Delta \to \mathbb{R}$. For $\Phi \subset \Delta$ we define $\mathbb{R}^\Delta(\Phi)$ to be the subset of $\mathbb{R}^\Delta$ consisting of $x \in \mathbb{R}^\Delta$ with $x_\alpha < 0$ for $\alpha \in \Phi$ and $x_\beta \geq 0$ for $\beta \in \Delta \setminus \Phi$. It is clear that $\mathbb{R}^\Delta$ is the disjoint union of the sets $\mathbb{R}^\Delta(\Phi)$. Consider the linear map $p : \mathfrak{a} \to \mathbb{R}^\Delta$ defined by $p(H)_\alpha = \alpha(H)$ for $\alpha \in \Delta$. Then p is a surjective linear map, hence $\mathfrak{a}$ is the disjoint union of the sets $p^{-1}(\mathbb{R}^\Delta(\Phi))$, for $\Phi \subset \Delta$.

We will finish the proof by showing that $p^{-1}(\mathbb{R}^\Delta(\Phi)) = \mathfrak{a}(\Phi)$. For this, suppose $H \in \mathfrak{a}$ and consider the decomposition $H = {}^{\backprime}H + H_\Phi$, according to (2.10). Then $H \in p^{-1}(\mathbb{R}^\Delta(\Phi))$ is equivalent to the assertion that $\alpha(H) < 0$ and $\beta(H) > 0$ for all $\alpha \in \Phi$ and $\beta \in \Delta \setminus \Phi$. This in turn is equivalent to the assertion that $\alpha({}^{\backprime}H) < 0$ and $\beta(H_\Phi) \geq 0$ for all $\alpha \in \Phi$ and $\beta \in \Delta \setminus \Phi$, hence to ${}^{\backprime}H \in {}^{\backprime}\mathfrak{a}_\Phi^-$ and $H_\Phi \in \mathrm{cl}(\mathfrak{a}_\Phi^+)$. By definition, the latter is equivalent to $H \in \mathfrak{a}(\Phi)$. □

Proof of Lemma 2.5. We may fix $k \in \mathbb{N}$ sufficiently large, such that for every $\alpha \in \Delta$ we have $k\alpha + \vartheta \leq \xi$ on $-h_\alpha$. For $\Phi \subset \Delta$ we put

$$\sigma_\Phi := \sum_{\alpha \in \Phi} \alpha.$$

Then it is readily verified that $k\sigma_\Phi + \vartheta \leq \xi$ on ${}^{\backprime}\mathfrak{a}_\Phi^-$. Since σ_Φ vanishes on $\mathfrak{a}_\Phi^+$, whereas $\mathfrak{a}_\Phi^+ \subset \mathrm{cl}(\mathfrak{a}^+)$, it follows from the hypothesis that the same estimate is valid on $\mathfrak{a}_\Phi^+$. Using the subadditivity of ϑ and the linearity of σ_Φ and ξ we now find that

$$k\sigma_\Phi + \vartheta \leq \xi \qquad \text{on} \quad \mathfrak{a}(\Phi). \tag{2.11}$$

The idea is now to derive suitable estimates on the set $A(\Phi) := \exp(\mathfrak{a}(\Phi))$ for $\Phi \subset \Delta$ fixed, by using Lemma 2.3. Put $u = u_\Phi = \prod_{\alpha \in \Phi} X_\alpha^k$, where an arbitrary fixed ordering in the product may be taken, and where $X_\alpha \in \mathfrak{g}_\alpha$ are such that $\chi_*(X_\alpha) = 1$.

Let $f \in C^\infty(G/N_0 : \chi)$ and $a \in A(\Phi)$, then it follows that

$$|f(a)| = a^{k\sigma_\Phi} |\chi_*(u)|^{-1} |L_u f(a)| = a^{k\sigma_\Phi} |L_u f(a)|.$$

Therefore,

$$a^{-\xi} |f(a)| = a^{-\xi + k\sigma_\Phi} |L_u f(a)| \leq e^{-\vartheta(\log a)} |L_u f(a)|.$$

As the sets $\mathfrak{a}(\Phi)$ cover $\mathfrak{a}$, we find the desired estimate with $\Theta = \{u_\Phi \mid \Phi \subset \Delta\}$. □

Corollary 2.7 *Assume that G has compact center and let (π, H) be an admissible Hilbert G-representation of finite length. Let $\lambda \in \mathrm{Wh}_\chi(H^\infty)$. Then there exists a $\xi \in \mathfrak{a}^*$ and a continuous seminorm n on H^∞ such that for all $v \in H^\infty$ and $a \in A$ we have*

$$|\mathrm{wh}_\lambda(v)(a)| \leq a^\xi \mathrm{n}(v).$$

Proof. It follows from (2.6) that there exists a continuous seminorm n_0 on H^∞ such that for all $v \in H^\infty$ and $a \in A$,

$$|\mathrm{wh}_\lambda(v)(a)| \leq e^{r|\log a|} \mathrm{n}_0(v).$$

Since G has compact center, $\mathfrak{a}_\Delta = 0$, so that $\mathrm{cl}(\mathfrak{a}^+)$ is a proper closed cone in $\mathfrak{a}$. Hence, there exists a linear functional $\xi \in \mathfrak{a}^*$ such that $\xi > 0$ on $\mathrm{cl}(\mathfrak{a}^+) \setminus \{0\}$. Let S be the unit sphere in $\mathfrak{a}^*$, then by compactness of $S \cap \mathrm{cl}(\mathfrak{a}^+)$ we may multiply ξ by a positive scalar to arrange that $\xi > r$ on $S \cap \mathrm{cl}(\mathfrak{a}^+)$. This implies that $r|\cdot| \leq \xi$ on $\mathrm{cl}(\mathfrak{a}^+)$. Let now $\Theta \subset U(\mathfrak{n}_0)$ be a finite subset as in Lemma 2.5. Then $L_u w(v) = w(\pi(u)v)$, so that

$$a^{-\xi}|\mathrm{wh}_\lambda(v)(a)| \leq \max_{u\in\Theta} e^{-r|\log a|}|\mathrm{wh}_\lambda(\pi(u)v)(a)| \leq \max_{u\in\Theta} n_0(\pi(u)v).$$

The required estimate now follows with the continuous seminorm defined by $n(v) := \max_{u\in\Theta} n_0(\pi(u)v)$. □

At a later stage we will also need the following result. We retain the assumption that G has compact center.

Lemma 2.8 *Let $\mu \in \mathfrak{a}^*$ be such that $\mu(h_\alpha) > 0$ for all $\alpha \in \Delta$. Then there exists a constant $s > 0$ and a finite subset $\Theta \subset U(\mathfrak{n}_0)$ such that for all $f \in C^\infty(G/N_0 : \chi)$ and all $a \in A$ we have the estimate*

$$a^{-\mu}|f(a)| \leq e^{-s|\log a|} \max_{u\in\Theta} |L_u f(a)|. \tag{2.12}$$

Proof. Since $\mathrm{cl}(\mathfrak{a}^+)$ is the cone spanned by the elements h_α, for $\alpha \in \Delta$, it follows that there exists $s > 0$ such that $\mu \geq s$ on $\mathrm{cl}(\mathfrak{a}^+) \cap S$, where S is the unit sphere in $\mathfrak{a}$. This implies that $\mu(H) \geq s|H|$ for all $H \in \mathfrak{a}^+$. The result now follows by application of Lemma 2.5. □

3 The Whittaker Schwartz space

We denote the map $G \to \mathfrak{a}$ associated with the Iwasawa decomposition $G = KAN_0$ by H. Thus, for $k \in K, a \in A$ and $n_0 \in N_0$,

$$H(kan_0) = \log a. \tag{3.1}$$

Let $\rho \in \mathfrak{a}^*$ be defined by $\rho(H) := \frac{1}{2}\mathrm{tr}\,[\mathrm{ad}(H)|_{\mathfrak{n}_0}]$.

Following Harish-Chandra [12, §1.3] and Wallach [21, §15.3.1] we define the Whittaker Schwartz space $\mathcal{C}(G/N_0 : \chi)$ to be the space of functions $f \in C^\infty(G/N_0 : \chi)$ such that for all $u \in U(\mathfrak{g})$ and $N > 0$

$$n_{u,N}(f) := \sup_{x\in G}(1 + |H(x)|)^N e^{\rho H(x)} \cdot |L_u f(x)| < \infty.$$

The indicated seminorms $n_{u,N}$ induce a Fréchet topology on $\mathcal{C}(G/N_0 : \chi)$. It is readily verified that $C_c^\infty(G/N_0 : \chi) \subset \mathcal{C}(G/N_0 : \chi) \subset C^\infty(G/N_0 : \chi)$, with continuous inclusion maps.

Lemma 3.1 *The space $C_c^\infty(G/N_0 : \chi)$ is dense in $\mathcal{C}(G/N_0 : \chi)$.*

Proof. For $t > 0$ we define

$$B_t := \{x \in G \mid |H(x)| \le t\}$$

Then B_t is right N_0-invariant, with compact image in G/N_0. Adapting the argument given in [19, p. 343, Lemma 1] in an obvious fashion, we infer that there exist left K-invariant functions $\psi_t \in C_c^\infty(G/N_0)$, for $t > 0$ such that $0 \le \psi_t \le 1$, $\psi_t = 1$ on B_t, $\operatorname{supp} \psi_t \subset B_{t+1}$ for all $t > 0$ and such that, in addition, for every $u \in U(\mathfrak{g})$ there exists a constant $C_u > 0$ such that

$$|L_u(\psi_t)(x)| \le C_u \quad \text{for all} \quad t > 0,\ x \in G.$$

Adapting the argument of [19, p. 343, Thm. 2], again in an obvious way, we deduce that for every $u \in U(\mathfrak{g})$, $N > 0$ there exists a finite subset $V \subset U(\mathfrak{g})$ such that for all $t \ge 1$,

$$n_{u,N}(f - \psi_t f) \le \sum_{v \in V} (1+t)^{-1} n_{v,N+1}(f).$$

From this it follows that $\psi_t f \to f$ in $C(G/N_0 \colon \chi)$ as $t \to \infty$. □

Lemma 3.2 *The space $C(G/N_0 \colon \chi)$ is invariant under left translation by elements of G. The associated left regular representation L of G on it is continuous.*

Proof. We start with the observation that for $x \in G/N_0$ and $g \in G$ one has

$$H(gx) = H(gk(x)) + H(x),$$

where $k(x)$ is determined by $x \in k(x)AN_0$. It follows from this that for every compact subset $S \subset G$ and every $N \in \mathbb{N}$ there exists a constant $C_{S,N} > 0$ such that

$$e^{\rho H(gx)}(1 + |H(gx)|)^N \le C_{S,N} e^{\rho H(x)} (1 + |H(x)|)^N.$$

This implies that, for $g \in S$ and $f \in C(G/N_0 \colon \chi)$,

$$n_{1,N}(L_g f) \le C_{S,N} n_{1,N+1}(f).$$

Noting that $L_u(L_g f) = L_{\mathrm{Ad}(g^{-1})u} f$ and observing that $\mathrm{Ad}(S^{-1})u$ is a bounded subset of a finite dimensional subspace of $U(\mathfrak{g})$, we deduce the existence of a finite subset $V \subset U(\mathfrak{g})$ such that for all $g \in S$ and $f \in C(G/N_0 \colon \chi)$ we have

$$n_{u,N}(L_g f) \le \sum_{v \in V} n_{v,N}(f).$$

This implies that $C(G/N_0 \colon \chi)$ is invariant for the left regular representation and that the set of linear maps L_g, for $g \in S$, is equicontinuous in $\mathrm{End}(C(G/N_0 : \chi))$. If $f_0 \in C_c^\infty(G/N_0 \colon \chi)$ then for $g \to e$, the function $L_g f_0$ tends to f_0 in $C_c^\infty(G/N_0 \colon \chi)$ hence in $C(G/N_0 \colon \chi)$. Using the density of $C_c^\infty(G/N_0 \colon \chi)$ in $C(G/N_0 \colon \chi)$ it follows by a standard argument that for all $f \in C(G/N_0 \colon \chi)$ we have

$$\lim_{g \to e} L_g f = f \quad \text{in} \quad C(G/N_0 \colon \chi).$$

Invoking the equicontinuity mentioned above, it now follows by a standard argument that the map $(g, f) \mapsto L_g f$ is continuous $G \times C(G/N_0 \colon \chi) \to C(G/N_0 \colon \chi)$. □

Lemma 3.3 *If $\ell > \dim A$ then*

$$\int_{G/N_0} (1 + |H(x)|)^{-\ell} e^{-2\rho H(x)}\, dx < \infty.$$

Proof. By substitution of variables a measurable function $\varphi : G/N_0 \to \mathbb{C}$ is absolutely integrable if and only if the function $(k, a) \mapsto \varphi(ka)a^{2\rho}$ is absolutely integrable over $K \times A$. If so, the integrals $\int_{G/N_0} \varphi(x)\, d\dot{x}$ and $\int_{K\times A} \varphi(ka)a^{2\rho}\, dkda$ are equal, provided the invariant measures are suitably normalized. From this, the proof is immediate. □

Corollary 3.4 $C(G/N_0 : \chi) \subset L^2(G/N_0 : \chi)$*, with continuous linear inclusion map.*

We end this section with a result that will be applied in the next section. It is assumed that G has compact center. Then Δ is a linear basis of $\mathfrak{a}^*$; the associated dual basis of $\mathfrak{a}$ is denoted by $\{h_\alpha \mid \alpha \in \Delta\}$.

Lemma 3.5 *Suppose that G has compact center and let $\xi \in \mathfrak{a}^*$ be such that $\xi(h_\alpha) < -\rho(h_\alpha)$ for all $\alpha \in \Delta$. Let (π, H) be an admissible continuous representation of finite length of G in a Hilbert space and let $\lambda \in \mathrm{Wh}_\chi(H^\infty)$. Assume there exist a continuous seminorm n on H^∞ and a constant $d \in \mathbb{N}$ such that, for all $v \in H^\infty$ and $a \in A$,*

$$|\mathrm{wh}_\lambda(v)(a)| \le a^\xi (1 + |\log a|)^d\, \mathrm{n}(v).$$

Then the Whittaker coefficient map wh_λ is continuous $H^\infty \to C(G/N_0{:}\chi)$.

Proof. We put $\mu = -\rho - \xi$. Then it follows that $\mu(h_\alpha) > 0$ for all $\alpha \in \Delta$. By Lemma 2.8 there exists a finite set $\Theta \subset U(\mathfrak{n}_0)$ and a constant $s > 0$ such that for all $f \in C^\infty(G/N_0{:}\chi)$ and all $a \in A$ we have the estimate

$$a^{-\mu}|f(a)| \le e^{-s|\log a|} \max_{u\in\Theta} |L_u f(a)|.$$

This implies that

$$\begin{aligned} a^\rho |f(a)| &= a^{-\xi} a^{-\mu} |f(a)| \\ &\le \max_{u\in\Theta} a^{-\xi} e^{-s|\log a|} |L_u f(a)| \\ &\le e^{-s|\log a|}(1 + |\log a|)^d \max_{u\in\Theta} a^{-\xi}(1 + |\log a|)^{-d} |L_u f(a)|. \end{aligned}$$

Using the above estimate for $f = \mathrm{wh}_\lambda(v)$, with $v \in H^\infty$, we find that

$$a^\rho |\mathrm{wh}_\lambda(v)(a)| \le a^{-s|\log a|}(1 + |\log a|)^d \max_{u\in\Theta} \mathrm{n}(\pi(u)v). \tag{3.2}$$

For $N \in \mathbb{N}$ we define the positive number

$$C_N = \sup_{t\ge 0} e^{-st}(1 + t)^{N+d}. \tag{3.3}$$

It follows from (3.2) and (3.3) that, for all $v \in H^\infty$ and $a \in A$,

$$a^\rho (1 + |\log a|)^N |\mathrm{wh}_\lambda(v)(a)| \leq C_N \, \mathrm{n}_1(v),$$

where n_1 is the continuous seminorm on H^∞ given by $\mathrm{n}_1(v) = \max_{u \in \Theta}(\pi(u)v)$. Finally, the last displayed estimate implies that, for $u \in U(\mathfrak{g})$ and for all $v \in H^\infty$ and $x \in G$,

$$\begin{aligned} e^{\rho H(x)}(1 + |H(x)|)^N |L_u(\mathrm{wh}_\lambda(v))(x)| &= e^{\rho H(x)}(1 + |H(x)|)^N |(\mathrm{wh}_\lambda(\pi(u)v))(x)| \\ &\leq \mathrm{n}_u(v), \end{aligned}$$

with n_u the continuous seminorm on H^∞ given by $\mathrm{n}_\mathrm{u}(v) = C_2 \sup_{k \in K} \mathrm{n}_1(\pi(k)\pi(u)v))$.
□

4 Sharp estimates for Whittaker coefficients

In this section we assume that G has compact center. We shall derive certain growth properties of Whittaker coefficients, building on results and ideas of Wallach [21] and Harish-Chandra [12].

We assume that (H, π) is an admissible continuous representation of finite length of G in a Hilbert space. Let $V = H_K$ be the associated Harish-Chandra module, and let $V^\sim$ denote the associated dual Harish-Chandra module. Then it is well known that the natural map $(H^\infty)' \to V^*$ induces an isomorphism $(H^\infty)'_K \simeq V^\sim$.

We agree to write $E(P_0, V)$ for the set of generalized weights of the finite dimensional $\mathfrak{a}$-module $V/\mathfrak{n}_0 V$.

Let Δ be the collection of simple roots in $\Sigma(\mathfrak{n}_0, \mathfrak{a})$ and let $\{h_\alpha \mid \alpha \in \Delta\}$ be the associated dual basis of $\mathfrak{a}$. We define $\Lambda_V \in \mathfrak{a}^*$ by

$$\Lambda_V(h_\alpha) := \max\{-\mathrm{Re}\,\mu(h_\alpha) \mid \mu \in E(P_0, V)\}. \tag{4.1}$$

Remark 4.1 At a later stage it will be of crucial importance to us that for π irreducible unitary, the following two conditions are equivalent

(a) π is equivalent to a direct sum of representations from the discrete series of G;

(b) $\Lambda_V(h_\alpha) < -\rho(h_\alpha)$ for all $\alpha \in \Delta$.

For a proof of this well known result, we refer to [20] as follows. The element Λ_V corresponds to $\Lambda_{V^\sim}$ as defined in [20, §4.3.5]. In the terminology of [20, §5.1.1] assertion (b) means that $V^\sim$ is rapidly decreasing. According to [20, Prop. 5.1.3 & Thm. 5.5.4] this is equivalent to the assertion that $(\pi^\vee, H')$ is a direct sum of square integrable representations, which in turn is equivalent to assertion (a).

The following result is due to Wallach [21, 15.2.2]. The regularity of χ is not required.

Proposition 4.2 *Put* $\Lambda := \Lambda_V$ *and let* $\lambda \in \mathrm{Wh}_\chi(H^\infty)$. *Then there exists a constant* $d \geq 0$ *and for every* $v \in H_K$ *a constant* $C_v > 0$ *such that*

$$|\lambda(\pi(a^{-1})v)| \leq C_v(1 + |\log a|)^d\, a^\Lambda.$$

for all $a \in \mathrm{cl}(A^+)$.

Wallach's proof of this result follows the lines of the proof of an earlier result, stated in [20, Thm. 4.3.5]. That result, applied to the contragredient representation $\pi^\vee$, asserts that for a given $v \in ((H')^\infty)'_K \simeq H_K$ there exists a constant $d > 0$, a continuous seminorm σ_v on $(H')^\infty$ such that for all $\lambda \in (H')^\infty$ one has

$$|v(\pi^\vee(a)\lambda)| \leq (1 + |\log a|)^d a^\Lambda \sigma_v(\lambda).$$

As $v(\pi^\vee(a)\lambda) = \lambda(\pi(a^{-1})v)$, the result [20, Thm. 4.3.5] implies that Proposition 4.2 is valid for λ a smooth vector in H'.

The proof of [20, Thm. 4.3.5] makes use of initial estimates and of estimate improvement through asymptotic behavior along maximal standard parabolic subgroups P_Φ, with $\Phi = \Delta \setminus \{\alpha\}$. It exploits a system of differential equations coming from the observation that $V/\mathfrak{n}_\Phi V$ is an admissible $(\mathfrak{m}_{1\Phi}, K_\Phi)$-module ($\mathfrak{m}_{1\Phi} = \mathfrak{m}_\Phi + \mathfrak{a}_\Phi$), so that $\mathfrak{a}_\Phi$ acts finitely on it. From the proof one sees that only estimates of $[\pi^\vee(U)\lambda](\pi(a)w)$ for $U \in U(\mathfrak{n}_0)$, $w \in H_K$ and $a \in A^+$ are needed to make the approach work. This is the condition of (P_0, A)-tameness of [21, §15.2.1]. If $\lambda \in \mathrm{Wh}_\chi(H^\infty)$ then $\pi^\vee(U)\lambda = \chi_*(U^\vee)\lambda$ so that the needed tameness is trivially guaranteed. Therefore, essentially the same approach gives the validity of Proposition 4.2. For further details, see the proof of [21, Thm. 15.2.2], assertion (1).

For regular χ the following refinement of Proposition 4.2 will be of crucial importance to us, since it gives an estimate for the Whittaker coefficient $\mathrm{wh}_\lambda(v)$ for every smooth $v \in H^\infty$ on all Weyl chambers of A. The key idea is to now focus on the $\mathfrak{a}_\Phi$-actions on the modules $U(\mathfrak{g})\lambda/\bar{\mathfrak{n}}_\Phi^k U(\mathfrak{g})\lambda$, for $k \geq 1$, making use of the information provided by Proposition 4.2 to exclude the contribution of $\mathfrak{a}_\Phi$-weights that are not dominated by $\Lambda|_{\mathfrak{a}_\Phi}$.

Theorem 4.3 *Suppose that G has compact center and let* $\Lambda := \Lambda_V$. *Assume that* χ *is a regular unitary character of* N_0 *and that* $\lambda \in \mathrm{Wh}_\chi(H^\infty)$. *Then there exists a constant* $d \geq 0$ *and a continuous seminorm* n *on* H^∞ *such that for every* $v \in H^\infty$,

$$|\lambda(\pi(a^{-1})v)| \leq (1 + |\log a|)^d\, a^\Lambda\, \mathrm{n}(v),$$

for all $a \in A$.

Remark 4.4 [21, Thm. 15.2.5] gives this estimate for $a \in \mathrm{cl}(A^+)$.

The rest of this section is devoted to the proof of Theorem 4.3. We start with the fact that there exists a $\xi \in \mathfrak{a}^*$, a constant $d \geq 0$ and a continuous seminorm n on H^∞ such that

$$|\mathrm{wh}_\lambda(v)(a)| \leq a^\xi (1 + |\log a|)^d \,\mathrm{n}(v) \qquad (v \in H^\infty,\ a \in A). \tag{4.2}$$

Indeed, according to Corollary 2.7 this estimate holds with $d = 0$ for a suitable choice of ξ.

In the following we shall say that $\xi \in \mathfrak{a}^*$ dominates the Whittaker coefficient wh_λ if there exist a $d \in \mathbb{N}$ and a continuous seminorm n on H^∞ such that the estimate (4.2) is valid for all $v \in H^\infty$ and $a \in A$. The following result concerning domination will be useful.

Lemma 4.5 *Let $\vartheta, \xi \in \mathfrak{a}^*$ be such that $\vartheta \leq \xi$ on $\mathfrak{a}^+$. If ϑ dominates the Whittaker coefficient* wh_λ, *then so does ξ.*

Proof. Let Θ be a finite subset of $U(\mathfrak{n}_0)$ with the properties guaranteed by Lemma 2.5. Assume ϑ is dominating. Then there exist a constant $d \geq 0$ and a continuous seminorm n on H^∞ such that for all $v \in H^\infty$ and all $a \in A$ we have

$$a^{-\vartheta} |\mathrm{wh}_\lambda(v)(a)| \leq (1 + |\log a|)^d \mathrm{n}(v).$$

By applying Lemma 2.5 with $f = \mathrm{wh}_\lambda(v)$, and using that $L_u(\mathrm{wh}_\lambda(v)) = \mathrm{wh}_\lambda(\pi_*(u)v)$ we infer that, for all $v \in H^\infty$ and $a \in A$,

$$\begin{aligned} a^{-\xi} |\mathrm{wh}_\lambda(v)(a)| &\leq \max_{u \in \Theta} a^{-\vartheta} |\mathrm{wh}_\lambda(\pi_*(u)v)(a)| \\ &\leq (1 + |\log a|)^d \max_{u \in \Theta} \mathrm{n}(\pi_*(u)v). \end{aligned}$$

Since $\mathrm{n}' : v \mapsto \max_{u \in \Theta} \mathrm{n}(\pi_*(u)v)$ is a continuous seminorm on H^∞ it follows that ξ dominates wh_λ. □

The idea is now to show that a dominating ξ can be improved by using asymptotic expansions along maximal standard parabolic subgroups derived from suitable differential equations. This method, inspired by [20, §4.4] and [21, §15.2], leads to the following lemma, which is the main step in our proof of Theorem 4.3. We write $\Lambda = \Lambda_{H_K}$.

Lemma 4.6 (estimate improvement) *Assume that the Whittaker coefficient* wh_λ *is dominated by $\xi \in \mathfrak{a}^*$. Let $\alpha \in \Delta$. Then* wh_λ *is also dominated by ξ', where $\xi' \in \mathfrak{a}^*$ is defined by*

(a) $\xi' = \xi$ *on* $\ker \alpha$;

(b) $\xi'(h_\alpha) = \min(\xi(h_\alpha), \Lambda(h_\alpha))$.

The proof of this lemma will be given in the next section. Here we note that using the lemma successively for all simple roots $\alpha \in \Delta$ we obtain the following corollary.

Corollary 4.7 (estimate improvement) *Assume that the Whittaker coefficient* wh_λ *is dominated by* $\xi \in \mathfrak{a}^*$*. Then* wh_λ *is also dominated by* ξ''*, where* $\xi'' \in \mathfrak{a}^*$ *is defined by* $\xi''(h_\alpha) = \min(\xi(h_\alpha), \Lambda(h_\alpha))$ *for every* $\alpha \in \Delta$*. In particular,* $\xi''(h_\alpha) \leq \Lambda(h_\alpha)$ *for all* $\alpha \in \Delta$.

The above results allow us to establish the main result of this section.

Proof of Theorem 4.3. According to Corollary 2.7 there exists a $\xi \in \mathfrak{a}^*$ which dominates wh_λ. By Corollary 4.7 the coefficient wh_λ is also dominated by ξ''. From the definition of ξ'' one sees that $\xi'' \leq \Lambda$ on $\mathfrak{a}^+$. By application of Lemma 4.5 it now follows that Λ dominates wh_λ. This establishes the assertion of Theorem 4.3. □

Corollary 4.8 *Suppose that* (π, H) *belongs to the discrete series of* G *and let* $\lambda \in \mathrm{Wh}_\chi(H^\infty)$*. Then the associated Whittaker coefficient* wh_λ *defines a continuous linear map* $H^\infty \to C(G/N_0 : \chi)$.

Proof. From Remark 4.1 it follows that $\Lambda(h_\alpha) < -\rho(h_\alpha)$ for $\alpha \in \Delta$. The result now follows by combining Theorem 4.3 with Lemma 3.5. □

5 Proof of Lemma 4.6: improvement of estimates

As in the previous section, we assume that G has compact center. Furthermore, (π, H) is an admissible continuous representation of finite length of G in a Hilbert space, and $V = H_K$ is the associated Harish-Chandra module. We define $\Lambda = \Lambda_V$ as in (4.1) and assume that $\lambda \in \mathrm{Wh}(H^\infty)$. Let $\mathrm{wh}_\lambda : H^\infty \to C^\infty(G/N_0 : \chi)$ be the associated Whittaker coefficient. The purpose of this section is to prove Lemma 4.6.

Our starting assumption is that wh_λ is dominated by $\xi \in \mathfrak{a}^*$. This means that there exists a constant $d \in \mathbb{N}$ and a continuous seminorm n on H^∞ such that for all $v \in H^\infty$ and all $a \in A$ we have

$$|m_\lambda(v)(a)| \leq (1 + |\log a|)^d a^\xi \mathrm{n}(v). \tag{5.1}$$

We will improve upon this estimate by using a system of differential equations satisfied by wh_λ. Our first goal is to set up this system.

Given a finite dimensional real linear space $\mathfrak{v}$ we denote by $S(\mathfrak{v})$ the symmetric algebra of its complexification $\mathfrak{v}_\mathbb{C}$ and by $P(\mathfrak{v})$ the algebra of polynomial functions $\mathfrak{v}_\mathbb{C} \to \mathbb{C}$.

Given a real Lie algebra $\mathfrak{l}$ we denote by $U(\mathfrak{l})$ the universal enveloping algebra of its complexification $\mathfrak{l}_\mathbb{C}$, and by $\mathfrak{Z}(\mathfrak{l})$ the center of $U(\mathfrak{l})$. Furthermore, $U(\mathfrak{l})$ is equipped with the standard filtration by order, $(U(\mathfrak{l})_n)_{n\geq 0}$. The center $\mathfrak{Z}(\mathfrak{l})$ is equipped with the induced filtration.

For a parabolic subgroup P of G we denote its Langlands decomposition by $P = M_P A_P N_P$ and we write $M_{1P} := M_P A_P$. We agree to use the abbreviated notation $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{g})$, $\mathfrak{Z}_{1P} = \mathfrak{Z}(\mathfrak{m}_{1P})$ and $\mathfrak{Z}_P := \mathfrak{Z}(\mathfrak{m}_P)$.

Given a subset $\Phi \subset \Delta$ we denote by P_Φ the associated standard parabolic subgroup of G. Its Langlands components are denoted by M_Φ, A_Φ, N_Φ. Furthermore, $M_{1\Phi} := M_\Phi A_\Phi$, $\mathfrak{Z}_{1\Phi} := \mathfrak{Z}(\mathfrak{m}_{1\Phi})$ and $\mathfrak{Z}_\Phi := \mathfrak{Z}(\mathfrak{m}_\Phi)$.

We consider the $U(\mathfrak{g})$-submodule

$$\mathcal{Y} := U(\mathfrak{g})\lambda$$

of $(H^\infty)'$. Since H is admissible and of finite length, $\mathfrak{Z}\lambda$ is a finite dimensional subspace of $\mathcal{Y}$.

We fix $\Phi \subset \Delta$. By the PBW theorem, $U(\mathfrak{g}) = U(\mathfrak{m}_{1\Phi}) \oplus (\bar{\mathfrak{n}}_\Phi U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_\Phi)$. The associated projection $U(\mathfrak{g}) \to U(\mathfrak{m}_{1\Phi})$, restricted to $\mathfrak{Z}$, defines an algebra homomorphism

$$p : \mathfrak{Z} \to \mathfrak{Z}_{1\Phi}. \tag{5.2}$$

It is well known that p is injective and preserves the filtrations induced by the standard filtration $(U(\mathfrak{g})_n)_{n\in\mathbb{N}}$ by order on $U(\mathfrak{g})$.

We fix a maximal torus $\mathfrak{t} \subset \mathfrak{m}$; then $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $W(\mathfrak{h})$ denote the Weyl group of the root system $R(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{h}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Furthermore, let $W_\Phi(\mathfrak{h})$ denote the Weyl group of $R(\mathfrak{m}_{1\Phi}, \mathfrak{h})$. Then $W_\Phi(\mathfrak{h})$ equals the centralizer of $\mathfrak{a}_\Phi$ in $W(\mathfrak{h})$.

We denote by $S(\mathfrak{h})^{W(\mathfrak{h})}$ and $S(\mathfrak{h})^{W_\Phi(\mathfrak{h})}$ the associated subalgebras of Weyl group invariants in $S(\mathfrak{h})$. Then it is well known that $S(\mathfrak{h})^{W_\Phi(\mathfrak{h})}$ is a free $S(\mathfrak{h})^{W(\mathfrak{h})}$-module of rank $\ell = [W(\mathfrak{h}) : W_\Phi(\mathfrak{h})]$ with free homogeneous generators, $q_1 = 1, q_2, \dots, q_\ell$., see [23, Thm 2.1.3.6].

Let $\gamma : \mathfrak{Z} \to S(\mathfrak{h})^{W(\mathfrak{h})}$ and $\gamma_\Phi : \mathfrak{Z}_{1\Phi} \to S(\mathfrak{h})^{W_\Phi(\mathfrak{h})}$ denote the associated Harish-Chandra isomorphisms. These are known to be isomorphisms of filtered algebras. Furthermore,

$$p = T_{-\rho_\Phi} \circ \gamma_\Phi^{-1} \circ \gamma,$$

where $T_{-\rho_\Phi}$ is the automorphism T of $\mathfrak{Z}_{1\Phi} \simeq \mathfrak{Z}_\Phi \otimes S(\mathfrak{a}_\Phi)$ determined by $T = I$ on $\mathfrak{Z}_\Phi$ and by $T(X) = X - \rho_\Phi(X)$, for $X \in \mathfrak{a}_\Phi$. Here $\rho_\Phi \in \mathfrak{a}^*$ is given by $\rho_\Phi(X) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}(X)|_{\mathfrak{n}_\Phi})$. For $1 \leq i \leq \ell$, put

$$u_i := T_{-\rho_\Phi}\gamma_\Phi^{-1}(q_i) \in \mathfrak{Z}_{1\Phi}.$$

Then we see that $\mathfrak{Z}_{1\Phi}$ is a free $p(\mathfrak{Z})$-module with basis $u_1 = 1, u_2, \dots, u_\ell$. Furthermore, since the q_i are homogeneous and since γ, γ_Φ and $T_{-\rho_\Phi}$ are isomorphisms of filtered algebras, the following is valid.

For every $u \in \mathfrak{Z}_{1\Phi} \cap U(\mathfrak{g})_n$, $(n \in \mathbb{N})$, let $Z_1, \dots, Z_\ell \in \mathfrak{Z}$ be the unique elements such that $u = \sum_{i=1}^\ell u_i p(Z_i)$. Then for all $1 \leq i \leq \ell$,

$$\mathrm{ord}(u_i) + \mathrm{ord}(Z_i) \leq n \tag{5.3}$$

Let E_Φ be the complex linear span of the elements $\{u_i \mid 1 \leq i \leq \ell\}$ in $\mathfrak{Z}_{1\Phi}$. Then the map $(u, Z) \mapsto up(Z)$ induces a linear isomorphism $E_\Phi \otimes \mathfrak{Z} \simeq \mathfrak{Z}_{1\Phi}$.

Lemma 5.1 *With $E_\Phi \subset \mathfrak{Z}_{1\Phi}$ as above, we have*

$$U(\mathfrak{g}) = U(\bar{\mathfrak{n}}_\Phi)U(\mathfrak{m}_\Phi)E_\Phi\mathfrak{Z}U(\mathfrak{n}_\Phi). \tag{5.4}$$

Proof. By induction on $n \in \mathbb{N}$ we will show that $U(\mathfrak{g})_n$ is contained in the space on the right-hand side of (5.4). For $n = 0$ the inclusion is obvious. Thus, let $n \geq 1$ and assume the inclusion has been established for strictly smaller values of n. We observe that by the PBW theorem, every element of $U(\mathfrak{g})_n$ may be written as a sum of an element of $U(\mathfrak{g})_{n-1}$ and a finite sum of products $wvuy$ with $w \in U(\bar{\mathfrak{n}}_\Phi)$, $v \in U(\mathfrak{m}_\Phi)$, $u \in U(\mathfrak{a}_\Phi)$ and $y \in U(\mathfrak{n}_\Phi)$ such that

$$\mathrm{ord}(w) + \mathrm{ord}(v) + \mathrm{ord}(u) + \mathrm{ord}(y) \leq n. \tag{5.5}$$

In view of the induction hypothesis, it suffices to show that each such product $wvuy$ with (5.5) belongs to the space on the right of (5.4).

Since $U(\mathfrak{a}_\Phi) \subset \mathfrak{Z}_{1\Phi}$, the element u may be expressed as a sum of elements $u_i p(Z_i)$ with $\mathrm{ord}(u_i)+\mathrm{ord}(Z_i) \leq \mathrm{ord}(u)$. Now $p(Z_i)-Z_i \in U(\mathfrak{g})_{n_i-1}\mathfrak{n}_\Phi$, where $n_i := \deg Z_i \leq n$. It follows that for every $1 \leq i \leq \ell$,

$$wvu_i(p(Z_i) - Z_i)y \in U(\mathfrak{g})_{n-1}\mathfrak{n}_\Phi.$$

Summing over i and applying the induction hypothesis, we find

$$wvuy \in wv\sum_i u_i Z_i y + U(\mathfrak{g})_{n-1}\mathfrak{n}_\Phi \subset U(\bar{\mathfrak{n}}_\Phi)U(\mathfrak{m}_\Phi)E_\Phi\mathfrak{Z}U(\mathfrak{n}_\Phi).$$

□

For $k \geq 1$, the quotient $\mathcal{M}_k := \mathcal{Y}/\bar{\mathfrak{n}}_\Phi^k\mathcal{Y}$ is a left $U(\mathfrak{m}_{1\Phi})$-module.

Lemma 5.2 *The subspace $E_\Phi\mathfrak{Z}\lambda$ of $\mathcal{Y}$ has a finite dimensional $\mathfrak{a}_\Phi$-invariant image in $\mathcal{M}_1$, which generates the $U(\mathfrak{m}_\Phi)$-module $\mathcal{M}_1$.*

Proof. Since E_Φ and $\mathfrak{Z}\lambda$ are finite dimensional, the mentioned image F_1 of $E_\Phi\mathfrak{Z}\lambda$ in $\mathcal{M}_1$ is finite dimensional. From

$$U(\mathfrak{a}_\Phi)E_\Phi \subset \mathfrak{Z}_{1\Phi} \subset E_\Phi p(\mathfrak{Z}) \subset E_\Phi\mathfrak{Z} + \bar{\mathfrak{n}}_\Phi U(\mathfrak{g})$$

it follows that F_1 is finite dimensional and $\mathfrak{a}_\Phi$-invariant. □

In particular, it follows that the set S_Φ of generalized $\mathfrak{a}_\Phi$-weights of $\mathcal{M}_1$ is finite and that $\mathcal{M}_1$ is the direct sum of the associated generalized $\mathfrak{a}_\Phi$-weight spaces.

Lemma 5.3 *For $k \geq 1$ the set $\mathrm{wt}(\mathcal{M}_k)$ of generalized $\mathfrak{a}_\Phi$-weights of $\mathcal{M}_k$ is finite. Each of its elements is of the form $\sigma - (\alpha_1 + \cdots + \alpha_l)$, with $\sigma \in S_\Phi$, $0 \leq l < k$ and $\alpha_j \in \Sigma(\mathfrak{n}_\Phi, \mathfrak{a}_\Phi)$ for all $1 \leq j \leq l$.*

Proof. For $k = 1$ the result has been established above. For $k \geq 2$ we notice that the natural map $q_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is a surjective morphism of $\mathfrak{a}_\Phi$-modules. Furthermore, its kernel is isomorphic as an $\mathfrak{a}_\Phi$-module to the quotient

$$Q_k := \bar{\mathfrak{n}}_\Phi^{k-1} U(\mathfrak{g})\lambda / \bar{\mathfrak{n}}_\Phi^k U(\mathfrak{g})\lambda.$$

Define $Q_1 := \mathcal{M}_1$, then $\mathrm{wt}(Q_1) = S_\Phi$ and Q_1 is the associated direct sum of generalized $\mathfrak{a}_\Phi$-weight spaces. The natural map $\bar{\mathfrak{n}}_\Phi \otimes Q_{k-1} \to Q_k$ is surjective, for $k \geq 1$. Thus, if the $\mathfrak{a}_\Phi$-module Q_{k-1} is the direct sum of its generalized weight spaces, then so is Q_k, and $\mathrm{wt}(Q_k) \subset \mathrm{wt}(\bar{\mathfrak{n}}_\Phi) + \mathrm{wt}(Q_{k-1})$. It follows by induction on k that each Q_k is the direct sum of its generalized weight spaces, and that for $k \geq 1$ each weight of Q_k is of the form $\sigma - (\alpha_1 + \cdots + \alpha_\ell)$, where $\sigma \in S_\Phi$, $\ell < k$ and $\alpha_j \in \Sigma(\mathfrak{n}_\Phi, \mathfrak{a}_\Phi)$, for $1 \leq j \leq \ell$.

For $k \geq 2$ we now have the short exact sequence of $\mathfrak{a}_\Phi$-modules

$$0 \to Q_k \to \mathcal{M}_k \to \mathcal{M}_{k-1} \to 0.$$

Here Q_k is the direct sum of finitely many generalized $\mathfrak{a}_\Phi$-weight spaces. If $\mathcal{M}_{k-1}$ is the direct sum of finitely many weight spaces, then it follows that the module $\mathcal{M}_k$ is the direct sum of finitely many weight spaces as well, while $\mathrm{wt}(\mathcal{M}_k) \subset \mathrm{wt}(Q_k) \cup \mathrm{wt}(\mathcal{M}_{k-1})$. The asserted result now follows by induction on k. □

After this preparation, we proceed with setting up the system of differential equations. Fix $k \geq 1$. At a later stage we shall impose a condition on the magnitude of k. The space $U(\mathfrak{a}_\Phi)\lambda$ maps to a finite dimensonal subspace F of $\mathcal{M}_k = \mathcal{Y}/\bar{\mathfrak{n}}_\Phi^k \mathcal{Y}$. We fix elements $u_1 = 1, u_2, \ldots, u_p$ of $U(\mathfrak{a}_\Phi)$ such that the images $[u_j\lambda]$ in $\mathcal{M}_k$ form a basis of F.

For $H \in \mathfrak{a}_\Phi$ we denote by $B(H)$ the transposed of the matrix of the action of H on F relative to this basis. Then there exist linear maps $y_j : \mathfrak{a}_\Phi \to \bar{\mathfrak{n}}_\Phi^k U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)$, $(\mathfrak{m}_1 := \mathfrak{m}+\mathfrak{a})$ such that for every $H \in \mathfrak{a}_\Phi$ we have the following identities in $\mathcal{Y} = U(\mathfrak{g})\lambda$,

$$Hu_j\lambda = \sum_{i=1}^{p} B(H)_{ji} u_i \lambda + y_j(H)\lambda, \qquad (1 \leq j \leq p).$$

We now define the functions $F : H^\infty \times A \to \mathbb{C}^p$ and $R : V \times A \times \mathfrak{a}_\Phi \to \mathbb{C}^p$ by

$$F_j(\varphi, a) = u_j\lambda(\pi(a)^{-1}v), \quad R_j(v, a, H) = y_j(H)\lambda(\pi(a)^{-1}v), \qquad (1 \leq j \leq p).$$

We use the decomposition (2.10) and put ${}^\backprime A_\Phi := \exp({}^\backprime\mathfrak{a}_\Phi)$, so that $A \simeq {}^\backprime A_\Phi A_\Phi$. Then,

$$\frac{d}{dt} F(v, {}^\backprime a \exp(tH)) = B(H)F(v, {}^\backprime a \exp tH) + R(v, {}^\backprime a \exp tH, H),$$

for all ${}^\backprime a \in {}^\backprime A_\Phi$, $H \in \mathfrak{a}_\Phi$ and $t \in \mathbb{R}$. This equation in turn leads to

$$\frac{d}{dt} e^{-tB(H)} F(v, {}^\backprime a \exp(tH)) = e^{-tB(H)} R(v, {}^\backprime a \exp tH, H).$$

Finally, by integrating with respect to t we obtain the equivalent equation

$$F(v, {}^{\backprime}a\exp(tH)) = e^{tB(H)}F(v, {}^{\backprime}a) + e^{tB(H)}\int_0^t e^{-\tau B(H)}R(v, {}^{\backprime}a\exp(\tau H), H)\,d\tau. \quad (5.6)$$

The domination estimate (5.1) leads to estimates of F and R.

Lemma 5.4 *There exists a constant $d \in \mathbb{N}$ and a continuous seminorm* n *on H^∞ such that for all $v \in H^\infty$ and all $a \in A$,*

$$\|F(v,a)\| \le a^\xi (1 + |\log a|)^d \ \mathrm{n}(v). \quad (5.7)$$

Proof. Using that $F_j(v,a) = u_j\lambda(\pi(a)^{-1}v) = \mathrm{wh}_\lambda(\pi(u_j^\vee)v)(a)$ in combination with the estimate (5.1) we find that the estimate (5.7) is valid with a seminorm of the form $\mathrm{n}(v) = C\max_j \mathrm{n}_0(\pi(u_j^\vee)v)$, where n_0 denotes the seminorm of (5.1). □

In the following we will use the abbreviated notation

$$|({}^{\backprime}a, H)| := (1 + |\log {}^{\backprime}a|)(1 + |H|), \qquad ({}^{\backprime}a \in {}^{\backprime}A_\Phi, H \in \mathfrak{a}_\Phi).$$

Lemma 5.5 *Let $\xi \in \mathfrak{a}^*$ be as in (5.1). Then there exist $N \in \mathbb{N}$ and a continuous seminorm* n *on H^∞ such that for all $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $H \in \mathfrak{a}_\Phi^+$, $t \ge 0$,*

$$\|R(v, {}^{\backprime}a\exp tH, H)\| \le |({}^{\backprime}a, tH)|^N (1 + |H|)({}^{\backprime}a)^\xi e^{(\xi - k\beta_\Phi)(tH)}\ \mathrm{n}(v). \quad (5.8)$$

Here $\beta_\Phi : \mathfrak{a}_\Phi^+ \to [0,\infty[$ is defined by $\beta_\Phi(H) = \min_{\alpha \in \Delta\setminus\Phi} \alpha(H)$.

Proof. Let $y \in \bar{\mathfrak{n}}_\Phi^k U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)$. Then we may express y as a finite sum of weight vectors U for the adjoint action of $\mathfrak{a}$. The weight associated with such a term U is of the form $-\mu = -\sum_{\alpha\in\Delta}\mu_\alpha\,\alpha$ with $\mu_\alpha \in \mathbb{N}$ and with $\sum_{\alpha\in\Delta\setminus\Phi}\mu_\alpha \ge k$. It follows that

$$U\lambda(\pi({}^{\backprime}a\exp tH)^{-1}v) = ({}^{\backprime}a\exp tH)^{-\mu}\lambda(\pi({}^{\backprime}a\exp tH)^{-1}U^\vee v).$$

Now there exists a finite sequence $\alpha_1, \dots, \alpha_q$ of simple roots from Φ such that $\mu = \alpha_1 + \cdots + \alpha_q$ on ${}^{\backprime}\mathfrak{a}_\Phi$. Let $X_{\alpha_j} \in \mathfrak{g}_{\alpha_j}$ be such that $\chi_*(X_{\alpha_j}) = 1$ for each j and put $X = X_{\alpha_1}\cdots X_{\alpha_q}$, then $\chi_*(X) = 1$. Hence, $X\lambda = \lambda$ and it follows that

$$\begin{aligned} U\lambda(\pi({}^{\backprime}a\exp tH)^{-1}v) &= UX\lambda(\pi({}^{\backprime}a\exp tH)^{-1}v) \\ &= (\exp tH)^{-\mu}\lambda(\pi({}^{\backprime}a\exp tH)^{-1}X^\vee U^\vee v). \end{aligned}$$

Now

$$-\mu(H) \le -k\beta_\Phi(H), \qquad (H \in \mathfrak{a}_\Phi^+),$$

and we see, by using the initial estimate (5.1) that there exists a continuous seminorm n_0 on H^∞ such that for all $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$, $H \in \mathfrak{a}_\Phi^+, t \ge 0$

$$\begin{aligned} |U\lambda(\pi({}^{\backprime}a\exp tH)^{-1}v)| &\le e^{-k\beta_\Phi(tH)}|\mathrm{wh}_\lambda(\pi({}^{\backprime}a\exp tH)X^\vee U^\vee v)| \\ &\le e^{-k\beta_\Phi(tH)}({}^{\backprime}a\exp tH)^\xi |({}^{\backprime}a, tH)|^d n_0(v). \end{aligned}$$

It follows that a similar estimate holds for each $y \in \bar{\mathfrak{n}}_\Phi^k U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)$. Using the linear dependence of $y_i(H)$ on H for $1 \le i \le p$ and using the above estimates with $y_i(H)$ in place of y, we infer the existence of a continuous seminorm n on H^∞ such that the asserted estimation (5.8) is valid. □

We now assume that $\alpha \in \Delta$ and $\Phi = \Delta \setminus \{\alpha\}$, so that P_Φ is a maximal standard parabolic subgroup. Clearly, $\mathfrak{a}_\Phi = \mathbb{R}h_\alpha$, where h_α is defined as in the beginning of Section 4.

We may and will assume $\Lambda(h_\alpha) < \xi(h_\alpha)$ since otherwise $\xi' = \xi$ and there would be nothing to prove. Accordingly,

$$\xi'(h_\alpha) = \Lambda(h_\alpha), \qquad \xi'|_{{}^\backprime\mathfrak{a}_\Phi} = \xi|_{{}^\backprime\mathfrak{a}_\Phi}. \tag{5.9}$$

We now observe that $\beta_\Phi(h_\alpha) = 1$ and impose the mentioned condition on $k \in \mathbb{N}$ that

$$(\xi - k\beta_\Phi)(h_\alpha) = \xi(h_\alpha) - k < \Lambda(h_\alpha). \tag{5.10}$$

The spectrum of $B(h_\alpha)$ is contained in the set

$$X := \{[\sigma - (\alpha_1 + \ldots + \alpha_\ell)](h_\alpha) \mid 0 \le \ell < k,\ \forall j : \alpha_j \in \Sigma(\mathfrak{n}_\Phi, \mathfrak{a}_\Phi)\}.$$

For $x \in X$ we denote by $P_x : \mathbb{C}^p \to \mathbb{C}^p$ the projection onto the associated generalized weight space of $B(h_\alpha)$, along the generalized weight spaces for the eigenvalues different from x. If x is not an eigenvalue for $B(h_\alpha)$, then $P_x = 0$. Then it is clear from (5.7), possibly after adaptation of d and n that,

$$\|P_x F(v, a)\| \le a^\xi (1 + |\log a|)^d \ \mathrm{n}(v), \tag{5.11}$$

for all $v \in H^\infty$ and $a \in A$. Likewise, for the components $P_x R$ we have estimates of the form (5.8), with adapted N' and n' if necessary.

In the following we agree to write

$$a = {}^\backprime a a_t, \quad {}^\backprime a \in {}^\backprime A_\Phi \quad \text{and} \quad a_t = \exp(t h_\alpha), \quad t \in \mathbb{R}.$$

Remark 5.6 Clearly, to finish the proof of Lemma 4.6 it suffices to prove the estimate (5.11) with ξ replaced by ξ'. Now $\xi'(h_\alpha) \le \xi(h_\alpha)$ implies that for ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \le 0$ we have

$$a^\xi = ({}^\backprime a)^\xi (a_t)^\xi \le ({}^\backprime a)^\xi (a_t)^{\xi'} = a^{\xi'},$$

so that the required estimate (5.11) with ξ' in place of ξ is automatically fulfilled for a outside ${}^\backprime A_\Phi A_\Phi^+$. It therefore suffices to prove the estimate for $a = {}^\backprime a a_t$, $t > 0$.

It is well known that there exist unique polynomial maps $Q_x : \mathbb{R} \to \mathrm{End}(\mathbb{C}^p)$, for $x \in X$, such that $Q_x(t)$ commutes with $B(h_\alpha)$ for all $t \in \mathbb{R}$, hence with all the weight space projections for $B(h_\alpha)$, and satisfies $P_x Q_x(t) = Q_x(t) = Q_x(t) P_x$ and

$$e^{tB(h_\alpha)} P_x = e^{tx} Q_x(t), \qquad (t \in \mathbb{R}). \tag{5.12}$$

Furthermore, the polynomial degrees of Q_x are at most p, so that there exists a suitable constant $C_0 > 0$ such that, for all $x \in X$,

$$\|Q_x(t)\|_{\mathrm{op}} \le C_0 (1 + |t|)^p, \qquad (t \in \mathbb{R}). \tag{5.13}$$

We recall that $\beta_\Phi(h_\alpha) = 1$ and that Lemma 5.5 with $H = h_\alpha$ implies the existence of a continuous seminorm n on H^∞ such that for all $v \in H^\infty$, ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \geq 0$ we have

$$\|R(v, {}^\backprime a a_t, h_\alpha)\| \leq |({}^\backprime a, th_\alpha)|^N ({}^\backprime a)^\xi e^{t[\xi(h_\alpha)-k]} \mathrm{n}(v). \tag{5.14}$$

Writing

$$R(v, {}^\backprime a \exp th_\alpha, h_\alpha) = e^{t[\xi(h_\alpha)-k]} R_0(v, {}^\backprime a \exp(th_\alpha))$$

we find that the estimate (5.14) becomes

$$\|P_x R_0(v, {}^\backprime a \exp(th_\alpha))\| \leq |({}^\backprime a, th_\alpha)|^N ({}^\backprime a)^\xi \mathrm{n}(v). \tag{5.15}$$

Finally, writing $F_x = P_x F$, formula (5.6) leads to

$$\begin{aligned} & F_x(v, {}^\backprime a \exp(th_\alpha)) \\ = \;& e^{tx} Q_x(t) F(v, {}^\backprime a) + e^{tx} Q_x(t) \int_0^t Q_x(\tau) e^{\tau(-x+[\xi(h_\alpha)-k])} R_0(v, {}^\backprime a \exp \tau h_\alpha)\, d\tau. \end{aligned} \tag{5.16}$$

We will need to distinguish two cases depending on the position of the real number x, namely:

(a) $-x + [\xi(h_\alpha) - k] < 0$,

(b) $-x + [\xi(h_\alpha) - k] \geq 0$.

In case (a) we will need to distinguish the subcases (a.1): $x \leq \Lambda(h_\alpha)$ and (a.2): $x > \Lambda(h_\alpha)$. In case (b) we automatically have $x < \Lambda(h_\alpha)$, in view of (5.10).

Case (a) In this case the integrand of (5.16) is integrable over $[0, \infty[$ and we find that the expression on the right-hand side of (5.16) becomes

$$e^{tx} Q_x(t) F(v, {}^\backprime a) + e^{tx} Q_x(t) I_0(v, {}^\backprime a) - e^{tx} Q_x(t) I_t(v, a'). \tag{5.17}$$

where, for $t \in [0, \infty[$,

$$I_t(v, {}^\backprime a) := \int_t^\infty Q_x(\tau) e^{\tau(-x+[\xi(h_\alpha)-k])} R_0(v, {}^\backprime a \exp \tau h_\alpha)\, d\tau\ .$$

From (5.11) and (5.13) we infer that

$$\|Q_x(t) F(v, {}^\backprime a)\| \leq C_0 (1 + |t|)^p (1 + |\log {}^\backprime a|)^d ({}^\backprime a)^\xi.$$

We select $\varepsilon > 0$ such that

$$-x + [\xi(h_\alpha) - k] + \varepsilon < 0 \quad \text{and} \quad [\xi(h_\alpha) - k] + \varepsilon < \Lambda(h_\alpha).$$

Then

$$\begin{aligned} \|I_t(v, {}^\backprime a)\| &\leq e^{(-x+[\xi(h_\alpha)-k]+\varepsilon)t} \int_t^\infty \|Q_x(\tau)\| e^{-\varepsilon\tau}\, |R_0(v, {}^\backprime a \exp \tau h_\alpha)| d\tau \\ &\leq C_1 e^{(-x+[\xi(h_\alpha)-k]+\varepsilon)t} (1 + |\log {}^\backprime a|)^N ({}^\backprime a)^\xi \mathrm{n}(v) \\ &\leq C_1 e^{(-x+\Lambda(h_\alpha))t} (1 + |\log {}^\backprime a|)^N ({}^\backprime a)^\xi \mathrm{n}(v) \end{aligned}$$

with

$$C_1 = \int_0^\infty \|Q_x(\tau)\| e^{-\varepsilon\tau} \, (1+|\tau|)^N d\tau < \infty.$$

For $t = 0$ this leads to

$$\|I_0(v, {}^\backprime a)\| \leq C_1(1 + |\log {}^\backprime a|)^N ({}^\backprime a)^\xi \mathrm{n}(v).$$

For general $t \geq 0$ this leads to

$$e^{xt}\|I_t(v, {}^\backprime a)\| \leq C_1 e^{t\Lambda(h_\alpha)}(1 + |\log {}^\backprime a|)^N ({}^\backprime a)^\xi \mathrm{n}(v) \tag{5.18}$$

We will now consider the two subcases (a.1) and (a.2).

Subcase (a.1) In this subcase, $x \leq \Lambda(h_\alpha)$. Then from the above estimates it follows that the norm of each term of (5.17) can be estimated by $C_2 := \max(C_0, C_1)$ times

$$e^{t\Lambda(h_\alpha)}(1+|t|)^p(1 + |\log {}^\backprime a|)^N ({}^\backprime a)^\xi \mathrm{n}(v)$$

There exists a constant $C_3 > 0$ such that the above expression can be estimated by

$$C_3 e^{t\Lambda(h_\alpha)}({}^\backprime a)^\xi (1 + |\log({}^\backprime a) + t h_\alpha|)^{p+N} \mathrm{n}(v) = C_3({}^\backprime a a_t)^\xi \, (1 + |\log({}^\backprime a a_t)|)^{p+N}\mathrm{n}(v).$$

for all ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \geq 0$. It follows that F_x satisfies the required estimate for all $v \in H^\infty$ and $a = {}^\backprime a a_t$ with $t \geq 0$. In view of Remark 5.6 this establishes the required estimate in the present subcase.

Subcase (a.2) In this case we have $x > \Lambda(h_\alpha)$. If follows from (5.16) that for ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \geq 0$ we have

$$F_x(v, {}^\backprime a a_t) - e^{tx} Q_x(t) I_t(v, a') = e^{tx} Q_x(t)[F(v, {}^\backprime a) + I_0(v, {}^\backprime a)] \tag{5.19}$$

In view of Proposition 4.2 there exists a constant $d' \in \mathbb{N}$ and for every $v \in H_K$ a constant $C_v > 0$ such that for ${}^\backprime a a_t \in \mathrm{cl}(A^+)$ we have

$$\|F_x(v, {}^\backprime a a_t)\| \leq C_v(1 + |\log {}^\backprime a|)^{d'} ({}^\backprime a)^\Lambda (1+|t|)^{d'} e^{t\Lambda(h_\alpha)}. \tag{5.20}$$

Write ${}^\backprime\mathfrak{a}_\Phi^+$ for the set of elements $H \in {}^\backprime\mathfrak{a}_\Phi$ such that $\beta(H) > 0$ for all $\beta \in \Phi$. Then this set has non-empty interior in ${}^\backprime\mathfrak{a}_\Phi$. Let ${}^\backprime A_\Phi^+ := \exp({}^\backprime\mathfrak{a}_\Phi^+)$. Then it follows that ${}^\backprime A_\Phi^+ a_t \subset A^+$ for all $t > 0$.

Combining (5.20) with the estimates (5.18) and (5.13) we infer that the norm of the sum on the left-hand side of (5.19) allows for every $v \in H_K$ and ${}^\backprime a \in \mathrm{cl}({}^\backprime A_\Phi^+)$ an estimation by a constant times $(1+|t|)^{N'} e^{t\Lambda(h_\alpha)}$, for all $t \geq 0$. On the other hand, the expression on the right-hand side is of exponential polynomial type with exponent $x > \Lambda(h_\alpha)$. By uniqueness of asymptotics this implies that for $v \in H_K$, ${}^\backprime a \in {}^\backprime A_\Phi^+$ and $t > 0$ the expression on the right-hand side in (5.19) is zero. Hence,

$$F_x(v, {}^\backprime a a_t) = e^{tx} Q_x(t) I_t(v, {}^\backprime a) \tag{5.21}$$

for $v \in H_K$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi^+$ and $t > 0$.

From the definitions of F_x and I_t it easily follows that for $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t > 0$,

$$F_x(v, {}^{\backprime}aa_t) = F_x(\pi({}^{\backprime}a)^{-1}v, a_t) \quad \text{and} \quad I_t(v, a') = I_t(\pi({}^{\backprime}a)^{-1}v, e). \tag{5.22}$$

Furthermore, from the estimates (5.11) and (5.18) it follows that for each $t > 0$ the maps $v \mapsto F_x(v, a_t)$ and $v \mapsto I_t(v, e)$ are continuous linear $V^\infty \to \mathbb{C}^p$. If v is K-finite in H then v is an analytic vector for H^∞, so that the map ${}^{\backprime}a \mapsto \pi({}^{\backprime}a)^{-1}v$ is analytic ${}^{\backprime}A_\Phi \to H^\infty$. We may now conclude that the maps ${}^{\backprime}a \mapsto F_x(\pi({}^{\backprime}a)^{-1}v, a_t)$ and ${}^{\backprime}a \mapsto I_t(\pi({}^{\backprime}a)^{-1}v, e)$ are analytic ${}^{\backprime}A_\Phi \to \mathbb{C}^p$. In view of (5.22) it now follows by analytic continuation in the variable ${}^{\backprime}a$ that the validity of the identity (5.21) extends to all $v \in H_K$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t > 0$. Using that both members of (5.21) depend continuous linearly on $v \in H^\infty$, whereas H_K is dense in the latter space, we conclude that (5.21) is actually valid for all $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t > 0$.

Using (5.13) and (5.18) to estimate the expression on the right-hand side of (5.21), we obtain, for $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t > 0$,

$$\begin{aligned} \|F_x(v, {}^{\backprime}aa_t)\| &\leq C_0C_1(1+|t|)^p e^{t\Lambda(h_\alpha)}(1+|\log {}^{\backprime}a|)^N ({}^{\backprime}a)^\xi \mathrm{n}(v) \\ &\leq C_3(1+|\log({}^{\backprime}aa_t)|)^{2(N+p)}({}^{\backprime}aa_t)^{\xi'}\mathrm{n}(v), \end{aligned}$$

with $C_3 > 0$ uniform with respect to $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t > 0$. This gives the required estimate for F_x on $H^\infty \times {}^{\backprime}A_\Phi A_\Phi^+$. In view of Remark 5.6 this completes the discussion in the present subcase.

Case (b) In this case, $-x + [\xi(h_\alpha) - k] \geq 0$, and $x < \Lambda(h_\alpha)$. The identity (5.16) can be rewritten as

$$F_x(v, {}^{\backprime}aa_t) = e^{tx}Q_x(t)[F(v, {}^{\backprime}a) + J(t, v, {}^{\backprime}a)]. \tag{5.23}$$

with

$$J(t, v, {}^{\backprime}a) := \int_0^t Q_x(\tau)e^{\tau(-x+[\xi(h_\alpha)-k])}R_0(v, {}^{\backprime}a\exp\tau h_\alpha)\, d\tau.$$

By a straightforward estimation of the integral defining J, we find, for $v \in H^\infty$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t \geq 0$, using (5.13) and (5.15) that

$$\begin{aligned} \|J(t, v, {}^{\backprime}a)\| &\leq C_0e^{t(-x+[\xi(h_\alpha)-k])}\int_0^t (1+|\tau|)^p\|R_0(v, {}^{\backprime}a\exp\tau h_\alpha)\|d\tau \\ &\leq C_0C^Ne^{t(-x+[\xi(h_\alpha)-k])}(1+|t|)^{N+p+1}(1+|\log {}^{\backprime}a|)^N({}^{\backprime}a)^\xi\mathrm{n}(v), \end{aligned}$$

with $C = \sup(1+|t||h_\alpha|)(1+|t|)^{-1}$. This implies that

$$\begin{aligned} \|e^{tx}Q_x(t)J(t, v, {}^{\backprime}a)\| &\leq C_0^2C^Ne^{t([\xi(h_\alpha)-k])}(1+|t|)^{2N+p+1}(1+|\log {}^{\backprime}a|)^N({}^{\backprime}a)^\xi\mathrm{n}(v) \\ &\leq C_0^2C^Ne^{t\Lambda(h_\alpha)}(1+|t|)^{2N+p+1}(1+|\log {}^{\backprime}a|)^N({}^{\backprime}a)^\xi\mathrm{n}(v) \\ &\leq C_2({}^{\backprime}aa_t)^{\xi'}(1+|\log {}^{\backprime}aa_t|)^{6N+2p+2}\mathrm{n}(v), \end{aligned}$$

see also (5.10), with $C_2 > 0$ uniform with respect to $v \in H^\infty$, ${}^\backprime a \in {}^\backprime A_\Phi$ and $t > 0$.

On the other hand, since $x < \Lambda(h_\alpha) = \xi'(h_\alpha)$, see (5.9),

$$\begin{aligned} \|e^{tx} Q_x(t) F(v, {}^\backprime a)\| &\leq C_0 e^{\Lambda(h_\alpha)} (1 + |t|)^p ({}^\backprime a)^\xi (1 + |\log {}^\backprime a|)^d \mathrm{n}(v) \\ &\leq C_3 ({}^\backprime a a_t)^{\xi'} (1 + |\log {}^\backprime a a_t|)^{2(p+d)} \mathrm{n}(v), \end{aligned}$$

with $C_3 > 0$ uniform with respect to $v \in H^\infty$, ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \geq 0$. Combining the two latter estimates we find that there exist a constant $N' > 0$ and a continuous seminorm n' on H^∞ such that

$$\|P_x F(v, {}^\backprime a a_t)\| \leq ({}^\backprime a a_t)^{\xi'} (1 + |\log({}^\backprime a a_t)|)^{N'} \mathrm{n}'(v)$$

for all $v \in H^\infty$, ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \geq 0$. In view of Remark 5.6 this gives the required estimates for F_x in case (b) and completes the proof of Lemma 4.6. □

6 Parabolically induced representations

In this section we will describe the space of smooth vectors for parabolically induced representations of the form

$$\mathrm{Ind}_P^G(\xi), \tag{6.1}$$

where $P = M_P A_P N_P$ is a parabolic subgroup of G with the indicated Langlands decomposition, and ξ a continuous representation of P in a Hilbert space H_ξ.

Remark 6.1 In particular we will be interested in the situation that $\xi = \sigma \otimes \nu \otimes 1$, with σ a unitary representation in H_σ and $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. The representation ξ is now given by $\xi(man)v = a^\nu \sigma(m) v$ for all $v \in H_\sigma$ and $(m, a, n) \in M_P \times A_P \times N_P$. For technical reasons we wish to have the possibility to tensor representations of this particular form with finite dimensional representations of P, whence the greater generality.

We put $K_P := K \cap M_P = K \cap P$. By averaging over K_P we may replace the inner product on H_ξ with a K_P-invariant inner product for which the associated norm is equivalent to the original norm. Accordingly, we may and will assume that $\xi|_{K_P}$ is unitary.

Let $\delta_P : P \to [0, \infty[$ be the character of P defined by

$$\delta_\delta(p) = |\det[\mathrm{Ad}(p)|_{\mathrm{Lie}(P)}]|^{1/2}, \qquad (p \in P).$$

Then

$$\delta_P(man) = a^{\rho_P}, \qquad ((m, a, n) \in M_P \times A_P \times N_P),$$

where $\rho_P \in \mathfrak{a}^*$ is defined by $\rho_P(X) = \frac{1}{2} \mathrm{tr}\,(\mathrm{ad}(X)|_{\mathfrak{n}_P})$, for $X \in \mathfrak{a}$.

We denote by $C(G/P : \xi)$ the Fréchet space of continuous functions $f : G \to H_\xi$ transforming according to the rule

$$f(gp) = \delta_P(p)^{-1} \xi(p)^{-1} f(x), \qquad (x \in G, p \in P).$$

This space is equipped with the left regular representation L of G. Let ${}^{`}H_\xi$ denote the Hilbert space $H_\xi$ equipped with the representation ${}^{`}\xi$ of P given by ${}^{`}\xi(p) := \delta_P(p)\xi(p)$. The group $P$ naturally acts on $C(G, {}^{`}H_\xi)$ by the formula

$$[p\varphi](g) = {}^{`}\xi(p)[\varphi(gp)], \qquad (g \in G, p \in P). \tag{6.2}$$

Accordingly,

$$C(G/P : \xi) = C(G, {}^{`}H_\xi)^P.$$

Remark 6.2 In the particular case $\xi = \sigma \otimes \nu \otimes 1$, we write

$$C(G/P : \xi) = C(G/P : \sigma : \nu).$$

We write $C(K/K_P : \xi)$ for the Fréchet space of continuous functions $\varphi : K \longrightarrow H_\xi$ such that

$$\varphi(km) = \xi(m)^{-1}\varphi(k), \qquad (k \in K, m \in K_P). \tag{6.3}$$

Using the decomposition $G \simeq K \times_{K_P} M_P \times A_P \times N_P$, given by the multiplication map, we readily see that restriction to K induces a K-equivariant topological linear isomorphism

$$r : C(G/P : \xi) \xrightarrow{\simeq} C(K/K_P : \xi), \quad f \mapsto f|_K. \tag{6.4}$$

Via this isomorphism we may transfer the representation L of G on $C(G/P : \xi)$ to a continuous representation of G in $C(K/K_P : \xi)$, denoted $\pi_{P,\xi}$.

Remark 6.3 In case $\xi = \sigma \otimes \nu \otimes 1$ we will use the notation $\pi_{P,\sigma,\nu} = \pi_{P,\xi}$.

Let $d\dot{k}$ be the choice of a K-invariant Radon measure on K/K_P normalized by $\int_{K/K_P} d\dot{k} = 1$. Then it follows from the isomorphism (6.4) that the sesquilinear pairing

$$C(G/P : \xi) \times C(G/P : \xi) \to \mathbb{C} \tag{6.5}$$

given by

$$\langle f \,,\, g\rangle := \int_{K/K_P} \langle f(k) \,,\, g(k)\rangle_\xi \, d\dot{k} \tag{6.6}$$

is a K-equivariant pre-Hilbert structure. We denote the associated norm by $\|\cdot\|_2$. The associated Hilbert completion is denoted by $L^2(G/P : \xi)$. The restriction map (6.4) induces an isometric isomorphism from this completion onto $L^2(K/K_P : \xi)$, the completion of $C(K/K_P : \xi)$ with respect to the pre-Hilbert structure given by (6.6). We note that a different but equivalent choice of K-invariant inner product on H_ξ gives rise to the same completed spaces (as topological linear spaces), with equivalence of the Hilbert inner products.

Out next objective is to show that the representation $(L, C(G/P : \xi))$ has a unique extension to a continuous representation of G in the Hilbert space $L^2(G/P : \xi)$. To prepare for the proof we start by recalling a well known result involving the representation in $\mathbb{C}$ given by the character δ_P of P. The associated space $C(G/P : \delta_P)$ consists of the continuous functions $\varphi : G \longrightarrow \mathbb{C}$ such that

$$\varphi(xman) = a^{-2\rho_P}\varphi(x), \qquad (x \in G, (m, a, n) \in M_P \times A_P \times N_P).$$

Lemma 6.4 *For all $\varphi \in C(G/P : \delta_P)$ and $g \in G$,*

$$\int_{K/K_P} \varphi(gk)\, d\dot{k} = \int_{K/K_P} \varphi(k)\, d\dot{k}.$$

There exists a choice of Haar measure $d\bar{n}$ on $\bar{N}_P$ such that for all $\varphi \in C(G/P : \delta_P)$,

$$\int_{K/K_P} \varphi(k)\, d\dot{k} = \int_{\bar{N}_P} \varphi(\bar{n})\, d\bar{n}.$$

The following corollary will be useful for the ongoing discussion. We define $\kappa_P : G \to K$ $\mu_P : G \to \exp(\mathfrak{m}_P \cap \mathfrak{p})$, $h_P : G \to A_P$ and $n_P : G \to N_P$ to be the unique maps determined by

$$x = \kappa_P(x)\mu_P(x)h_P(x)n_P(x), \qquad (x \in G).$$

These maps are all analytic maps between the indicated analytic manifolds.

Corollary 6.5 *Let ω be a bounded subset of G. Then there exists a constant $C_\omega > 0$ such that for all $\psi \in C(K/K_P)$ and $g \in \omega$,*

$$\int_{K/K_P} |\psi(\kappa_P(gk))|\, d\dot{k} \leq C_\omega \int_{K/K_P} |\psi(k)|\, d\dot{k}.$$

Proof. Consider the function $\varphi : G \to H_\xi$ defined by $\varphi(kp) = \delta_P(p)^{-2}\psi(k)$. Then $\varphi \in C(G/P : \delta_P)$. Put $C_\omega := \sup_{x \in \omega K} \delta_P(x)^{-2}$. Then it follows by application of Lemma 6.4 that, for $g \in \omega$,

$$\begin{aligned}
\int_{K/K_P} |\psi(\kappa_P(gk))|\, d\dot{k} &= \int_{K/K_P} \delta_P(k)^{-2}|\varphi(gk))|\, d\dot{k} \\
&\leq C_\omega \int_{K/K_P} |\varphi(gk)|\, d\dot{k} = C_\omega \int_{K/K_P} |\varphi(k)|\, d\dot{k} \\
&= C_\omega \int_{K/K_P} |\psi(k)|\, d\dot{k}.
\end{aligned}$$

□

The following result as well as its proof are contained in [5, III.7].

Proposition 6.6 *The left regular representation L of G in $C(G/P : \xi)$ has a unique extension to a continuous representation of G in the Hilbert space $L^2(G/P : \xi)$.*

Proof. By the principle of uniform boundedness, the operator norm $\|{}^{\backprime}\xi(p)\|_{\mathrm{op}}$ of ${}^{\backprime}\xi(p) = \delta_P(p)\xi(p) \in \mathrm{End}(H_\xi)$ is locally bounded as a function of $p \in P$. For the purpose of this proof, we define the analytic map $p_P : G \to P$ by $p_P = \mu_P h_P n_P$. Then $\|{}^{\backprime}\xi(p_P(g)\|_{\mathrm{op}}$ is locally bounded as a function of $g \in G$. For a bounded subset $S \subset G$ let ${}^{\backprime}C_S > 0$ be the supremum of the values $\|{}^{\backprime}\xi(p_P(x^{-1}k)^{-1})\|_{\mathrm{op}}$ for $x \in S$ and $k \in K$.

Let $f \in C(G/P : \xi)$ and $x \in S$. Since $\xi|_{K_P}$ is unitary, it follows that the function $\psi : k \mapsto \|f(k)\|_\xi^2$ belongs to $C(K/K_P)$. Hence, by application of Corollary 6.5 with $\omega = S^{-1}$,

$$\begin{aligned}\int_{K/K_P} \|L_x f(k)\|_\xi^2 \, d\dot{k} &= \int_{K/K_P} \|`\xi(p_P(x^{-1}k)^{-1}) f(\kappa_P(x^{-1}k)\|_\xi^2 \, d\dot{k} \\ &\leq `C_S^2 \int_{K/K_P} |\psi(\kappa_P(x^{-1}k)| \, d\dot{k} \\ &\leq `C_S^2 C_\omega \int_{K/K_P} |\psi(k)| \, d\dot{k} = `C_S^2 C_\omega \int_{K/K_P} \|f(k)\|_\xi^2 \, d\dot{k}.\end{aligned}$$

Let $\|\cdot\|_2$ denote the norm associated with the Hilbert structure on $L^2(G/P : \xi)$. It follows from the estimate above that the map L_x is continuous with respect to the norm $\|\cdot\|_2$ on $C(G/P : \xi)$ with operator norm that is locally bounded relative to $x \in G$. This implies that L_x has a unique continuous linear extension to a bounded map of the Hilbert completion $L^2(G/P : \xi)$, with locally bounded operator norm. It remains to be shown that the associated action map $G \times L^2(G/P : \xi) \to L^2(G/P : \xi)$ is continuous.

Let $f \in C(G/P : \xi)$. Then it is readily checked that $\sup_{k \in K} \|L_x f(k) - f(k)\|_\xi \to 0$ for $G \ni x \to e$. This implies that $L_x f \to f$ in $L^2(G/P : \xi)$. If $f \in L^2(G/P : \xi)$ and $f_0 \in C(G/P : \xi)$ then

$$\begin{aligned}\|L_x f - f\|_2 &\leq \|L_x(f - f_0)\|_2 + \|L_x f_0 - f_0\|_2 + \|f_0 - f\|_2 \\ &\leq (\|L_x\|_{\mathrm{op}} + 1) \|f - f_0\|_2 + \|L_x f_0 - f_0\|_2.\end{aligned}$$

Using density of $C(G/P : \xi)$ in $L^2(G/P : \xi)$ we infer from the results obtained in the first part of this proof that $\|L_x f - f\|_2 \to 0$ for $x \to e$ in G.

Finally, let $f_0 \in L^2(G/P : \xi)$. Then

$$\begin{aligned}\|L_x f - f_0\|_2 &\leq \|L_x f - L_x f_0\|_2 + \|L_x f_0 - f_0\|_2 \\ &\leq \|L_x\|_{\mathrm{op}} \|f - f_0\|_2 + \|L_x f_0 - f_0\|_2.\end{aligned}$$

By what we have established above it follows that both terms in the latter sum tend to zero as $(x, f) \to (e, f_0)$ in $G \times L^2(G/P : \xi)$. Thus the action map $G \times L^2(G/P : \xi) \to L^2(G/P : \xi)$ is continuous at every point of $\{e\} \times L^2(G/P : \xi)$.

Since the operator norm of L_x is locally bounded relative to $x \in G$, the continuity of the action map at any point of $G \times L^2(G/P : \xi)$ follows. □

The representation (6.1) is defined to be the unique extended representation of Proposition 6.6. Under the (unitary) restriction map $\varphi \mapsto \varphi|_K$, this representation is transfered to a continuous representation of G in $L^2(K/K_P : \xi)$ which extends $\pi_{P,\xi}$ and is denoted by the same symbol. The latter representation is called the compact picture of (6.1).

We are now prepared to determine the space of smooth vectors of the representation (6.1) . We define $C^\infty(G/P:\xi) := C^\infty(G, {}^\backprime H_\xi) \cap C(G/P, \xi)$. Equivalently, this can be expressed in terms of the action of P on $C^\infty(G, {}^\backprime H_\xi)$ given by formula (6.2):

$$C^\infty(G/P:\xi) = C^\infty(G, {}^\backprime H_\xi)^P \tag{6.7}$$

This is a closed subspace of the Fréchet space $C^\infty(G, {}^\backprime H_\xi)$, hence a Fréchet space of its own right. The left regular representation of G in the first space in (6.7) is smooth, hence restricts to a smooth representation of G in the second space.

Theorem 6.7 *The space of smooth vectors in $(L, L^2(G/P:\xi))$, equipped with its usual Fréchet topology is given by*

$$L^2(G/P:\xi)^\infty = C^\infty(G/P:\xi). \tag{6.8}$$

Proof. By Fubini's theorem and compactness of G/P it follows that

$$L^2(G/P:\xi) = L^2_{\mathrm{loc}}(G, {}^\backprime H_\xi)^P, \tag{6.9}$$

with equality of the usual locally convex topologies; here superscript P indicates the space of invariants for the obvious action of P on $L^2_{\mathrm{loc}}(G, H_\xi)$, described by the formula given in (6.2). This P-action is by continuous linear maps which commute with the left regular action of G. Hence, the space on the right of (6.9) is a closed G-invariant subspace of $L^2_{\mathrm{loc}}(G, H_\xi)$. From this it readily follows that

$$L^2(G/P:\xi)^\infty = L^2_{\mathrm{loc}}(G, {}^\backprime H_\xi)^P \cap L^2_{\mathrm{loc}}(G, {}^\backprime H_\xi)^\infty. \tag{6.10}$$

By the Sobolev embedding theorem we have that

$$L^2_{\mathrm{loc}}(G, {}^\backprime H_\xi)^\infty = C^\infty(G, {}^\backprime H_\xi), \tag{6.11}$$

with equality of the usual locally convex topologies. Combining (6.10) and (6.11) we find that $L^2(G/P:\xi)^\infty = C^\infty(G, {}^\backprime H_\xi)^P$. In view of (6.7) this completes the proof. □

Corollary 6.8 *The left regular representation L of G in $C^\infty(G/P:\xi)$ is smooth.*

Remark 6.9 For $\xi = \sigma \otimes \nu \otimes 1$ with σ a unitary representation of M_P and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the characterisation (6.8) was used in [4], with a reference to [5, III.7.9]. However, the characterization of $L^2(G/P:\xi)^\infty$ in [5] was different. It is presented in the lemma below.

Lemma 6.10 *The following equality of locally convex spaces is valid:*

$$C^\infty(G, {}^\backprime H_\xi)^P = C^\infty(G, {}^\backprime H_\xi^\infty)^P. \tag{6.12}$$

Proof. We consider the Fréchet space $C^\infty(P, {}^\backprime H_\xi)$ equipped with the P-action given by $p\psi = {}^\backprime\xi(p)L_p\psi$. The action is by continuous linear maps so that the subspace of P-invariants is closed, hence Fréchet.

The proof is motivated by the observation that

$$ {}^\backprime H_\xi^\infty \simeq C^\infty(P, {}^\backprime H_\xi)^P \tag{6.13}$$

as topological linear spaces. The isomorphism from left to right is given by $\alpha : v \mapsto \psi_v$, where $\psi_v : P \to {}^\backprime H_\xi$ is given by $\psi_v(p) = {}^\backprime\xi(p)v$. The inverse ε is given by evaluation at the identity. The isomorphism intertwines the P-action given by ${}^\backprime\xi$ on the first space with the P-action obtained from restriction of the right regular P-action on $C^\infty(P, {}^\backprime H_\xi)$.

The idea of the proof is to establish the following sequence of topological linear isomorpisms:

$$\begin{aligned} C^\infty(G, {}^\backprime H_\xi)^P &\simeq C^\infty(G\times P, {}^\backprime H_\xi)^{P\times P} && (6.14)\\ &\simeq C^\infty(G, C^\infty(P, {}^\backprime H_\xi)^P)^P && (6.15)\\ &\simeq C^\infty(G, {}^\backprime H_\xi^\infty)^P && (6.16)\end{aligned}$$

The isomorphism (6.14) is obtained by restriction of the map

$$S : C^\infty(G, {}^\backprime H_\xi) \to C^\infty(G\times P, {}^\backprime H_\xi)$$

given by $S(\varphi)(g, q) = \varphi(gq)$, for $\varphi \in \mathrm{dom}(S)$ and $(g, q) \in G \times P$. The image of S consists of the space of invariants for the P action on its codomain given by $[p_1 \cdot \psi](g, q) = \psi(gp_1, p_1^{-1}q)$, for $\psi \in C^\infty(G\times P, {}^\backprime H_\xi)$, $g \in G$ and $p_1, q \in P$. As the action takes place by continuous linear maps, the image of S is closed hence Fréchet and it is readily verified that S is a topological isomorphism onto its image.

There is a second action of P on the codomain of S, given by $p_2 \cdot \psi(g, q) = {}^\backprime\xi(p_2)\psi(g, qp_2)$, for $g \in G$ and $q, p_2 \in P$. This second action commutes with the first one, hence leaves $\mathrm{im}(S)$ invariant. The map S intertwines the usual P-action on its domain with the second P-action on its codomain. The associated spaces of invariants are closed, hence Fréchet, and it follows that S induces a topological linear isomorphism between these spaces. This is the isomorphism (6.14).

To understand the map (6.15) we consider the map

$$T : C^\infty(G\times P, {}^\backprime H_\xi) \xrightarrow{\simeq} C^\infty(G, C^\infty(P, {}^\backprime H_\xi)),$$

given by $[T\psi](g)(p) = \varphi(g, p)$, It is well known that this map is a topological linear isomorphism of Fréchet spaces. It is readily verified that T intertwines the given action of $P \times P$ on its domain with the action on its codomain given by

$$(p_1, p_2) \cdot \vartheta(g)(q) := {}^\backprime\xi(p_2)\vartheta(gp_1)(p_1^{-1}qp_2),$$

for $\vartheta \in C^\infty(G, C^\infty(P, {}^\backprime H_\xi))$, $g \in G$ and $q, p_1, p_2 \in P$. The associated space of invariants for $\{e\} \times P$ equals $C^\infty(G, C^\infty(P, {}^\backprime H_\xi)^P)$. Taking the remaining action of $P \times \{e\}$ into account we infer that T induces a topological linear isomorphism (6.15). Finally, the isomorphism (6.13) induces the topological linear isomorphism (6.16). □

Remark 6.11 From the proof it is clear that Lemma 6.10 is valid for any triple $(G, P, {}^{\backprime}\xi)$, with G a Lie group, P a closed subgroup and $({}^{\backprime}\xi, {}^{\backprime}H_\xi)$ a continuous Hilbert representation of P.

Remark 6.12 If $\xi = \sigma \otimes \nu \otimes 1$ as in Remark 6.1, then the space of smooth vectors for σ in H_σ equals the space of smooth vectors for ξ in $H_\xi = H_\sigma$. Hence, in this setting the equality (6.12) becomes

$$C^\infty(G/P : \sigma : \nu) = C^\infty(G, H^\infty_{\sigma,\nu})^P$$

where $H^\infty_{\sigma,\nu}$ denotes H^∞_σ equipped with the representation of $P = M_P A_P N_P$ given by $(man, v) \mapsto a^{\nu+\rho_P}\sigma(m)v$.

In the sequel we will need the description of the space of smooth vectors for parabolically induced representations in the compact picture. To accomodate this we agree to write $\mathcal{R}(P)$ for the set of (equivalence classes of) continuous Hilbert space representations (ξ, H_ξ) of P such that the space of smooth vectors for P in H_ξ equals the space of smooth vectors for its compact subgroup K_P.

Lemma 6.13

(a) *Any $\xi = \sigma \otimes \nu \otimes 1$ with σ a unitary representation of M_P and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ belongs to $\mathcal{R}(P)$.*

(b) *If $\xi \in \mathcal{R}(P)$ and if (π, F) is a finite dimensional continuous representation of P, then $\xi \otimes \pi \in \mathcal{R}(P)$.*

Proof. Assertion (a) follows from [21, p. 3]. For (b) we first note that if L is any Lie group, ξ a continuous representation of L in a Hilbert (or more generally quasi-complete locally convex) space E and (π, F) a finite dimensional continuous representation of L, then

$$(E \otimes F)^\infty = E^\infty \otimes F. \tag{6.17}$$

To prove this, we first note that $\pi : L \to \mathrm{End}(F)$ is smooth. By finite dimensionality of F it now follows that $(1 \otimes \pi) : L \to \mathrm{End}(E) \otimes \mathrm{End}(F) \simeq \mathrm{End}(E \otimes F)$ is smooth. Let $T \in E \otimes F$ be a smooth vector. Then it follows that

$$x \mapsto (\xi(x) \otimes 1)T = (1 \otimes \pi(x^{-1}))(\xi(x) \otimes \pi(x))T$$

is smooth from L to $E \otimes F$. By finite dimensionality of F this implies that $T \in E^\infty \otimes F$. This shows that the space on the left in (6.17) is contained in the space on the right.

For the converse inclusion, let $e \in E^\infty$ and $f \in F$, then $x \mapsto \xi(x)e$, $L \to F$ and $x \mapsto \pi(x)f$, $L \to F$ are smooth. By finite dimensionality of F it now follows that $e \otimes f$ is smooth. The claim follows.

Returning to (b), and letting $\infty(P)$ indicate the smooth vectors for P and $\infty(K_P)$ those for K_P, we see that

$$(H_\xi \otimes F)^{\infty(P)} = H_\xi^{\infty(P)} \otimes F = H_\xi^{\infty(K_P)} \otimes F = (H_\xi \otimes F)^{\infty(K_P)}.$$

□

We denote by $C^\infty(K/K_P : \xi)$ the Fréchet space of smooth functions $\varphi : K \to H_\xi$ transforming according to the rule (6.3). Then clearly the isomorphism (6.4), induced by restriction to K, restricts to an injective continuous linear map from $C^\infty(G/P : \xi)$ into $C^\infty(K/K_P : \xi)$.

Lemma 6.14 *If $\xi \in \mathcal{R}(P)$, then the restriction map $f \mapsto f|_K$ defines a K-equivariant topological linear isomorphism*

$$r : C^\infty(G/P : \xi) \xrightarrow{\simeq} C^\infty(K/K_P : \xi). \tag{6.18}$$

See also [21, §10.1.1], or [5, Cor. III.7.9]).

Before proceeding with the proof we notice that there exists a unique representation $\pi^\infty_{P,\xi}$ of G in $C^\infty(K/K_P : \xi)$ which makes the map (6.18) G-equivariant. This representation is called the compact picture of the induced representation on the level of smoooth vectors.

Proof. Suppose $\xi \in \mathcal{R}(P)$. In view of the closed graph theorem for Fréchet spaces, it suffices to prove the surjectivity of the map above. Let V denote the space of $v \in H_\xi$ which are smooth for the restricted unitary representation $\xi|_{K_P}$, equipped with its natural topology, and let H_ξ^∞ denote the space of smooth vectors for ξ. Then H_ξ^∞ is contained in V with continuous inclusion map.

By assumption on ξ all elements of V are smooth for P, so that $V = H_\xi^\infty$ as sets. By application of the closed graph theorem for Fréchet spaces, it now follows that $V = H_\xi^\infty$ as Fréchet spaces. By application of Lemma 6.10 and Remark 6.11 we have

$$C^\infty(K/K_P : \xi) = C^\infty(K, H_\xi^\infty)^{K_P}.$$

Hence, if $\varphi \in C^\infty(K/K_P : \xi)$, we infer that the function $F : K \times P \to H_\xi$ given by

$$F(k, p) = {}^{\backprime}\xi(p^{-1})[\varphi(k)]$$

is smooth. It factors through a smooth function

$$f : G \simeq K \times_{K_P} P \to H_\sigma.$$

Hence, $f : G \to H_\xi$ is smooth and belongs to $C^\infty(G/P : \xi)$. Moreover, $f|_K = \varphi$. □

Finally, we will need a few results related the nilpotent picture of $\mathrm{Ind}_P^G(\xi)$. Given a compact subset $S \subset \bar{N}_P$, we denote by $C_S^\infty(\bar{N}_P, H_\xi^\infty)$ the subspace of functions in $C^\infty(\bar{N}_P, H_\xi^\infty)$ with support contained in S. Let

$$C_S^\infty(G/P : \xi) := \{f \in C^\infty(G/P : \xi) \mid \mathrm{supp}(f) \subset SP\}.$$

This space is a closed subspace of $C^\infty(G/P : \xi) = C^\infty(G, {}^{\backprime}H_\xi^\infty)^P$. It follows that restriction from G to $\bar{N}_P$ induces a continuous linear map

$${}^{n}r : C_S^\infty(G/P : \xi) \to C_S^\infty(\bar{N}_P, H_\xi^\infty) \tag{6.19}$$

Since the multiplication map $\bar{N}_P \times P \to G$ is a diffeomorphism onto the dense open subset $\bar{N}_P P$ of G, it is readily seen that the map (6.19) is injective.

Proposition 6.15 *The map (6.19) is a topological linear isomorphism of Fréchet spaces.*

Proof. By the closed graph theorem for Fréchet spaces, it is sufficient to show that ${}^n r$ is surjective. Let $\psi \in C_S^\infty(N_P, H_\xi^\infty)$. The map $\iota_\xi : H_\xi^\infty \to C^\infty(P, H_\xi)$ defined by

$$\iota_\xi(v)(p) = {}^\backprime\xi(p)^{-1}v, \qquad (v \in H_\xi^\infty, p \in P),$$

is a continuous embedding onto the closed subspace $C^\infty(P, {}^\backprime H_\xi)^P$. It follows that

$$\iota_\xi \circ \psi : \bar{N}_P \to C^\infty(P, H_\xi)$$

is a smooth map. This implies that the function $\bar{N}_P \times P \to H_\xi$,

$$(\bar{n}, p) \mapsto \iota_\xi(\psi(\bar{n}))(p) = {}^\backprime\xi(p)^{-1}\varphi(\bar{n})$$

is smooth. This function has support contained in $S \times P$. It follows from this that the function $\varphi : G \to H_\xi$ defined by $\varphi = 0$ on $G \setminus SP$ and by

$$\varphi(\bar{n}p) = {}^\backprime\xi(p)^{-1}\varphi(\bar{n}), \qquad ((\bar{n}, p) \in S \times P),$$

belongs to $C_{SP}^\infty(G, H_\xi)$. It is now readily seen that $\varphi \in C_S^\infty(G : \xi)$ and that ${}^n r(\varphi) = \psi$. □

The inverse of the above map ${}^n r$ will be denoted by $i_{\xi,S} : C_S^\infty(\bar{N}_P, H_\sigma^\infty) \to C_S^\infty(G/P : \sigma : \nu)$. It is a continuous linear isomorphism of Fréchet spaces. We define $C_c^\infty(\bar{N}_P, H_\xi^\infty)$ as the locally convex direct limit of the Fréchet spaces $C_S^\infty(\bar{N}_P, H_\xi^\infty)$. Then it follows that the maps $i_{\xi,S}$, for all $S \subset \bar{N}_P$ compact, are the restrictions of a single continuous linear map

$$i_\xi : C_c^\infty(\bar{N}_P, H_\sigma^\infty) \to C^\infty(G/P : \xi). \tag{6.20}$$

For every compact set $S \subset \bar{N}_P$ the natural bilinear map $C_S^\infty(\bar{N}_P) \times H_\sigma^\infty \to C_S^\infty(\bar{N}_P, H_\xi)$, $(\psi, v) \mapsto \psi \otimes v$ is jointly continuous. However, we warn the reader that this need not be true for the similar bilinear map $C_c^\infty(\bar{N}_P) \times H_\xi^\infty \to C_c^\infty(\bar{N}_P, H_\xi^\infty)$, see [18, Cor. 4.18].

7 Generalized vectors for induced representations

We retain the notation of the previous section. In particular, $P = M_P A_P N_P$ is a parabolic subgroup of G containing A and (ξ, H_ξ) is a continuous Hilbert space representation of P. Without loss of generality we may assume that $\xi|_{K_P}$ is unitary, see the text subsequent to (6.6).

We will now introduce a G-equivariant pairing between induced representations which is well known for the particular case $\xi = \sigma \otimes \nu \otimes 1$.

We denote by ξ^* the conjugate to ξ in H_ξ, which is the representation of P in H_ξ defined by

$$\langle \xi^*(p)v\,,\, w\rangle = \langle v\,,\, \xi(p^{-1})w\rangle \qquad (p \in P,\ v, w \in H_\xi).$$

Thus, $\xi^*(p)$ equals the Hilbert adjoint of $\xi(p^{-1})$. Clearly, ξ is unitary if and only if $\xi^* = \xi$. In Remark 1.3 we mentioned that ξ^* is a continuous representation of G in H_ξ.

If $f \in C(G/P : \xi)$ and $g \in C(G/P : \xi^*)$, we define the function $\langle f\,,\, g\rangle_\xi : G \to \mathbb{C}$ by

$$\langle f\,,\, g\rangle_\xi(x) = \langle f(x)\,,\, g(x)\rangle_\xi, \qquad (x \in G),$$

where the expression $\langle \cdot\,,\, \cdot\rangle_\xi$ on the right denotes the inner product of H_ξ. Since the restriction of ξ to K_P is unitary, $\xi^*|_{K_P} = \xi|_{K_P}$, and we see that restriction of the function $\langle f\,,\, g\rangle_\xi$ to K belongs to $C(K/K_P)$. This allows us to define the sesquilinear pairing

$$\langle \cdot\,,\, \cdot\rangle : C(G/P : \xi) \times C(G/P : \xi^*) \to \mathbb{C}, \tag{7.1}$$

by the formula

$$\langle f\,,\, g\rangle := \int_{K/K_P} \langle f(k)\,,\, g(k)\rangle_\xi \; d\dot{k}. \tag{7.2}$$

Lemma 7.1 *The sesquilinear pairing (7.1) is G-equivariant.*

Proof. Let $f, g \in C(G/P : \xi) \times C(G/P : \xi^*)$ and define $\varphi : G \to \mathbb{C}$ by $\varphi(y) = \langle f(y)\,,\, g(y)\rangle_\xi$. Then one readily verifies that $\varphi \in C(G/P : \delta_P)$. Using Lemma 6.4 we infer, for $x \in G$, that

$$\langle L_x f\,,\, L_x g\rangle = \int_{K/K_P} \varphi(x^{-1}k)\; d\dot{k} = \int_{K/K_P} \varphi(k)\; d\dot{k} = \langle f\,,\, g\rangle. \qquad \square$$

The pairing (7.1) obviously extends to a continuous sesquilinear pairing

$$L^2(G/P : \xi) \times L^2(G/P : \xi^*) \to \mathbb{C}, \tag{7.3}$$

given by the same formula. By density and continuity, the extended pairing is G-equivariant. In particular, we see again that if ξ is unitary then $\xi^* = \xi$ and the representation $(L, L^2(G/P : \xi))$ is unitary. In general, without the requirement that ξ be unitary, the following result is valid.

Lemma 7.2 *The Hermitian pairing (7.3) is a perfect pairing of Hilbert spaces, realizing each of them G-equivariantly as the conjugate dual of the other one.*

Proof. Since $\xi|_{K_P}$ is unitary, ξ^* and ξ are equal on K_P. Accordingly, restriction to K induces isometric isomorphisms $L^2(G/P : \xi) \simeq L^2(K/K_P : \xi|_{K_P})$ and $L^2(G/P : \xi^*) \simeq L^2(K/K_P : \xi|_{K_P})$. Via these isomorphisms, the pairing ((7.3) becomes the Hermitian pairing of $L^2(K/K_P : \xi|_{K_P})$ with itself, given by (6.5). As that pairing is perfect, so is (7.3). The final assertion follows from the G-equivariance of (7.3). $\square$

At various points in this article we will need the following description of the equivariant pairing in terms of the nilpotent group $\bar{N}_P$.

Lemma 7.3 *If $f \in C(G/P : \xi)$ and $g \in C(G/P : \xi^*)$ then*

$$\langle f\,,\, g\rangle = \int_{\bar{N}_P} \langle f(\bar{n})\,,\, g(\bar{n})\rangle_\xi \; d\bar{n}.$$

Proof. Since $\varphi := \langle f\,,\, g\rangle_\xi$ belongs to $C(G/P : \delta_P)$, see also the proof of Lemma 7.1, we have, by application of Lemma 6.4, that

$$\langle f\,,\, g\rangle = \int_{K/K_P} \varphi(k)\; dk = \int_{\bar{N}_P} \varphi(\bar{n})\; d\bar{n},$$

whence the required identity. □

Being perfect, the Hermitian pairing (7.3) induces a G-equivariant topological linear isomorphism

$$L^2(G/P : \xi) \xrightarrow{\simeq} \overline{L^2(G/P : \xi^*)\,'}. \tag{7.4}$$

Since $C^\infty(G/P : \xi^*)$ is a dense subspace of $L^2(G/P : \xi^*)$, the transpose of the associated inclusion map induces an injective continuous linear map

$$\overline{L^2(G/P : \xi^*)\,'} \hookrightarrow \overline{C^\infty(G/P : \xi^*)\,'}. \tag{7.5}$$

Here the second space is equipped with the strong dual (locally convex) topology. The map is given by restriction to $C^\infty(G/P : \xi^*)$.

Definition 7.4 Let (ξ, H_ξ) be a continuous representation of P in a Hilbert space. Then by $C^{-\infty}(G/P : \xi)$ we denote the conjugate continuous linear dual $C^\infty(G/P : \xi^*)'$, given as the second space in (7.5), equipped with the strong dual topology.

Remark 7.5 Being the strong dual of a Fréchet space, $C^{-\infty}(G/P : \xi)$ is a complete locally convex Hausdorff space. Since the induced representation $\pi_\xi = L$ of G in $C^\infty(G/P : \xi)$ is smooth, it follows that the natural representation $\pi_\xi^{-\infty}$ of G in $C^{-\infty}(G/P : \xi)$ is continuous and even smooth. See the text around (1.2).

Furthermore, the associated derived representation of $U(\mathfrak{g})$, also denoted $\pi_\xi^{-\infty}$, is given by

$$\pi_\xi^{-\infty}(u) : \varphi \mapsto \varphi \circ \pi_\xi^\infty(\bar{u}^\vee)$$

for $u \in U(\mathfrak{g})$. Here $u \mapsto \bar{u}^\vee$ is the conjugate linear automorphism of $U(\mathfrak{g})$ that for (real) $X \in \mathfrak{g}$ is given by $\bar{X}^\vee = -X$.

The composition of the two maps (7.4) and (7.5) leads to the G-equivariant continuous linear embedding

$$L^2(G/P : \xi) \hookrightarrow C^{-\infty}(G/P : \xi). \tag{7.6}$$

given by $f \mapsto \langle f\,,\,\cdot\,\rangle$. The elements of the latter space will be called generalized vectors for the induced representation $\mathrm{Ind}_P^G(\xi)$. In the sequel, we will use the map (7.6) to identify $L^2(G/P : \xi)$ with a subspace of $C^{-\infty}(G/P : \xi)$.

Remark 7.6 In particular, if $\xi = \sigma \otimes \nu \otimes 1$, with (σ, H_σ) a unitary representation of M_P and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we obtain that $\xi^* = \sigma \otimes -\bar{\nu} \otimes 1$ so that $C^{-\infty}(G/P : \sigma : \nu)$ is defined as the conjugate continuous linear dual of $C^\infty(G/P : \sigma : -\bar{\nu})$.

If (ξ_j, H_j) are continuous representations of P in Hilbert spaces, for $j = 1, 2$, then any continuous linear intertwining operator $\varphi : H_1 \to H_2$ induces the G-equivariant continuous linear map

$$\mathrm{Ind}_P^G(\varphi) : L^2(G/P : \xi_1) \to L^2(G/P : \xi_2), \quad f \mapsto \varphi \circ f.$$

This map restricts to a G-equivariant continuous linear map $C^\infty\mathrm{Ind}_P^G(\varphi)$ from the space $C^\infty(G/P : \xi_1)$ to the space $C^\infty(G/P : \xi_2)$.

Lemma 7.7 *The map* $\mathrm{Ind}_P^G(\varphi)$ *has a unique continuous linear extension to a map*

$$C^{-\infty}\mathrm{Ind}_P^G(\varphi) : C^{-\infty}(G/P : \xi_1) \to C^{-\infty}(G/P : \xi_2).$$

The extension is G-equivariant.

Proof. The conjugate map $\varphi^* : H_2 \to H_1$ is continuous linear and intertwines ξ_2^* with ξ_1^*. We consider the transpose T' of the map $T := C^\infty\mathrm{Ind}_P^G(\varphi^*)$. This map, given by the formula $\vartheta \mapsto \vartheta \circ T$, is G-equivariant and continuous linear $C^{-\infty}(G/P : \xi_1) \to C^{-\infty}(G/P : \xi_2)$ (use that $\xi_j^{**} = \xi_j$). We will proceed by establishing the claim that this map restricts to $\mathrm{Ind}_P^G(\varphi)$. Indeed, let $f \in L^2(G/P : \xi_1)$. Then for $g \in C^\infty(G/P : \xi_2)$ we have

$$\begin{aligned} \langle T'(f)\,,\, g\rangle &= \langle f\,,\, Tg\rangle = \int_{K/K_P} \langle f(k)\,,\, \varphi^* \circ g(k)\rangle_1 \, d\dot{k} \\ &= \int_{K/K_P} \langle \varphi \circ f(k)\,,\, g(k)\rangle_2 \, d\dot{k} = \langle \mathrm{Ind}_P^G(\varphi)(f)\,,\, g\rangle. \end{aligned}$$

Hence $T'f = \mathrm{Ind}_P^G(\varphi)(f)$ for $f \in L^2(G/P : \xi_1)$, establishing the claim. This settles the existence of the continuous linear extension of $\mathrm{Ind}_P^G(\xi)$. The uniqueness and G-equivariance follow from the density of $C^\infty(G/P : \xi_1)$ in $C^{-\infty}(G/P : \xi_1)$. □

We agree to identify the open right P-invariant subsets of G with the open subsets of G/P via the canonical projection $G \to G/P$. Likewise, the closed right P-invariant subsets of G are identified with the closed subsets of G/P. Accordingly, if $S \subset G/P$ is closed, we denote by $C_S^\infty(G/P : \xi^*)$ the closed subspace of $f \in C^\infty(G/P : \xi^*)$ such that $\mathrm{supp}\, f \subset S$. For a given open subset $\Omega \subset G/P$, we write $\mathrm{CPT}(\Omega)$ for the collection right P-invariant subsets of Ω which are closed, hence compact, as subsets of G/P, and put

$$C_c^\infty(\Omega : \xi^*) := \cup_{S \in \mathrm{CPT}(\Omega)} C_S^\infty(G/P : \xi^*).$$

Accordingly, we equip $C_c^\infty(\Omega : \xi^*)$ with the direct limit locally convex topology. Thus, a seminorm σ on $C_c^\infty(\Omega : \xi^*)$ is continuous if and only if for every $S \in \mathrm{CPT}(\Omega)$ the restriction of σ to $C_S^\infty(G/P : \xi^*)$ is continuous. For $\Omega \subset G/P$ open we define

$$C^{-\infty}(\Omega : \xi) := \overline{C_c^\infty(\Omega : \xi^*)'}, \tag{7.7}$$

equipped with the strong dual topology. We will view $C(\Omega : \xi)$ as a linear subspace of $C^{-\infty}(\Omega : \xi)$ via the map $g \mapsto \langle \cdot \,, g\rangle$ given by the sesquilinear pairing

$$C_c^\infty(\Omega : \xi^*) \times C(\Omega : \xi) \to \mathbb{C}, \quad (f, g) \mapsto \langle f \,, g\rangle = \int_{K/K_P} \langle f(k) \,, g(k)\rangle_\xi \, d\dot{k}.$$

Accordingly, the natural sesquilinear pairing associated with (7.7) will be denoted by

$$\langle \cdot \,, \cdot \rangle : \;\; C_c^\infty(\Omega : \xi^*) \times C^{-\infty}(\Omega : \xi) \to \mathbb{C}$$

We denote by $C^\infty(\Omega)$ the space of smooth right P-invariant functions $\Omega \to \mathbb{C}$. For each φ in this space the multiplication map $g \mapsto \varphi g$, $C(\Omega : \xi) \to C(\Omega : \xi)$ has a unique continuous linear extension to a map $C^{-\infty}(\Omega : \xi) \to C^{-\infty}(\Omega : \xi)$. This map, denoted $u \mapsto \varphi u$, is given by

$$\langle g \,, \varphi u\rangle = \langle \bar{\varphi} g \,, u\rangle, \quad (u \in C^{-\infty}(\Omega : \xi), \; g \in C^\infty(G/P : \xi^*)).$$

If $\Omega_1 \subset \Omega_2$ are open subsets of G/P, the transpose of the inclusion map $C_c^\infty(\Omega_1 : \xi) \hookrightarrow C_c^\infty(\Omega_2 : \xi)$ gives us the continuous linear restriction map

$$\rho^{\Omega_2}_{\Omega_1} : u \mapsto u|_{\Omega_1}, \;\; C^{-\infty}(\Omega_2 : \xi) \to C^{-\infty}(\Omega_1 : \xi).$$

Together with these restriction maps, the assignment $\Omega \mapsto C^{-\infty}(\Omega : \xi)$ defines a presheaf of $C^\infty(G/P)$-modules on G/P. By using smooth partitions of 1 over G/P, is readily seen that this presheaf is in fact a sheaf, as it has the following required restriction and glueing properties, for any open covering $\{\Omega_i \mid i \in I\}$ of G/P.

Restriction poperty. If $u \in C^{-\infty}(G/P : \xi)$ and $u|_{\Omega_i} = 0$ for all $i \in I$, then $u = 0$.

Glueing property. If $u_i \in C^{-\infty}(\Omega_i : \xi)$ for $i \in I$ and $u_i|_{\Omega_i \cap \Omega_j} = u_j|_{\Omega_i \cap \Omega_j}$ for all $i, j \in I$, then there exists a $u \in C^{-\infty}(G/P : \xi)$ such that $u|_{\Omega_i} = u_i$ for all $i \in I$.

We will finish this section by introducing a certain direct limit topology on the spaces $C^{-\infty}(G/P : \xi)$, assuming that both ξ and ξ^* belong to $\mathcal{R}(P)$. By Lemma 6.14 restriction to K induces a topological linear isomorphism $r : C^\infty(G/P : \xi^*) \to C^\infty(K/K_P : \xi^*)$. Denote the conjugate continuous linear dual of the latter space by $C^{-\infty}(K/K_P : \xi)$. Since $\xi|_K$ is unitary, it follows that ξ^* and ξ are equal on K, so that the latter two spaces do not change if ξ is replaced by ξ^*. By transposition we obtain a topological linear isomorphism

$$r^* : C^{-\infty}(K/K_P : \xi) \;\xrightarrow{\simeq}\; C^{-\infty}(G/P : \xi). \tag{7.8}$$

The map r^* is equivariant for a uniquel representation of G in the space $C^{-\infty}(K/K_P : \xi)$ which we denote by $\pi_{P,\xi}^{-\infty}$. This representation is called the compact picture of the induced representation $\mathrm{Ind}_P^G(\xi)$ on the level of generalized vectors.

In the sequel it will be important to use a specific set of continuous norms on $C^\infty(K/K_P : \xi)$, hence on $C^\infty(G/P : \xi)$. These are introduced as follows. We fix a basis $X_1, \dots, X_m$ of $\mathfrak{k}$ and use the notation $X^\mu := X_1^{\mu_1} \cdots X_n^{\mu_n} \in U(\mathfrak{k})$, for $\mu \in \mathbb{N}^n$ a multi-index. For $s \in \mathbb{N}$ the space $C^s(K, H_\xi)$ of C^s-functions $K \to H_\xi$ is a Banach space for the norm

$$f \mapsto \|f\|_s := \sum_{|\mu| \leq s} \sup_{k \in K} \|R_{X^\mu} f(k)\|_{H_\xi}. \tag{7.9}$$

The space $C^s(K/K_P : \xi) := C^s(K, H_\xi) \cap C(K/K_P : \xi)$ is a closed subspace. Hence, equipped with the norm $\| \cdot \|_s$ it is a Banach space of its own right. Clearly, the Fréchet topology on $C^\infty(K/K_P : \xi^*)$ is induced by the restrictions of the norms $\| \cdot \|_s$, for $s \in \mathbb{N}$.

For each $s \in \mathbb{N}$, the conjugate continuous linear dual of $C^s(K/K_P : \xi^*)$ will be denoted by $C^{-s}(K/K_P : \xi)$. Equipped with the dual norm $\| \cdot \|_{-s}$, it is a Banach space of its own right. The transpose of the inclusion $C^\infty(K/K_P : \xi^*) \to C^s(K/K_P : \xi^*)$ is an injective continuous linear map

$$C^{-s}(K/K_P : \xi) \hookrightarrow C^{-\infty}(K/K_P : \xi) \tag{7.10}$$

via which we shall identify elements. As the norms $\| \cdot \|_s, (s \in \mathbb{N})$, induce the topology of $C^\infty(K/K_P : \xi^*)$, it follows that $C^{-\infty}(K/K_P : \xi)$ is the union of the spaces $C^{-s}(K/K_P : \xi)$, for $s \in \mathbb{N}$. The latter spaces increase with s, and constitute the so-called filtration by order. The associated inclusion maps $C^{-s} \to C^{-s-t}$, for $s, t \in \mathbb{N}$, are continuous, so that the spaces form a directed family of locally convex spaces. We now observe that, as a linear space, $C^{-\infty}(K/K_P : \xi)$ is the direct limit of the directed family consisting of the spaces $C^{-s}(K/K_P : \xi)$, for $s \in \mathbb{N}$. We may therefore equip $C^{-\infty}(K/K_P : \xi)$ with the associated direct limit locally convex topology. Since the natural maps $C^{-s}(K/K_P : \xi) \to C^{-\infty}(K/K_P : \xi)$ are continuous for the strong dual topologies, it follows that the direct limit locally convex topology on $C^{-\infty}(K/K_P : \xi)$ is finer than (or equal to) the strong dual topology.

8 Whittaker vectors for induced representations

In this section we will initiate our study of Whittaker vectors for induced representations of the form $\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes \nu \otimes 1)$, with P a standard parabolic subgroup of G, i.e., $P \supset P_0 = MAN_0$. Here (σ, H_σ) is an irreducible unitary representation of M_P and $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. At a later stage, σ will be assumed to belong to the discrete series $\widehat{M}_{P,\mathrm{ds}}$ of M_P, i.e. the set of equivalence classes of irreducible square integrable representations of M_P. Implicitly it is then assumed that P is cuspidal. From Remark 7.6 we recall that $(\sigma \otimes \nu \otimes 1)^* = \sigma \otimes (-\bar{\nu}) \otimes 1$.

We note that $\mathfrak{a}_P \subset \mathfrak{a}$, $M_P \supset M$ and $N_P \subset N_0$. Accordingly, the set $N_0\bar{P}$ is open (and dense) in G. The space of Whittaker functionals for the induced representation $\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes -\bar{\nu} \otimes 1)$ is denoted

$$\mathrm{Wh}_\chi(L^2(G/\bar{P} : \sigma : -\bar{\nu})^\infty), \tag{8.1}$$

cf. (1.5). In view of and (1.5) and Theorem 6.7 the space (8.1) consists of the continuous linear functionals $\eta \in C^\infty(G/\bar{P} : \sigma : -\bar{\nu})'$ such that

$$\eta \circ L_n = \chi(n)\eta, \qquad (n \in N_0).$$

In view of (1.14) and Definition 7.4 with the subsequent remark, the space of Whittaker vectors of $\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes -\bar{\nu} \otimes 1)$ equals the space

$$C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi, \tag{8.2}$$

consisting of $j \in C^{-\infty}(G/\bar{P} : \sigma : \nu)$ transforming according to the rule

$$L_n j = \chi(n) j, \qquad (n \in N_0),$$

in the sense of generalized functions, see also (1.14).

The following result of Harish-Chandra [12, Thm. 1, p. 143], see also [16] for a proof, will be crucial for the determination of the space (8.2). In fact, it is crucial for the entire Whittaker theory. If Q is a parabolic subgroup of G we write

$$M_{1Q} = M_Q A_Q$$

for its θ-stable Levi component.

Theorem 8.1 (Harish-Chandra [12]) *Let Q be a standard parabolic subgroup. Then $N_0 M_{1Q} \bar{N}_Q$ is open in G. Let χ be regular and suppose that u is a distribution on G such that*

$$R_{\bar{n}} u = u \ \text{ and } \ L_{n_0} u = \chi(n_0) u, \qquad (\bar{n} \in \bar{N}_Q, n_0 \in N_0).$$

If $u = 0$ on $N_0 M_{1Q} \bar{N}_Q$, then $u = 0$ on G.

Remark 8.2 In the above, the space $\mathcal{D}'(G)$ of distributions on G is defined to be the continuous linear dual of the complete locally convex space $\mathcal{D}(G) := C_c^\infty(G)$. The left and right regular actions are defined by

$$L_g u = u \circ L_g^{-1}, \ R_g u = u \circ R_g^{-1}, \qquad (u \in \mathcal{D}'(G), g \in G).$$

The following corollary is of immediate importance for our discussion. We retain the assumption that σ is an irreducible unitary representation of M_P. We will say that an element $\psi \in C^{-\infty}(G/\bar{P} : \sigma : \nu)$ vanishes on an open subset $\mathcal{O}$ of $G/\bar{P}$ if $\langle f , \psi \rangle = 0$ for all $f \in C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$ with $\mathrm{supp} f \subset \mathcal{O}$.

Corollary 8.3 *Let χ be regular, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and $j \in C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$. If j vanishes on the orbit $N_P\bar{P}$ in $G/\bar{P}$ then $j = 0$.*

Proof. Fix $v \in H_\sigma \setminus \{0\}$. For $\varphi \in C^\infty_c(G)$ we define the function $T\varphi : G \to H_\sigma$ by

$$T\varphi(x) = \int_{M_P}\int_{A_P}\int_{\bar{N}_P} a^{-\bar{\nu}-\rho_P}\varphi(xma\bar{n})[\sigma(m)v]\ dmdad\bar{n}.$$

Then it is readily verified that T defines a continuous linear operator $C^\infty_c(G) \to C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$ which intertwines the left regular actions of G on these spaces. It follows that

$$u : \varphi \mapsto \langle T\varphi\,,\ j\rangle$$

defines a distribution on G. It is clear that u is right $\bar{N}_P$-invariant. By equivariance of T we see that, for $\varphi \in C^\infty_c(G)$ and $n \in N_0$,

$$L_n u(\varphi) = u(L_n^{-1}\varphi) = \langle L_{n^{-1}}T\varphi\,,\ j\rangle = \langle T\varphi\,,\ L_n j\rangle = \chi(n)^{-1}u(\varphi)$$

Now assume that $j = 0$ on $N_P\bar{P}$. If $\operatorname{supp}\varphi \subset N_P\bar{P}$ then $\operatorname{supp}(T\varphi) \subset N_P\bar{P}$, from which it follows that $u = 0$ on $N_P\bar{P}$. Since χ^{-1} is a regular character, it now follows from Theorem 8.1 that $u = 0$. This implies that j gives zero when applied to the space $T(C^\infty_c(G))$. We will finish the proof by showing that the latter space is dense in $C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$. In view of the natural decomposition $G \simeq K \times_{K_P} M_P \times A_P \times \bar{N}_P$ it suffices to show that the operator

$$S : C^\infty_c(K \times_{K_P} M_P) \to C^\infty(K/K_P : \sigma)$$

defined by

$$S\psi(k) = \int_{M_P} \psi(k,m)[\sigma(m)v]\ dm$$

has dense image. Let $f \in C^\infty(K/K_P : \sigma)$ be K-finite from the left. Then it suffices to show that $f \in \operatorname{im}(S)$. There exists a bi K-invariant finite dimensional subspace $F \subset C^\infty(K)$ such that f belongs to $F \otimes (H_\sigma)_{K_P}$ and is fixed under $R_{k_P} \otimes \sigma(k_P)$ for all $k_P \in K_P$. Thus, f is a finite sum of terms $f_j \otimes v_j$, with $f_j \in F$ and $v_j \in (H_\sigma)_{K_P}$. By irreducibility of σ, there exist left K_P-finite $\psi_j \in C^\infty_c(M_P)$ such that $\sigma(\psi_j)v = v_j$. Put

$$\psi(k,m) = \sum_j \int_{K_P} f_j(kk_P)\psi_j(k_P^{-1}m)\ dk_P,$$

where dk_P denotes normalized Haar measure on K_P. Then it is readily verified that ψ defines an element of $C^\infty_c(K \times_{K_P} M_P)$ which has image f. □

In the following it is not required that χ is regular. We denote by $C(G, {}'H^{-\infty}_{\sigma,\nu})^{\bar{P}}$ the space of continuous functions $f : G \to H^{-\infty}_\sigma$ such that

$$f(xma\bar{n}) = a^{-\nu+\rho_P}\sigma^{-\infty}(m^{-1})f(x),$$

for $x \in G$ and $(m, a, \bar{n}) \in M_P \times A_P \times \bar{N}_P$. Furthermore, we define

$$C(G, {}^{\backprime}H^{-\infty}_{\sigma,\nu})^{\bar{P}}_{\mathrm{fs}} \tag{8.3}$$

to be the subspace consisting of $f \in C(G, {}^{\backprime}H^{-\infty}_{\sigma,\nu})^{\bar{P}}$ such that the support of f is contained in $N_P\bar{P}$ and

$$\dim \operatorname{span} f(N_P) < \infty.$$

Lemma 8.4 *If $f \in C(G, {}^{\backprime}H^{-\infty}_{\sigma,\nu})^{\bar{P}}$, then for every $\varphi \in C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$ the function $x \mapsto \langle \varphi(x)\,,\, f(x)\rangle_\sigma$ belongs to $C(G/\bar{P} : \delta_{\bar{P}})$.*

Proof. This is straightforward. □

It follows that for f as in the lemma, we may define the linear functional

$$^{\vee}f_* : C^\infty(G/\bar{P} : \sigma : -\bar{\nu}) \to \mathbb{C}, \quad \varphi \mapsto \int_{K/K_P} \langle \varphi(k)\,,\, f(k)\rangle_\sigma \, dk.$$

Lemma 8.5 *If $f \in C(G, {}^{\backprime}H^{-\infty}_{\sigma,\nu})^{\bar{P}}_{\mathrm{fs}}$, then ${}^{\vee}f_*$ is continuous hence is the image of a unique $f_* \in C^{-\infty}(G/P : \sigma : \nu)$. Furthermore, the associated map $f \mapsto f_*$ is a linear injection*

$$C(G, {}^{\backprime}H^{-\infty}_{\sigma,\nu})^{\bar{P}}_{\mathrm{fs}} \hookrightarrow C^{-\infty}(G/P : \sigma : \nu).$$

Proof. For brevity, we write ξ for the continuous representation $\sigma \otimes -\bar{\nu} \otimes 1$ of P in H_σ. There exists a compact subset $S \subset N_P$ such that $\operatorname{supp} f \subset S\bar{P}$. From Lemmas 8.4 and 6.4 it follows that, for every $\varphi \in C^\infty(G/\bar{P} : \xi)$,

$$^{\vee}f_*(\varphi) = \int_{N_P} \langle \varphi(n)\,,\, f(n)\rangle_\sigma \, dn = \int_S \langle \varphi(n)\,,\, f(n)\rangle_\sigma \, dn. \tag{8.4}$$

In view of the hypothesis, the span E of $f(N_P)$ is a finite dimensional subspace of $H^{-\infty}_\sigma$. The natural pairing $E \times H^\infty_\sigma \to \mathbb{C}$ is continuous. It follows that there exist a norm $\|\cdot\|_E$ on E and a continuous seminorm q on H^∞_σ such that $|\langle v\,,\, \mu\rangle| \leq \|\mu\|_E q(v)$ for all $\mu \in E$ and $v \in H^\infty_\sigma$. Using (8.4) we now infer that, for $\varphi \in C^\infty(G/\bar{P} : \xi)$,

$$|{}^{\vee}f_*(\varphi)| \leq C \sup_S q(\varphi(n))$$

where $C := \sup_{n \in S} \|f(n)\|_E$. Let $\kappa : G \to K$ be the Iwasawa map associated with the decomposition $G = KA\bar{N}_0$, and let $p_{\bar{P}} : G \to \bar{P}$ be defined by $p_{\bar{P}}(x) = \kappa(x)^{-1}x$. Then it follows that $p_{\bar{P}}(S)$ is a compact subset of P. By uniform boundedness, there exists a continuous seminorm r on the Fréchet space H^∞_σ such that

$$q({}^{\backprime}\xi(p_{\bar{P}}(n))^{-1}v) \leq r(v), \quad (n \in S, v \in H^\infty_\sigma).$$

It now follows that, for $\varphi \in C^\infty(G/\bar{P} : \xi)$ and $n \in S$,

$$q(\varphi(n)) \leq r(\varphi(\kappa(n))) \leq \sup_{k \in K} r(\varphi(k)). \tag{8.5}$$

Via the natural isomorphisms $C^\infty(K, H_\sigma^\infty)^{K_P} \simeq C^\infty(K/K_P : \xi) \simeq C^\infty(G/\bar{P} : \xi)$ we see that the expression on the right of (8.5) defines a continuous seminorm on $C^\infty(G/\bar{P} : \xi)$. The asserted continuity of ${}^\vee f_*$ follows.

The element f_* is now defined by ${}^\vee f_* = \langle \cdot\ ,\ f_* \rangle$. Since $f \mapsto {}^\vee f_*$ is conjugate linear, linearity of $f \mapsto f_*$ is obvious. For injectivity, assume $f_* = 0$. Let $\varphi \in C_c^\infty(N_P)$ and $v \in H_\sigma^\infty$ be arbitrary. Then there exists a unique $\tilde{\varphi} \in C^\infty(G/\bar{P} : \xi)$ with support in $N_P\bar{P}$ such that $\tilde{\varphi}|_{N_P} = \varphi \otimes v$. From $\langle \tilde{\varphi}\ ,\ f_* \rangle = 0$ it follows by using (8.4) that

$$\int_{N_P} \varphi(n) \langle v\ ,\ f(n) \rangle_\sigma \, dn = 0.$$

Hence, for every $v \in H_\sigma^\infty$ the continuous function $n \mapsto \langle v\ ,\ f(n) \rangle_\sigma$ vanishes on N_P and we conclude that $f = 0$. □

Theorem 8.6 *Let $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ and $j \in C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$. There exists a unique element $\eta \in H_\sigma^{-\infty}$ such that the restriction of j to $N_P\bar{P}$ equals the continuous function $j_\eta : N_P\bar{P} \to H_\sigma^{-\infty}$ given by*

$$j_\eta(n_P m_P a_P \bar{n}_P) = a_P^{-\nu+\rho_P} \chi(n_P)^{-1} \sigma(m_P)^{-1} \eta. \tag{8.6}$$

Moreover,

$$\eta \in (H_\sigma^{-\infty})_{\chi|_{N_0 \cap M_P}}. \tag{8.7}$$

Remark 8.7 By the assertion that the restriction of j to $N_P\bar{P}$ equals the continuous function $j_\eta : N_P\bar{P} \to H_\sigma^{-\infty}$ it is meant that for every $\psi \in C^\infty(G/\bar{P})$ with $\mathrm{supp}\,\psi \subset N_P\bar{P}$ we have $(\psi j_\eta)_* = \psi j$.

Proof. Let $i_{\sigma,-\bar{\nu}} : C_c^\infty(N_P, H_\sigma^\infty) \to C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$ be the continuous linear map defined as in (6.20) with $\bar{P}$ in place of P and with $\xi = \sigma \otimes (-\bar{\nu})$. Thus, if $f \in C_c^\infty(N_P, H_\sigma^\infty)$ then $i_{\sigma,-\bar{\nu}}(f) \in C^\infty(G : \sigma : -\bar{\nu})$ is uniquely determined by $\mathrm{supp}\, i_{\sigma,-\bar{\nu}}(f) \subset N_P\bar{P}$ and by

$$i_{\sigma,-\bar{\nu}}(f)|_{N_P} = f.$$

Given $v \in H_\sigma^\infty$ and $\varphi \in C_c^\infty(N_P)$ we define

$${}_v j(\varphi) := \langle i_{\sigma,-\bar{\nu}}(\varphi \otimes v)\ ,\ j \rangle.$$

Then ${}_v j$ defines a continuous linear functional on $C_c^\infty(N_P)$, i.e., a distribution on N_P, which depends linearly on $v \in H_\sigma^\infty$.

Lemma 8.8 *The map $(\varphi, v) \mapsto {}_v j(\varphi)$ is continuous bilinear $C_c^\infty(N_P) \times H_\sigma^\infty \to \mathbb{C}$.*

Proof. Let us denote the above bilinear map by b. For every compact set $S \subset N_P$ the bilinear map $(\varphi, v) \mapsto \varphi \otimes v$ evidently has a continuous restriction to $C_S^\infty(N_P) \times H_\sigma^\infty$. This implies that the restricted bilinear map $b : C_S^\infty(N_P) \times H_\sigma^\infty \to \mathbb{C}$ is continuous.

Since $L_n j = \chi(n) j$, $(n \in N_P)$, it follows that

$$b(L_n\varphi, v) = \chi(n) b(\varphi, v), \qquad ((\varphi, v) \in C_c^\infty(N_P) \times H_\xi^\infty,\ n \in N_P).$$

Taking into account that the manifold N_P is diffeomorphic to $\mathbb{R}^n$ for a certain n, it now follows by application of Lemma 8.9 that b is continuous. □

Lemma 8.9 *Let V be a (Hausdorff) locally convex space, and $b : C_c^\infty(\mathbb{R}^n) \times V \to \mathbb{C}$ a bilinear map. Suppose that there exists a compact neighborhood $\mathcal{K}$ of 0 in $\mathbb{R}^n$ such that the restriction of b to $C^\infty_{\mathcal{K}}(\mathbb{R}^n) \times V$ is continuous. Suppose in addition that for every $x \in \mathbb{R}^n$ exists a diffeomorphism ℓ_x from an open neighborhood ω'_x of 0 to an open neighborhood ω_x of x and a constant $C_x > 0$ such that*

$$|b(\varphi, v)| \leq C_x |b(\ell_x^* \varphi, v)| \tag{8.8}$$

for all $\varphi \in C_c^\infty(\omega_x)$ and $v \in V$. Then b is (jointly) continuous.

Proof. By hypothesis, there exist continuous seminorms p on $C^\infty_{\mathcal{K}}(\mathbb{R}^n)$ and q on V such that

$$|b(\varphi, v)| \leq p(\varphi) q(v), \qquad (\varphi \in C^\infty_{\mathcal{K}}(\mathbb{R}^n),\ v \in V). \tag{8.9}$$

For $S \subset \mathbb{R}^n$ compact and $k \geq 0$ we define the seminorm $p_{S,k}$ on $C^\infty(\mathbb{R}^n)$ by

$$p_{S,k}(\varphi) := \max_{|\alpha| \leq k} \sup_S |\partial^\alpha \varphi|.$$

There exist constants $k \in \mathbb{N}$ and $c > 0$ such that $p \leq c p_{\mathcal{K},k}$ on $C^\infty_{\mathcal{K}}(\mathbb{R}^n)$. From now on we will keep k fixed and write p_S for $p_{S,k}$.

Let $x \in \mathbb{R}^n$ be arbitrary and fix $\ell_x : \omega'_x \to \omega_x$ with the property mentioned in the hypothesis. We select a compact neighborhood S_x of x such that $S_x \subset \ell_x(\omega_x \cap \mathcal{K})$. Then there exists a constant $D_x > 0$ such that for all $\varphi \in C^\infty_{S_x}(\mathbb{R}^n)$

$$p_{\mathcal{K}}(\ell_x^* \varphi) \leq D_x p_{S_x}(\varphi).$$

Combining this with (8.8) and (8.9) we find that, for all $\varphi \in C^\infty_{S_x}(\mathbb{R}^n)$ and $v \in V$,

$$|b(\varphi, v)| \leq c\, C_x\, p_{\mathcal{K}}(\ell_x^* \varphi) q(v) \leq c\, C_x D_x\, p_{S_x}(\varphi) q(v).$$

The sets $\mathrm{int}(S_x)$ cover $\mathbb{R}^n$. By paracompactness of the latter space, there exists a C^∞ partition of unity $\{\psi_i \mid i \in I\} \subset C_c^\infty(\mathbb{R}^n)$ subordinate to the mentioned cover. Thus, for each $i \in I$, there exists $x_i \in \mathbb{R}^n$ such that $\sigma_i := \mathrm{supp}\, \psi_i \subset \mathrm{int}(S_{x_i})$. Moreover, $\sum_{i \in I} \psi_i = 1$, with locally finite sum. Write $S_i := S_{x_i}$. Then for each $i \in I$ it follows by application of the Leibniz formula to $\partial^\alpha(\psi_i \varphi)$ that there exists a constant $d_i > 0$ such that for all $\varphi \in C^\infty(\mathbb{R}^n)$,

$$p_{S_i}(\psi_i \varphi) \leq d_i p_{\sigma_i}(\varphi).$$

Noting that $\{\sigma_i | i \in I\}$ is a locally finite collection of compact sets, we define the seminorm $\tilde{p}$ on $C_c^\infty(\mathbb{R}^n)$ by

$$\tilde{p} := \sum_{i \in I} c\, C_{x_i} D_{x_i} d_i\, p_{\sigma_i}.$$

By a simple argument it follows that $\tilde{p}$ is continuous on $C_S^\infty(\mathbb{R}^n)$ for every compact subset $S \subset \mathbb{R}^n$. Hence $\tilde{p}$ is continuous on $C_c^\infty(\mathbb{R}^n)$. Furthermore, for every $\varphi \in C_c^\infty(R^n)$ and $v \in V$ we have

$$\begin{aligned} b(\varphi, v) &= \sum_{i \in I} b(\psi_i \varphi, v) \\ &\leq \sum_{i \in I} c C_{x_i} D_{x_i} p_{S_i}(\psi_i \varphi) q(v) \\ &\leq \sum_{i \in I} c C_{x_i} D_{x_i} d_i p_{\sigma_i}(\varphi) q(v) = \tilde{p}(\varphi) q(v). \end{aligned}$$

This establishes the continuity of b. □

Completion of the proof of Theorem 8.6. From Lemma 8.9 it follows that the map $v \mapsto {}_vj$ is continuous linear $H_\xi^\infty \to \mathcal{D}'(N_P) = C_c^\infty(N_P)'$. Let $L^\vee$ denote the contragredient of the left regular representation L of N_P in $C_c^\infty(N_P)$. Since the map $i_{\sigma, -\bar{\nu}}$ is equivariant for the left regular actions of N_P, it follows that

$$L_n^\vee({}_vj) = \chi(n)^{-1} {}_vj, \quad (n \in N_P).$$

We fix $\psi \in C_c^\infty(N_P)$ such that

$$\int_{N_P} \psi(n) \chi(n)^{-1} \, dn = 1.$$

Then it follows that

$${}_vj = \int_{N_P} \psi(n) L_n^\vee({}_vj) \, dn = L^\vee(\psi)({}_vj).$$

In view of Lemma 8.10 below, it now follows that for every $v \in H_\sigma^\infty$ there exists a unique function $J_v \in C^\infty(N_P)$ such that

$${}_vj(\varphi) = \int_{N_P} J_v(n) \varphi(n) \, dn, \qquad (\varphi \in C_c^\infty(N_P)).$$

Moreover, by the same lemma, the map $v \mapsto J_v$ is continuous linear $H_\sigma^\infty \to C^\infty(N_P)$. By uniqueness, $J_v(n'n) = \chi(n') J_v(n)$, for all $n, n' \in N_P$, hence $J_v(n) = \chi(n) J_v(e)$. Define $\lambda : H_\sigma^\infty \to \mathbb{C}$ by

$$\lambda(v) := J_v(e), \qquad (v \in H_\sigma^\infty).$$

Since $\delta_e : C^\infty(N_P) \to \mathbb{C}$ is continuous, it follows that $\lambda \in (H_\sigma^\infty)'$. We now apply the isomorphism (1.12) in the setting $\pi_1 = \pi_2 = \sigma$ and with (1.7) given by the inner product on H_σ which makes σ unitary. It follows that there exists a unique $\eta \in H_\sigma^{-\infty}$ such that

$$\lambda(v) = \langle v \,,\, \eta \rangle, \qquad (v \in H_\sigma^\infty).$$

Define $j_\eta : N_P\bar{P} \to H_\sigma^{-\infty}$ as in (8.6). Then it easily follows that j_η is continuous. Furthermore, $j_\eta(N_P) \subset \mathbb{C}\eta$. Let $\psi \in C^\infty(G/\bar{P})$ have support contained in $N_P\bar{P}$. Then it follows that $\psi j_\eta \in C(G, H^\infty_{\sigma,\nu})^{\bar{P}}_{\mathrm{fs}}$, see (8.3). In view of Remark 8.7 it suffices to show that $(\psi j_\eta)_* = \psi j$. Let $\varphi \in C_c^\infty(N_P)$ and $v \in H_\sigma^\infty$. Then

$$\begin{aligned}
\langle i_{\sigma,-\bar{\nu}}(\varphi \otimes v)\,,\, (\psi j_\eta)_*\rangle &= \langle i_{\sigma,-\bar{\nu}}(\varphi \otimes v)\,,\, \psi j_\eta\rangle = \int_{N_P} \overline{\psi(n)}\langle \varphi(n)v\,,\, j_\eta(n)\rangle\, dn \\
&= \int_{N_P} \overline{\psi(n)}\varphi(n)\chi(n)\langle v\,,\, \eta\rangle\, dn = \int_{N_P} \overline{\psi(n)}\varphi(n)\chi(n)J_v(e)\, dn \\
&= \int_{N_P} \overline{\psi(n)}\varphi(n)J_v(n)\, dn = {}_v j(\bar{\psi}\varphi) \\
&= \langle i_{\sigma,-\bar{\nu}}(\bar{\psi}\varphi \otimes v)\,,\, j\rangle = \langle i_{\sigma,-\bar{\nu}}(\varphi \otimes v)\,,\, \psi j\rangle.
\end{aligned}$$

Therefore, the continuous linear functional $h : C^\infty(G/\bar{P} : \sigma : -\bar{\nu}) \to \mathbb{C}$ defined by $h := \langle \cdot\,,\, \psi j - (\psi j_\eta)_*\rangle$ vanishes on $i_{\sigma,-\bar{\nu}}(C_c^\infty(N_P) \otimes H_\sigma^\infty)$; here the algebraic tensor product has been taken.

Let $S_1 \subset N_P$ be compact. We select $S_2 \subset N_P$ compact such that $S_1 \subset \mathrm{int}\,(S_2)$; then $C^\infty_{S_1}(N_P, H_\xi^\infty)$ is contained in the closure of $C^\infty_{S_2}(N_P) \otimes H_\xi^\infty$ in $C^\infty_{S_2}(N_P, H_\xi^\infty)$. Since $i_{\sigma,-\bar{\nu}}$ restricts to a topological isomorphism from $C^\infty_{S_j}(N_P, H_\xi^\infty)$ onto the space $C^\infty_{S_j}(G/\bar{P} : \sigma : -\bar{\nu})$, it follows that h vanishes on $C^\infty_{S_1}(G/\bar{P} : \sigma : -\bar{\nu})$. As this is valid with S_1 an arbitrary compact subset of N_P, we may assume that S_1 contains an open neighborhood ω of $\mathrm{supp}\,\psi$ in N_P. Then $\Omega_1 := \omega\bar{P}$ and $\Omega_2 := G \setminus \mathrm{supp}\,\psi\,\bar{P}$ form an open cover of $G/\bar{P}$ such that $\psi j - (\psi j_\eta)_*$ restricts to 0 on Ω_j, $(j = 1, 2)$. By the restriction property mentioned in the text following (7.7) it follows that $\psi j = (\psi j_\eta)_*$.

This establishes the existence of η such that (8.6) is valid. If η' is a similar element, let $\psi \in C^\infty(G/\bar{P})$ have support in $N_P\bar{P}$ and satisfy $\psi([e]) \neq 0$. Then $[\psi(j_\eta - j_{\eta'})]_* = 0$. By injectivity of the map $f \mapsto f_*$ it follows that $\psi(j_\eta - j_{\eta'}) = 0$. Evaluating this identity at e we obtain $\psi([e])(\eta - \eta') = 0$ and conclude $\eta = \eta'$. Uniqueness of η follows.

It remains to show (8.7). For this we note that, for $n_0 \in N_0 \cap M_P$, conjugation by n_0 leaves N_P invariant and

$$L_{n_0} i_{\sigma,\nu}(\varphi \otimes v) = i_{\sigma,\nu}(L_{n_0}R_{n_0}\varphi \otimes \sigma(n_0)v).$$

From the definition of ${}_v j$ it now follows that

$$\chi(n_0)\ {}_v j(\varphi) = {}_{\sigma(n_0)v} j(L_{n_0}R_{n_0}\varphi).$$

This implies, in turn,

$$\chi(n_0)J_v(n) = J_{\sigma(n_0)v}(n_0 n n_0^{-1}), \qquad (n \in N_P).$$

Evaluation at $n = e$ gives

$$\chi(n_0)\langle v\,,\, \eta\rangle_\sigma = \langle \sigma(n_0)v\,,\, \eta\rangle_\sigma,$$

for all $v \in H_\sigma^\infty$ and $n_0 \in N_0 \cap M_P$. Finally, this gives

$$\sigma^{-\infty}(n_0)^{-1}\eta = \chi(n_0)^{-1}\eta, \quad (n_0 \in N_0 \cap M_P),$$

and (8.7) follows. This finishes the proof of Theorem 8.6. □

In the next lemma, we assume that H is a Lie group, equipped with a left Haar measure dx. We equip $C_c^\infty(H)$ with the left regular representation L of H, and its dual $\mathcal{D}'(H)$ with the contragredient representation $L^\vee$. Accordingly, we define, for $\psi \in C_c^\infty(H)$, the continuous linear map $L^\vee(\psi) : \mathcal{D}'(H) \to \mathcal{D}'(H)$ by

$$L^\vee(\psi)(u) = \int_H \psi(x)\, u \circ L_x^{-1}\, dx.$$

For $v \in C^\infty(H)$ we define $i(v) \in \mathcal{D}'(H)$ by

$$i(v)(f) = \int_H f(x)v(x)dx, \qquad (f \in C_c^\infty(H)).$$

Then $i : C^\infty(H) \to \mathcal{D}'(H)$ is an equivariant injective continuous linear map with dense image. In particular, $i \circ L(\psi) = L^\vee(\psi) \circ i$.

Lemma 8.10 *If $\psi \in C_c^\infty(H)$, then $L^\vee(\psi)$ is a smoothing operator in the sense that there exists a unique continuous linear map $T_\psi : \mathcal{D}'(H) \to C^\infty(H)$ such that*

$$L^\vee(\psi) = i \circ T_\psi. \tag{8.10}$$

Proof. Uniqueness is obvious, since i is injective. For $y \in H$ we define $R_y(\check{\psi}) \in C_c^\infty(H)$ by

$$R_{y^{-1}}(\check{\psi})(x) := \check{\psi}(xy^{-1}) = \psi(yx^{-1}), \qquad (x \in H).$$

Let $\Delta : H \to]0, \infty[$ be defined by $\Delta(x) = |\det \mathrm{Ad}(x)|$. Then the map $y \mapsto \Delta R_{y^{-1}}(\check{\psi})$ is smooth $H \to C_c^\infty(H)$. Thus, if $u \in \mathcal{D}'(H)$ then $T_\psi(u) : y \mapsto u(\Delta R_{y^{-1}}(\check{\psi}))$ is a smooth function on H. Moreover, the map $u \mapsto T_\psi(u)$ is continuous linear $\mathcal{D}'(H) \to C^\infty(H)$. We will show that it satisfies (8.10). Since the expressions on both sides of (8.10) are continuous linear endomorphisms of $\mathcal{D}'(H)$, it suffices to establish the equality on the dense subspace of elements of the form $u = i(v)$, with $v \in C^\infty(H)$. Since $L^\vee(\psi) \circ i = i \circ L(\psi)$ and since i is injective, it suffices to show that

$$L(\psi)(v) = T_\psi \circ i(v).$$

This identity of functions in $C^\infty(H)$ is established as follows. If $y \in H$, then

$$\begin{aligned} L(\psi)(v)(y) &= \int_H \psi(x)v(x^{-1}y)\, dx = \int_H \psi(yx)v(x^{-1})dx \\ &= \int_H \psi(yx^{-1})v(x)\Delta(x)dx = \int_H \Delta(x)R_{y^{-1}}(\check{\psi})(x)v(x)\, dx \\ &= i(v)(\Delta R_{y^{-1}}(\check{\psi})) = T_\psi(i(v))(y). \end{aligned}$$

□

We return to the setting of Theorem 8.6. Given $j \in C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$ we denote by by $\mathrm{ev}_e(j)$ the associated element $\eta \in H_\sigma^{-\infty}$ such that $j = j_\eta$ on $N_P\bar{P}$. Then by uniqueness of η combined with (8.7), we find that ev_e defines a linear map

$$\mathrm{ev}_e : C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi \to (H_\sigma^{-\infty})_{\chi|_{N_0 \cap M_P}}. \tag{8.11}$$

Corollary 8.11 *If χ is regular, then the map (8.11) is injective.*

Proof. Let $j \in C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$ and suppose that $\mathrm{ev}_e(j) = 0$. Then it follows from Theorem 8.6 that $j = 0$ on $N_P\bar{P}$. By Corollary 8.3 this implies that $j = 0$. □

We retain the assumption that χ is regular. Then $\chi|_{N_0 \cap M_P}$ is regular with respect to the roots of ${}^*\mathfrak{a}_P$ in $\mathfrak{n}_0 \cap \mathfrak{m}$. Indeed, for each such root α, the associated root space satisfies $\mathfrak{g}_\alpha \subset \mathfrak{m}_P \cap \mathfrak{n}_0$. We agree to use the abbreviated notation

$$H_{\sigma,\chi_P}^{-\infty} := (H_\sigma^{-\infty})_{\chi|_{N_0 \cap M_P}}. \tag{8.12}$$

For $R \in \mathbb{R}$ we put

$$\mathfrak{a}_{P\mathbb{C}}^*(P, R) := \{\nu \in \mathfrak{a}_{P\mathbb{C}}^* \mid \forall \alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}) : \langle \mathrm{Re}\, \nu\, ,\, \alpha \rangle > R\}. \tag{8.13}$$

Given $\eta \in H_{\sigma,\chi_P}^{-\infty}$ and $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(P, 0)$, we define the function $j_\nu = j(P, \sigma, \nu, \eta) : G \to H_\sigma^{-\infty}$ by $j_\nu = 0$ on $G \setminus N_P\bar{P}$ and by

$$j(P, \sigma, \nu, \eta)(nma\bar{n}) = a^{-\nu+\rho_P}\chi(n)^{-1}\sigma(m)^{-1}\eta, \tag{8.14}$$

for $n \in N_P$, $m \in M_P$, $a \in A_P$ and $\bar{n} \in \bar{N}_P$, see also (8.6).

Proposition 8.12 *Suppose that $\sigma \in \widehat{M}_{P,\mathrm{ds}}$ and let $\eta \in H_{\sigma,\chi_P}^{-\infty}$.*

(a) *If $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(P, 0)$ then the function $j_\nu = j(P, \sigma, \nu, \eta) : G \to H_\sigma^{-\infty}$ satisfies*

$$j_\nu(n_0 x m a \bar{n}) = \chi(n_0)^{-1}\, a^{-\nu+\rho_P}\sigma^{-\infty}(m)^{-1} j_\nu(x), \tag{8.15}$$

for all $x \in G$, $n_0 \in N_0$, $m \in M_P$, $a \in A_P$, and $\bar{n} \in \bar{N}_P$.

(b) *The function j_ν is continuous $H_\sigma^{-\infty}$-valued on $\mathcal{K} := K \cap N_P\bar{P}$. There exists a continuous seminorm s_σ on H_σ^∞, and for every $R > 0$ a Lebesgue integrable function $L_R : K \to [0, \infty[$ such that*

$$|\langle v\, ,\, j(\bar{P}, \sigma, \nu, \eta)(k)\rangle_\sigma| \le L_R(k) s_\sigma(v),$$

for all $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(P, R)$, $k \in K$ and $v \in H_\sigma^\infty$.

Proof. Since $N_P\bar{P}$ is left N_0-invariant, so is the complement $G \setminus N_P\bar{P}$, where j_ν equals zero. Hence j_ν satisfies (8.15) for $x \in G \setminus N_P\bar{P}$ and all values of $n_0, m, a, \bar{n}$.

For $x \in N_P\bar{P}$, the rule (8.14) implies (8.15) for $m \in M_P, a \in A_P, \bar{n} \in \bar{N}_P$ and all $n_0 \in N_P$. To obtain the rule for all $n_0 \in N_0$ we use that $N_0 = (N_0 \cap M_P)N_P$, and note that for $n_0 \in N_0 \cap M_P$ and $nma\bar{n} \in N_P M_P A_P \bar{N}_P$ we have, taking into account that $n_0 n n_0^{-1} \in N_P$,

$$\begin{aligned} j_\nu(n_0nman) &= j_\nu(n_0nn_0^{-1}(n_0m)an) \\ &= = a^{-\nu+\rho_P}\chi(n_0nn_0^{-1})^{-1}\sigma^{-\infty}(n_0m)^{-1}\eta \\ &= a^{-\nu+\rho_P}\chi(n)^{-1}\sigma^{-\infty}(m)^{-1}[\chi(n_0)^{-1}\eta] \\ &= \chi(n_0)^{-1}j_\nu(nman). \end{aligned}$$

Rule (8.15) now follows, and we turn to proving (b).

Let P_0' be the minimal parabolic subgroup containing A with $N_{P_0'} = (N_0 \cap M_P)\bar{N}_P$. We consider the maps $\kappa' : G \to K,\ h' : G \to A$ and $n' : G \to N_{P_0'}$ associated with the Iwasawa decomposition

$$G = KAN_{P_0'}, \tag{8.16}$$

and put $H' := \log \circ h' : G \to \mathfrak{a}$.

The natural map $N_P \to G/\bar{P}$ is an embedding onto a dense open subset. Via the natural diffeomorphism $K/K_P \simeq G/\bar{P}$ we have a corresponding open embedding $N_P \to K/K_P$. Since $P_0' \subset \bar{P}$, this open embedding is given by $n \mapsto \kappa'(n)K_P$. The associated open embedding $N_P \times K_P \to K$, given by $(n, k_P) \mapsto \kappa'(n)k_P$, has image $\mathcal{K}$. By transformation of variables it is well known that a function $f : K \to \mathbb{C}$ is absolutely integrable if and only if

$$\int_{N_P \times K_P} |f(\kappa'(n)k_P)| e^{2\rho_P H'(n)}\, dk_P dn < \infty \tag{8.17}$$

Furthermore, if (8.17) holds then $\|f\|_{L^1(K)}$ equals the given integral (up to a positive scalar factor, depending on the normalization of measures).

In the following we put ${}^*A := M_P \cap A$ and ${}^*N_0 := M_P \cap N_0$, so that

$$M_P = K_P {}^*A {}^*N_0 \tag{8.18}$$

is the Iwasawa decomposition of M_P associated with its minimal parabolic subgroup $P_0' \cap M_P = P_0 \cap M_P$. Therefore, this Iwasawa decomposition for M_P is compatible with the decomposition (8.16). Let ${}^*\rho \in \mathfrak{a}^*$ and $\rho' \in \mathfrak{a}^*$ be defined by

$${}^*\rho(H) = \tfrac{1}{2}\mathrm{tr}\,[\mathrm{ad}(H)|_{{}^*\mathfrak{n}_0}], \quad \rho'(H) = \tfrac{1}{2}\mathrm{tr}\,[\mathrm{ad}(H)|_{\mathfrak{n}_{P_0'}}],$$

for $H \in \mathfrak{a}$. Then

$$\rho' = {}^*\rho - \rho_P \tag{8.19}$$

and this decomposition is compatible with $\mathfrak{a} = {}^*\mathfrak{a}_P \oplus \mathfrak{a}_P$ in the sense that ${}^*\rho = 0$ on $\mathfrak{a}_P$ and $\rho_P = 0$ on ${}^*\mathfrak{a}_P$.

We chose $N \in \mathbb{N}$ sufficiently large (the precise condition will appear later). By Corollary 4.8 applied to M_P in place of G, and with $\lambda = \langle \cdot \,, \eta \rangle \in \mathrm{Wh}_{\chi_P}(H_\sigma^\infty)$, there exists a continuous seminorm q on H_σ^∞ such that

$$|\langle \sigma(m)^{-1} v \,, \eta \rangle| \leq (1 + |{}^*H(m)|)^{-N} e^{-{}^*\rho({}^*H(m))} q(v) \tag{8.20}$$

for all $v \in H_\sigma^\infty$ and $m \in M_P$. Here ${}^*H$ denotes the Iwasawa projection $M_P \to {}^*\mathfrak{a}$ associated with the decomposition (8.18).

In accordance with the decomposition $\mathfrak{a} = {}^*\mathfrak{a} \oplus \mathfrak{a}_P$, we decompose the Iwasawa projection $H'(x)$ of an element $x \in G$ as

$$H'(x) = {}^*H'(x) + H'_{\bar{P}}(x). \tag{8.21}$$

Furthermore, we agree to write ${}^*h' = \exp \circ {}^*H'$ and $h'_{\bar{P}} = \exp \circ H'_{\bar{P}}$. By compatibility of the decompositions (8.16) and (8.18) we note that ${}^*H'|_{M_P} = {}^*H$.

We now observe that for $n \in N_P$ we have $\kappa'(n) = n n'(n)^{-1} h'(n)^{-1}$. For $v \in H_\sigma^\infty$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ the function $x \mapsto \langle v \,, j_\nu(x) \rangle_\sigma$ is left N_0-equivariant and right $\bar{N}_P$-invariant. Hence, for $n \in N_P$ and $k_P \in K_P$,

$$\begin{aligned} \langle v \,, j_\nu(\kappa'(n) k_P) \rangle_\sigma &= \chi(n)^{-1} \, \langle \sigma(k_P) v \,, j_\nu(h'(n)^{-1}) \rangle_\sigma \\ &= \chi(n)^{-1} \, e^{(\nu - \rho_P) H'_{\bar{P}}(n)} \, \langle \sigma({}^*h'(n))^{-1} \sigma(k_P) v \,, \eta \rangle_\sigma \end{aligned} \tag{8.22}$$

Fom (8.21) we see that $(\nu - \rho_P) H'_{\bar{P}}(n) = (\nu - \rho_P) H'(n)$ for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$. Furthermore, ${}^*\rho {}^*H'(n) = {}^*\rho H'(n)$. Applying the estimate (8.20) to (8.22) we now find that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ and $v \in H_\sigma^\infty$, all $n \in N_P$ and $k_P \in K_P$,

$$\begin{aligned} & |\langle v \,, j_\nu(\kappa'(n) k_P) \rangle_\sigma| \, e^{2\rho_P H'(n)} \\ \leq \;& e^{(\mathrm{Re}\,\nu + \rho_P) \, H'(n)} (1 + |{}^*H'(n)|)^{-N} e^{-{}^*\rho {}^*H'(n)} \, q(\sigma(k_P) v) \\ \leq \;& e^{(\mathrm{Re}\,\nu) \, H'(n)} (1 + |{}^*H'(n)|)^{-N} e^{-\rho' H'(n)} s_\sigma(v). \end{aligned} \tag{8.23}$$

For the last inequality we have applied (8.19). Furthermore, s_σ is a continuous seminorm on H_σ^∞ such that $q(\sigma(k_P) v) \leq s_\sigma(v)$ for all $v \in H_\sigma^\infty$ and $k_P \in K_P$; it exists by compactness of K_P.

Since $\bar{P}$ contains P_0' it follows by Lemma 8.13 below, with P_0' and $\bar{P}$ in place of P_0 and Q respectively, that $H'(N_P)$ equals the closed convex cone Γ spanned by the root vectors H_α for $\alpha \in \Sigma(\bar{\mathfrak{n}}_P, \mathfrak{a})$ (here $H_\alpha \in \mathfrak{a} \cap (\ker \alpha)^\perp$ and $\alpha(H_\alpha) = 2$). If $\alpha \in \Sigma(\bar{\mathfrak{n}}_P, \mathfrak{a})$, then $\rho_P(H_\alpha) < 0$ and for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ we have

$$\rho_P(H_\alpha)^{-1} \mathrm{Re}\,\nu(H_\alpha) = \langle \rho_P \,, -\alpha \rangle^{-1} \langle \mathrm{Re}\,\nu \,, -\alpha \rangle > \langle \rho_P \,, -\alpha \rangle^{-1} R \geq \varepsilon R,$$

where $\varepsilon > 0$ is the minimal value of $\langle \rho_P \,, \beta \rangle^{-1}$, for $\beta \in \Sigma(\mathfrak{n}_P, \mathfrak{a})$. It follows that

$$\mathrm{Re}\,\nu(H) \leq \varepsilon R \rho_P(H)$$

for all $H \in \Gamma$. This implies the existence of a constant $C_{R,N} > 0$ such that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ and all $H \in \Gamma$ we have

$$e^{\mathrm{Re}\,\nu(H)} \leq C_{R,N}(1 - \rho_P(H))^{-N}.$$

Since $-\rho_P > 0$ on $\Gamma \setminus \{0\}$ there exists a constant $\gamma > 0$ such that

$$\gamma|H| \leq -\rho_P(H), \quad (H \in \Gamma).$$

If $H \in \mathfrak{a}$, we write $H = {}^*H + H_P$ according to the (orthogonal) decomposition $\mathfrak{a} = {}^*\mathfrak{a} + \mathfrak{a}_P$. Then $|H_P| \leq |H|$. It follows that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ and all $H \in \Gamma$ we have

$$e^{\mathrm{Re}\,\nu(H)} \leq C_{R,N}(1 + \gamma|H_P|)^{-N} \leq \tilde{C}_{R,N}(1 + |H_P|)^{-N};$$

here $\tilde{C}_{R,N} = C_{R,N} \sup_{t \geq 1} |(1+t)^{-1}(1+\gamma t)|^{-N}$. Finally, we infer that for $H \in \Gamma$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ we have the estimate

$$e^{\mathrm{Re}\,\nu(H)}(1 + |{}^*H|)^{-N} \leq \tilde{C}_{R,N}(1 + |H_P|)^{-N}(1 + |{}^*H|)^{-N} \leq \tilde{C}_{R,N}(1 + |H|)^{-N}.$$

Observing that in (8.23) the element $H'(n)$ belongs to Γ, we infer that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ and all $v \in H^\infty_\sigma$ we have

$$|\langle v\,,\, j_\nu(\kappa'(n)k_P)\rangle_\sigma|\, e^{2\rho_P H'(n)} \leq \tilde{C}_{R,N} e^{-\rho' H'(n)}\,(1 + |\log H'(n)|)^{-N} s_\sigma(v), \qquad (8.24)$$

If N is sufficiently large, then the integral of the latter function over N_P is absolutely integrable, see Lemma 8.13 (b). For such a choice of N the function $L : \mathcal{K} \to [0, \infty[$ defined by

$$L_R(\kappa'(n)k_P) = \tilde{C}_{R,N} e^{-\rho' H'(n)}(1 + |\log H'(n)|)^{-N} e^{-2\rho_P H'(n)} \qquad (8.25)$$

satsfies the required conditions. □

Given a root $\alpha \in \Sigma$ we denote by H_α the element of $\mathfrak{a}$ determined by $H_\alpha \perp \ker \alpha$ and $\alpha(H_\alpha) = 2$.

Lemma 8.13 *Let Q be a standard parabolic subgroup. Then*

(a) *$H(\bar{N}_Q)$ equals the cone $\Gamma(\Sigma(\mathfrak{n}_Q, \mathfrak{a}))$ spanned by the elements H_α for $\alpha \in \Sigma(\mathfrak{n}_Q, \mathfrak{a})$.*

(b) *There exists a constant $m \in \mathbb{N}$ such that*

$$\int_{\bar{N}_Q} e^{-\rho H(\bar{n})}(1 + |H(\bar{n})|)^{-m}\, d\bar{n} < \infty.$$

Proof. We consider the minimal parabolic subgroup R of G determined by $N_R = (M_Q \cap N_0)\bar{N}_Q$. Then it is well-known, see e.g. [6, Lemma 4.9], that $H(N_R \cap \bar{N}_0)$ equals the cone spanned by the elements H_α for $\alpha \in \Sigma(\bar{\mathfrak{n}}_R \cap \mathfrak{n}_0)$. Now $N_R \cap \bar{N}_0 = \bar{N}_Q$ and $\bar{\mathfrak{n}}_R \cap \mathfrak{n}_0 = \mathfrak{n}_Q$ and (a) follows.

The validity of (b) is due to Harish-Chandra, see e.g. [10, §31]. □

It follows from Proposition 8.12 that for every $\varphi \in C^\infty(K/K_P : \sigma)$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ the function

$$k \mapsto \langle \varphi(k) \,,\, j(\bar{P}, \sigma, \nu, \eta)(k) \rangle_\sigma$$

is continuous on $\mathcal{K} = K \cap N_P \bar{P}$ and dominated by a Lebesgue integrable function, hence integrable over K. Accordingly, we define the linear functional ${}^\vee j_*(\bar{P}, \sigma, \nu, \eta) : C^\infty(K/K_P : \sigma) \to \mathbb{C}$ by

$${}^\vee j_*(\bar{P}, \sigma, \nu, \eta)(\varphi) := \int_K \langle \varphi(k) \,,\, j(\bar{P}, \sigma, \nu, \eta)(k) \rangle_\sigma \, dk. \tag{8.26}$$

It follows from the estimate (8.24) that

$$|{}^\vee j_*(\bar{P}, \sigma, \nu, \eta)(\varphi)| \leq I(L_R) \sup_{k \in K} s_\sigma(\varphi(k)), \qquad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)), \tag{8.27}$$

where $I(L_R) := \int_K L_R(k)\, dk$. In particular, we see that ${}^\vee j_*(\bar{P}, \sigma, \nu, \eta) \in C^\infty(K/K_P : \sigma)'$ hence equals $\langle \cdot \,,\, j_*(\bar{P}, \sigma, \nu, \eta) \rangle$ for a unique element $j_*(\bar{P}, \sigma, \nu, \eta) \in C^{-\infty}(K/K_P : \sigma)$, for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$. From the text subsequent to (7.9) we recall that $C^{-\infty}(K/K_P : \sigma)$ is the union of the Banach spaces $C^{-s}(K/K_P : \sigma)$, for $s \in \mathbb{N}$.

Proposition 8.14 *If $R > 0$ there exists a constant $r > 0$ such that the following assertions hold for all $\eta \in H^{-\infty}_{\sigma,\chi_P}$.*

(a) *There exists a bounded subset of $C^{-r}(K/K_P : \sigma)$ to which $j_*(\bar{P}, \sigma, \nu, \eta)$ belongs for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$.*

(b) *The map $\nu \mapsto j_*(\bar{P}, \sigma, \nu, \eta)$ is holomorphic as a function on $\mathfrak{a}^*_{P\mathbb{C}}(P, R)$ with values in the Banach space $C^{-r}(K/K_P : \sigma)$.*

Proof. Let η and R be fixed. Let s_σ and L_R be as in Proposition 8.12. Then is it readily verified that $\varphi \mapsto \sup_{k \in K} s_\sigma(\varphi(k))$ is a continuous seminorm on $C^\infty(K/K_P : \sigma)$. As the topology on the latter space is generated by the seminorms $\| \cdot \|_r$, for $r \in \mathbb{N}$, see the text accompanying (7.9), there exists a constant $C > 0$ such that

$$\sup_{k \in K} s_\sigma(\varphi(k)) \leq C \|\varphi\|_r \tag{8.28}$$

for all $\varphi \in C^\infty(K/K_P : \sigma)$. Thus, from the estimate (8.27) it follows that there exists a constant $r \in \mathbb{N}$ such that $j_*(\bar{P}, \sigma, \nu, \eta) \in C^{-r}(K/K_P : \sigma)$ for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$. By linearity in η and finite dimensionality of $H^{-\infty}_{\sigma,\chi_P}$ the constant r may be taken the same for all η, and (a) follows, with the mentioned boundedness.

For (b) we will first show that the map $\nu \mapsto j_*(\bar{P}, \sigma, \nu, \eta)$ is continuous from $\mathfrak{a}^*_{P\mathbb{C}}(P, R)$ to $C^{-r}(K/K_P : \sigma)$. Let $\nu_0 \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ be fixed. Then it suffices to show that

$$\|j_*(\bar{P}, \sigma, \nu, \eta) - j_*(\bar{P}, \sigma, \nu_0, \eta)\|_{-r} \to 0 \qquad (\nu \to \nu_0). \tag{8.29}$$

In the notation of the proof of Proposition 8.12 we define, for $\mu \in \mathfrak{a}^*_{P\mathbb{C}}$ the function $\varepsilon_\mu : K \to \mathbb{C}$ by $\varepsilon_\mu = 0$ outside $\mathcal{K}$ and by

$$\varepsilon_\mu(\kappa'(n)k_P) = e^{\mu H'(n)}, \qquad (n \in N_P, k_P \in K_P).$$

If $\mu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ then $\operatorname{Re}\mu \leq 0$ on the cone $\Gamma = H'(N_P)$ so that $|\varepsilon_\mu(k)| \leq 1$ for all $k \in K$. From (8.22) combined with the definitions in the proof of Proposition 8.12 it is now readily checked that, for $0 < c < 1$,

$$j_*(\bar{P}, \sigma, \nu, \eta) - j_*(\bar{P}, \sigma, \nu_0, \eta) = \left[\varepsilon_{\nu - c\nu_0} - \varepsilon_{\nu_0 - c\nu_0}\right] j_*(\bar{P}, \sigma, c\nu_0, \eta).$$

Fix c sufficiently close to 1, so that $c\nu_0 \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$. Then combining (8.24) and (8.25) it follows that, for $v \in H_\sigma$,

$$|\langle\left[\varepsilon_{\nu - c\nu_0} - \varepsilon_{\nu_0 - c\nu_0}\right] j_*(\bar{P}, \sigma, c\nu_0, \eta)\,,\, v\rangle| \leq \left|\varepsilon_{\nu - c\nu_0} - \varepsilon_{\nu_0 - c\nu_0}\right| L_R \cdot s_\sigma(v).$$

Substituting $v = \varphi(k)$, integrating over K and using the estimate (8.28) we find, for all $\varphi \in C^\infty(K/K_P : \sigma)$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ that

$$|\langle j_*(\bar{P}, \sigma, \nu, \eta) - j_*(\bar{P}, \sigma, \nu_0, \eta)\,,\, \varphi\rangle| \leq CI(|\varepsilon_{\nu - c\nu_0} - \varepsilon_{\nu_0 - c\nu_0}| \cdot L_R)\, \|\varphi\|_r,$$

where I denotes the integral over K. It follows by application of the dominated convergence theorem that

$$I(|\varepsilon_{\nu - c\nu_0} - \varepsilon_{\nu_0 - c\nu_0}| \cdot L_R) \to 0, \qquad (\nu \to \nu_0).$$

The continuity (8.29) now follows.

Now that the continuity has been established, it follows by a simple application of the Cauchy integral formula that it suffices to prove the holomorphy of

$$\nu \mapsto \langle j_*(\bar{P}, \sigma, \nu, \eta)\,,\, \varphi\rangle, \quad \mathfrak{a}^*_{P\mathbb{C}}(P, R) \to \mathbb{C} \tag{8.30}$$

for a fixed $\varphi \in C^\infty(K/K_P : \sigma)$. According to (8.26) we have that

$$\langle j_*(\bar{P}, \sigma, \nu, \eta)\,,\, \varphi\rangle = \int_K \langle j(\bar{P}, \sigma, \nu, \eta)(k)\,,\, \varphi(k)\rangle_\sigma\, dk. \tag{8.31}$$

By Proposition 8.12 (b) the integrand is Lebesgue integrable in k for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$, holomorphic in $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ for every $k \in K$ and uniformly dominated by the Lebesgue integrable function $[\sup_{k \in K} s_\sigma(\varphi(k))] \cdot L_R$ for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$. This implies that the integral in (8.31) defines a holomorphic function of $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$. □

From now on we will omit the $*$ in the notation of the functional defined by (8.26), thus identifying the function $j(\bar{P}, \sigma, \nu, \eta) : K \to H^{-\infty}_\sigma$ with an element of $C^{-\infty}(K/K_P : \sigma)$ for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$. For such ν we will also write $j(\bar{P}, \sigma, \nu)$ for the element of

$$(H^{-\infty}_{\sigma, \chi_P})^* \otimes C^{-\infty}(K/K_P : \sigma) \simeq \operatorname{Hom}(H^{-\infty}_{\sigma, \chi_P}, C^{-\infty}(K/K_P : \sigma)) \tag{8.32}$$

defined by $\eta \mapsto j(\bar{P}, \sigma, \nu, \eta)$.

Proposition 8.15 *Let $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$. Then the following assertions are valid.*

(a) *If $\eta \in H^{-\infty}_{\sigma,\chi_P}$, then $j(\bar{P}, \sigma, \nu, \eta) \in C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$.*

(b) *The map*

$$j(\bar{P}, \sigma, \nu) : H^{-\infty}_{\sigma,\chi_P} \to C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi \tag{8.33}$$

is bijective with inverse ev_e.

Proof. Let $\eta \in H^{-\infty}_{\sigma,\chi_P}$. Then for (a) it suffices to show that

$$\pi^{-\infty}_{\bar{P},\sigma,\nu}(n_0)\, j(\bar{P}, \sigma, \nu, \eta) = \chi(n_0)\, j(\bar{P}, \sigma, \nu, \eta), \qquad (n_0 \in N_0).$$

Let $\varphi \in C^\infty(K/K_P : \sigma)$ and let $\varphi_{-\bar{\nu}}$ denote the unique function in $C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$ which restricts to φ on K. For $n_0 \in N_0$ we may write $n_0 = n_1 n_2$ with $n_1 \in M_P \cap N_0$ and $n_2 \in N_P$. Accordingly, in view of Lemma 6.4,

$$\begin{aligned}
\langle \varphi\,,\, \pi^{-\infty}_{\bar{P},\sigma,\nu}(n_0)\, j(\bar{P}, \sigma, \nu, \eta)\rangle &= \langle \pi_{\bar{P},\sigma,-\bar{\nu}}(n_0)^{-1}\varphi\,,\, j(\bar{P}, \sigma, \nu, \eta)\rangle \\
&= \int_{N_P} \langle \varphi_{-\bar{\nu}}(n_0 n)\,,\, j(\bar{P}, \sigma, \nu, \eta)(n)\rangle_\sigma\, dn \\
&= \int_{N_P} \chi(n)\, \langle \varphi_{-\bar{\nu}}(n_1 n_2 n)\,,\, \eta\rangle_\sigma\, dn \\
&= \int_{N_P} \chi(n_2^{-1} n)\, \langle \varphi_{-\bar{\nu}}(n_1 n)\,,\, \eta\rangle_\sigma\, dn.
\end{aligned}$$

Using that n_1 normalizes N_P with Jacobian 1, whereas $\chi(n_2^{-1}n) = \chi(n_2^{-1}n_1^{-1}nn_1)$, we infer that

$$\begin{aligned}
\langle \varphi\,,\, \pi^{-\infty}_{\bar{P},\sigma,\nu}(n_0)\, j(\bar{P}, \sigma, \nu, \eta)\rangle &= \int_{N_P} \chi(n_2^{-1} n)\, \langle \varphi_{-\bar{\nu}}(n)\,,\, \sigma(n_1)\eta\rangle_\sigma\, dn \\
&= \chi(n_0)^{-1} \langle \varphi\,,\, j(\bar{P}, \sigma, \nu, \eta)\rangle.
\end{aligned}$$

This establishes (a). For (b) we note that from (8.14) and the definition of ev_e in (4.5) it follows that $\mathrm{ev}_e \circ j(\bar{P}, \sigma, \nu, \eta) = \eta$. This implies that the map (8.33) is injective, with ev_e as left inverse. Therefore, ev_e is surjective onto $H^{-\infty}_{\sigma,\chi_P}$. In view of Cor. 8.11 it now follows that ev_e is bijective with two-sided inverse $j(\bar{P}, \sigma, \nu)$. □

9 The Whittaker integral

We will now reformulate the results of the previous section in terms of what Wallach [21, §15.4.1] calls the Jacquet integral. Given $f \in C^\infty(K/K_P : \sigma)$ we write $f_{\bar{P},\nu}$ for the unique function in $C^\infty(G/\bar{P} : \sigma : \nu)$ whose restriction to K equals f. Thus,

$$f_{\bar{P},\nu}(kman) = a^{-\nu+\rho_P} \sigma(m)^{-1} f(k), \tag{9.1}$$

for $k \in K$, $(m, a, n) \in M_P \times A_P \times \bar{N}_P$. We recall the definition of the continuous linear isomorphism $\eta \mapsto {}^\vee\eta$ from $H^{-\infty}_{\sigma,\chi_P}$ onto $\mathrm{Wh}_{\chi_P}(H^\infty_\sigma)$ as given in (1.13) with $(M_P, N_0 \cap P, \chi_P)$ in place of (G, N_0, χ), and with $\pi_1 = \pi_2 = \sigma$.

Lemma 9.1 *Let χ be regular, $\sigma \in \widehat{M}_{P\mathrm{ds}}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$. Then for all $f \in C^\infty(K/K_P : \sigma)$ we have*

$$\langle f_{\bar{P},-\bar{\nu}}\,,\, j(\bar{P}, \sigma, \nu, \eta)\rangle = \int_{N_P} \chi(n)\, {}^\vee\eta(f_{\bar{P},-\bar{\nu}}(n))\; dn,$$

with absolutely convergent integral.

Proof. By the substitution of variables used in (8.17) it follows that, with absolutely convergent integrals,

$$\begin{aligned}
\langle f_{\bar{P},-\bar{\nu}}\,,\, j(\bar{P}, \sigma, \nu, \eta)\rangle &= \int_{K/K_P} \langle f(k)\,,\, j(\bar{P}, \sigma, \lambda, \eta)(k)\rangle_\sigma \; dk \\
&= \int_{N_P} \langle f(\kappa'(n)\,,\, j(\bar{P}, \sigma, \nu, \eta)(\kappa'(n))\rangle_\sigma \; e^{2\rho_P H'(n)}\; dn \\
&= \int_{N_P} \langle f_{\bar{P},-\bar{\nu}}(n)\,,\, j(\bar{P}, \sigma, \nu, \eta)(n)\rangle_\sigma \; dn \\
&= \int_{N_P} \langle f_{\bar{P},-\bar{\nu}}(n)\,,\, \chi(n)^{-1}\eta\rangle_\sigma \; dn \\
&= \int_{N_P} \chi(n)\, {}^\vee\eta(f_{\bar{P},-\bar{\nu}}(n))\; dn.
\end{aligned}$$

□

If $\mathrm{Re}\,\nu$ is $\bar{P}$-dominant, and $\lambda \in \mathrm{Wh}_{\chi_P}(H^\infty_\sigma)$, this motivates the definition of the Jacquet integral

$$J(\bar{P}, \sigma, \nu, \lambda)(f) := \int_{N_P} \chi(n)\lambda(f_{\bar{P},\nu}(n))\; dn, \tag{9.2}$$

for $f \in C^\infty(K/K_P : \sigma)$.

From Lemma 9.1 we see that this integral is absolutely convergent for $\bar{P}$-dominant $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and defines a Whittaker functional for $C^\infty(G/\bar{P} : \sigma : \nu)$. In fact, the assertion of that lemma may be reformulated as

$${}^\vee j(\bar{P}, \sigma, \nu, \eta) = J(\bar{P}, \sigma, -\bar{\nu}, {}^\vee\eta). \tag{9.3}$$

Here the expression on the left is viewed as an element of $C^\infty(K/K_P : \sigma)'$ according to the compact picture, see (6.18). It follows from Wallach's work [21, Thm 15.4.1] that the Jacquet integral has a weakly holomorphic extension to a map $\mathfrak{a}^*_{P\mathbb{C}} \to C^\infty(K/K_P : \sigma)'$. Furthermore, for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the extension gives a linear isomorphism

$$\mathrm{Wh}_{\chi_P}(H^\infty_\sigma) \to \mathrm{Wh}_\chi(C^\infty(G/\bar{P}, \sigma, \nu)), \;\; \xi \mapsto J(P, \sigma, \lambda, \xi)$$

At a later stage we will strengthen this result by deriving a functional equation for $j(P,\sigma,\nu)$ and applying it to show that for every $\eta \in H^{-\infty}_{\sigma,\chi_P}$ the function $j(P,\sigma,\cdot,\eta)$ extends to a holomorphic function $\mathfrak{a}^*_{P\mathbb{C}} \to C^{-\infty}(K/K_P,\sigma)$, where the image space is equipped with the direct limit topology. By analytic continuation of the N_0-equivariance of $j(P,\sigma,\cdot,\eta)$ combined with Corollary 8.11, it then follows that for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the map

$$j(P,\sigma,\nu,\,\cdot\,) : H^{-\infty}_{\sigma,\chi_P} \to C^{-\infty}(G/\bar{P},\sigma,\nu)_\chi$$

is a linear isomorphism.

Given a representation π of the discrete series of G, we note that by Lemma 1.2 and Corollary 4.8 the linear map $\mu_\pi : H^\infty_\pi \otimes \overline{H^{-\infty}_{\pi,\chi}} \to C^\infty(G/N_0 : \chi)$ given by $\mu_\pi(v\otimes\eta)(x) = \langle \pi(x)^{-1}v\,,\,\eta\rangle$ for $x \in G$, is actually continuous linear into $C(G/N_0 : \chi)$. Its image is denoted by $C(G/N_0 : \chi)_\pi$. We denote the closure of this subspace of $L^2(G/N_0 : \chi_P)$ by $L^2(G/N_0 : \chi_P)_\pi$. Clearly the latter space is invariant under the left regular representation L of G. The following result is contained in [21, Thm. 15.3.4].

Lemma 9.2 *The map μ_π has a unique extension to a topological linear isomorphism*

$$H_\pi \otimes \overline{H^{-\infty}_{\pi,\chi}} \xrightarrow{\ \simeq\ } L^2(G/N_0 : \chi)_\pi. \tag{9.4}$$

This extension intertwines $\pi \otimes I$ with L.

Corollary 9.3 *If $\pi_1,\pi_2 \in \widehat{G}_{\rm ds}$ and $\pi_1 \not\sim \pi_2$ then $L^2(G/N_0 : \chi)_{\pi_1} \perp L^2(G/N_0 : \chi)_{\pi_2}$*

Proof. Let ι_1 denote the inclusion of the first of the spaces into $L^2(G/N_0 : \chi)$ and let p_2 denote the orthogonal projection onto the second of these spaces. Then $p_2 \circ i_1$ is a G-equivariant operator $L^2(G/N_0 : \chi)_{\pi_1} \to L^2(G/N_0 : \chi)_{\pi_2}$. From Lemma 9.2 and the inequivalence of π_1 and π_2 it readily follows that $p_2 \circ i_1 = 0$. □

Lemma 9.4 *Let (π, H) be an irreducible unitary representation of a Lie group L, and let V be a finite dimensional linear space. Suppose that $H \otimes V$ is equipped with a Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{H\otimes V}$ for which $\pi \otimes 1_V$ is a unitary representation of L. Then there exists a unique inner product $\langle\,\cdot\,,\,\cdot\,\rangle_V$ on V such that*

$$\langle x_1 \otimes v_1\,,\, x_2 \otimes v_2\rangle_{H\otimes V} = \langle x_1\,,\,x_2\rangle_H\,\langle v_1\,,\,v_2\rangle_V, \qquad (x_1,x_2 \in H,\ v_1,v_2 \in V). \tag{9.5}$$

Proof. We equip $\bar{H}'$ with the contragredient conjugate representation $\bar{\pi}^\vee$ of L, and V and $\bar{V}'$ with the trivial representation of L.

By finite dimensionality of V, it readily follows that the natural map $(A,B) \mapsto A\otimes B$ induces a linear isomorphism

$$\mathrm{Hom}_L(H,\bar{H}') \otimes \mathrm{Hom}(V,\bar{V}') \simeq \mathrm{Hom}_L(H\otimes V, \bar{H}' \otimes \bar{V}').$$

By Schur's lemma, the space on the left equals

$$\mathbb{C}i \otimes \mathrm{Hom}(V,\bar{V}') \simeq \mathrm{Hom}(V,\bar{V}'),$$

where $i : H \to \bar{H}', x \mapsto \langle x\,,\ \cdot\,\rangle_H$. We consider the L-equivariant linear map $h : H \otimes V \to \bar{H}' \otimes \bar{V}'$ determined by $h(x \otimes v)(y \otimes w) = \langle x \otimes v\,,\ y \otimes w\rangle$. By the above isomorphism there exists a unique linear map $j : V \to \bar{V}'$ such that $i \otimes j$ is mapped onto h. It is now readily checked that for all $v, w \in V$ we have $j(v)(w) = \overline{j(w)(v)}$ and $j(v)(v) > 0$. Thus, $\langle\,\cdot\,,\ \cdot\,\rangle_V : V \times \bar{V} \to \mathbb{C}, (v, w) \mapsto j(v)(w)$ defines a Hermitian inner product which satisfies the requirement. Conversely, if such an inner product is given then $j : v \mapsto \langle v\,,\ \cdot\,\rangle$ is such that $i \otimes j$ is mapped onto h, and uniqueness of $\langle\,\cdot\,,\ \cdot\,\rangle_V$ follows. □

We retain the notation of Lemma 9.2.

Corollary 9.5 *There exists a unique Hermitian inner product on $\overline{H^{-\infty}_{\pi,\chi}}$ such that the isomorphism (9.4) becomes an isometry.*

Proof. Use Lemma 1.2 and apply Lemma 9.4 with $L = G$, $V = \overline{H^{-\infty}_{\pi,\chi}} \simeq \mathrm{Wh}_\chi(H^\infty_\pi)$ and with $H_\pi \otimes V$ equipped with the pull-back of the L^2-inner product under μ_π. □

From now on, we assume that the finite dimensional spaces of Whittaker vectors $H^{-\infty}_{\sigma,\chi_P}$, for $\sigma \in \widehat{M}_{P,\mathrm{ds}}$, are equipped with the Hermitian inner products satisfying the assertion of Corollary 9.5.

We are now prepared to introduce Harish-Chandra's Whittaker integral, which is an appropriate analogue of the Eisenstein integral for groups and symmetric spaces. Let τ be a unitary representation of K in a finite dimensional complex Hilbert space V_τ.

We write $C^\infty(\tau : G/N_0 : \chi)$ for the space of smooth functions $f : G \to V_\tau$ such that $f(kxn) = \chi(n)^{-1}\tau(k)f(x)$, for all $x \in G, k \in K, n \in N_0$. Via the inverse of the natural isomorphism $C^\infty(G) \otimes V_\tau \to C^\infty(G, V_\tau)$ we have

$$C^\infty(\tau : G/N_0 : \chi) \simeq (C^\infty(G/N_0 : \chi) \otimes V_\tau)^K.$$

Accordingly, we define the associated space of τ-spherical Whittaker Schwartz functions by

$$\mathcal{C}(\tau : G/N_0 : \chi) := C^\infty(\tau : G/N_0 : \chi) \cap [\mathcal{C}(G/N_0 : \chi) \otimes V_\tau]$$

Furthermore, we put

$$\mathcal{A}_2(\tau : G/N_0 : \chi)_\pi := \mathcal{C}(\tau : G/N_0 : \chi) \cap [\mathcal{C}(G/N_0 : \chi)_\pi \otimes V_\tau].$$

and

$$\mathcal{A}_2(\tau : G/N_0 : \chi) := \oplus_{\pi \in \widehat{G}_{\mathrm{ds}}} \mathcal{A}(\tau : G/N_0 : \chi)_\pi. \tag{9.6}$$

Note that this direct sum has only finitely many non-zero terms, since $(V_\tau \otimes H_\pi)^K \neq 0$ for only finitely many $\pi \in \widehat{G}_{\mathrm{ds}}$. Moreover, each of the components is finite dimensional since π is admissible and $H^{-\infty}_{\pi,\chi}$ is finite dimensional. It follows that the space (9.6) is finite dimensional. In particular, for each π the corresponding summand is a closed subspace of $L^2(G/N_0 : \chi)_\pi \otimes V_\tau$. In view of Corollary 9.3 it follows that (9.6) is a finite orthogonal direct sum of finite dimensional subspaces of the Hilbert space $L^2(G/N_0 : \chi)) \otimes V_\tau$. Accordingly, we equip the space $\mathcal{A}_2(\tau : G/N_0 : \chi)$ with the restricted Hilbert structure.

Remark 9.6 It is a result of both Harish-Chandra [12] and Wallach [21] that the space (9.6) equals the space of $\mathfrak{Z}$-finite functions in $\mathcal{C}(\tau : G/N_0 : \chi)$. Equivalently, this means that the irreducible unitary representations which appear discretely in $L^2(G/N_0 : \chi)$ belong to $\widehat{G}_{\rm ds}$. However, we shall not need this in the present paper.

Let $P = M_P A_P N_P$ be a standard parabolic subgroup of G. We recall that $\chi_P := \chi|_{M_P \cap N_0}$ is regular relative to $(M_P, M_P \cap N_0)$, put $\tau_P = \tau|_{K_P}$ and define the finite dimensional space

$$\mathcal{A}_{2,P} := \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P) \tag{9.7}$$

as above. Then

$$\mathcal{A}_{2,P} = \oplus_{\sigma \in \widehat{M}_{P,\rm ds}} \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)_\sigma. \tag{9.8}$$

To keep notation manageable we will denote the summands by $\mathcal{A}_{2,P,\sigma}$. Since τ will be kept fixed, this will not cause any ambiguity.

Remark 9.7 Note that for $P = P_0$ minimal we have $M_P \cap N_0 = \{e\}$ so that

$$\mathcal{A}_{2,P_0} = C^\infty(\tau_M : M) \simeq V_\tau.$$

At the other extreme, for $P = G$ we have $M_P = {}^\circ G$, so that $M_P \cap N_0 = N_0$ and

$$\mathcal{A}_{2,G} = \oplus_{\sigma \in \widehat{{}^\circ G}_{\rm ds}} \mathcal{C}(\tau : {}^\circ G/N_0 : \chi)_\sigma.$$

We return to the setting of a parabolic subgroup P containing A. For $\sigma \in \widehat{M}_{P,\rm ds}$ we define $C^\infty(\tau_P : K/K_P : \sigma)$ to be the space of smooth functions $\varphi : K \to H_\sigma^\infty \otimes V_\tau$ such that

$$\varphi(k_1 k m) = [\tau(k_1) \otimes \sigma(m)^{-1}]\varphi(k), \qquad (k \in K,\ k_1, m \in K_P). \tag{9.9}$$

We equip this space with the pre-Hilbert structure induced by the L^2-inner product on $L^2(K, H_\sigma \otimes V_\tau)$, with respect to the Haar measure dk on K normalized by $\int_K dk = 1$.

For a finite subset $\vartheta \subset \widehat{K_P}$ we denote by $H_{\sigma,\vartheta}$ the sum of the K_P-isotypical components of H_σ for the K_P-types in ϑ. We note that (9.9) implies that $\varphi(e) \in (H_\sigma^\infty \otimes V_\tau)^{K_P} \subset H_{\sigma,\vartheta} \otimes V_\tau$, with

$$\vartheta = \{\delta \mid \mathrm{Hom}_{K_P}(\delta^\vee, \tau) \neq 0\}.$$

By sphericality this implies that $C^\infty(\tau_P : K/K_P : \sigma)$ equals the space of smooth $\varphi : K \to H_{\sigma,\vartheta} \otimes V_\tau$ such that (9.9). In particular it is finite dimensional, hence Hilbert for the given pre-Hilbert structure.

We define, for $\sigma \in \widehat{M}_{P,\mathrm{ds}}$ and $T = f \otimes \eta \in C^\infty(\tau : K/K_P : \sigma) \otimes \overline{H^{-\infty}_{\sigma,\chi_P}}$, the function $\psi_T : M_P \to V_\tau$ by

$$\psi_T(m) = \gamma \circ ({}^\vee\eta \otimes I) \circ (\sigma(m)^{-1} \otimes I)(f(e)). \tag{9.10}$$

Here f is viewed as a function with values in $H^\infty_\sigma \otimes V_\tau$, ${}^\vee\eta \in \mathrm{Wh}_\chi(H^\infty_\sigma)$ is defined by $v \mapsto \langle v\,,\, \eta\rangle$ and γ denotes the canonical linear map $\mathbb{C} \otimes V_\tau \to V_\tau$. It is readily verified that $\psi_T \in \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)_\sigma$.

Lemma 9.8 *The linear map $T \mapsto \psi_T$ is an isometric linear isomorphism*

$$C^\infty(\tau : K/K_P : \sigma) \otimes \overline{H^{-\infty}_{\sigma,\chi_P}} \xrightarrow{\simeq} \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)_\sigma. \tag{9.11}$$

This is analogous to a result of Harish-Chandra in the case of the group, see [11, Lemmas 7.1, 9.1]. It is also analogous to [1, Lemma 4.1] in the setting of symmetric spaces.

Proof. We equip $L^2(K, V_\tau)$ with the natural L^2-inner product corresponding to the fixed normalized Haar measure dk. By restriction this induces an inner product on $C^\infty(\tau : K)$. Clearly, the map $C^\infty(\tau : K) \to V_\tau$, $f \mapsto f(e)$, is a linear isomorphism which is K-equivariant for R and τ. Furthermore, for $f, g \in C^\infty(\tau : K)$ we have, by sphericality,

$$\langle f\,,\, g\rangle = \int_K \langle f(e)\,,\, g(e)\rangle\, dk = \langle f(e)\,,\, g(e)\rangle.$$

Thus $f \mapsto f(e)$ defines an isometric linear isomorphism $C^\infty(\tau : K) \xrightarrow{\simeq} V_\tau$.

It now follows that

$$\begin{aligned} C^\infty(\tau : K/K_P : \sigma) \otimes \overline{H^{-\infty}_{\sigma,\chi_P}} &= [C^\infty(\tau : K) \otimes H_\sigma]^{K_P} \otimes \overline{H^{-\infty}_{\sigma,\chi_P}} \\ &\simeq [V_\tau \otimes H_\sigma]^{K_P} \otimes \overline{H^{-\infty}_{\sigma,\chi_P}} \\ &\simeq [V_\tau \otimes C(M_P/M_P \cap N_0)_\sigma]^{K_P} \\ &= \mathcal{A}_{2,P}(\tau_P : M_P/M_P \cap N_0)_\sigma. \end{aligned}$$

In the above array the identity signs indicate isometric isomorphisms via which spaces are naturally identified. The composition of the first two isomorphisms is given by $f \otimes \eta \mapsto f(e) \otimes \eta$. By what we said in the above, this is an isometric isomorphism. The application of the third isomorphism maps $f(e) \otimes \eta$ to the function $M_P \to V_\tau$ given by $m \mapsto {}^\vee\eta(\sigma(m)^{-1} f(e)) = \langle \sigma(m)^{-1} f(e)\,,\, \eta\rangle$, which gives an isometric isomorphism in view of Corollary 9.5. From these descriptions it follows that the composition of the isomorphisms in the array is isometric and gives $T \mapsto \psi_T$. □

We now assume that P is a standard parabolic subgroup of G. For $\psi \in \mathcal{A}_{2,P}$, see (9.7), we define the associated Whittaker integral by

$$\mathrm{Wh}(P, \psi, \nu)(x) = \int_{N_P} \chi(n)\, \psi_{\bar{P},-\nu}(xn)\, dn, \tag{9.12}$$

where $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and where $\psi_{\bar{P},-\nu} \in C^\infty(G/\bar{P}, \sigma, -\nu) \otimes V_\tau$ is defined by

$$\psi_{\bar{P},-\nu}(kma\bar{n}) = a^{\nu+\rho_P}\tau(k)\psi(m),$$

for $k \in K, (m, a, \bar{n}) \in M_P \times A_P \times \bar{N}_P$. This is precisely the definition given by Harish-Chandra, [12, §1.7, p.147]. By rewriting this integral in terms of the Jacquet integral, we will see that it converges absolutely for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with $\langle \mathrm{Re}\,\nu\,,\,\alpha\rangle > 0$ for all $\nu \in \Sigma(P, \mathfrak{a}_P)$.

The Whittaker integral can be related to matrix coefficients, hence to the Jacquet integral, as follows. For σ a discrete series representation of M_P and $T = f \otimes \eta \in C^\infty(\tau : K/K_P : \sigma) \otimes \overline{H^{-\infty}_{\sigma,\chi_P}}$, let $\psi_T : M_P \to V_\tau$ be defined as in (9.10).

We note that $\gamma \circ (J(P, \sigma, \nu, \lambda) \otimes I_{V_\tau})$ defines a continuous linear map from $C^\infty(\tau : K/K_P : \sigma)$ to V_τ which we shall denote by

$$J(P, \sigma, \nu, \lambda)_\tau : C^\infty(\tau : K/K_P : \sigma) \to V_\tau.$$

Accordingly, we have the following relation of the Whittaker integral with the Jacquet integral.

Lemma 9.9 *Let $P = M_PA_PN_P$ be standard and $\sigma \in \widehat{M}_{P,\mathrm{ds}}$. Let $f \in C^\infty(\tau : K/K_P : \sigma)$ and $\eta \in H^{-\infty}_{\sigma,\chi_P}$. If $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$, then*

$$\mathrm{Wh}(P, \psi_{f\otimes\eta}\,, \nu)(x) = J(\bar{P}, \sigma, -\nu, {}^\vee\eta)_\tau(\pi_{\bar{P},\sigma,-\nu}(x)^{-1}f),$$

with absolutely convergent integral for the Whittaker integral on the left. Here we have abused notation, by writing $\pi_{\bar{P},\sigma,-\nu}(x)$ for $\pi_{\bar{P},\sigma,-\nu}(x) \otimes I_{V_\tau}$.

Proof. We put $\psi := \psi_{f\otimes\eta}$ and define $\psi_{\bar{P},-\nu} : G \to V_\tau$ by

$$\psi_{\bar{P},-\nu}(kman) := a^{\nu+\rho_P}\tau(k)\psi(m).$$

Furthermore, we define $f_{\bar{P},-\nu} : G \to H_\sigma \otimes V_\tau$ by

$$f_{\bar{P},-\nu}(kman) := a^{\nu+\rho_P}\,[\sigma(m)^{-1} \otimes I]f(k).$$

Then

$$\begin{aligned}
\psi_{\bar{P},-\nu}(kman) &= a^{\nu+\rho_P}\tau(k)\gamma[{}^\vee\eta \circ \sigma(m)^{-1} \otimes I]f(e) \\
&= \gamma[{}^\vee\eta \otimes I]a^{\nu+\rho_P}[\sigma(m)^{-1} \otimes \tau(k)]f(e) \\
&= \gamma[{}^\vee\eta \otimes I]a^{\nu+\rho_P}[\sigma(m)^{-1} \otimes I]f(k) \\
&= \gamma[{}^\vee\eta \otimes I]f_{\bar{P},-\nu}(kman).
\end{aligned}$$

This in turn implies that

$$L_{x^{-1}}\psi_{\bar{P},-\nu} = \gamma[{}^\vee\eta \otimes I]([\pi_{\bar{P},\sigma,-\nu}(x^{-1}) \otimes I]f)_{\bar{P},-\nu}. \tag{9.13}$$

The function on the right-hand side is integrable over N_P with integral

$$J(\bar{P}, \sigma, -\nu, {}^\vee\eta)_\tau([\pi_{\bar{P},\sigma,-\nu}(x^{-1}) \otimes I]f)$$

(abusing notation). It follows that the function on the left-hand side of (9.13) is also integrable over N_P, with integral being equal to $\mathrm{Wh}(P, \psi_{f\otimes\xi}, \nu)(x)$, see (9.12). □

Corollary 9.10 *Let the setting be as in Lemma 9.9. Then*

$$\mathrm{Wh}(P, \psi_{f\otimes\eta}, \nu)(x) = \langle \pi_{\bar{P},\sigma,-\nu}(x)^{-1} f\,,\; j(\bar{P}, \sigma, \bar{\nu}, \eta)\rangle.$$

Here we have abused notation in the expression on the left, by suppressing trivial actions on the tensor component V_τ and the role of the canonical isomorphism $\gamma : \mathbb{C}\otimes V_\tau \to V_\tau$.

Proof. This follows from Lemma 9.9 combined with (9.3). □

Corollary 9.11 *Let $P = M_P A_P N_P$ be standard. If $\psi \in \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)$, then for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ we have*

$$\mathrm{Wh}(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi).$$

Furthermore, $\nu \mapsto \mathrm{Wh}(P, \psi, \nu)$ is a holomorphic function on $\mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ with values in $C^\infty(\tau : G/N_0 : \chi)$.

Proof. By decomposition (9.8) and linearity, we may assume that ψ belongs to the space $\mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)_\sigma$ with σ a representation of the discrete series of M_P. In view of the isomorphism we may further assume that $\psi = \psi_{f\otimes\eta}$, with f and η as in Lemma 9.9. The result now follows by application of Corollary 9.10 and Proposition 8.14 (recall that in the compact picture, j_* is j viewed as an element of $C^{-\infty}(K/K_P : \sigma)$). □

For $Z \in \mathfrak{Z}$ we note that the endomorphism R_Z of $C^\infty(G)$ leaves the subspace $C^\infty(G/N_0 : \chi)$ invariant and induces a differential operator on that space, viewed as the space of smooth section of the associated bundle $G \times_{N_0} \mathbb{C}_\chi$. This differential operator is denoted R_Z as well. The associated endomorphism $I \otimes R_Z$ of $V_\tau \otimes C^\infty(G/N_0 : \chi)$ restricts to an endomorphism of $C^\infty(\tau : G/N_0 : \chi)$. In a similar fashion we may equip the latter space with a left action $I \otimes L_Z$. Since $I \otimes L_{Z^\vee} = I \otimes R_Z$, we see that the latter operator preserves $\mathcal{A}_2(\tau : G/N_0 : \chi)$.

Let $P \in \mathcal{P}(A)$. We agree to equip $\mathcal{A}_{2,P}$ with the structure of $\mathfrak{Z}(\mathfrak{m}_P)$-module induced by the right regular representation of $\mathfrak{m}_P$ on $C^\infty(M_P)$, as in the preceding text with G replaced by M_P.

Let

$$\mu_P : \mathfrak{Z} \to \mathfrak{Z}(\mathfrak{m}_{1P})$$

be the canonical embedding. The decomposition $\mathfrak{m}_{1P} = \mathfrak{m}_P \oplus \mathfrak{a}_P$ induces the canonical isomorphisms

$$\mathfrak{Z}(\mathfrak{m}_{1P}) \simeq S(\mathfrak{a}_P) \otimes \mathfrak{Z}(\mathfrak{m}_P) \simeq P(\mathfrak{a}^*_P) \otimes \mathfrak{Z}(\mathfrak{m}_P).$$

Thus, if $Z \in \mathfrak{Z}$ then $\mu_P(Z)$ may be viewed as a polynomial function on $\mathfrak{a}^*_{P\mathbb{C}}$ with values in $\mathfrak{Z}(\mathfrak{m}_P)$. Accordingly, for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we put $\mu_P(Z, \nu) := \mu_P(Z)(\nu)$. We agree to denote by $\underline{\mu}_P(Z, \nu)$ the endomorphism by which $\mu_P(Z, \nu)$ acts on $\mathcal{A}_{2,P}$. Then $\underline{\mu}_P(\,\cdot\,, \nu)$ may be viewed as an algebra homomorphism $\mathfrak{Z} \to \mathrm{End}(\mathcal{A}_{2,P})$, with polynomial dependence on ν.

Lemma 9.12 *Let $P \in \mathcal{P}(A)$ be standard, $Z \in \mathfrak{Z}$, $\psi \in \mathcal{A}_{2,P}$. Then for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ we have*

$$R_Z \mathrm{Wh}(P, \psi, \nu) = \mathrm{Wh}(P, \underline{\mu}_P(Z, \nu)\psi, \nu). \tag{9.14}$$

Proof. By linearity it suffices to fix a representation σ of the discrete series of M_P and to establish the identity for $\psi = \psi_{f\otimes\eta}$, with $f \in C^\infty(\tau_P : K/K_P : \sigma)$ and $\eta \in H^{-\infty}_{\sigma,\chi_P}$. From Corollary 9.10 it follows that

$$\begin{aligned} R_Z \mathrm{Wh}(P, \psi, \nu)(x) &= \langle \pi_{\bar{P},\sigma,-\nu}(x)^{-1}\,{}^{\backprime}\! f\,,\ j(\bar{P}, \sigma, \bar{\nu}, \eta)\rangle \\ &= \mathrm{Wh}(P, \psi_{{}^{\backprime}\! f\otimes\eta}, \nu)(x), \end{aligned} \tag{9.15}$$

with

$${}^{\backprime}\! f(k) = \pi_{\bar{P},\sigma,-\nu}(Z^\vee) f(k) = f_{\bar{P},\sigma,-\nu}(k; Z) = \sigma(\mu_{\bar{P}}(Z, \nu)^\vee) f(k).$$

From the definition of ψ, see (9.10) it follows that

$$\psi_{{}^{\backprime}\! f\otimes\xi} = R_{\mu_{\bar{P}}(Z,\nu)}\psi = R_{\mu_P(Z,\nu)}\psi = \underline{\mu}_P(Z, \nu)\psi.$$

Substituting this in (9.15), we obtain (9.14). □

As mentioned in the introduction, the main purpose of the present paper is to show that the Whittaker integrals extend holomorphically in the variable $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and, for imaginary ν, satisfy estimates of a uniformly tempered type.

A first step into this direction is the following estimate, for $\mathrm{Re}\,\nu$ P-dominant. The proof given below corresponds to the proof in [12, Lemma 9.22.1].

Lemma 9.13 *For every $\psi \in \mathcal{A}_{2,P}$ there exists a constant $m > 0$ and for every $R > 0$ a constant $C > 0$ such that for all $a \in A$ and all $\nu \in \mathfrak{a}^*_{\mathbb{C}}(P, R)$,*

$$\|\mathrm{Wh}(P, \psi, \nu)(a)\|_\tau \le C(1 + |\log a|)^m a^{\mathrm{Re}\,\nu - \rho}.$$

Proof. Since $\psi \in \mathcal{A}_{2,P}$ there exists for every $m > 0$ a constant $C_m > 0$ such that

$$\|\psi({}^*a)\| \le C_m(1 + |\log {}^*a|)^{-m}({}^*a)^{-{}^*\rho_P}, \tag{9.16}$$

for ${}^*a \in {}^*A_P := M_P \cap A$.

In the following we write $\psi_{-\nu} = \psi_{\bar{P},-\nu}$. Furthermore, we write $a = {}^*a a_P$ according to the decomposition $A = {}^*A_P A_P$. Then from (9.12) with $x = a$, we find, by substituting ana^{-1} for n that

$$\mathrm{Wh}(P, \psi, \nu)(a) = a_P^{\nu-\rho_P} \int_{N_P} \chi(a^{-1}na))\psi_{-\nu}(n\,{}^*a)\,dn.$$

Let P_0' be the minimal parabolic subgroup $MA(M_P \cap N_0)\bar{N}_P$ as in (8.16) and let κ', h', n' be the projection maps for the associated Iwasawa decomposition. Decomposing

$n = \kappa'(n)h'(n)n'(n)$ and $h'(n) = {}^*h'(n)h'_P(n)$ according to $A = {}^*A_PA_P$, as well as $n'(n) = {}^*n'(n)\bar{n}'_P(n)$ according to $N_{P'_0} = (M_P \cap N_0)\bar{N}_P$, we find

$$\begin{aligned} \psi_{-\nu}(n^*a) &= \tau(\kappa'(n))\,\psi_{-\nu}(h'(n)n'(n)^*a) \\ &= \tau(\kappa'(n))\,\psi_{-\nu}({}^*h'(n){}^*n'(n)h'_P(n)^*a) \\ &= h'_P(n)^{\nu+\rho_P}\,\chi({}^*a^{-1}{}^*n'(n){}^*a)^{-1}\,\tau(\kappa'(n))\,\psi({}^*h'(n){}^*a). \end{aligned}$$

In view of the unitarity of τ and χ this leads to the estimate

$$\|\mathrm{Wh}(P,\psi,\nu)(a)\| \leq a_P^{\mathrm{Re}\,\nu-\rho_P}\int_{N_P}\|\psi({}^*h'(n){}^*a)\|\;h'_P(n)^{\mathrm{Re}\,\nu+\rho_P}\,dn. \tag{9.17}$$

From (9.16) it follows, taking account that $a_P^\nu = a^\nu$ and $a_P^{\rho_P}({}^*a)^{{}^*\rho} = a^\rho$, that

$$\begin{aligned} &a_P^{\mathrm{Re}\,\nu-\rho_P}\|\psi({}^*h'(n){}^*a)\|\;h'_P(n)^{\mathrm{Re}\,\nu+\rho_P} \\ &\leq\; C_m\,a^{\mathrm{Re}\,\nu-\rho}(1+|\log{}^*a|)^m(1+|\log{}^*h'(n)|)^{-m}h'(n)^{-\rho'}h'_P(n)^{\mathrm{Re}\,\nu}. \end{aligned}$$

Applying Lemma 8.13 (a) with P'_0 in place of P_0 and $\bar{P}$ in place of Q, we see that the image $\log\circ h'_P(N_P)$ is contained in the cone in $\mathfrak{a}_P$ spanned by the elements $-\mathrm{pr}_P H_\alpha$, for $\alpha \in \Sigma(\mathfrak{n}_P,\mathfrak{a})$; here pr_P denote the orthogonal projection $\mathfrak{a} \to \mathfrak{a}_P$. This implies that there exists a constant $C_{m,R} > 0$ such that, for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P,R)$,

$$h'_P(n)^{\mathrm{Re}\,\nu} \leq C_{m,R}(1+|\log h'_P(n)|)^{-m}, \qquad (n \in N_P).$$

Therefore,

$$\begin{aligned} &a_P^{\mathrm{Re}\,\nu-\rho_P}\|\psi({}^*h'(n){}^*a)\|\;h'_P(n)^{\mathrm{Re}\,\nu+\rho_P} \\ &\leq\; C_mC_{m,R}\,a^{\mathrm{Re}\,\nu-\rho}(1+|\log{}^*a|)^m(1+|\log h'(n)|)^{-m}h'(n)^{-\rho'}. \end{aligned} \tag{9.18}$$

In view of Lemma 8.13 (b) we may fix $m > 0$ such that

$$I_m := \int_{N_P}(1+|\log h'(n)|)^{-m}h'(n)^{-\rho'}\;dn < \infty. \tag{9.19}$$

Combining (9.17) with (9.18) and (9.19) we find that, for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P,R)$ and $a \in A$,

$$\|\mathrm{Wh}(P,\psi,\nu)(a)\| \leq C_mC_{m,R}I_m\;a_P^{\mathrm{Re}\,\nu-\rho_P}(1+|\log{}^*a|)^m.$$

□

10 Finite dimensional spherical representations

We assume that $\mathfrak{h}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = \mathfrak{t}\oplus\mathfrak{a}$ with $\mathfrak{t}$ a maximal torus in $\mathfrak{m}$. Accordingly, we may naturally identify $\mathfrak{a}^*_\mathbb{C}$ with the space of $\lambda \in \mathfrak{h}^*_\mathbb{C}$ such that $\lambda|_\mathfrak{t} = 0$.

The recall the definition of B from (2.1), and denote its restriction to $\mathfrak{h}$ as well as the complex bilinear extension to $\mathfrak{h}_\mathbb{C}$ by $(\,\cdot\,,\,\cdot\,)$. The latter restricts to a positive definite inner product on $\mathfrak{h}_\mathbb{R} := i\mathfrak{t} \oplus \mathfrak{a}$. Its complexified dual, denoted by $(\,\cdot\,,\,\cdot\,)$ as well, is positive definite on $\mathfrak{h}_\mathbb{R}^* := i\mathfrak{t}^* + \mathfrak{a}^*$. The restriction of $(\,\cdot\,,\,\cdot\,)$ to $\mathfrak{a}^*$ coincides with the dual of the restriction of the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ defined by (2.2).

We denote by $R(\mathfrak{h}) \subset \mathfrak{h}_\mathbb{C}^*$ the root system of $\mathfrak{h}$ in $\mathfrak{g}_\mathbb{C}$ and select a positive system $R^+(\mathfrak{h})$ which is compatible with Σ^+. The latter means that if $\alpha \in R(\mathfrak{h})$ and $\alpha|_\mathfrak{a} \in \Sigma^+$ then $\alpha \in R^+(\mathfrak{h})$.

Let $\Lambda(\mathfrak{h})$ denote the collection of weights of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, i.e., the collection of $\lambda \in \mathfrak{h}_\mathbb{C}^*$ such that $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in R(\mathfrak{h})$. Let $\Lambda^+(\mathfrak{h}) \subset \mathfrak{h}_\mathbb{C}^*$ be the associated collection of dominant weights, i.e., the weights $\lambda \in \Lambda(\mathfrak{h})$ such that $(\lambda, \alpha) \geq 0$ for all $\alpha \in R^+(\mathfrak{h})$.

By the Cartan–Helgason classification [14, Ch. 5, Thm. 4.1], a finite dimensional irreducible representation π of G is spherical, i.e., has a K-fixed vector, if and only if M acts trivially on its highest weight space. Furthermore, the latter condition implies that the highest weight of π is an element of the set

$$\Lambda^+(\mathfrak{a}) = \{\mu \in \mathfrak{a}_\mathbb{C}^* \mid \forall \alpha \in \Sigma^+ : \frac{(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{N}\}.$$

Conversely, $\Lambda^+(\mathfrak{a}) \subset \Lambda^+(\mathfrak{h})$ and if $\mu \in \Lambda^+(\mathfrak{a})$, then up to equivalence there is a unique spherical representation of G of highest weight μ.

In [14] these results are proven for G connected semisimple with finite center. The extension of this result to groups of the Harish-Chandra class is straightforward. Given an element $\mu \in \Lambda(\mathfrak{a})$ we denote by π_μ the associated irreducible spherical representation of G.

The following result is well-known.

Lemma 10.1 $2(\Lambda^+(\mathfrak{h}) \cap \mathfrak{a}_\mathbb{C}^*) \subset \Lambda^+(\mathfrak{a})$.

Proof. Let $\alpha \in \Sigma$ and let $\tilde{\alpha} \in R(\mathfrak{h})$ be such that $\alpha = \tilde{\alpha}|_\mathfrak{a}$. Then $(\tilde{\alpha}, \tilde{\alpha}) = m(\alpha, \alpha)$, for a certain $m \in \{1, 2, 4\}$; if $m = 4$ then $2\alpha \in \Sigma$, see [13, Ch. VII, Lemma 8.4]. Let $\lambda \in \Lambda^+(\mathfrak{h}) \cap \mathfrak{a}_\mathbb{C}^*$. Then $(\lambda, \alpha) = (\lambda, \tilde{\alpha})$, so that

$$\frac{(2\lambda, \alpha)}{(\alpha, \alpha)} = 2m \frac{(\lambda, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})} \in m\mathbb{N} \subset \mathbb{N}.$$

□

In the rest of this section we assume that P is a standard parabolic subgroup of G. We write ${}^*\mathfrak{h}_P = \mathfrak{h} \cap \mathfrak{m}_P$. Then ${}^*\mathfrak{h}_P$ is a real θ-stable Cartan subspace of $\mathfrak{m}_P$, which decomposes as ${}^*\mathfrak{h}_P = \mathfrak{t} \oplus (\mathfrak{a} \cap \mathfrak{m}_P)$ Note that $\mathfrak{h} = {}^*\mathfrak{h}_P \oplus \mathfrak{a}_P$, so that we may identify ${}^*\mathfrak{h}_{P\mathbb{C}}^*$ and $\mathfrak{a}_{P\mathbb{C}}^*$ with subspaces of $\mathfrak{h}_\mathbb{C}^*$. We denote by $R({}^*\mathfrak{h}_P)$ the root system of ${}^*\mathfrak{h}_P$ in $\mathfrak{m}_{P\mathbb{C}}$. Then $R({}^*\mathfrak{h}_P)$ consists of the roots in $R(\mathfrak{h})$ which vanish on $\mathfrak{a}_P$. Furthermore, $R^+({}^*\mathfrak{h}_P) = R({}^*\mathfrak{h}_P) \cap R^+(\mathfrak{h})$ is a positive system. The associated weight lattice is denoted by $\Lambda({}^*\mathfrak{h}_P)$ and the subset of dominant ones by $\Lambda^+({}^*\mathfrak{h}_P)$.

Via the decomposition $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{m}_P) \oplus \mathfrak{a}_P$ we view $\mathfrak{a}^*_{P\mathbb{C}}$ as the linear subspace of $\mathfrak{a}^*_\mathbb{C}$ consisting of all $\mu \in \mathfrak{a}^*_\mathbb{C}$ that vanish on $(\mathfrak{a} \cap \mathfrak{m}_P)$. Accordingly, we define

$$\Lambda^+(\mathfrak{a}_P) := \Lambda^+(\mathfrak{a}) \cap \mathfrak{a}^*_{P\mathbb{C}}.$$

Lemma 10.2 *Let $\mu \in \Lambda^+(\mathfrak{a})$ and let π_μ be the irreducible spherical representation of G of highest weight μ. Then the following assertions are equivalent.*

(a) *M_P acts trivially on the highest weight space of π_μ;*

(b) $\mu \in \Lambda^+(\mathfrak{a}_P)$.

Proof. Let F be a finite dimensional complex linear space on which π_μ is realized. Let $e_\mu \in F_\mu \setminus \{0\}$ be a non-zero highest weight vector.

Assume (a). Then $A \cap M_P$ acts trivially e_μ, hence $\mu = 0$ on $\mathfrak{a} \cap \mathfrak{m}_P$, which implies (b).

For the converse, assume (b). Then $M_P = (M_P)_e M$, so that it suffices to show that $\mathfrak{m}_{P\mathbb{C}}$ annihilates e_μ. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{m}_{P\mathbb{C}}$ determined by the positive system $R^+({}^*\mathfrak{h}_P)$. Then $\mathfrak{b}$ is contained in $\mathfrak{m} + (\mathfrak{a} \cap \mathfrak{m}_P) + \mathfrak{n}_0$ hence annihilates e_μ. This implies that $U(\mathfrak{m}_P)e_\mu$ is a finite dimensional cyclic highest weight $\mathfrak{m}_{P\mathbb{C}}$-module of highest weight 0. Therefore, $U(\mathfrak{m}_P)e_\mu = \mathbb{C}e_\mu$ from which we obtain $\mathfrak{m}_P e_\mu = 0$. □

We define

$$\Lambda^{++}(\mathfrak{a}_P) := \{\mu \in \Lambda^+(\mathfrak{a}_P) \mid \forall \alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P) : \quad (\mu, \alpha) > 0\}. \tag{10.1}$$

The following lemma guarantees in particular that the set (10.1) is non-empty.

Lemma 10.3 *The element $4\rho_P$ belongs to $\Lambda^{++}(\mathfrak{a}_P)$.*

Proof. Let $\theta_\mathbb{C}$ denote the complex linear extension of the Cartan involution to $\mathfrak{g}_\mathbb{C}$. It restricts to a linear automorphism of $\mathfrak{h}_\mathbb{C}$ whose inverse transpose $\mathfrak{h}^*_\mathbb{C} \to \mathfrak{h}^*_\mathbb{C}$ is denoted by $\theta_\mathbb{C}$ as well. The latter map preserves both $R(\mathfrak{h})$ and $R({}^*\mathfrak{h}_P)$. Since $R^+(\mathfrak{h})$ is compatible with Σ^+ it follows that $-\theta_\mathbb{C}$ preserves the set $\widetilde{\Sigma}^+ := \{\alpha \in R^+(\mathfrak{h}) \mid \alpha|_\mathfrak{a} \neq 0\}$.

We define $\delta_P = \delta - \delta_{\mathfrak{m}_P}$ where δ and $\delta_{\mathfrak{m}_P}$ are half the sums of the positive roots from $R^+(\mathfrak{h})$ and $R^+({}^*\mathfrak{h}_P)$, respectively. Then δ_P equals half the sum of the positive roots from $R^+(\mathfrak{h}) \setminus R^+({}^*\mathfrak{h}_P)$. The latter set equals $\widetilde{\Sigma}^+ \setminus R({}^*\mathfrak{h}_P)$ hence is invariant under the map $-\theta_\mathbb{C}$. It follows that $-\theta_\mathbb{C}\delta_P = \delta_P$, so that $\delta_P \in \mathfrak{a}^*_\mathbb{C}$. Since clearly $\delta_P|_\mathfrak{a} = \rho_P$, we find that

$$\delta_P = \rho_P.$$

In particular this implies that $2\rho_P \in \Lambda(\mathfrak{h})$. Let β be a simple root from $R^+(\mathfrak{h})$. If it vanishes on $\mathfrak{a}_P$, then clearly, $(\rho_P, \beta) = 0$. If β does not vanish on $\mathfrak{a}_P$ then the simple roots γ from $R^+({}^*\mathfrak{h}_P)$ are simple for $R^+(\mathfrak{h})$ and not equal to β, hence satisfy $(\gamma, \beta) \leq 0$. For such a root β we thus have $(\beta, \delta_{\mathfrak{m}_P}) \leq 0$ so that

$$(\beta, \delta_P) \geq (\beta, \delta) = (\beta, \beta) > 0.$$

We thus conclude that $2\rho_P \in \Lambda^+(\mathfrak{h}) \cap \mathfrak{a}^*_{\mathbb{C}}$. By application of Lemma 10.1 it now follows that $4\rho_P \in \Lambda^+(\mathfrak{a}) \cap \mathfrak{a}^*_{P\mathbb{C}} = \Lambda^+(\mathfrak{a}_P)$.

We finish the proof by establishing the inequalities of (10.1). Let $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a})$. Then α is the restriction to $\mathfrak{a}$ of a root $\hat{\alpha} \in R^+(\mathfrak{h})$ which does not vanish on $\mathfrak{a}_P$. Now $\hat{\alpha}$ can be written as a sum of simple roots $\beta \in R^+(\mathfrak{h})$. For all these we have $(\delta_P, \beta) \geq 0$, see above. For those not vanishing on $\mathfrak{a}_P$ we have $(\delta_P, \beta) > 0$. Therefore,

$$(\rho_P, \alpha) = (\delta_P, \hat{\alpha}) > 0$$

□

11 Projection along infinitesimal characters

Let V be an admissible $(\mathfrak{g}, K)$-module and suppose that $\mathfrak{Z}$, the center of $U(\mathfrak{g})$, acts on V in a finite way. By this we mean that V decomposes into a finite direct sum of generalized weight spaces for $\mathfrak{Z}$. If ξ belongs to the set $\widehat{\mathfrak{Z}}$ of characters of $\mathfrak{Z}$, we denote the associated generalized weight space by $V[\xi]$. Obviously, $V[\xi]$ is an admissible $(\mathfrak{g}, K)$-submodule of V. Let X be the set of $\xi \in \widehat{\mathfrak{Z}}$ such that $V[\xi] \neq 0$; then X is finite and V is the direct sum of the weight spaces $V[\xi]$ for $\xi \in X$. For each $\xi \in \widehat{\mathfrak{Z}}$ the associated $\mathfrak{Z}$-equivariant projection map $V \to V$ with image $V[\xi]$ is denoted $p^V_\xi = p_\xi$. It is readily checked that p_ξ is $(\mathfrak{g}, K)$-equivariant. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$, then we agree to write $V[\lambda] := V[\xi_\lambda]$ and $p_\lambda := p_{\xi_\lambda}$; here $\xi_\lambda : Z \mapsto \gamma(Z, \lambda)$ is the character of $\mathfrak{Z}$ defined via the canonical isomorphism $\gamma : \mathfrak{Z} \to P(\mathfrak{a}^*)$.

Similar definitions can be given if V is a complete locally convex space on which G has a smooth admissible representation π such that $\mathfrak{Z}$ acts finitely. This leads again to a finite decomposition into a sum of generalized weight spaces

$$V = \oplus_{\xi \in X} V[\xi]$$

with $\mathfrak{Z}$-equivariant projection maps $p^V_\xi : V \to V$ with image $V[\xi]$. For a character $\xi \in \widehat{\mathfrak{Z}}$ the associated generalized weight space is the intersection of the spaces $\ker(Z - \xi(Z))^p$, for $Z \in \mathfrak{Z}$, and $p \geq 1$. As these spaces are all G-invariant and closed, it follows that $V[\xi]$ is G-invariant and closed. This in turn implies that $p_\xi : V \to V$ is a G-equivariant continuous projection, for every $\xi \in \widehat{\mathfrak{Z}}$. In view of admissibility we note that

$$V[\xi] \cap V_K = V_K[\xi], \qquad V[\xi] = \mathrm{cl} V_K[\xi]).$$

Furthermore, $p_\xi|_{V_K}$ is the projection $p^{V_K}_\xi$ associated with V_K and p_ξ.

We now assume (ρ, E) to be a smooth representation of G in a complete locally convex space which is admissible and of finite length. The following result follows immediately from [17, Thm. 5.1]. We assume that (π, F) is a finite dimensional irreducible representation of G of highest weight $\mu \in \mathfrak{h}^*_{\mathbb{C}}$.

Lemma 11.1 *Let (ρ, E) be as above and have infinitesimal character $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. Let $\{\mu_1 = \mu, \mu_2, \dots, \mu_m\} \subset \mathfrak{h}_{\mathbb{C}}^*$ be the set of distinct weights of the finite dimensional representation (π, F). Then for every $Z \in \mathfrak{Z}$,*

$$\prod_{k=1}^{m} (Z - \gamma(Z, \lambda + \mu_k)) \quad \text{acts by zero on} \quad E \otimes F.$$

Proof. The reference [17, Thm. 5.1] gives this result for $E_K \times F$, where E_K is the $U(\mathfrak{g})$-module of K-finite vectors in E. The required result now follows by density of E_K in E and continuity of the action of $\mathfrak{Z}$ on E. □

Let Q a parabolic subgroup of G containing A and let ω be a continuous representation of Q in a Hilbert space H_ω. If (π, F) is a finite dimensional representation of G, we have a natural G-equivariant topological linear isomorphism

$$\varphi : C^\infty(G/Q : \omega) \otimes F \xrightarrow{\simeq} C^\infty(G/Q : \omega \otimes \pi|_Q) \tag{11.1}$$

given by the formula

$$\varphi(f \otimes v)(x) = f(x) \otimes \pi(x)^{-1} v, \qquad (x \in G),$$

for $f \in C^\infty(G/Q : \omega)$ and $v \in F$. The inverse to this isomorphism is given by $\varphi^{-1}(f)(x) = (1 \otimes \pi(x)) f(x)$, for $f \in C^\infty(G/Q : \omega \otimes \pi|_Q)$ and $x \in G$. Clearly, all these assertions also hold with the bigger spaces of continuous functions that arise from replacing C^∞ by C everywhere.

Lemma 11.2 *The isomorphism (11.1) has a unique extension to a continuous linear map*

$$\varphi^{-\infty} : C^{-\infty}(G/Q : \omega) \otimes F \to C^{-\infty}(G/Q : \omega \otimes \pi|_Q). \tag{11.2}$$

This extension is a G-equivariant topological linear isomorphism.

Proof. Uniqueness is obvious, by density and continuity. For existence, let ${}^*\varphi$ denote the isomorphism (11.1) for the conjugate representations (ω^*, H_ω) and (π^*, F) in place of (ω, H_ω) and (π, F) (we assume F to be equipped with a K_Q-invariant inner product). Then by taking the transpose of the isomorphism

$$({}^*\varphi)^{-1} : C^\infty(G/Q : \omega^* \otimes \pi^*|_Q) \xrightarrow{\simeq} C^\infty(G/Q : \omega^*) \otimes F$$

one obtains an extension of (11.1) to a G-equivariant topological linear isomorphism. □

At a later stage we will use the notation φ_ν for the map φ of (11.1) in the case that $\omega = \sigma \otimes \nu \otimes 1$, with σ a unitary representation of M_Q and $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$.

Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and retain the notation of the beginning of Section 10. In particular, $\mathfrak{t} = \mathfrak{m} \cap \mathfrak{h}$ and

$$\mathfrak{h}_{\mathbb{R}}^* := \{\xi \in \mathfrak{h}_{\mathbb{C}}^* \mid \xi(i\mathfrak{t} + \mathfrak{a}) \subset \mathbb{R}\} = i\mathfrak{t}^* \oplus \mathfrak{a}^*.$$

We denote by $W_Q(\mathfrak{h})$ the centralizer of $\mathfrak{a}_Q$ in $W(\mathfrak{h})$.

Recall the definition of the complex bilinear form $\langle \cdot\,,\,\cdot \rangle$ on $\mathfrak{a}_{\mathbb{C}}^*$ in the text following (2.2). We denote by $\Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q)$ the set of $\mathfrak{a}_Q$-weights in $\mathfrak{n}_Q$.

Definition 11.3 By an affine $\Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q)$-hyperplane in $\mathfrak{a}_{Q\mathbb{C}}^*$ we mean a hyperplane of the form

$$\mathfrak{H}_{\alpha,c} = \{\nu \in \mathfrak{a}_{Q\mathbb{C}}^* \mid \langle \nu\,,\,\alpha \rangle = c\}, \qquad (\alpha \in \Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q), c \in \mathbb{C}). \tag{11.3}$$

The hyperplane is said to be real if $\mathfrak{H}_{\alpha,c} \cap \mathfrak{a}_Q^* \neq \emptyset$, which is equivalent to $c \in \mathbb{R}$.

Lemma 11.4 *Let P be a parabolic subgroup of G containing A and let $X \subset \mathfrak{h}_{\mathbb{C}}^*$ be finite. Then there exists a finite collection $\mathcal{H} = \mathcal{H}(X)$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes such that for each $\xi_1, \xi_2 \in X$, $w \in W(\mathfrak{h})$, and all $\nu \in \mathfrak{a}_{P\mathbb{C}}^* \setminus \cup\mathcal{H}$,*

$$w(\xi_1 + \nu) = \xi_2 + \nu \Rightarrow w \in W_P(\mathfrak{h}).$$

If $X \subset \mathfrak{h}_{\mathbb{R}}^*$, then $\mathcal{H}$ may be chosen to consist of real $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes.

Proof. It is easily verified that it suffices to prove the lemma for the case that $\mathfrak{g}$ is semisimple. Assume this to be the case.

Let $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$, $w \in W_P^c := W(\mathfrak{h}) \setminus W_P(\mathfrak{h})$, and $\xi_1, \xi_2 \in X$, and assume that $w(\xi_1 + \nu) = \xi_2 + \nu$. Put $X_w := w(X) - X$. Then $(I - w)(\nu) \in X_w$, hence $\nu \in (I - w)^{-1}(X_w) \cap \mathfrak{a}_{P\mathbb{C}}^*$. It is sufficient to show that the latter set is contained in a finite collection $\mathcal{H}_w$ of $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes. Then $\mathcal{H} = \cup_{w \in W_P^c} \mathcal{H}_w$ fulfils the requirements.

By our assumption on w, the linear space $\ker(I - w) \cap \mathfrak{a}_P^*$ is properly contained in $\mathfrak{a}_P^*$, and therefore has a non-zero linear complement T in $\mathfrak{a}_P^*$. We find that

$$\nu \in (T_{\mathbb{C}} \cap (I - w)^{-1}(X_w)) + (\ker(I - w) \cap \mathfrak{a}_{P\mathbb{C}}^*). \tag{11.4}$$

We claim that the first of these sets is finite. For this we note that the map $(I - w) \in \mathrm{End}(\mathfrak{h}_{\mathbb{C}}^*)$ preserves $\mathfrak{h}_{\mathbb{R}}^*$. Since $T \subset \mathfrak{a}_P^* \subset \mathfrak{h}_{\mathbb{R}}^*$ and $\ker(I - w) \cap T = 0$, it follows that $\ker(I - w) \cap T_{\mathbb{C}} = 0$ as well. This implies that the first set in (11.4) has cardinality at most $\#X_w$ and establishes the claim.

If $\xi \in \ker(I - w) \cap \mathfrak{a}_{P\mathbb{C}}^*$, then w can be written as a product of reflections in $\mathfrak{h}$-roots vanishing on ξ. At least one of these roots, say $\widehat{\alpha}$, does not vanish on $\mathfrak{a}_P$, so that $\alpha = \widehat{\alpha}|_{\mathfrak{a}_P^*} \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$. We see that $\xi \in \ker \alpha$. It follows that the set in (11.4) is contained in $\cup\mathcal{H}_w$, where $\mathcal{H}_w$ is the collection of all hyperplanes of the form $\eta + \mathfrak{H}_{\alpha,0}$, with $\eta \in (T_{\mathbb{C}} \cap (I - w)^{-1}(X_w))$ and $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$.

For the proof of the last assertion assume that $X \subset \mathfrak{h}_{\mathbb{R}}^*$. Then $X_w \subset \mathfrak{h}_{\mathbb{R}}^*$. Let

$$\eta \in T_{\mathbb{C}} \cap (I - w)^{-1}(X_w).$$

Then it suffices to show that $\eta \in T$. Write $\eta = \eta_1 + i\eta_2$. Then $(I - w)(\eta_j) \in \mathfrak{h}_{\mathbb{R}}^*$ and by considering real and imaginary parts we conclude that $(I - w)(\eta_2) = 0$. Now $T \cap \ker(I - w) = 0$ and we infer $\eta_2 = 0$. □

We now fix a standard parabolic subgroup P and a unitary representation (σ, H_σ) of M_P, which is admissible and of finite length, and such that H_σ^∞ is quasi-simple with infinitesimal character determined by $\Lambda \in (\mathfrak{h} \cap \mathfrak{m}_P)^*_\mathbb{C}$.

Furthermore, we fix $\mu \in \Lambda^{++}(\mathfrak{a}_P)$, see (10.1), and denote the associated irreducible finite dimensional representation of highest weight μ by (π, F). Let $\{\mu_1, \dots, \mu_m\}$ be the set of $\mathfrak{h}$-weights of π, ordered in such a way that $\mu_1 = \mu$.

Our goal is to describe the projection $p_{\Lambda+\nu+\mu}$ on the space $C^{-\infty}(G/Q : \sigma : \nu) \otimes F$ of generalized vectors of the representation $\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi$, for $Q \subset G$ a parabolic subgroup with split component $A_Q = A_P$. The (finite) set of all such parabolic subgroups is denoted by $\mathcal{P}(A_P)$.

Lemma 11.5 *For every $j > 1$ the elements $\Lambda + \mu_j$ and $\Lambda + \mu$ are not conjugate under $W_P(\mathfrak{h})$, the centralizer of $\mathfrak{a}_P$ in $W(\mathfrak{h})$.*

Proof. Let e_μ be a (non-zero) highest-weight vector of F. Then M_P acts trivially on e_μ, see Lemma 10.2, so that $U(\bar{\mathfrak{n}}_P)e_\mu = U(\mathfrak{g})e_\mu$ is a subspace of F which is invariant under the action of G_e and under the action of $M_P A_P$, hence under the action of G. By irreducibility it follows that $U(\bar{\mathfrak{n}}_P)e_\mu = F$. We thus see that each μ_j, for $j > 1$, is of the form $\mu_j = \mu - \xi_j$, where $\xi_j \in \Sigma(\mathfrak{n}_P, \mathfrak{h}) \setminus \{0\}$. The latter implies that ξ_j does not vanish identically on $\mathfrak{a}_P$.

If $w \in W_P(\mathfrak{h})$ and $j > 1$, then $w(\Lambda + \mu) - (\Lambda + \mu_j) = w(\Lambda) - \Lambda + \xi_j$. Now $w(\Lambda) - \Lambda$ vanishes identically on $\mathfrak{a}_P$, and ξ_j does not, so that $w(\Lambda + \mu) - (\Lambda + \mu_j) \neq 0$. □

Corollary 11.6 *There exists a finite collection $\mathcal{H}$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes such that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$ and all $j > 1$ the element $\Lambda + \nu + \mu_j$ does not belong to $W(\mathfrak{h})(\Lambda + \nu + \mu)$.*

If $\Lambda \in \mathfrak{h}^*_\mathbb{R}$, the assertion is valid with the additional requirement that $\mathcal{H}$ consists of real $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes.

Proof. Put $X = \{\Lambda + \mu_j \mid j \geq 1\}$. Then $X \subset \Lambda + \mathfrak{h}^*_\mathbb{R} \subset \mathfrak{h}^*_\mathbb{C}$. Let $\mathcal{H}$ be the finite collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes satisfying the conclusions of Lemma 11.4.

If $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$ and $\Lambda + \nu + \mu_j \in W(\mathfrak{h})(\Lambda + \nu + \mu)$ for $j > 1$, then it would follow that $\Lambda + \mu_j \in W_P(\mathfrak{h})(\Lambda + \mu)$, violating the assertion of Lemma 11.5. □

Let $\gamma : \mathfrak{Z} \to P(\mathfrak{h}^*_\mathbb{C})^{W(\mathfrak{h})}$ be Harish-Chandra's canonical isomorphism. Following the notation of [1] we define, for $Z \in \mathfrak{Z}$, the polynomial map $\Pi(Z) = \Pi_\mu(Z) : \mathfrak{a}^*_{P\mathbb{C}} \to \mathfrak{Z}$ by

$$\Pi(Z, \nu) := \prod_{j>1} [Z - \gamma(Z, \Lambda + \nu + \mu_j)] \tag{11.5}$$

Lemma 11.7 *Let $\mathcal{H}$ be a finite collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes as in Corollary 11.6. Let $Q \in \mathcal{P}(A_P)$. If $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$, then*

$$[\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi](\Pi(Z, \nu)) = 0 \quad \text{on} \quad \ker(p_{\Lambda+\nu+\mu}). \tag{11.6}$$

Proof. Fix $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$ and put $\rho = \mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi$. Write $\Pi(Z) := \Pi(Z, \nu)$, for $Z \in \mathfrak{Z}$. For each $Z \in \mathfrak{Z}$ the operator $\rho(Z)$ commutes with $p_{\Lambda+\nu+\mu}$ hence leaves the kernel of the latter invariant. Let $\delta \in \widehat{K}$. Since the operator $\rho(Z)$ is G-equivariant, it restricts to an endomorphism $\rho(Z)_\delta$ of the finite dimensional isotypical component $\mathcal{K}_\delta := \ker(p_{\Lambda+\nu+\mu})_\delta$. For each character ξ of $\mathfrak{Z}$ let $\mathcal{K}_{\delta,\xi}$ denote the associated union of the spaces $\ker(\rho(Z)_\delta - \xi(Z))^k$, for $k \geq 1$. Let $\mathfrak{X}$ be the set of characters χ for which $\mathcal{K}_{\delta,\xi}$ is non-zero. Then $\mathcal{K}_\delta$ is the finite direct sum of the generalized weight spaces $\mathcal{K}_{\delta,\xi}$, for $\xi \in \mathfrak{X}$.

Fix $\xi \in \mathfrak{X}$. For each $Z \in \mathfrak{Z}$, the endomorphism $\rho(Z)$ restricts to an endomorphism $\rho(Z)_{\delta,\xi}$ of $\mathcal{K}_{\delta,\xi}$. It suffices to show that $\rho(\Pi(Z))_{\delta,\xi} = 0$ for all $Z \in \mathfrak{Z}$. Let $\mathcal{E}$ be a finite dimensional linear subspace of $\mathfrak{Z}$ which generates the algebra $\mathfrak{Z}$. Since $\mathfrak{Z}$ is the union of such subspaces, it suffices to show that $\rho(\Pi(Z))_{\delta,\xi} = 0$ for all $Z \in \mathcal{E}$.

For $Z \in \mathfrak{Z}$ we define $l(Z) = Z - \gamma(Z, \Lambda + \nu + \mu)$. Then it follows from Lemma 11.1 with $\Lambda + \nu$ in place of λ that

$$\rho(l(Z))_{\delta,\xi}\rho(\Pi(Z))_{\delta,\xi} = 0. \tag{11.7}$$

For $Z \in \mathfrak{Z}$ the endomorphism $\rho(Z)_{\delta,\xi}$ has the single eigenvalue $\xi(Z)$, so that the endomorphism $\rho(l(Z))_{\delta,\xi} = \rho(Z)_{\delta,\xi} - \gamma(Z, \Lambda + \nu + \mu)$ has the single eigenvalue $\xi(Z) - \gamma(Z, \Lambda + \nu + \mu)$.

By definition of $p_{\Lambda+\nu+\mu}$ each character from $\mathfrak{X}$ is different from $Z \mapsto \gamma(Z, \Lambda+\nu+\mu)$. Since $\mathcal{E}$ generates the algebra $\mathfrak{Z}$ there must be an element $Z \in \mathcal{E}$ such that $\gamma(Z, \Lambda + \nu + \mu) \neq \xi(Z)$. It follows that the subspace $\mathcal{E}_0$ of $Z \in \mathcal{E}$ such that $\gamma(Z, \Lambda + \nu + \mu) = \xi(Z)$ is a proper hyperplane in $\mathcal{E}$. For $Z \in \mathcal{E} \setminus \mathcal{E}_0$ the endomorphism $\rho(l(Z))_{\delta,\xi}$ has a single non-zero eigenvalue, hence is invertible. Taking (11.7) into account we infer that $\rho(\Pi(Z))_{\delta,\xi} = 0$ for all $Z \in \mathcal{E} \setminus \mathcal{E}_0$. By density this extends to all $Z \in \mathcal{E}$. □

From now on we assume that σ is a representation from the discrete series of M_P. In particular, its infinitesimal character Λ belongs to $\mathfrak{h}^*_{\mathbb{R}}$ and is regular.

Lemma 11.8 *Let $Q \in \mathcal{P}(\mathfrak{a}_P)$ and $\mu \in \Lambda^{++}(\mathfrak{a}_P)$. Then there exists a locally finite collection $\mathcal{H} = \mathcal{H}(Q, \sigma, \mu)$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes in $\mathfrak{a}^*_{P\mathbb{C}}$ such that, for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$,*

$$p_{\Lambda+\nu+\mu}[C(G/Q : \sigma : \nu)_K \otimes F] \simeq C(G/Q : \sigma : \nu + \mu)_K$$

as $(\mathfrak{g}, K)$-modules.

Proof. We first assume that $Q = P$ and denote by φ_ν the isomorphism (11.1) for $\omega = \xi_\nu := \sigma \otimes \nu \otimes 1$. Then φ_ν restricts to an equivariant isomorphism

$$C(G/P : \sigma : \nu)_K \otimes F \xrightarrow{\simeq} C(G/P : \xi_\nu \otimes \pi|_P)_K$$

and therefore $p_{\Lambda+\nu+\mu} \circ \varphi_\nu = \varphi_\nu \circ p_{\Lambda+\nu+\mu}$. Since μ is P-dominant, the highest weight space F_μ is a P-submodule of F. Let $\mathcal{H}_0$ be the finite collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes of Cor. 11.6. We claim that for $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}_0$ we have

$$p_{\Lambda+\nu+\mu}(C(G/P : H_{\sigma,\nu} \otimes F)_K) = C(G/P : H_{\sigma,\nu} \otimes F_\mu)_K.$$

As the latter space is isomorphic to $C(G/P : \sigma : \nu + \mu)_K$, the result for $Q = P$ will follow from the claim. To prove the claim, we use exactness of the induction functor $\omega \mapsto \mathrm{Ind}_P^G(\omega)$ from the category of admissible $(\mathfrak{m}_{1P} \oplus \mathfrak{n}_P, K_P)$-modules to the category of $(\mathfrak{g}, K)$-modules. Let $F_1 \subset F_2$ be a sequence of P-submodules of F containing F_μ, such that F_2/F_1 is irreducible. Then by the mentioned exactness it suffices to show that $p_{\Lambda+\nu+\mu} = 0$ on $\mathrm{Ind}_P^G(H_{\sigma,\nu} \otimes (F_2/F_1))$.

It follows from the irreducibility that N_P acts trivially on F_2/F_1 and that the M_{1P}-action is irreducible, with a set of $\mathfrak{h}$-weights of the form $\{\mu_j \mid j \in J\}$ with $J \subset \{2, \dots, m\}$. Note that $\mu_j|_{\mathfrak{a}_P}$ is independent of $j \in J$. It follows from Lemma 11.1 that the infinitesimal characters of $\mathfrak{Z}(\mathfrak{m}_{1P})$ in $H_\sigma \otimes (F_2/F_1)$ are all of the form $\Lambda + \mu_j$, with $j \in J$. We conclude that the infinitesimal characters appearing in $\mathrm{Ind}_P^G(H_{\sigma,\nu} \otimes (F_2/F_1))$ are all of the form $\gamma(\,\cdot\,, \Lambda + \mu_j + \nu)$, with $j > 1$. By our choice of $\mathcal{H}_0$ these characters are different from $\gamma(\,\cdot\,, \Lambda + \mu + \nu)$ for $\nu \in \mathfrak{a}_{P\mathbb{C}}^* \setminus \cup\mathcal{H}_0$. Hence $p_{\Lambda+\nu+\mu}$ vanishes on $\mathrm{Ind}_P^G(H_{\sigma,\nu} \otimes (F_2/F_1))$. This establishes the result for $Q = P$.

Let now $Q \in \mathcal{P}(A_P)$ be arbitrary. Then by the rank one product formula there exists a locally finite collection $\mathcal{H}_Q$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes such that the standard intertwining operator $A(Q, P, \sigma, \nu)$ is regular and invertible for $\nu \in \mathfrak{a}_{P\mathbb{C}}^* \setminus \cup\mathcal{H}_Q$.

Then

$$\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi \simeq \mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1) \otimes \pi$$

and

$$\mathrm{Ind}_Q^G(\sigma \otimes (\nu + \mu) \otimes 1) \simeq \mathrm{Ind}_P^G(\sigma \otimes (\nu + \mu) \otimes 1)$$

for $\nu \notin \cup\mathcal{H}_Q \cup (-\mu + \cup\mathcal{H}_Q)$. The required result now follows with the hyperplane collection

$$\mathcal{H}(Q, \sigma, \mu) = \mathcal{H}_Q \cup (-\mu + \mathcal{H}_Q) \cup \mathcal{H}_0;$$

here we have written $-\mu + \mathcal{H}_Q = \{-\mu + H \mid H \in \mathcal{H}_Q\}$. □

Remark 11.9 In this paper we shall not need the deep result that the collection $\mathcal{H}_Q$ in the preceding proof may be chosen to consist of real affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes. Indeed, the singular sets of the standard intertwining operators $\nu \mapsto A(Q, P, \sigma, \nu)$ are locally finite unions of real affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes. For this it is only required that σ is irreducible unitary with real infinitesimal character, see [15, Thm. 6.6]. The mentioned deep result requires in addition that the zero set of $\nu \mapsto \eta(Q, P, \sigma, \nu)$ be contained in a locally finite union or real affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes. If σ belongs to the discrete series of M_P, then Harish-Chandra's explicit determination of the Plancherel measure, see [11, Lemma 35.3], guarantees this.

For $Z \in \mathfrak{Z}$ we define the polynomial function $b(Z) : \mathfrak{a}_{P\mathbb{C}}^* \to \mathbb{C}$ by

$$b(Z, \nu) := \prod_{j>1} [\gamma(Z, \Lambda + \nu + \mu) - \gamma(Z, \Lambda + \nu + \mu_j)].$$

Corollary 11.10 *Let $\mathcal{H}$ be a finite set of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes in $\mathfrak{a}^*_{P\mathbb{C}}$ as in Corollary 11.6. If $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ is such that the function $b(\,\cdot\,, \nu)$ vanishes identically on $\mathfrak{Z}$, then $\nu \in \cup\mathcal{H}$.*

Proof. Assume that ν satisfies the hypothesis, then it follows that $b(Z, \nu)$ vanishes for all $Z \in \mathfrak{Z}$. Since $\mathfrak{Z}$ is finitely generated, we may fix a finite dimensional linear subspace $\mathfrak{Z}_0$ of $\mathfrak{Z}$ which generates $\mathfrak{Z}$. Since $Z \mapsto b(Z, \nu), \mathfrak{Z}_0 \to \mathbb{C}$ is a polynomial function on $\mathfrak{Z}_0$ which vanishes identically on $\mathfrak{Z}_0$ it follows that there exists $j > 1$ such that the factor

$$Z \mapsto \gamma(Z, \Lambda + \nu + \mu) - \gamma(Z, \Lambda + \nu + \mu_j)$$

vanishes identically on $\mathfrak{Z}_0$. Since γ is an algebra homomorphism, whereas $\mathfrak{Z}_0$ generates $\mathfrak{Z}$, it follows that the above factor vanishes identically on $\mathfrak{Z}$. In turn, this implies that $\Lambda + \nu + \mu$ and $\Lambda + \nu + \mu_j$ are $W(\mathfrak{h})$ conjugate. By application of Corollary 11.6 it now follows that $\nu \in \cup\mathcal{H}$. □

Lemma 11.11 *There exists a locally finite collection $\mathcal{H} = \mathcal{H}_{\sigma,\mu}$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes in $\mathfrak{a}^*_{P\mathbb{C}}$ such that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$ and $Z \in \mathfrak{Z}$,*

$$b(Z, \nu)\ p_{\Lambda+\nu+\mu} = [\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi](\Pi(Z, \nu))$$

on $C^{-\infty}(G/Q : \sigma : \nu) \otimes F$.

Proof. The two mentioned maps are continuous linear, hence by density and continuity it suffices to prove the identity on the level of K-finite vectors. Let $\mathcal{H}_1$ be a finite collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes as in Lemma 11.7. Let $\mathcal{H}_2$ be a locally finite collection of such hyperplanes as in Lemma 11.8. We will prove the result with $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. Let $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$. Then by Lemma 11.7 the required identity is valid on $\ker p_{\Lambda+\nu+\mu}$. By Lemma 11.8 the image of $p_{\Lambda+\nu+\mu}$ is isomorphic to $C(G/Q : \sigma : \nu+\mu)_K$ on which $Z \in \mathfrak{Z}$ acts by the scalar $\gamma(Z, \Lambda + \nu + \mu)$. Therefore, the identity is also valid on the image of $p_{\Lambda+\nu+\mu}$. Since $p_{\Lambda+\nu+\mu}$ is a projection (on the level of K-finite vectors), the result follows. □

The induced representation $\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1)$ has infinitesimal character $\Lambda + \nu$, hence it follows from Lemma 11.1 that

$$(Z - \gamma(\Lambda + \nu + \mu))\Pi(Z, \nu) = 0 \quad \text{on} \;\; C^{-\infty}(G/Q : \sigma : \nu) \otimes F.$$

If $Q \in \mathcal{P}(A_P)$ we define the algebra homomorphism

$$I_{Q,\nu} : \mathfrak{Z} \to \mathrm{End}(C(K/K_P : \sigma)_K \otimes F), \;\; Z \mapsto [\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi](Z).$$

Note that $I_{Q,\nu}(Z)$ depends polynomially on ν, for fixed $Z \in \mathfrak{Z}$.

Lemma 11.12 *There exists a polynomial map* $\underline{Z}_\mu : \mathfrak{a}^*_{P\mathbb{C}} \to \mathfrak{Z}$ *and a polynomial function* $q \in P(\mathfrak{a}^*_P)$ *which is a finite product of factors of the form* $\langle \alpha\,,\ \cdot\, \rangle - c$*, with* $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ *and* $c \in \mathbb{C}$*, such that for every* $Q \in \mathcal{P}(A_P)$*,*

$$b(Z, \nu)\, \underline{Z}_\mu(\nu) - q(\nu)\Pi(Z, \nu) \in \ker(I_{Q,\nu})$$

for all $Z \in \mathfrak{Z}$ *and* $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$*.*

Proof. We first assume that $Q = P$ and follow the ideas of [1, proof of Prop. 8.3]. It follows from Lemma 11.11 that for all $Z_1, Z_2 \in \mathfrak{Z}$

$$b(Z_2, \nu)\Pi(Z_1, \nu) - b(Z_1, \nu)\Pi(Z_2, \nu) \in \ker(I_{P,\nu})$$

for ν in an open dense subset of $\mathfrak{a}^*_{P\mathbb{C}}$. By continuity the above identity actually holds for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$.

Let $\mathcal{I}$ be the ideal in the ring $S(\mathfrak{a}_P) \simeq P(\mathfrak{a}^*_P)$ generated by the polynomials $b(Z)$, for $Z \in \mathfrak{Z}$. Let $V_\mathcal{I}$ be the associated common zero set in $\mathfrak{a}^*_{P\mathbb{C}}$. Let $\mathcal{H}$ be a finite collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes in $\mathfrak{a}^*_{P\mathbb{C}}$ as in Corollary 11.10. Then it follows from the mentioned corollary that

$$V_\mathcal{I} \subset \cup\mathcal{H}.$$

We select $\tilde{q} \in P(\mathfrak{a}^*_P)$ a product of linear factors of the form $\langle \alpha\,,\ \cdot\, \rangle - c$, with $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ and $c \in \mathbb{C}$, such that $\tilde{q}$ vanishes on $\cup\mathcal{H}$. Then it follows that $\tilde{q}$ vanishes on $V_\mathcal{I}$. By Hilbert's Nulstellen Satz, there exists a positive integer N such that $q := \tilde{q}^N$ belongs to $\mathcal{I}$. By the Noetherian property, the ideal $\mathcal{I}$ is already generated by finitely many of its elements, say $b(Z_1), \ldots, b(Z_l)$. It follows that there exist $a_j \in S(\mathfrak{a}_P)$ such that, for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$q(\nu) = \sum_{j=1}^{l} a_j(\nu) b(Z_j, \nu).$$

We define

$$\underline{Z}_\mu(\nu) := \sum_{j=1}^{l} a_j(\nu)\, \Pi(Z_j, \nu).$$

Then for all $Z \in \mathfrak{Z}$ and $\nu \in \mathfrak{a}^*_\mathbb{C}$ we have that, modulo $\ker(I_{P,\nu})$,

$$\begin{aligned} q(\nu)\, \Pi(Z, \nu) &= \sum_{j=1}^{l} a_j(\nu) b(Z_j, \nu)\, \Pi(Z, \nu) \\ &\equiv \sum_{j=1}^{l} a_j(\nu) b(Z, \nu)\, \Pi(Z_j, \nu) \\ &= b(Z, \nu)\, \underline{Z}_\mu(\nu). \end{aligned}$$

This establishes the result for $Q = P$. The general result follows from an easy application of the standard intertwining operator $A(\nu) = A(Q, P, \sigma, \nu)$, by noting that $I_{Q,\nu} = A(\nu) \circ I_{P,\nu}$ for generic ν, combined with a density argument. □

Corollary 11.13 *Let $Q \in \mathcal{P}(\mathfrak{a}_P)$. Then there exists a locally finite collection $\mathcal{H}$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes in $\mathfrak{a}^*_{P\mathbb{C}}$ such that, for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$,*

$$q(\nu)p_{\Lambda+\nu+\mu} = [\mathrm{Ind}_Q^G(\sigma \otimes \nu \otimes 1) \otimes \pi](\underline{Z}_\mu(\nu))$$

on $C^{-\infty}(K/K_Q : \sigma) \otimes F$.

Proof. By a simple density argument we see that it suffices to establish the identity on the K-finite level. In that case, let $\mathcal{H}_{\sigma,\mu}$ be the collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes of Lemma 11.11. Then it follows from combining that lemma with Lemma 11.12 that

$$q(\nu)b(Z, \nu)p_{\Lambda+\nu+\mu} = b(Z, \nu)I_{Q,\nu}(\underline{Z}_\mu(\nu))$$

on $C(K/K_Q : \sigma)\otimes F$, for all $Z \in \mathfrak{Z}$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}_{\sigma,\nu}$. Let $\mathcal{H}_2$ be a finite collection of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes as in Corollary 11.10. Then for $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}_2$ there exists $Z \in \mathfrak{Z}$ such that $b(Z, \nu) \neq 0$. The required result now follows with $\mathcal{H} = \mathcal{H}_{\sigma,\mu} \cup \mathcal{H}_2$. □

Remark 11.14 It follows from Corollary 11.13 that if $\underline{Z}'_\mu : \mathfrak{a}^*_{P\mathbb{C}} \to \mathfrak{Z}$ is a second polynomial map as in Lemma 11.12, then for all $Q \in \mathcal{P}(\mathfrak{a}_P)$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we have $\underline{Z}_\mu(\nu) - \underline{Z}'_\mu(\nu) \in \ker(I_{Q,\nu})$.

12 The functional equation

We retain the assumption of the previous section that P is a standard parabolic subgroup of G, that σ is a representation of the discrete series of M_P, that $\mu \in \Lambda^{++}(\mathfrak{a}_P)$, and that (π, F) is the irreducible finite dimensional spherical representation of highest weight μ.

Let $e_K \in F$ be a nonzero K-fixed vector, and $e^{-\mu} \in F^*$ a non-zero lowest weight vector. We define the matrix coefficient map $i_\mu : F \to C^\infty(G)$ by

$$i_\mu(v)(x) = \langle v\,,\, \pi(x)e^{-\mu}\rangle, \qquad (x \in G). \tag{12.1}$$

Then i_μ intertwines π with the left regular representation and is readily seen to define a G-equivariant embedding

$$i_\mu : F \hookrightarrow C^\infty(G/\bar{P} : 1 : \mu + \rho_P). \tag{12.2}$$

In particular, we note that $i_\mu(e_K)(kan) = a^{-\mu}\langle e_K\,,\, e^{-\mu}\rangle$, for $(k, a, n) \in K \times A \times \bar{N}_0$, and see that $i_\mu(e_K)$ is a nowhere vanishing function on G. By renormalizing e_K we may arrange that $\langle e_K\,,\, e^{-\mu}\rangle = i_\mu(e_K)(1) = 1$. Accordingly, we define the map

$$\mathcal{M}_\mu : C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu) \to C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F \tag{12.3}$$

by $f \mapsto i_\mu(e_K)^{-1} f \otimes e_K$. In the compact picture of the induced representations, $\mathcal{M}_\mu$ corresponds to the map

$$C^{-\infty}(K/K_P : \sigma) \to C^{-\infty}(K/K_P : \sigma) \otimes F, \quad f \mapsto f \otimes e_K.$$

We fix a non-zero highest weight vector $e_\mu \in F$. In order to emphasize the feature that it is N_0-fixed, we agree to also write $e_{N_0} = e_\mu$. Likewise, we denote by e^{N_0} a fixed choice of non-zero highest weight vector in the complex linear dual space F^*.

We now define the map

$$\varepsilon^{N_0} := m \circ (I \otimes e^{N_0}) : C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F \to C^{-\infty}(G/\bar{P} : \sigma : \nu), \tag{12.4}$$

where m denotes the natural linear isomorphism from $C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes \mathbb{C}$ onto $C^{-\infty}(G/\bar{P} : \sigma : \nu)$, induced by multiplication. In the compact picture, this becomes a map $\varepsilon^{N_0} : C^{-\infty}(K/K_P : \sigma) \otimes F \to C^{-\infty}(K/K_P : \sigma)$, constant in the variable $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$.

For $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we define the endomorphism $\underline{Z}_{\bar{P},\mu}(\nu)$ of $C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F$ by

$$\underline{Z}_{\bar{P},\mu}(\nu) := [\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes \nu \otimes 1) \otimes \pi](\underline{Z}_\mu(\nu)), \tag{12.5}$$

with $\underline{Z}_\mu : \mathfrak{a}^*_{P\mathbb{C}} \to \mathfrak{Z}$ a polynomial map as in Lemma 11.12. Note that the endomorphism (12.5) is independent of the particular choice of $\underline{Z}_\mu$, in view of Remark 11.14.

Finally, we define the operator

$$D_\mu(\sigma, \nu) : C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu) \to C^{-\infty}(G/\bar{P} : \sigma : \nu)$$

by

$$D_\mu(\sigma, \nu) = \varepsilon^{N_0} \circ \underline{Z}_{\bar{P},\mu}(\nu) \circ \mathcal{M}_\mu, \tag{12.6}$$

Proposition 12.1 *The operator $D_\mu(\sigma, \nu)$, viewed as an endomorphism of the space $C^{-\infty}(K/K_P : \sigma)$ is continuous and depends polynomially on $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$. There exists a constant $d \in \mathbb{N}$ such that the following is valid.*

There exists an $r \in \mathbb{N}$ and for every $s \in \mathbb{N}$ a constant $C > 0$ such that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the endomorphism $D_\mu(\sigma, \nu)$ maps the Banach space $C^{-s}(K/K_P : \sigma)$ continuously to the Banach space $C^{-s-r}(K/K_P : \sigma)$ with operator norm satisfying the estimate

$$\|D_\mu(\sigma, \nu)\|_{\mathrm{op}} \leq C(1 + |\nu|)^d.$$

For the proof we need the following lemma.

Lemma 12.2 *Let $u \in U(\mathfrak{g})$ be an element of degree at most d. Then for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the endomorphism $\pi^\infty_{\bar{P},\sigma,\nu}(u)$ of $C^\infty(K/K_P : \sigma)$ is continuous and support preserving. Furthermore, the following assertions are valid.*

(a) *The function $\nu \mapsto \pi^\infty_{\bar{P},\sigma,\nu}(u)$ is polynomial* $\mathrm{End}(C^\infty(K/K_P : \sigma))$*-valued of degree at most d.*

(b) *There exists a constant $t \in \mathbb{N}$ and for every $s \in \mathbb{N}$ a constant $C > 0$ such that for all $f \in C^\infty(K/K_P : \sigma)$,*

$$\|\pi^\infty_{\bar{P},\sigma,\nu}(u) f\|_s \le C(1 + |\nu|)^d \|f\|_{s+t}.$$

Proof. See [4, Lemma 2.1]. □

Corollary 12.3 *Let $u \in U(\mathfrak{g})$ be an element of degree at most d. Then for every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the endomorphism $\pi^{-\infty}_{\bar{P},\sigma,\nu}(u)$ of $C^{-\infty}(K/K_P : \sigma)$ is continuous and support preserving. Furthermore,*

(a) *The map $\nu \mapsto \pi^{-\infty}_{\bar{P},\sigma,\nu}(u)$ is polynomial* $\mathrm{End}(C^{-\infty}(K/K_P : \sigma))$*-valued of degree at most d.*

(b) *There exists a constant $t \in \mathbb{N}$ and for every $s \in \mathbb{N}$ a constant $C > 0$ such that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the endomorphism $\pi^{-\infty}_{\bar{P},\sigma,\nu}(u)$ maps $C^{-s}(K/K_P : \sigma)$ continuous linearly into $C^{-s-t}(K/K_P : \sigma)$ with operator norm*

$$\|\pi^{-\infty}_{\bar{P},\sigma,\nu}(u)\|_{\mathrm{op}} \le C(1 + |\nu|)^d.$$

Proof. This follows from Lemma 12.2 by taking adjoints. □

Proof of Proposition 12.1. We start by observing that $\underline{Z}_\mu : \mathfrak{a}^*_{P\mathbb{C}} \to \mathfrak{Z}$ is polynomial in the variable ν. For $Z \in \mathfrak{Z}$ we define

$$D(Z)(\nu) := \varepsilon^{N_0} \circ [\pi^{-\infty}_{\bar{P},\sigma,\nu} \otimes \pi_\mu](Z) \circ \mathcal{M}_\mu.$$

Then it clearly suffices to prove the assertions of the proposition for $D(Z)$ in place of $D_\mu(\nu)$. □

In terms of the above maps we can now present the functional equation for the Whittaker vectors. We will stay close to the notation of [1, Thm. 9.3] in order to emphasize the strong analogy. Recall the definition of $D_\mu(\sigma, \nu)$ in (12.6).

Theorem 12.4 (Functional equation) *Let $\mu \in \Lambda^{++}(\mathfrak{a}_P)$. Then there exists a rational* $\mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$*-valued function $\nu \mapsto R_\mu(\sigma, \nu)$ on $\mathfrak{a}^*_{P\mathbb{C}}$ such that*

$$j(\bar{P}, \sigma, \nu) = D_\mu(\sigma, \nu) \circ j(\bar{P}, \sigma, \nu + \mu) \circ R_\mu(\sigma, \nu). \tag{12.7}$$

Remark 12.5 In the next section we will show that $\nu \mapsto p(\nu) R_\mu(\sigma, \nu)$ is polynomial for a suitable polynomial function $p : \mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$ which can be written as a product of linear factors of the form $\nu \mapsto \langle \nu\, , \alpha \rangle + c$, $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ and $c \in \mathbb{C}$.

We will prove Theorem 12.4 in a sequence of lemmas occupying the rest of this section and the next. A key ingredient in our proof is the map

$$\Phi_\mu(\nu) : C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F \to C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu), \quad f \otimes v \mapsto i_\mu(v) f,$$

with i_μ as in (12.2), which is readily verified to be G-equivariant.

Definition 12.6 We will say that an assertion depending on a parameter $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ holds for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic ν, if there exists a locally finite collection $\mathcal{H}$ of $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes in $\mathfrak{a}_{P\mathbb{C}}^*$ such that the assertion is valid for all $\nu \in \mathfrak{a}_{P\mathbb{C}}^* \setminus \cup\mathcal{H}$.

Lemma 12.7 *For $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$,*

$$\Phi_\mu(\nu) \circ p_{\Lambda+\nu+\mu} = \Phi_\mu(\nu). \tag{12.8}$$

Proof. By equivariance of Φ_μ, we have

$$\Phi_\mu(\nu) \circ p_{\Lambda+\nu+\mu} = p_{\Lambda+\nu+\mu} \circ \Phi_\mu(\nu).$$

The map on the right of this equation equals $\Phi_\mu(\nu)$, since $\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes (\nu + \mu) \otimes 1)$ has infinitesimal character $\Lambda + \nu + \mu$. □

The following identity, for $\nu \in \mathfrak{a}_\mathbb{C}^*$, is a straightforward consequence of the definitions,

$$\Phi_\mu(\nu) \circ \mathcal{M}_\mu = I \qquad \text{on} \quad C^{-\infty}(G/\bar{P} : \sigma : \nu). \tag{12.9}$$

The map $\mathcal{M}_\mu$ is not equivariant. However, for the map

$$\Psi_\mu(\nu) := q(\nu)^{-1}\, \underline{Z}_{\bar{P},\mu}(\nu) \circ \mathcal{M}_\mu \tag{12.10}$$

we have the following result.

Lemma 12.8 *The map $\nu \mapsto q(\nu)\Psi_\mu(\nu)$ is polynomial as a map with values in the space of equivariant continuous linear operators from $(C^{-\infty}(K/K_P : \sigma), \pi_{\bar{P},\sigma,\nu})$ to $(C^{-\infty}(K/K_P : \sigma) \otimes F, \pi_{\bar{P},\sigma,\nu} \otimes \pi)$. Furthermore, for all $\nu \in \mathfrak{a}_{P\mathbb{C}}^* \setminus q^{-1}(0)$,*

$$\Phi_\mu(\nu) \circ \Psi_\mu(\nu) = I, \tag{12.11}$$

$$\Psi_\mu(\nu) \circ \Phi_\mu(\nu) = q(\nu)^{-1}\, \underline{Z}_{\bar{P},\mu}(\nu). \tag{12.12}$$

Proof. We first observe that from the definitions it follows that $\nu \mapsto q(\lambda)\Psi_\mu(\nu)$ is polynomial as a map into the space of continuous linear operators $C^{-\infty}(K/K_P : \sigma) \to C^{-\infty}(K/K_P : \sigma) \otimes F$. The equivariance of the operators in the image of that polynomial map will be addressed in a moment.

In view of Corollary 11.13 we have

$$q(\nu)^{-1}\, \underline{Z}_{\bar{P},\mu}(\nu) = p_{\Lambda+\nu+\mu} \quad \text{on} \quad C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F, \tag{12.13}$$

for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}_\mathbb{C}^*$. From (12.10) it now follows that

$$\Phi_\mu(\nu) \circ \Psi_\mu(\nu) = \Phi_\mu(\nu) \circ \mathrm{pr}_{\Lambda+\nu+\mu} \circ \mathcal{M}_\mu.$$

Taking (12.8) into account we infer the validity of (12.11), for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. By analytic continuation the identity (12.11) follows for all $\nu \in \mathfrak{a}_{P\mathbb{C}}^* \setminus q^{-1}(0)$.

It follows from (12.9) that $\Phi_\mu(\nu)$ is surjective for all ν. Furthermore, for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic ν, the image $\mathrm{im}(p_{\Lambda+\nu+\mu}) \subset C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F$ is a closed G-invariant subspace, which satisfies

$$\mathrm{im}(p_{\Lambda+\nu+\mu})_K \simeq C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu)_K \tag{12.14}$$

as $(\mathfrak{g}, K)$-modules, in view of Lemma 11.8. From (12.8) we see, still for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic ν, that $\Phi_\mu(\nu)$ is a G-equivariant surjective continuous linear map from the space $\mathrm{im}(p_{\Lambda+\sigma+\nu})$ onto $C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu)$, which by (12.14) is injective, hence bijective.

It follows from (12.13) and (12.10) that $\Psi_\mu(\nu)$ maps into $\mathrm{im}(p_{\Lambda+\nu+\mu})$, for generic ν, so the equivariance of $\Psi_\mu(\nu)$ follows from (12.11) and the equivariance of $\Phi_\mu(\nu)$, for generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$. By analytic continuation the equivariance of $q(\nu)\Psi_\mu(\nu)$ follows for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$.

We finally turn to proving the identity (12.12). By analytic continuation, it suffices to establish that identity for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$. Since the maps on both sides of (12.12) map into the space $\mathrm{im}(p_{\Lambda+\nu+\mu})$, on which $\Phi_\mu(\nu)$ restricts to an injective map, for generic ν, it suffices to check that

$$\Phi_\mu(\nu) \circ \Psi_\mu(\nu) \circ \Phi_\mu(\nu) = \Phi_\mu(\nu) \circ q(\nu)^{-1} \underline{Z}_{\bar{P},\mu}(\nu).$$

The expression on the left simplifies to $\Phi_\mu(\nu)$, in view of (12.11). The expression on the right equals $\Phi_\mu(\nu) \circ p_{\Lambda+\nu+\mu}$ by (12.13), and $\Phi_\mu(\nu)$ by (12.8). □

Recall from (12.1) that $e^{-\mu} \in F^*$ is a non-zero lowest weight vector (of $\mathfrak{a}$-weight $-\mu$) and put

$$m_\mu := i_\mu(e_{N_0})(1) = \langle e_{N_0} , e^{-\mu} \rangle.$$

From $m_\mu = 0$ it would follow that $i_\mu(e_{N_0})$ vanishes on $N_0\bar{P}$ hence on G, contradicting the injectivity of i_μ. Therefore, m_μ is a nonzero complex number.

Lemma 12.9 *For every $\eta \in H_\sigma^{-\infty,\chi}$ and all $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ with $\langle \mathrm{Re}\, \nu , \alpha \rangle > 0$, ($\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a})$),*

$$\Phi_\mu(\nu)[j(\bar{P}, \sigma, \nu, \eta) \otimes e_{N_0}] = j(\bar{P}, \sigma, \nu + \mu)(m_\mu \eta). \tag{12.15}$$

Proof. It is readily verified that the expression on the left belongs to $C^{-\infty}(G/\bar{P} : \sigma : \nu + \mu)^\chi$ hence is of the form $j(\bar{P}, \sigma, \nu + \mu)(\eta')$ for some $\eta' \in H_\sigma^{-\infty,\chi}$. On the other hand, on $N_P\bar{P}$ the expression on the left hand side is the continuous $H_\sigma^{-\infty}$-valued function whose value at the unit element $e = 1$ is

$$\mathrm{ev}_e j(\bar{P}, \sigma, \nu, \eta)\, i_\mu(e_{N_0})(1) = m_\mu \eta.$$

It follows that $\eta' = m_\mu \eta$. □

Completion of the proof of Theorem 12.4. Applying the operator $q(\nu)\varepsilon^{N_0} \circ \Psi_\mu(\nu)$ to (12.15) and taking into account Lemma 12.8 we obtain

$$\varepsilon^{N_0} \circ \underline{Z}_{\bar{P},\mu}(\nu)[j(\bar{P},\sigma,\nu)\eta \otimes e_{N_0}] = q(\nu)\varepsilon^{N_0} \circ \Psi_\mu(\nu) j(\bar{P},\sigma,\nu+\mu)(m_\mu \eta).$$

From (12.6) and (12.10) we see that $q(\nu)\varepsilon^{N_0} \circ \Psi_\mu(\nu) = D_\mu(\sigma,\nu)$. Hence,

$$\varepsilon^{N_0} \circ \underline{Z}_{\bar{P},\mu}(\nu)[j(\bar{P},\sigma,\nu)\eta \otimes e_{N_0}] = D_\mu(\sigma,\nu) j(\bar{P},\sigma,\nu+\mu)(m_\mu \eta),$$

for all $\eta \in H^{-\infty}_{\sigma,\chi_P}$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with $\mathrm{Re}\,\nu$ P-dominant. The functional equation now follows with $R_\mu(\sigma,\nu) = M_\mu(\sigma,\nu)^{-1} m_\mu$ by application of Proposition 12.10 below. □

Proposition 12.10 *There exists a unique polynomial function $M_\mu = M_\mu(\sigma,\cdot) : \mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$ such that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P,0)$ we have*

$$\varepsilon^{N_0} \circ \underline{Z}_{\bar{P},\mu}(\nu)[j(\bar{P},\sigma,\nu)\eta \otimes e_{N_0}] = j(\bar{P},\sigma,\nu)(M_\mu(\nu)\eta).$$

The polynomial function $\det \circ M_\mu : \mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$ *is not identically zero.*

To prepare for the proof, we introduce the space $C^\infty_{c,N_P}(G/\bar{P} : \sigma : \nu)$ of functions $f \in C^\infty(G/\bar{P} : \sigma : \nu)$ whose support $\mathrm{supp} f$ has compact intersection with N_P. Restriction to N_P induces a linear isomorphism r_ν from $C^\infty_{c,N_P}(G/\bar{P} : \sigma : \nu)$ onto $C^\infty_c(N_P, H^\infty_\sigma)$. We note that $C^\infty_{c,N_P}(G/\bar{P} : \sigma : \nu)$ is invariant under the left regular action by $U(\mathfrak{g})$ and denote by ${}^n\pi_{\bar{P},\sigma,\nu}$ the unique representation of $U(\mathfrak{g})$ in $C^\infty_c(N_P, H^\infty_\sigma)$ such that r_ν intertwines the left regular representation of $U(\mathfrak{g})$ with ${}^n\pi_{\bar{P},\sigma,\nu}$.

Our first step in the proof of Proposition 12.10 is the following observation.

Lemma 12.11 *Let $\varphi \in C^\infty_c(N_P, H^\infty_\sigma)$. Then for each element $u \in U(\mathfrak{g})$ the function $\mathfrak{a}^*_{P\mathbb{C}} \to C^\infty_c(N_P, H^\infty_\sigma)$, $\nu \mapsto {}^n\pi_{\bar{P},\sigma,\nu}(u)\varphi$ is polynomial, i.e., it belongs to the space $P_k(\mathfrak{a}^*_{P\mathbb{C}}) \otimes C^\infty_c(N_P, H^\infty_\sigma)$ for some $k \in \mathbb{N}$.*

Proof. It suffices to prove this assertion for $u = X \in \mathfrak{g}$, with $k = 1$. Let Ω be a bounded open neighborhood of $\mathrm{supp}\,\varphi$ in N_P. Let $X \in \mathrm{Lie}(\bar{P})$. Then there exists an open interval $I \ni 0$ in $\mathbb{R}$ such that for every $t \in I$ and $n \in \Omega$ we have $\exp(-tX)n \in N_P\bar{P}$. Consequently, there exist smooth functions $U : I \times \Omega \to N_P$ and $V : I \times \Omega \to \bar{P}$ such that

$$\exp(-tX)n = nU(t,n)V(t,n), \qquad (n \in \Omega,\ t \in I).$$

We note that $U(0,n) = e = V(0,n)$ for all $n \in N_P$. Let $\varphi_\nu \in C^\infty(G/\bar{P},\sigma,\nu)$ be defined by $\mathrm{supp}\varphi_\nu \subset \Omega\bar{P}$ and

$$\varphi_\nu(nma\bar{n}) := a^{-\nu+\rho_P}\sigma(m)^{-1}\varphi(n), \qquad ((n,m,a,\bar{n}) \in N_P \times M_P \times A_P \times \bar{N}_P).$$

Then $\varphi_\nu|_{N_P} = \varphi$, so that ${}^n\pi_{\bar{P},\sigma,\nu}(X)\varphi(n) = L_X\varphi_\nu(n)$. For $n \in N_P$ and $t \in I$ we have

$$\begin{aligned} \varphi_\nu(\exp(-tX)n) &= \varphi_\nu(nU(t,n)V(t,n)) \\ &= [\sigma \otimes (-\nu + \rho_P) \otimes 1](V(t,n))^{-1}\varphi(nU(t,n)). \end{aligned}$$

Differentiating this expression in t at $t = 0$ we find, for $n \in \Omega$,

$$\begin{array}{rcl} {}^n\pi_{\bar{P},\sigma,\nu}(X)\varphi(n) & = & L_X(\varphi_\nu)(n) \\ & = & -[\sigma \otimes (-\nu + \rho_P) \otimes 1](\partial_t V(0, n))\varphi(n) + R_{\partial_t U(0,n)}\varphi(n). \end{array}$$

We thus see that $\nu \mapsto {}^n\pi_{\bar{P},\sigma,\nu}(X)\varphi$ belongs to $P_1(\mathfrak{a}^*_{P\mathbb{C}}) \otimes C^\infty_c(\Omega, H^\infty_\sigma)$. □

The next step in the proof of Proposition 12.10 is formulated in the following lemma.

Lemma 12.12 *For $Z \in \mathfrak{Z}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ there exists a unique endomorphism $m(Z, \nu) \in \mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$ such that, for all $\eta \in H^{-\infty}_{\sigma,\chi_P}$,*

$$\varepsilon^{N_0}[\pi_{\bar{P},\sigma,\nu} \otimes \pi_\mu](Z)[j(\bar{P}, \sigma, \nu, \eta) \otimes e_{N_0}] = j(\bar{P}, \sigma, \nu, m(Z, \nu)(\eta)). \tag{12.16}$$

The map $\nu \mapsto m(Z, \nu)$ is polynomial on $\mathfrak{a}^*_{P\mathbb{C}}$ with values in $\mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$.

Proof. For ν as stated, the expression on the left-hand side belongs to the space $C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$, hence by Proposition 8.15 (b) can be written as the expression on the right-hand side, with a uniquely determined $m(Z, \nu)(\eta) \in H^{-\infty}_{\sigma,\chi_P}$. By uniqueness, $m(Z, \nu)(\eta)$ depends linearly on η, hence, $m(Z, \nu) \in \mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$.

It remains to be shown that $m(Z, \cdot\,)$ is polynomial. Let $\eta \in H^{-\infty}_{\sigma,\chi_P}$ and let $v \in H^\infty_\sigma$ be an arbitrary element. Then it suffices to show that $\nu \mapsto \langle m(Z, \nu)\eta\, ,\, v\rangle_\sigma$ is polynomial in the indicated range. See (1.11) for the definition of the pairing used. For this we recall that for ν in that range, $j(\bar{P}, \sigma, \nu)(\eta)$ restricted to N_P is the continuous function $N_P \to H^{-\infty}_{\sigma,\chi_P}$ given by $n \mapsto \chi(n)^{-1}\eta$. Fix a function $\psi \in C^\infty_c(N_0)$ such that

$$\int_{N_P} \chi(n)\psi(n)\, dn = 1.$$

We define $f \in C^\infty_c(N_P, H^\infty_\sigma)$ by $f(n) = \psi(n)v$ for $n \in N_P$. Furthermore, we denote by $f_{-\bar{\nu}}$ the extension of f to an element of $C^\infty(G/\bar{P} : \sigma : -\bar{\nu})$ with support contained in $N_P\bar{P}$. Then, for $\eta' \in H^{-\infty}_{\sigma,\chi_P}$,

$$\langle j(\bar{P}, \sigma, \nu)(\eta')\, ,\, f_{-\bar{\nu}}\rangle = \int_{N_P} \chi(n)^{-1}\langle \eta'\, ,\, \psi(n)v\rangle_\sigma\, dn = \langle \eta'\, ,\, v\rangle_\sigma.$$

Substituting $\eta' = m(Z, \nu)\eta$ and combining the result with (12.16) we now find that

$$\langle v\, ,\, m(Z, \nu)\eta\rangle_\sigma = \langle \varepsilon^{N_0}[\pi_{\bar{P},\sigma,\nu} \otimes \pi_\mu](Z)[j(\bar{P}, \sigma, \nu, \eta) \otimes e_{N_0}]\, ,\, f_{-\bar{\nu}}\rangle.$$

By the Leibniz rule for tensors, the expression on the right-hand side is a sum of terms of the form

$$\langle \varepsilon^{N_0}\pi_{\bar{P},\sigma,\nu}(U)j(\bar{P}, \sigma, \nu)(\eta) \otimes \pi_\mu(V)e_{N_0}\, ,\, f_{-\bar{\nu}}\rangle,$$

with (U, V) ranging over a subset of $U(\mathfrak{g}) \times U(\mathfrak{g})$ independent of ν. The above term may be rewritten as

$$\varepsilon^{N_0}(\pi_\mu(V)e_{N_0}) \cdot \langle \pi_{\bar{P},\sigma,\nu}(U) j(\bar{P}, \sigma, \nu)(\eta)\,,\ f_{-\bar{\nu}}\rangle.$$

Thus, it suffices to show that the latter expression is polynomial in ν. We now observe that

$$\begin{aligned}
\langle \pi_{\bar{P},\sigma,\nu}(U) j(\bar{P}, \sigma, \nu)(\eta)\,,\ f_{-\bar{\nu}}\rangle &= \langle j(\bar{P}, \sigma, \nu)(\eta)\,,\ \pi_{\bar{P},\sigma,-\bar{\nu}}(U^*) f_{-\bar{\nu}}\rangle \\
&= \langle j(\bar{P}, \sigma, \nu)(\eta)\,,\ ({}^n\pi_{\bar{P},\sigma,-\bar{\nu}}(U^*) f)_{-\bar{\nu}}\rangle \\
&= \int_{N_P} \chi(n)\langle \eta\,,\ {}^n\pi_{\bar{P},\sigma,-\bar{\nu}}(U^*) f(n)\rangle_\sigma \, dn.
\end{aligned}$$

By virtue of Lemma 12.11 the latter integral depends polynomially on ν. ☐

Proof of Proposition 12.10. We note that the map $\mathfrak{a}^*_{P\mathbb{C}} \to \mathfrak{Z}$, $\nu \mapsto \underline{Z}_\mu(\nu)$ is polynomial. Moreover,

$$\underline{Z}_{\bar{P},\mu}(\nu) = [\pi_{\bar{P},\sigma,\nu} \otimes \pi_\mu](\underline{Z}_\mu(\nu)).$$

By application of Lemma 12.12 the first assertion now follows with

$$M_\mu(\nu) = m(\underline{Z}_\mu(\nu), \nu).$$

For completing the proof of Proposition 12.10 it thus remains to establish the lemma below. ☐

Lemma 12.13 *The polynomial function $\mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$, $\nu \mapsto \det M_\mu(\nu)$ is not identically zero.*

Proof. It suffices to show that $\nu \mapsto \det M_\mu(\nu)$ is non-zero for a suitable ν. For this it suffices to show that there exists a $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ such that $M_\mu(\nu)$ is an injective endomorphism of $H^{-\infty}_{\sigma,\chi_P}$. Taking the characterization of M_μ in Proposition 12.10 into account and using that

$$\underline{Z}_{\bar{P},\mu}(\nu) = q(\nu) p_{\Lambda+\nu+\mu} \quad \text{on} \quad C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F \tag{12.17}$$

we infer that it suffices to show that for generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the map $j \mapsto \varepsilon^{N_0} p_{\Lambda+\nu+\mu}(j \otimes e_{N_0})$ is an injective endomorphism from $C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$ to itself. The latter statement follows from Lemma 12.14 below, which will be proven in the next section. ☐

Lemma 12.14 *Let $Q \in \{P, \bar{P}\}$. Then for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the map*

$$j \mapsto \varepsilon^{N_0} \circ p_{\Lambda+\nu+\mu}(j \otimes e_{N_0}) \tag{12.18}$$

is an injective linear endomorphism of $C^{-\infty}(G/Q : \sigma : \nu)_\chi$.

13 Proof of Lemma 12.14

We retain the notation of the previous section. The following lemma serves as a first step in the proof of Lemma 12.14.

Lemma 13.1 *Let $Q \in \{P, \bar{P}\}$. For $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$, the map*

$$j \mapsto p_{\Lambda+\nu+\mu}[j \otimes e_{N_0}] : \; C^{-\infty}(G/Q : \sigma : \nu)_\chi \to C^{-\infty}(G/Q : \sigma : \nu) \otimes F$$

is injective with values in $(C^{-\infty}(G/Q : \sigma : \nu) \otimes F)_\chi$.

Proof. That the given map attains values in $(C^{-\infty}(G/Q : \sigma : \nu) \otimes F)_\chi$ is a straightforward consequence of the G-equivariance of $p_{\lambda+\nu+\mu}$. We therefore focus on the asserted injectivity.

There exists a locally finite union $\mathcal{H}$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes such that for $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$ the standard intertwining operator

$$A(\nu) = A(\bar{Q}, Q, \sigma, \nu) : C^{-\infty}(G/Q : \sigma : \nu) \to C^{-\infty}(G/\bar{Q} : \sigma : \nu)$$

is bijective and maps the subspace $C^{-\infty}(G/Q : \sigma : \nu)_\chi$ bijectively onto $C^{-\infty}(G/\bar{Q} : \sigma : \nu)_\chi$. Furthermore, by the intertwining property of $A(\nu)$ we have, for such ν, that

$$(A(\nu) \otimes I) \circ p_{\Lambda+\nu+\mu}[j \otimes e_{N_0}] = p_{\Lambda+\nu+\mu}[A(\nu) j \otimes e_{N_0}].$$

We thus see that it suffices to establish the assertion for $Q = \bar{P}$. Then by Lemma 12.7 we have

$$\Phi_\mu(\nu) p_{\Lambda+\nu+\mu}(j \otimes e_{N_0}) = \Phi_\mu(\nu)(j \otimes e_{N_0}) = i_\nu(e_{N_0}) j,$$

for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and $j \in C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$. Since the function $i_\nu(e_{N_0})$ is non-zero on $N_0\bar{P}$, it follows that the expression on the right of the above equality is zero if and only if $j|_{N_0\bar{P}} = 0$. By Corollary 8.3 this in turn is equivalent to $j = 0$. The asserted injectivity follows. $\square$

To prepare for the proof of Lemma 12.14, we need to introduce certain particular subspaces of generalized vectors of induced representations. Let Q be any parabolic subgroup of G containing A. We consider a continuous Hilbert representation (ξ, H_ξ) of Q.

We denote by $W_Q(\mathfrak{a})$ the centralizer of $\mathfrak{a}_Q$ in $W(\mathfrak{a})$. From the Bruhat decompositions for G and M_{1Q} it follows that the map $v \mapsto N_0 v Q$, $N_K(\mathfrak{a}) \to N_0\backslash G/Q$ induces a bijection from $W(\mathfrak{a})/W_Q(\mathfrak{a})$ onto the double coset space $N_0\backslash G/Q$. Precisely one of these cosets is open in G; it will be denoted by O_Q. In fact, for $v \in W(\mathfrak{a})$ we have

$$N_0 v Q = O_Q \iff v\bar{Q}v^{-1} \supset P_0.$$

Definition 13.2

$$C^{-\infty}(G/Q : \xi)_{N_0} \subset C^{-\infty}(G/Q : \xi) \tag{13.1}$$

is defined to be the subspace of elements $u \in C^{-\infty}(G/Q : \xi)$ such that

(a) u is N_0-finite from the left;

(b) $u|_{O_Q}$ is a continuous function $O_Q \to H_\xi^{-\infty}$

Assertion (b) means that there exists a continuous function $\tilde{u} : O_Q \to H_\xi^{-\infty}$ such that $\tilde{u}(nvq) = {}^\backprime\xi^{-\infty}(q)^{-1}\tilde{u}(nv)$ $(q \in Q, n \in N_0)$ and such that for all $\psi \in C^\infty(G/Q : \xi^*)$ with $\operatorname{supp}\psi \subset O_Q$, the following identity is valid:

$$\langle u\,,\,\psi\rangle = \langle\tilde{u}\,,\,\psi\rangle_K := \int_{K/K_Q} \langle\tilde{u}\,,\,\psi\rangle_\xi(k)\;d\dot{k}. \tag{13.2}$$

Note that the integrand is a continuous complex valued function with support contained in $K \cap O_Q$. Note also that $\tilde{u}$ is uniquely determined.

If $v \in O_Q$, the evaluation map

$$\underline{\mathrm{ev}}_v : u \mapsto \tilde{u}(v), \qquad C^{-\infty}(G/Q : \xi)_{N_0} \to \mathcal{H}_\xi^{-\infty}$$

is well defined and linear.

In the special setting $\xi = \sigma \otimes \nu$ with σ a unitary representation of M_Q and $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$, the space on the left in (13.1) is also denoted by $C^\infty(G/Q : \sigma : \nu)_{N_0}$.

The following observation will allow us to connect to the case that $\bar{Q}$ is standard. Let $v \in N_K(\mathfrak{a})$ be such that $v\bar{Q}v^{-1}$ is standard. Then $O_Q = N_0 vQ$. We write $v\xi$ for the representation of vQv^{-1} in H_ξ given by $v\xi(q) = \xi(v^{-1}qv)$. Then the right regular action by v induces a G-equivariant topological linear isomorphism

$$R_v : C^{-\infty}(G/Q : \xi) \ \xrightarrow{\simeq}\ C^{-\infty}(G/vQv^{-1} : v\xi). \tag{13.3}$$

If f belongs to the subspace $C^\infty(G/Q : \xi)$ then $R_v f$ is given by $x \mapsto f(xv)$ and belongs to $C^\infty(G/vQv^{-1} : v\xi)$.

Lemma 13.3 *The isomorphism (13.3) restricts to a linear isomorphism*

$$\underline{R}_v : C^{-\infty}(G/Q : \xi)_{N_0} \ \xrightarrow{\simeq}\ C^{-\infty}(G/vQv^{-1} : v\xi)_{N_0}.$$

Furthermore, $\underline{\mathrm{ev}}_e \circ R_v = \underline{\mathrm{ev}}_v$.

Proof. This is a straightforward consequence of the definitions. □

Lemma 13.4 *Let $Q \in \mathcal{P}(A)$, σ a unitary representation of M_Q and $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$. Then*

(a) $C^{-\infty}(G/Q : \sigma : \nu)_\chi \subset C^{-\infty}(G/Q : \sigma : \nu)_{N_0}$.

(b) *If $\bar{Q}$ is standard, then the evaluation map* ev_e *defined on the space on the left is the restriction of* $\underline{\mathrm{ev}}_e$ *defined on the space on the right.*

(c) *Let $v \in N_K(\mathfrak{a})$ be such that $O_Q = N_0 v Q$. Then the evaluation map $\underline{\mathrm{ev}}_v$ restricts to a linear map*

$$\mathrm{ev}_v : C^{-\infty}(G/Q : \sigma : \nu)_\chi \to (H_{v\sigma}^{-\infty})_{\chi|_{N_0 \cap v M_Q v^{-1}}}. \tag{13.4}$$

If χ is regular, the map (13.4) is injective.

Proof. We will first prove (a) - (c) under the assumption that $Q = \bar{P}$ with P standard. Then assertions (a) and (b) follow by application of Theorem 8.6, Equation (13.1) with $\xi = \sigma \otimes \nu$ and the definitions of ev_e and $\underline{\mathrm{ev}}_e$. Assertion (c) follows from Corollary 8.11.

Now assume that Q is general and fix $v \in N_K(\mathfrak{a})$ as in (c). Then $vQv^{-1} = \bar{P}$, with P standard. We consider the isomorphism R_v of (13.3) for $\xi := \sigma \otimes \nu$. Note that $v\xi = v\sigma \otimes v\nu$. By Lemma 13.3, R_v maps $C^{-\infty}(G/Q : \sigma : \nu)_{N_0}$ onto the space $C^{-\infty}(G/vQv^{-1} : v\sigma : v\nu)_{N_0}$ and the evaluation maps in v and e respectively are related by $\underline{e}v_e \circ R_v = \underline{e}v_v$.

By G-equivariance it also follows that R_v restricts to a linear isomorphism

$$R_v : C^{-\infty}(G/Q : \sigma : \nu)_\chi \xrightarrow{\simeq} C^{-\infty}(G/vQv^{-1} : v\sigma : v\nu)_\chi.$$

As the space on the right is contained in $C^{-\infty}(G/vQv^{-1} : v\sigma : v\nu)_{N_0}$ by the first part of the proof, it follows that the space on the left is contained in $C^{-\infty}(G/Q : \sigma : \nu)_{N_0}$ and we have the following commutative diagram with evaluation maps:

$$\begin{array}{ccc} C^{-\infty}(G/Q : \sigma : \nu)_\chi & \xrightarrow{R_v} & C^{-\infty}(G/vQv^{-1} : v\sigma : v\nu)_\chi \\ \mathrm{ev}_v \downarrow & & \downarrow \mathrm{ev}_e \\ (H_{v\sigma}^{-\infty})_{\chi|_{N_0 \cap v M_Q v^{-1}}} & \xrightarrow{I} & (H_{v\sigma}^{-\infty})_{\chi|_{N_0 \cap v M_Q v^{-1}}} \end{array}$$

If χ is regular, then ev_e is injective, and the injectivity of ev_v follows. □

We return to the setting that $Q \in \mathcal{P}(A)$ and that (ξ, H_ξ) is a continuous Hilbert representation of Q. The representation ξ^* of Q in H_ξ is defined as in Remark 1.3.

Proposition 13.5 *The subspace $C^{-\infty}(G/Q : \xi)_{N_0}$ of $C^{-\infty}(G/Q : \xi)$ is invariant under the left action by $\mathfrak{g}$.*

Proof. We fix $v \in N_K(\mathfrak{a})$ such that $vQv^{-1} = \bar{P}$, with P-standard. By Lemma 13.3 and since R_v of (13.3) is $\mathfrak{g}$-equivariant, it suffices to establish the assertion of the proposition with vQv^{-1} in place of Q. In other words, without loss of generality we may and will assume from the start that $Q = \bar{P}$, with P standard.

We recall that the left action by an element $X \in \mathfrak{g}$ on $u \in \mathcal{F} := C^{-\infty}(G/\bar{P} : \xi)$ is defined by

$$\langle L_X u \,,\, \psi \rangle = \langle u \,,\, L_{X^\vee} \psi \rangle,$$

for all $\psi \in C^\infty(G/Q : \xi^*)$.

It is sufficient to show that for $u \in \mathcal{F}_0 := \mathcal{F}_{N_0}$ and $X \in \mathfrak{g}$ the element $L_X u \in \mathcal{F}$ restricts to a continuous $H_\xi^{-\infty}$-valued function on $N_0\bar{P}$. Indeed the N_0-finiteness is obvious, since $L_n L_X = L_{\mathrm{Ad}(n)X} L_n$ for all $n \in N_0$. For the first statement we need a suitable interpretation of L_X on the space $\mathcal{F}_{N_0}$.

Let $u \in \mathcal{F}_0$. The span E of the left N_0-translates of u is a finite dimensional subspace of $\mathcal{F}_0$. The restriction of $L|_{N_0}$ to E is a finite dimensional representation of N_0, which we denote by ω. As ω is the restriction of the continuous representation $L^{-\infty}|_{N_0}$ to E, it is continuous. By finite dimensionality it follows that (ω, E) is smooth. We claim that for $X \in \mathfrak{n}_0$, we have $L_X = \omega(X)$ on E. To see this, note that for $\psi \in C^\infty(G/\bar{P} : \xi^*)$,

$$\begin{aligned} \langle L_X u\,,\,\psi\rangle &= \langle u\,,\,-L_X\psi\rangle = \frac{d}{dt}\Big|_{t=0} \langle u\,,\,L^{-1}_{\exp tX}\psi\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle L_{\exp tX} u\,,\,\psi\rangle = \frac{d}{dt}\Big|_{t=0} \langle \omega(\exp tX)u\,,\,\psi\rangle \\ &= \langle \omega(X)u\,,\,\psi\rangle. \end{aligned}$$

It follows from this that $L_X u \in \mathcal{F}_0$, hence $L_X u$ is continuous on $N_0\bar{P}$, for all $u \in \mathcal{F}_0$ and $X \in \mathfrak{n}_0$.

We now fix a general element $X \in \mathfrak{g}$ and will establish the continuity asssertion for $L_X u$. Since $N_0\bar{P} = N_P\bar{P}$, the assertion of continuity of $u \in \mathcal{F}$ on $N_0 P$ means that there exists a unique continuous function $\tilde{u} : N_P\bar{P} \to H_\xi^{-\infty}$ such that $\tilde{u}(x\bar{p}) = {}^{`}\xi(\bar{p})^{-1}\tilde{u}(x)$ $(x \in N_P\bar{P}, \bar{p} \in \bar{P})$ and such that (13.2) is valid. By the usual transformation of variables corresponding to the open embedding $N_P \to G/\bar{P} \simeq K/K_P$ that equation is equivalent to

$$\langle u\,,\,\psi\rangle = \langle \tilde{u}\,,\,\psi\rangle_{N_P} := \int_{N_P} \langle \tilde{u}\,,\,\psi\rangle_\xi(n)\; dn.$$

Since $\mathfrak{g} = \mathfrak{n}_P \oplus \mathrm{Lie}(\bar{P})$ it follows that for every $n \in N_P$ we may write

$$\mathrm{Ad}(n)^{-1}X = \mathrm{Ad}(n)^{-1}Y(n) + Z(n)$$

with $Y : N_P \to \mathfrak{n}_P$ and $Z : N_P \to \mathrm{Lie}(\bar{P})$ smooth functions. Let $Y_1, \ldots, Y_k$ be a basis for $\mathfrak{n}_P$. Then we see that $Y(n) = \sum_i y^i(n)Y_i$ with $y^i : N_P \to \mathbb{R}$ smooth functions.

Let now $\psi \in C^\infty(G/\bar{P} : \xi^*)$ have support contained in $N_P\bar{P}$. Then

$$\langle L_X u\,,\,\psi\rangle = -\langle u\,,\,L_X\psi\rangle = -\langle \tilde{u}\,,\,L_X\psi\rangle.$$

Furthermore, for $n \in N_P$,

$$\begin{aligned} L_X\psi(n) &= [L_{Y(n)}\psi](n) + [L_{\mathrm{Ad}(n)Z(n)}\psi](n) \\ &= \sum_i y^i(n) L_{Y_i}\psi(n) - [R_{Z(n)}\psi](n) \\ &= \sum_i L_{Y_i}[y^i\psi](n) - \sum_i L_{Y_i}(y^i)(n)\psi(n) + {}^{`}\xi(Z(n))\psi(n). \end{aligned}$$

By what we established above, $L_{Y_i}u = \omega(Y_i)u$ is given by a continuous function $\tilde{u}_i$ on $N_0\bar{P}$. Let $\hat{y}^i$ denote the unique element of $C^\infty(N_P\bar{P})$ given by $\hat{y}^i(n\bar{p}) = y^i(n)$. Then $\hat{y}^i\psi \in C^\infty(G/\bar{P} : \xi^*)$ (extension by zero outside $N_P\bar{P}$), and we see that, for each $1 \le i \le k$,

$$-\langle \tilde{u}\,,\, L_{Y_i}[y^i\psi]\rangle_{N_P} = \langle L_{Y_i}u\,,\, \hat{y}^i\psi\rangle = \langle \tilde{u}_i\,,\, \hat{y}^i\psi\rangle_{N_0} = \langle y^i\tilde{u}_i\,,\, \psi\rangle_{N_0}.$$

This leads to

$$-\langle L_X u\,,\, \psi\rangle = \langle \sum_i y^i\tilde{u}_i - L_{Y_i}(y^i)\tilde{u} - [{}^\backprime\xi \circ Z]\tilde{u}\,,\, \psi\rangle_{N_0}.$$

As $\psi|_{N_P}$ ranges over all functions of $C_c^\infty(N_P, H_\xi^\infty)$, it follows that on $N_0\bar{P} = N_P\bar{P}$, the generalized function $-L_X u$ is represented by

$$\sum_i [y^i\tilde{u}^i - L_{Y_i}(y^i)\tilde{u}] - [{}^\backprime\xi \circ Z]\tilde{u}.$$

The latter function is obviously continuous $N_0\bar{P} \to H_\xi^{-\infty}$. □

We now assume that $Q \in \mathcal{P}(A)$, that σ is a unitary representation of M_Q, and that $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$, and define the representation ξ_ν of $Q = M_Q A_Q N_Q$ in H_σ by

$$\xi_\nu = \sigma \otimes \nu \otimes 1.$$

Furthermore, we assume that (π, F) is a continuous finite dimensional representation of G. From (11.2) we recall the existence of a unique G-equivariant topological linear isomorphism

$$\varphi_\nu^{-\infty} : C^{-\infty}(G/Q : \xi_\nu) \otimes F \xrightarrow{\simeq} C^{-\infty}(G/Q : \xi_\nu \otimes \pi|_Q) \tag{13.5}$$

determined by

$$\varphi_\nu^{-\infty}(u \otimes e) = (1 \otimes \pi(\,\cdot\,)^{-1})(u \otimes e)$$

on the subspaces with C^∞ in place of $C^{-\infty}$. The inverse is given by $w \mapsto (I \otimes \pi(\,\cdot\,))w$ on the mentioned subspaces.

Corollary 13.6 *For $\Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q)$-generic $\nu \in \mathfrak{a}_{Q\mathbb{C}}^*$, the endomorphism $p_{\Lambda+\mu+\nu}$ of the space $C^{-\infty}(G/Q : \xi_\nu \otimes \pi|_Q)$ preserves the subspace $C^{-\infty}(G/Q : \xi_\nu \otimes \pi|_Q)_{N_0}$.*

Proof. This follows by combining Proposition 13.5 applied to $\xi = \xi_\nu$ and the characterization of $p_{\Lambda+\nu+\mu}$ in Corollary 11.13. □

Lemma 13.7 *Let $Q \in \mathcal{P}(A)$ and let $v \in N_K(\mathfrak{a})$ be such that $v\bar{Q}v^{-1}$ is standard. The map $\varphi_\nu^{-\infty}$ of (13.5) restricts to a $(\mathfrak{g}, N_0)$-equivariant linear isomorphism*

$$\underline{\varphi}_\nu : C^{-\infty}(G/Q : \sigma : \nu)_{N_0} \otimes F \xrightarrow{\simeq} C^{-\infty}(G/Q : \xi_\nu \otimes \pi|_Q)_{N_0} \tag{13.6}$$

which satisfies

$$\underline{\mathrm{ev}}_v \circ \underline{\varphi}_\nu = \underline{\mathrm{ev}}_v \otimes \pi(v)^{-1}. \tag{13.7}$$

Proof. We note that $(\xi_\nu)^* = \sigma \otimes -\bar{\nu} = \xi_{-\bar{\nu}}$. We assume that F is equipped with a Hermitian inner product, and F_* denote the finite dimensional Hilbert space F, equipped with the conjugate representation π^*. Let ${}^*\varphi_\nu$ be the equivariant isomorphism from $C^\infty(G/Q : \xi_{-\bar{\nu}}) \otimes F_*$ onto $C^\infty(G/Q : \xi_{-\bar{\nu}} \otimes \pi^*|_Q)$ as defined in (11.1).

Let $f \in C^{-\infty}(G/Q : \sigma : \nu)_{N_0} \otimes F$. Then the restriction of f to $\mathcal{O}_Q = N_0 vQ$ is continuous $\mathcal{O}_Q \to H_\sigma^{-\infty} \otimes F$. It follows from the definition of $\varphi_\nu^{-\infty}$ as in (11.2) that

$$\langle \varphi_\nu^{-\infty}(f)\, ,\, g\rangle = \langle f\, ,\, ({}^*\varphi_\nu)^{-1}(g)\rangle,$$

for all $g \in C^\infty(G/Q : \xi_{-\bar{\nu}}) \otimes F_*$. In particular this is true for all such g with support contained in $N_0 vQ$. In that case the above equality tells us that

$$\begin{aligned}\langle \varphi_\nu^{-\infty}(f)\, ,\, g\rangle &= \int_{K/K_Q} \langle f(k)\, ,\, ({}^*\varphi_\nu)^{-1}(g)(k)\rangle_\sigma \, d\dot{k} \\ &= \int_{K/K_Q} \langle f(k)\, ,\, (I \otimes \pi^*(k))g(k)\rangle_\sigma \, d\dot{k} \\ &= \int_{K/K_Q} \langle (I \otimes \pi(k)^{-1})f(k)\, ,\, g(k)\rangle_\sigma \, d\dot{k}.\end{aligned}$$

It follows from this that, for $x \in K \cap \mathcal{O}_Q$,

$$\varphi_\nu^{-\infty}(f)(x) = (1 \otimes \pi(x))^{-1}(\mathrm{ev}_x \otimes I)(f). \tag{13.8}$$

By Q-equivariance this equality is true for all $x \in \mathcal{O}_Q$. From this it is immediately clear that $\varphi_\nu^{-\infty}(f)$ belongs to the space on the right in (13.6). Moreover, by substituting $x = v$ in (13.8) we obtain (13.7) when applied to f.

By using a similar argument involving the maps $[\varphi_\nu^{-\infty}]^{-1}$ and ${}^*\varphi_\nu$ one sees that the map $\underline{\varphi}_\nu$ is a linear isomorphism as asserted.

Finally, the $(\mathfrak{g}, N_0)$-equivariance of $\underline{\varphi}_\nu$ follows from the similar equivariance of φ_ν combined with Proposition 13.5. □

Corollary 13.8 *For $\Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q)$-generic $\nu \in \mathfrak{a}^*_{Q\mathbb{C}}$, the endomorphism $p_{\Lambda+\mu+\nu}$ of the space $C^{-\infty}(G/Q : \sigma : \nu) \otimes F$ preserves the subspace $C^{-\infty}(G/Q : \sigma : \nu)_{N_0} \otimes F$.*

Proof. Since the map $\varphi_\nu^{-\infty}$ of (13.5) is a G-equivariant isomorphism from the space $C^{-\infty}(G/Q : \sigma : \nu) \otimes F$ onto $C^{-\infty}(G/Q : (\sigma \otimes \nu) \otimes \pi|_Q)$, we have $\varphi_\nu^{-\infty} \circ p_{\Lambda+\mu+\nu} = \varphi_\nu^{-\infty} \circ p_{\Lambda+\mu+\nu}$. The result now follows by combining Corollary 13.6 with Lemma 13.7. □

Finally, we are prepared to complete the proof announced in the title of this section.

Proof of Lemma 12.14. By application of standard intertwining operators as in the proof of Lemma 13.1, we may reduce to the case that $Q = P$ (recall that P is standard). In this case, we argue as follows. For $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we denote by ${}^{\backprime}H_{\sigma,\nu}$ the space H_σ on which $P = M_P A_P N_P$ acts by $man \mapsto a^{\nu+\rho_P}\sigma(m)$. The space $\mathbb{C}e_{N_0}$ is a P-invariant subspace of F, on which P acts by $man \mapsto a^{\mu}$. We write $\pi|_{P,\mathbb{C}e_{N_0}}$ for the restriction of $\pi|_P$ to this subspace. The inclusion map $\mathbb{C}e_{N_0} \to F$ is P-equivariant, hence induces a G-equivariant continuous linear map

$$\iota_\nu : C^{-\infty}(G/P : \xi_\nu \otimes \pi|_{P,\mathbb{C}e_{N_0}}) \hookrightarrow C^{-\infty}(G/P : \xi_\nu \otimes \pi|_P),$$

see Lemma 7.7 for details.

In the sequel we shall briefly write p_ν for $p_{\Lambda+\mu+\nu}$. Since $H_{\sigma,\nu} \otimes \mathbb{C}e_{N_0}$ is naturally isomorphic to $H_{\sigma,\nu+\mu}$ as a P-module, it follows that for $(\Sigma(\mathfrak{n}_P, \mathfrak{a}_P))$-generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the projection

$$p_\nu \in \mathrm{End}(C^{-\infty}(G/P : \xi_\nu \otimes \pi|_P)) \tag{13.9}$$

equals the identity on the image of ι_ν. On the other hand, it follows from Lemma 11.8 combined with the isomorphism $\mathrm{Ind}_P^G(\xi_\nu) \otimes \pi \simeq \mathrm{Ind}_P^G(\xi_\nu \otimes \pi|_P)$ that for generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$\mathrm{im}(p_\nu)_K \simeq C(G/P : \sigma : \mu + \nu)_K \simeq \mathrm{im}(\iota_\nu)_K,$$

as $(\mathfrak{g}, K)$-modules. Since (13.9) is a continuous projection its image $\mathrm{im}(p_\nu)$ is a closed subspace of $C^{-\infty}(G/P : \xi_\nu \otimes \pi|_P)$. We now infer that for generic $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we have

$$\mathrm{im}(p_\nu) = \mathrm{cl}(\mathrm{im}(\iota_\nu)_K),$$

where cl indicates that the closure in $C^{-\infty}(G/P : \xi_\nu \otimes \pi|_P)$ is taken. To characterize this closure in a useful way, we fix a Hermitian inner product on F and define the continuous representation π^* of G on it by $\pi^*(p) = \pi(p^{-1})^*$. The restriction of $\pi^*|_P$ to the $\pi^*(P)$-invariant subspace $E = (\mathbb{C}e_{N_0})^\perp$ of F is denoted by π^*_E. Clearly, $\mathbb{C}e_{N_0} = E^\perp$. We view $C^\infty(G/P : \xi_{-\bar{\nu}} \otimes \pi^*_E)$ as an invariant subspace of $C^\infty(G/P : \xi_{-\bar{\nu}} \otimes \pi^*_E)$. Via the sesquilinear pairing

$$C^{-\infty}(G/P : \xi_\nu \otimes \pi) \times C^\infty(G/P : \xi_{-\bar{\nu}} \otimes \pi^*) \to \mathbb{C} \tag{13.10}$$

we accordingly define Ann_ν to be the annihilator of $C^\infty(G/P : \xi_{-\bar{\nu}} \otimes \pi^*_E)$ in the first of the spaces in (13.10). This annihilator is closed and on the K-finite level it is readily seen that $(\mathrm{Ann}_\nu)_K = \mathrm{im}(\iota_\nu)_K$. It follows that the annihilator equals the closure of $\mathrm{im}(\iota_\nu)_K$. Hence,

$$\mathrm{im}(p_\nu) = \mathrm{Ann}_\nu.$$

We now select $v \in N_K(\mathfrak{a})$ such that $\mathcal{O}_P = N_0 v P$ is open in G. Note that $\mathcal{O}_P = v\bar{N}_P P$.

Lemma 13.9 *Let* $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$, *write* $\xi_\nu = \sigma \otimes \nu \otimes 1$ *and let* $u \in C^{-\infty}(G/P : \xi_\nu \otimes \pi|_P)_{N_0}$. *If* $u \in \mathrm{Ann}_\nu$ *then* $u|_{\mathcal{O}_P} \in C(\mathcal{O}_P, H_\sigma^{-\infty} \otimes \mathbb{C}e_{N_0})^P$.

Proof. Let u fulfill the hypothesis. Then it follows from Definition 13.2 that the restriction of u to $\mathcal{O}_P$ is continuous with values in $H_\sigma^{-\infty} \otimes F$. This means that there exists a continuous function $\tilde{u} \in N_0 v P \to H_\sigma^{-\infty} \otimes F$ such that $\tilde{u}(nvp) = [`\xi^{-\infty}(p) \otimes \pi(p)]^{-1}\tilde{u}(nv)$ $(p \in P, n \in N_0)$, and such that for all $\psi \in C^\infty(G/P : \xi_{-\bar{\nu}} \otimes \pi^*)$

with support contained in $\mathcal{O}_P$ we have (13.2), which by the substitution of variables $k = v\kappa_P(\bar{n})K_P$ may be rewritten as

$$\langle u\,,\,\psi\rangle = \int_{\bar{N}_P} \langle \tilde{u}(v\bar{n})\,,\,\psi(v\bar{n})\rangle_{\sigma\otimes\pi}\, d\bar{n}.$$

Let now $h \in H_\sigma^\infty$, $f \in E = (\mathbb{C}e_{N_0})^\perp$ and $\phi \in C_c^\infty(\bar{N}_P)$, and define $\psi \in C^\infty(G/P : \xi_{-\bar{\nu}} \otimes \pi^*)$ by the requirements

$$\psi(v\bar{n}) = \phi(\bar{n})(h \otimes f), \qquad \psi = 0 \;\text{ on }\; G \setminus v\bar{N}_P P.$$

Then it follows from $u \in \mathrm{Ann}_\nu$ that

$$\int_{\bar{N}_P} \phi(n)\langle \tilde{u}(v\bar{n})\,,\, h \otimes f\rangle\, d\bar{n} = 0.$$

As this is valid for all ϕ as above, the continuous function $\bar{n} \mapsto \langle \tilde{u}(v\bar{n})\,,\, h \otimes f\rangle$ is zero. This implies that for each $\bar{n} \in \bar{N}_P$ the element $\tilde{u}(v\bar{n}) \in H_\sigma^{-\infty} \otimes F$ satisfies

$$\langle \tilde{u}(v\bar{n})\,,\, h \otimes f)\rangle = 0, \qquad (h \in H_\sigma^\infty, f \in E).$$

This in turn implies that $u(v\bar{n}) \in H_\sigma^{-\infty} \otimes E^\perp = H_\sigma^{-\infty} \otimes \mathbb{C}e_{N_0}$. Since $v\bar{N}_P P = N_0 vP$ this finishes the proof. $\square$

We proceed with the completion of the proof of Lemma 12.14. From Lemma 13.1 it follows that for generic $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ the map

$$j \mapsto \varphi_\nu^{-\infty} \circ p_\nu(j \otimes e_{N_0}) \tag{13.11}$$

is injective from $C^{-\infty}(G/P : \sigma : \nu)_\chi$ to $C^{-\infty}(G/P : (\sigma \otimes \nu) \otimes \pi|_P)_\chi$. Furthermore, since φ_ν is an intertwining isomorphism, it follows that

$$\varphi_\nu^{-\infty} \circ p_\nu = p_\nu \circ \varphi_\nu^{-\infty}.$$

We thus see that the map (13.11) maps $C^{-\infty}(G/P : \sigma : \nu)_\chi$ injectively to

$$\mathrm{im}(p_\nu) \cap C^{-\infty}(G/P : (\sigma \otimes \nu) \otimes \pi|_P)_\chi \subset \mathrm{Ann}_\nu \cap C^{-\infty}(G/P : (\sigma \otimes \nu) \otimes \pi|_P)_{N_0}.$$

Applying Lemmas 13.4 and 13.9 we now see that the map

$$j \mapsto \mathrm{ev}_v[\varphi_\nu^{-\infty} \circ p_{\Lambda+\nu+\mu}(j \otimes e_{N_0})]$$

is injective from $C^{-\infty}(G/P : \sigma : \nu)_\chi$ to $H_\sigma^{-\infty} \otimes \mathbb{C}e_{N_0}$. Put

$$e^v := e^{N_0} \circ \pi(v) \in F^*; \tag{13.12}$$

then $e^v(e_{N_0}) \neq 0$, since otherwise the matrix coefficient $x \mapsto e^v(\pi(x)e_{N_0})$ would be zero on $N_0 vP$ hence on G, contradicting the irreducibility of π. It thus follows that

$$j \mapsto (I \otimes e^v)[\mathrm{ev}_v[\varphi_\nu^{-\infty} \circ p_\nu(j \otimes e_{N_0})]] \tag{13.13}$$

is injective $C^{-\infty}(G/P : \sigma : \nu)_\chi \to H_\sigma^{-\infty} \otimes \mathbb{C}$. On the other hand, $j \mapsto j \otimes e_{N_0}$ maps $C^{-\infty}(G/P : \sigma : \nu)_\chi$ into $C^{-\infty}(G/P : \sigma : \nu)_{N_0} \otimes F$. By Lemmas 13.4, 13.7 and Corollary 13.8 it now follows that (13.13) equals

$$j \mapsto (I \otimes e^{\upsilon}) \circ \underline{\mathrm{ev}}_{\upsilon} \circ \underline{\varphi}_{\nu} \circ \underline{p}_{\nu}(j \otimes e_{N_0}) \qquad (13.14)$$

which is therefore an injective map $C^{-\infty}(G/P : \sigma : \nu)_\chi \to H_{\sigma,\nu}^{-\infty}$. By Lemma 13.7 the above map (13.14) equals

$$\begin{aligned} j \quad &\mapsto \quad (I \otimes e^{\upsilon}) \circ (\underline{\mathrm{ev}}_{\upsilon} \otimes \pi(\upsilon)^{-1}) \circ \underline{p}_{\nu}(j \otimes e_{N_0}) \\ &= \quad (\underline{\mathrm{ev}}_{\upsilon} \otimes I) \circ (I \otimes e^{N_0}) \circ \underline{p}_{\nu}(j \otimes e_{N_0}). \end{aligned}$$

The injectivity of the latter map implies the injectivity of

$$j \mapsto (I \otimes e^{N_0}) \circ \underline{p}_{\nu}(j \otimes e_{N_0}) = (I \otimes e^{N_0}) \circ p_\nu(j \otimes e_{N_0}).$$

as a map from $C^{-\infty}(G/P : \sigma : \nu)_\chi$ to $C^{-\infty}(G/P : \sigma : \nu) \otimes \mathbb{C}$. Since $\varepsilon^{N_0} = m \circ (I \otimes e^{N_0})$, with m injective, see (12.4), the required injectivity of the map (12.18) follows. □

14 Holomorphy and uniformly moderate estimates

In this section, we assume that P is a standard parabolic subgroup and (σ, H_σ) a discrete series representation of M_P. We will first prove the following result which is inspired by Wallach [21, Thm. 15.4]. Let $\delta > 0$.

Theorem 14.1 *For every $R \leq \delta$ the function $\nu \mapsto j(\bar{P}, \sigma, \nu)$, originally defined for $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(P, \delta)$ allows a meromorphic extension to $\mathfrak{a}_{P\mathbb{C}}^*(P, R)$ as a function with values in the space $(H_{\sigma,\chi_P}^{-\infty})^* \otimes C^{-\infty}(K/K_P : \sigma)$.*

Furthermore, there exists a non-trivial polynomial function $p_R \in P(\mathfrak{a}_P^*)$ and constants $s, N \in \mathbb{N}$ and $C > 0$ such that for all $\eta \in H_{\sigma,\chi_P}^{-\infty}$ the extended function $\nu \mapsto p_R(\nu) j(\bar{P}, \sigma, \nu, \eta)$ is holomorphic $C^{-s}(K/K_P : \sigma)$-valued on $\mathfrak{a}_{P\mathbb{C}}^*(P, R)$ and satisfies the estimate

$$\|p_R(\nu) j(\bar{P}, \sigma, \nu, \eta)\|_{-s} \leq C(1 + |\nu|)^N \|\eta\|, \qquad (\nu \in \mathfrak{a}_{P\mathbb{C}}^*(P, R)).$$

Proof. First of all, by Proposition 8.14 the above result is true for $R = \delta$, with $s = N = 0$ Let $\mu \in \Lambda^{++}(\mathfrak{a}_P)$. Then $\langle \mu , \alpha \rangle > 0$ for all $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$. Let $m > 0$ be fixed and strictly smaller than the minimum of the numbers $\langle \mu , \alpha \rangle$, for $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$. Then

$$\mathfrak{a}_{P\mathbb{C}}^*(P, R - m) + \mu \subset \mathfrak{a}_{P\mathbb{C}}^*(P, R).$$

We will show that if the assertions of the theorem are valid for $R \leq \delta$, they are also valid with R replaced by $R - m$. The result then follows by induction.

Assume the result to be proven for a given R, with constants s, N_R and C_R in place of s, N, C. Let $D_\mu(\sigma, \nu)$ and $R_\mu(\sigma, \nu)$ be as in Theorem 12.4. By holomorphic continuation, this functional equation is still valid on $\mathfrak{a}^*_{P\mathbb{C}}(P, R)$ for the extended function $j(\bar{P}, \sigma, \cdot)$. Let $q : \mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$ be a non-trivial polynomial function such that $qR_\mu(\sigma, \cdot)$ is polynomial of degree d', with values in $\mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$.

For $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(R - r)$ we define

$$j_e(\bar{P} : \sigma : \nu)(\eta) = D_\mu(\sigma, \nu) j(P, \sigma, \nu + \mu) R_\mu(\sigma, \nu)\eta.$$

Let $r, d, C \in \mathbb{N}$ be the constants of Proposition 12.1. Then it follows by application of the mentioned proposition that $\nu \mapsto q(\nu) p_R(\nu + \mu) j_e(\bar{P} : \sigma : \nu)(\eta)$ is holomorphic on $\mathfrak{a}^*_{P\mathbb{C}}(P, R - m)$ with values in $C^{-s-r}(K/K_P : \sigma)$. Furthermore,

$$\begin{aligned}
&\|q(\nu) p_R(\nu + \mu) j_e(\bar{P} : \sigma : \nu)(\eta)\|_{-s-r} \\
&\quad\leq\ C(1 + |\nu|)^d \|p_R(\mu + \nu) j(P, \sigma, \nu + \mu, q(\nu) R_\mu(\sigma, \nu)\eta)\|_{-s} \\
&\quad\leq\ CC_R(1 + |\nu|)^d (1 + |\nu + \mu|)^{N_R} \|q(\nu) R_\mu(\sigma, \nu)\|_{\mathrm{op}} \|\eta\| \\
&\quad\leq\ C_{R-m}(1 + |\nu|)^{d+N_R+d'} \|\eta\|,
\end{aligned}$$

with $C_{R-m} > 0$ a constant which is uniform for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R - m)$. By the functional equation of Theorem 12.4 it follows that

$$q(\nu) p_R(\nu + \mu) j_e(\bar{P}, \sigma, \nu)(\eta) = q(\nu) p_R(\nu + \mu) j(\bar{P}, \sigma, \nu)(\eta)$$

for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$. This shows that the $(H^{-\infty}_{\sigma\chi_P})^* \otimes C^{-\infty}(K/K_P : \sigma)$-valued function $\nu \mapsto j_e(\bar{P}, \sigma, \nu)$ is the meromorphic extension of the original $\mathcal{H}^{-\infty}_{\sigma,\chi_P}$-valued function $j(\bar{P}, \sigma, \cdot)$ defined on $\mathfrak{a}^*_{P\mathbb{C}}(P, \delta)$. The proof is complete. □

It follows from the above result that as a $(H^{-\infty}_{\sigma,\chi_P})^* \otimes C^{-\infty}(K/K_P : \sigma)$-valued function, the function $\nu \mapsto j(\bar{P}, \sigma, \nu)$ has a meromorphic extension to all of $\mathfrak{a}^*_{P\mathbb{C}}$. This meromorphic extension will be denoted by the same symbol.

Lemma 14.2 *For $\eta \in H^{-\infty}_{\sigma,\chi_P}$ and a regular point $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the element $j(\bar{P}, \sigma, \nu, \eta) \in C^{-\infty}(K/K_P : \sigma)$ satisfies the transformation rule*

$$\pi^{-\infty}_{\bar{P},\sigma,\nu}(n) j(\bar{P}, \sigma, \nu, \eta) = \chi(n) j(\bar{P}, \sigma, \nu, \eta), \qquad (n \in N_0).$$

Proof. This follows by analytic continuation. □

Lemma 14.3 *Let $\varphi \in C^\infty(K/K_P : \sigma)$ have compact support contained in the set $K \cap N_0\bar{P}$. Then for every $\eta \in H^{-\infty}_{\sigma,\chi_P}$, the meromorphic function $\nu \mapsto \langle j(\bar{P}, \sigma, \nu)(\eta)\,,\, \varphi\rangle$ is holomorphic.*

Proof. It follows from the definition of $j(\bar{P},\sigma,\nu,\eta)$ for $\nu\in\mathfrak{a}^*_{P\mathbb{C}}(P,0)$ that the restriction $j(\bar{P},\sigma,\nu,\eta)|_{K\cap N_P\bar{P}}$ is given by the continuous function $K\cap N_0\bar{P}\to\mathcal{H}_\sigma^{-\infty}$ described by the formula

$$j_\eta(\nu): k\mapsto a(k)^{-\nu+\rho_P}\chi(n_P(k))\sigma(\mu_P(k))^{-1}\eta. \tag{14.1}$$

By this we mean that for a function $\varphi\in C^\infty(K/K_P:\sigma)$ with compact support contained in $K\cap N_0\bar{P}$ we have

$$\langle j(\bar{P},\sigma,\nu)(\eta)\,,\,\varphi\rangle=\int_{K/K_P}\langle j_\eta(\nu)(k)\,,\,\varphi(k)\rangle_\sigma dk$$

From

$$j_\eta(\lambda)=a(\,\cdot\,)^{(\nu-\lambda)}j_\eta(\nu),\qquad(\lambda\in\mathfrak{a}^*_{P\mathbb{C}}),$$

we see that j_η extends to a holomorphic function from $\mathfrak{a}^*_{P\mathbb{C}}$ to $C(K\cap N_0\bar{P},H_\sigma^{-\infty})$. Accordingly, it follows that for $\varphi\in C^{-\infty}(K/K_P:\sigma)$ with compact support in $K\cap N_P\bar{P}$ the $\mathbb{C}$-valued function

$$\nu\mapsto\langle j(\bar{P},\sigma,\nu)(\eta)\,,\,\varphi\rangle$$

is holomorphic on $\mathfrak{a}^*_{P\mathbb{C}}$. □

Theorem 14.4 *The map $\nu\mapsto j(\bar{P},\sigma,\nu)$ is holomorphic as a function on $\mathfrak{a}^*_{P\mathbb{C}}$ with values in the complete locally convex space $(H^{-\infty}_{\sigma,\chi_P})^*\otimes C^{-\infty}(K/K_P:\sigma)$. Here $C^{-\infty}(K/K_P:\sigma)$ is understood to be equipped with the direct limit topology, see the text below (7.10).*

Proof. Let $R\le 0$ and let $p=p_R$ be as in Theorem 14.1. Let $\Omega:=\mathfrak{a}^*_{P\mathbb{C}}(P,R)$ and let X be the zero set of p in Ω. Fix $\eta\in H^{-\infty}_{\sigma,\chi_P}$. It follows from the theorem that $j:\nu\mapsto j(\bar{P},\sigma,\nu)$ is holomorphic as a function from $\Omega\setminus X$ to the complete locally convex space $V:=C^{-\infty}(K/K_P:\sigma)$.

By Theorem 18.1 (Appendix) it suffices to show that j admits an extension to a holomorphic function $\Omega\setminus X_s\to V$, with $X_s=X\setminus X_r$, where X_r is the set of points $\nu_0\in X$ at which X is a complex differentiable submanifold of co-dimension 1. It is readily verified that X_r is open in X; therefore, X_s is closed in X hence in Ω.

Let $\nu_0\in X_r$. Then it suffices to show that there exists an open neighborhood Ω_0 of ν_0 in Ω such that $j|_{\Omega_0\setminus X}$ admits an extension to a holomorphic function $\Omega_0\to V$. See also Lemma 18.3 (Appendix).

By definition of X_r there exists an open neighborhood ω of ν_0 in Ω such that $X_0:=X\cap\omega$ is a connected complex differentiable submanifold of codimension 1. Let $\xi\in\mathfrak{a}^*_{P\mathbb{C}}$ be such that

$$T_{\nu_0}X_0\,\oplus\,\mathbb{C}\xi=\mathfrak{a}^*_{P\mathbb{C}}.$$

Then it follows that the map $\varphi:(\nu,z)\mapsto\nu+z\xi$ is a local holomorphic diffeomorphism at $(\nu_0,0)$. Replacing ω by a smaller neighborhood if necessary, and taking $r>0$

sufficiently small, we arrive at the situation that $\varphi : X_0 \times D(0,r) \to \mathfrak{a}^*_{P\mathbb{C}}$ is a holomorphic diffeomorphism onto an open neighborhood Ω_0 of ν_0 in $\mathfrak{a}^*_{P\mathbb{C}}$ and that $\varphi(X_0 \times \{0\}) = X \cap \Omega_0$.

Put $D = D(0,r)$. Then $j^* = j \circ \varphi|_{X_0 \times D \setminus \{0\}}$ is a holomorphic function $X_0 \times D \setminus \{0\} \to V$ and it suffices to show that this function extends to a holomorphic function $X_0 \times D \to V$.

Since $j^* : (\nu, z) \mapsto j(\nu + z\xi)$ is holomorphic on $X_0 \times (D \setminus \{0\})$ it has a Laurent series expansion in z of the form

$$j(\nu + z\xi) = \sum_{k \in \mathbb{Z}} c_k(\nu) z^k,$$

with $c_k : X_0 \to V$ holomorphic, for all $k \in \mathbb{Z}$. Thus, it suffices to show that $c_k = 0$ for $k < 0$.

The zero set of $p \circ \varphi$ equals $\varphi^{-1}(X) = X_0 \times \{0\}$. Hence, there exists a constant $d \geq 1$ such that $p(\nu + z\xi) = z^d q(\nu, z)$, with $q : X_0 \times D \to \mathbb{C}$ a holomorphic function that is not identically zero on $X_0 \times \{0\}$. Let X_0' be the open dense subset of $\nu \in X_0$ such that $q(\nu, 0) \neq 0$. Fix $\nu \in X_0'$, an open neighborhood X_1 of ν whose closure is contained in D_0' and a disk $D' \subset D$ centered at 0 such that $q(\nu, z) \neq 0$ for all $\nu \in X_1$ and $z \in D'$. Then for every $\nu \in X_1$ the function $z \mapsto z^d j(\nu + z\xi)$ extends to a holomorphic function $D' \to V$. It follows that $c_k(\nu) = 0$ for $k < -d$ and $\nu \in X_1$. By analytic continuation it now follows that $c_k = 0$ on X_0 for $k < -d$.

Let $m \in \mathbb{Z}$ be the maximal number such that $c_{-m} \neq 0$. Arguing by contradiction we will show that $m \leq 0$, thereby completing the proof. Thus, suppose $m > 0$. Then there exists $\nu_1 \in X_0$ such that $c_{-m}(\nu_1) \in V \setminus \{0\}$. We claim that for $n \in N_0$ we have

$$\pi^{-\infty}_{\nu_1}(n) c_{-m}(\nu_1) = \chi(n) c_{-m}(\nu_1). \tag{14.2}$$

Indeed, fix $n \in N_0$ and $\psi \in C^\infty(K/K_P : \sigma)$. Then it suffices to prove the identity evaluated at ψ. Writing $\pi_\nu = \pi_{\bar{P},\sigma,\nu}$, we start with the known identity expressing that $j(\nu)$ is a Whittaker vector in the induced representation:

$$\langle j(\nu)\,,\, \pi^\infty_{-\bar{\nu}}(n^{-1})\psi\rangle = \chi(n)\langle j(\nu)\,,\,\psi\rangle,$$

for $\nu \in \Omega \setminus X$. Substituting $\nu = \nu_1 + z\xi$ for $z \in D \setminus \{0\}$, we obtain the identity

$$\langle z^m j(\nu_1 + z\xi)\,,\, \pi^\infty_{-\bar{\nu}_1 - \bar{z}\bar{\xi}}(n^{-1})\psi\rangle = \chi(n)\langle z^m j(\nu_1 + z\xi)\,,\,\psi\rangle \tag{14.3}$$

of holomorphic functions on $D(0,r) \setminus \{0\}$. We observe that we may write

$$z^m j(\nu_1 + z\xi) = c_{-m}(\nu_1) + zR(z),$$

as an identity of holomorphic V-valued functions in $z \in D(0,r)$, with $R : D(0,r) \to V$ holomorphic. We may also write

$$\pi^\infty_{\nu_1 - z\bar{\xi}}(n^{-1})\psi = \pi^\infty_{\nu_1}(n^{-1})\psi + z\Psi(z), \qquad (z \in D(0,r)),$$

with Ψ a holomorphic function $D(0, R) \to C^\infty(K/K_P : \sigma)$. Substituting these expressions in (14.3) we obtain

$$\langle c_{-m}(\nu_1)\,,\, \pi^\infty_{\nu_1}(n^{-1})\psi\rangle = \chi(n)\langle c_{-m}(\nu_1)\,,\, \psi\rangle + zF(z), \tag{14.4}$$

where $F : D(0, r) \to \mathbb{C}$ is given by

$$F(z) = \langle R(z)\,,\, \chi(n)\psi - \pi_{\nu_1}(n^{-1})\psi - \bar{z}\Psi(\bar{z})\rangle - \langle c_{-m}(\nu_1)\,,\, \Psi(\bar{z})\rangle.$$

If z is restricted to a compact neighborhood of 0 in $D(0, r)$, then $\Psi(z)$ stays in a bounded subset of $C^\infty(K/K_P : \sigma)$ and $R(z)$ stays in a bounded subset of $C^{-\infty}(K/K_P : \sigma)$. This implies that $F(z)$ remains bounded, so that $\lim_{z\to 0} zF(z) = 0$. By taking the limit of (14.4) we find that (14.2) is valid after pairing both sides with $\psi \in C^\infty(K/K_P : \sigma)$. Since n and ψ were arbitrary, (14.2) follows. Thus, in the induced picture we have

$$c_{-m}(\nu_1) \in C^{-\infty}(\bar{P} : \sigma : \nu_1)_\chi.$$

Furthermore, by application of Lemma 14.3, the generalized function $c_{-m}(\nu_1)$ vanishes on the open orbit $N_0\bar{P}$. In view of Corollary 8.3 applied to $c_{-m}(\nu_1)$ in place of j it finally follows that $c_{-m}(\nu_1) = 0$, contradicting the condition involved in the choice of ν_1. □

Corollary 14.5 *For every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the map*

$$j(\bar{P}, \sigma, \nu) : \quad H^{-\infty}_{\sigma,\chi_P} \to C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$$

is a linear isomorphism with inverse equal to the map ev_e *defined in (8.11).*

Proof. It follows from the definition of $j(\bar{P}, \sigma, \nu)(\eta)$ for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ that the restriction of $j(\bar{P}, \sigma, \nu)(\eta)$ to $K \cap N_P\bar{P}$ is equal to the continuous function $j_\eta(\nu)$: $K \cap N_P\bar{P} \to H^{-\infty}_\sigma$ defined in (14.1). In the proof of Lemma 14.3 it is shown that $j_\eta(\nu)$ extends to a holomorphic function of $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with values in $C(K \cap N_P\bar{P}, H^{-\infty}_\sigma)$. By analytic continuation it follows that $j(\bar{P}, \sigma, \nu, \eta)|_{K\cap N_P\bar{P}}$ is given by the function $j_\eta(\nu)$. It now follows that $\mathrm{ev}_e j(\bar{P}, \sigma, \nu, \eta) = \eta$. Hence, ev_e is a left inverse to $j(\bar{P}, \sigma, \nu)$ and we see that ev_e is surjective $C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi \to H^{-\infty}_{\sigma,\chi_P}$. If we combine this with the injectivity of ev_e, asserted in Corollary 8.11, the required result follows. □

Lemma 14.6 *Let $M_\mu = M_\mu(\sigma, \cdot\,) : \mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$ be the polynomial function $\mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$ introduced in Proposition 12.10. Then the function* $\det M_\mu$ *is a non-zero constant times a finite product of first order polynomial functions on $\mathfrak{a}^*_{P\mathbb{C}}$ of the form $\nu \mapsto \langle \nu\,,\, \alpha\rangle + c$ with $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ and $c \in \mathbb{C}$.*

Proof. We observe that $\det M_\mu$ is a polynomial function $\mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$. By Proposition 17.1 it suffices to show that $\det M_\mu$ is non-zero on the complement of a locally finite collection $\mathcal{H}$ of affine $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-hyperplanes. Increasing $\mathcal{H}$ we may assume that $\cup\mathcal{H}$

contains the zero set of the polynomial function $q : \mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$ introduced in Lemma 11.12. From Lemma 12.14 we know that there exists such a collection of hyperplanes such for $\nu \in \mathfrak{a}^*_{P\mathbb{C}} \setminus \cup\mathcal{H}$ the map

$$\psi_\nu : j \mapsto \varepsilon^{N_0} \circ p_{\Lambda+\nu+\mu}(j \otimes e_{N_0}) \tag{14.5}$$

is a bijective endomorphism of $C^{-\infty}(G/\bar{P} : \sigma : \nu)_\chi$. By Corollary 14.5 this implies the existence of a unique linear automorphism b_ν of $H^{-\infty}_{\sigma,\chi_P}$ such that

$$\psi_\mu(j(\bar{P}, \sigma, \nu)) = j(\bar{P}, \sigma, \nu) \circ b_\nu.$$

Since $q(\nu) \neq 0$ it follows from (12.17) that $p_{\Lambda+\mu+\nu} = q(\nu)^{-1} Z_{\bar{P},\mu(\nu)}$ on $C^{-\infty}(G/\bar{P} : \sigma : \nu) \otimes F$. If we combine this with (14.5) and Proposition 12.10 we find that

$$\psi_\mu(j(\bar{P}, \sigma, \nu)) = j(\bar{P}, \sigma, \nu) \circ [q(\nu)^{-1} M_\mu(\nu)],$$

for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic ν. By uniqueness, this implies $q(\nu)^{-1} M_\mu(\nu) = b_\nu$ for $\Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$-generic ν. We conclude that $\det M_\mu(\nu) \neq 0$ for ν outside $\cup\mathcal{H}$. □

Corollary 14.7 *There exists a polynomial function $p : \mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$ which is a finite product of linear factors of the form $\langle \alpha , \cdot \rangle - c$, with $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ and $c \in \mathbb{C}$ such that $pR_\mu(\sigma, \cdot)$ is a polynomial* $\mathrm{End}(H^{-\infty}_{\sigma,\chi_P})$*-valued map.*

Proof. Let $p = \det M_\mu$, then by Lemma 14.6 the function p has the required form. By application of Cramer's rule the result now follows from the formula $R_\mu(\sigma, \nu) = M_\mu(\nu)^{-1} m_\mu$, given at the end of the proof of Theorem 12.4 just before Proposition 12.10. □

Theorem 14.8 *For every $R \leq 0$ there exist constants $C > 0, N \in \mathbb{N}$ and $r \in \mathbb{N}$ such that for all $\eta \in H^{-\infty}_{\sigma,\chi_P}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ we have $j(\bar{P}, \sigma, \nu)\eta \in C^{-r}(K/K_P : \sigma)$ and*

$$\|j(\bar{P}, \sigma, \nu)\eta\|_{-r} \leq C(1 + |\nu|)^N \|\eta\|.$$

Proof. We agree to write $j(\nu, \eta) = j(\bar{P}, \sigma, \nu)\eta$. Following the induction in the proof of Theorem 14.1 one sees that its assertion is valid with $p = p_R$ a polynomial function $\mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$ which is a finite product of linear factors of the form $l_{\alpha,c} : \nu \mapsto \langle \nu , \alpha \rangle - c$ with $\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$ and $c \in \mathbb{C}$. From the mentioned theorem we know that there exist constants $C' > 0$ and $s, N \in \mathbb{N}$ such that for all $\eta \in H^{-\infty}_{\sigma,\chi_P}$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ we have that $\nu \mapsto p(\nu) j(\nu, \eta)$ is holomorphic on $\mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ with values in $C^{-s}(k/K_P : \sigma)$ and satisfies the estimate

$$\|p(\nu) j(\nu, \eta)\|_{-s} \leq C'(1 + |\nu|)^N \|\eta\|. \tag{14.6}$$

Let $l : \mathfrak{a}^*_{P\mathbb{C}} \to \mathbb{C}$, $\nu \mapsto \langle \nu , \alpha \rangle - c$ be a linear polynomial dividing p. Then it suffices to prove the assertion and estimate of the above type with ${}^{\backprime}p = p/l$ in place of p and $2^{N+1} C'$ in place of C'.

Put $\xi = \alpha/|\alpha|$ and let $H = l^{-1}(0)$. Then $\varphi : H \times \mathbb{C} \to \mathfrak{a}^*_{P\mathbb{C}}$, $(\nu_0, z) \mapsto \nu_0 + z\xi$ is an affine isomorphism such that $l \circ \varphi(\nu_0, z) = z$. Then clearly, $(\nu_0, z) \mapsto {}^{\backprime}p(\nu_0 + z\xi)j(\nu_0 + z\xi, \eta)$ is holomorphic on $H \times \mathbb{C} \setminus \{0\}$. Furthermore, for ν in the complement of $\varphi(H \times D(0, \frac{1}{2}))$ we have the estimate (14.6) with ${}^{\backprime}p$ in place of p and $2C'$ in place of C'.

Let now $(\nu_0, z) \in H \times D(0, \frac{1}{2})$. Then by the Cauchy integral formula we have

$$\begin{aligned} {}^{\backprime}p(\nu_0 + z\xi)j(\nu_0 + z\xi, \eta) &= \frac{1}{2\pi i}\int_{|w|=1} \frac{{}^{\backprime}p(\nu_0 + w\xi)j(\nu_0 + w\xi, \eta)}{(w - z)}\, dw \\ &= \frac{1}{2\pi i}\int_{|w|=1} \frac{p(\nu_0 + w\xi)j(\nu_0 + w\xi, \eta)}{w(w - z)}\, dw. \end{aligned}$$

The formula holds a priori as an integral formula of $C^{-\infty}(K/K_P : \sigma)$-valued functions. However, as the integrand has values in $C^{-s}(K/K_P : \sigma)$, it readily follows that ${}^{\backprime}pj(\,\cdot\,, \eta)$ is $C^{-s}(K/K_P : \sigma)$-valued. Furthermore, by a straightforward estimation we obtain:

$$\begin{aligned} \|{}^{\backprime}p(\nu_0 + z\xi)j(\nu_0 + z\xi, \eta)\|_{-s} &\leq \sup_{|w|=1} \frac{C'(1 + |\nu_0 + w\xi|)^N}{|w| - |z|} \leq 2C'(2 + |\nu_0|)^N \\ &\leq 2^{N+1}C'(1 + |\nu_0 + z\xi|)^N. \end{aligned}$$

□

The above estimates give rise to the following uniformly moderate estimates for matrix coefficients of Whittaker vectors with smooth vectors.

Theorem 14.9 *Let $R \in \mathbb{R}$. Then there exist constants $N \in \mathbb{N}$, $r, s > 0$ and a continuous seminorm n on $C^\infty(K/K_P : \sigma)$ such that for all $f \in C^\infty(K/K_P : \sigma)$, all $\eta \in H^{\infty,\chi}_\sigma$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ and $x \in G$,*

$$|\langle f\,,\, \pi_{\bar{P},\sigma,\nu}(x)j(\bar{P}, \sigma, \nu, \eta)\rangle| \leq (1 + |\nu|)^N e^{s|\mathrm{Re}\,\nu||H(x)|}e^{r|H(x)|}n(f).$$

We prepare for the proof with a few lemmas.

Lemma 14.10 *There exists a constant $s > 0$ such that for all $g \in C^\infty(K/K_P : \sigma)$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and $a \in A$,*

$$\|\pi_{\bar{P},\sigma,\nu}(a^{-1})g\|_0 \leq e^{s(|\mathrm{Re}\,\nu|+|\rho_P|)|\log a|}\,\|g\|_0.$$

Proof. The constant s is built in to make the assertion independent of the choice of norms on $\mathfrak{a}$ and $\mathfrak{a}^*$. Here we will need that $|\nu(H)| \leq s|\nu||H|$ for $\nu \in \mathfrak{a}^*_P$ and $H \in \mathfrak{a}$. Define $g_\nu : G \to H_\sigma$ by

$$g_\nu(kma_P\bar{n}) = a_P^{-\nu+\rho_P}\sigma(m^{-1})g(k)$$

for $(k, m, a_P, \bar{n}) \in K \times M_P \times A_P \times \bar{N}_P$. Then for $k \in K$ we have $[\pi_{\bar{P},\sigma,\nu}(a^{-1})g](k) = g_\nu(ak)$ from which it follows that

$$\begin{aligned} \|[\pi_{\bar{P},\sigma,\nu}(a^{-1})g](k)\|_\sigma &\leq e^{(-\mathrm{Re}\,\nu+\rho_P)(H_{\bar{P}}(ak))}\|g(\kappa_{\bar{P}}(ak))\|_\sigma \\ &\leq e^{s|\mathrm{Re}\,\nu-\rho_P||H_{\bar{P}}(ak)|}\|g\|_0. \end{aligned}$$

Since $|H_{\bar{P}}(ak)| \leq |\log a|$ the required estimate follows. □

Lemma 14.11 *Let $u \in U(\mathfrak{g})$. Then there exist constants $N \in \mathbb{N}$, $r > 0$ and $t \in \mathbb{N}$ such that for all $f \in C^\infty(K/K_P : \sigma)$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and $a \in A$,*

$$\|\pi_{\bar{P},\sigma,\nu}(u)\pi_{\bar{P},\sigma,\nu}(a^{-1})f\|_0 \leq (1 + |\nu|)^N e^{s|\mathrm{Re}\,\nu||\log a|} e^{r|\log a|}\|f\|_t.$$

Proof. In view of the PBW theorem we may assume that $\mathfrak{a}$ acts via the adjoint action by a weight ξ on u. Accordingly we have, for $a \in A$,

$$\begin{aligned}\|\pi_{\bar{P},\sigma,\nu}(u)\pi_{\bar{P},\sigma,\nu}(a^{-1})f\|_0 &\leq a^\xi |\pi_{\bar{P},\sigma,\nu}(a^{-1})\pi_{\bar{P},\sigma,\nu}(u)f\|_0 \\ &\leq e^{s(|\mathrm{Re}\,\nu|+|\rho_P|)|\log a|} e^{\xi(\log a)} \|\pi_{\bar{P},\sigma,\nu}(u)f\|_0\end{aligned}$$

by application of the previous lemma. The proof is finished by application of Lemma 12.2. □

Completion of the proof of Theorem 14.9. It is clear that it suffices to prove the estimate for $x = a \in A$. By Theorem 14.8 there exists a constant $N > 0$ and a finite collection $F \subset U(\mathfrak{g})$ such that for all $f \in C^\infty(K/K_P : \sigma)$, all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, R)$ and all $a \in A$,

$$|\langle f\,,\, \pi_{\bar{P},\sigma,\nu}(a) j(\bar{P}, \sigma, \nu, \eta)\rangle| \leq (1 + |\nu|)^N \|\eta\| \max_{u \in F} \|\pi_{\bar{P},\sigma,\nu}(u)\pi_{\bar{P},\sigma,\nu}(a^{-1})f\|_0.$$

The proof is now readily completed by application of Lemma 14.11. □

15 Uniformly tempered estimates

The purpose of this section is to obtain uniformly tempered estimates for holomorphic families of Whittaker functions satisfying requirements of moderate growth.

Let P be a standard parabolic subgroup of G and (σ, H_σ) a representation of the discrete series of M_P. For $\varepsilon > 0$ we put

$$\mathfrak{a}^*_{P\mathbb{C}}(\varepsilon) = \{\nu \in \mathfrak{a}^*_{P\mathbb{C}} \mid |\mathrm{Re}\,(\nu)| < \varepsilon\}.$$

Definition 15.1 By a holomorphic family of Whittaker maps associated with (P, σ) and $\varepsilon_0 > 0$ we mean a family of maps

$$\mathrm{wh}_\nu : C^\infty(K/K_P : \sigma) \to C^\infty(G/N_0 : \chi), \qquad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)), \tag{15.1}$$

given by the matrix coefficient formula

$$\mathrm{wh}_\nu(f)(x) = \langle \pi_{\bar{P},\sigma,-\nu}(x)^{-1} f\,,\, j_{\bar{\nu}}\rangle \tag{15.2}$$

with $j_{\bar{\nu}} \in C^{-\infty}(\bar{P} : \sigma : \bar{\nu})_\chi$, $(\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0))$, such that $\nu \mapsto j_\nu$ is holomorphic as a $C^{-\infty}(K/K_P : \sigma)$-valued function.

Remark 15.2 Let $(\mathrm{wh}_\nu)_{\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)}$ be a holomorphic family of Whittaker maps as above. We note that the following assertions are valid.

(a) The map $(\nu, f) \mapsto \mathrm{wh}_\nu(f)$, $\mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0) \times C^\infty(K/K_P : \sigma) \to C^\infty(G/N_0 : \chi)$ is continuous and holomorphic in the variable ν.

(b) For each $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)$ the map (15.1) intertwines the generalized principal series $\pi_{\bar{P},\sigma,-\nu}$ with the left regular representation L.

Remark 15.3 It follows from Theorem 14.4 that for any $\varepsilon_0 > 0$ and all $\xi \in H^{-\infty}_{\sigma,\chi_P}$ the family $(\mathrm{wh}_\nu)_{\nu\in\mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)}$ of maps $H^\infty_\sigma \to C^\infty(G/N_0 : \chi)$ defined by

$$\mathrm{wh}_\nu(f)(x) = \langle \pi_{\bar{P},\sigma,-\nu}(x)^{-1} f \, , \, j(\bar{P}, \sigma, \bar{\nu}, \xi)\rangle$$

is a holomorphic family of Whittaker maps associated with (P, σ). Furthermore, by Theorem 14.9 it satisfies the condition of uniform moderate growth mentioned below.

To keep notation manageable, we will write

$$I^\infty_{P,\sigma} := C^\infty(K/K_P : \sigma). \tag{15.3}$$

Furthermore, we will use the notation

$$|(\nu, a)| := (1 + |\nu|)(1 + |\log a|),$$

for $a \in A$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$.

Definition 15.4 A Whittaker family (wh_ν) as in Definition 15.1 is said to have *uniform moderate growth* if there exist constants $r, s, N > 0$ and a continuous seminorm n on $I^\infty_{P,\sigma}$ such that

$$|\mathrm{wh}_\nu(f)(a)| \le |(\nu, a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} e^{r|\log a|} n(f) \tag{15.4}$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)$ and $a \in A$.

If we combine the estimate (15.4) with Lemma 2.5, then we see that for any linear functional $\xi \in \mathfrak{a}^*$ with $\xi \ge r|\cdot|$ on $\mathfrak{a}^+$ we may adapt the continuous seminorm n so that for all $f \in I^\infty_{P,\sigma}$ we have

$$|\mathrm{wh}_\nu(f)(a)| \le |(\nu, a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^\xi n(f) \tag{15.5}$$

for all $a \in A$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)$. We fix such a choice of ξ and observe that in particular $\xi \ge -\rho$ on $\mathfrak{a}^+$.

The above estimate can be improved to a much sharper estimate of *uniform tempered growth* for $\varepsilon_0 > 0$ taken sufficiently small. More precisely, we have the following result.

Theorem 15.5 *Let $G = {}^\circ G$, and let $(\mathrm{wh}_\nu)_{\nu\in\mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)}$ be a holomorphic family of Whittaker maps as in Definition 15.1. Assume the family satisfies the condition of uniform moderate growth formulated in (15.4). Then for $\varepsilon > 0$ sufficiently small there exist constants $s > 0, N > 0$ and a continuous seminorm n on $I^\infty_{P,\sigma}$ such that for all $f \in I^\infty_{P,\sigma}$, all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and all $a \in A$,*

$$|\mathrm{wh}_\nu(f)(a)| \le |(\nu, a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^{-\rho} n(f). \tag{15.6}$$

Before turning to its proof, we first formulate a useful consequence of the above result.

Corollary 15.6 *Let hypotheses be as in Theorem 15.5 and let $\varepsilon > 0$ be such that the conclusions of the theorem are valid.*

Let $u \in U(\mathfrak{g})$. Then there exist $s > 0, N > 0$ and a continuous seminorm n on $I^\infty_{P,\sigma}$ such that for all $f \in I^\infty_{P,\sigma}$, all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and all $x \in G$,

$$|L_u[\mathrm{wh}_\nu(f)](x)| \le (1+|\nu|)^N(1+|H(x)|)^N e^{s|\mathrm{Re}\,\nu||H(x)|}e^{-\rho H(x)}n(f). \tag{15.7}$$

Proof. In view of Remark 15.2 (b) and since $\pi^\infty_{\bar{P},\sigma,-\nu}(u)$ acts continuously on $I^\infty_{P,\sigma}$, with polynomial dependence on ν, the estimate (15.6) gives rise to an estimate

$$|L_u[\mathrm{wh}_\nu(f)](a)| \le |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|}a^{-\rho}n(f), \tag{15.8}$$

for all $f \in I^\infty_{P,\sigma}$, all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and all $a \in A$, provided N and n are suitably enlarged.

For a given finite subset $S \subset U(\mathfrak{g})$ we may enlarge N and n further to arrange that the estimate (15.8) is valid for all $u \in S$ and all f, ν, a as before.

For $k \in K, a \in A$ and $n \in N_0$ we have

$$|L_u(\mathrm{wh}_\nu(f))(kan)| = |L_{k^{-1}}L_u(\mathrm{wh}_\nu(f))(a)| = |L_{\mathrm{Ad}(k^{-1})u}\mathrm{wh}_\nu(L_{k^{-1}}f)(a)|.$$

We may write $\mathrm{Ad}(k^{-1})u = \sum_i c_i(k)u_i$ with $S = \{u_i\}$ a finite subset of $U(\mathfrak{g})$ and such that $c_i : K \longrightarrow \mathbb{C}$ are functions with sup-norm bounded by $C > 0$.

We may enlarge N and n so that the estimate (15.8) is valid with u replaced by any element of S and for all f, ν, a. There exist a continuous seminorm n' on $I^\infty_{P,\sigma}$ such that $n \circ L_k \le n'$ for all $k \in K$. From the last estimate mentioned above we now readily infer that for all $f \in I^\infty_{P,\sigma}$, all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and all $(k,a,n) \in K \times A \times N_0$,

$$|L_u(\mathrm{wh}_\nu(f))(kan)| \le |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|}e^{-\rho\log a}\, C|S|n'(f).$$

Enlarging N and n once more, we obtain the required estimate (15.7). □

In the proof of Theorem 15.5 the following terminology will be useful.

Definition 15.7 We will say that a functional $\xi \in \mathfrak{a}^*$ dominates the given Whittaker family (wh_ν) if there exist $\varepsilon > 0, s > 0, N > 0$ and n as above such that the estimate (15.5) is valid for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and $a \in A$.

Proof of Theorem 15.5. Clearly it is sufficient to prove that $-\rho$ dominates the Whittaker family (wh_ν). We will achieve this by improving ξ in a finite number of steps, each step corresponding to a simple root $\alpha \in \Delta$, by using the asymptotic behavior of $\mathrm{wh}_\nu(f)$ along standard maximal parabolic subgroups.

Since ${}^\circ G = G$, the collection Δ of simple roots in Σ^+ is a basis of $\mathfrak{a}^*$. Let $(h_\alpha)_{\alpha\in\Delta}$ be the dual basis in $\mathfrak{a}$. We will now establish an improvement step for each $\alpha \in \Delta$.

Put $\Phi = \Delta \setminus \{\alpha\}$. Then P_Φ is a maximal parabolic subgroup with split component $A_\Phi = \exp(\mathbb{R}h_\alpha)$.

We define the partial ordering $\preceq$ on $\mathfrak{a}^*$ by

$$\lambda \preceq \mu \iff \lambda(H) \le \mu(H) \quad \text{for all } H \in \mathfrak{a}^+. \tag{15.9}$$

The condition on the right is equivalent to $\lambda(h_\alpha) \le \mu(h_\alpha)$ for all $\alpha \in \Delta$.

Given $\xi \in \mathfrak{a}^*$ we write $i_\alpha(\xi)$ for the element in $\xi + \mathbb{R}\alpha$ satisfying $i_\alpha(\xi)(h_\alpha) = -\rho(h_\alpha)$. Equivalently, $i_\alpha(\xi) \in \mathfrak{a}^*$ is determined by

$$i_\alpha(\xi)(h_\beta) = \begin{cases} \xi(h_\beta) & \text{for} \quad \beta \in \Delta \setminus \{\alpha\}; \\ -\rho(h_\alpha) & \text{for} \quad \beta = \alpha. \end{cases}$$

Lemma 15.8 *If $\xi \succeq -\rho$, then for every simple root $\alpha \in \Delta$ it holds that $i_\alpha(\xi) \succeq -\rho$.*

Proof. This is straightforward. □

Lemma 15.9 (Improvement step) *Suppose that $\xi \in \mathfrak{a}^*$ dominates the Whittaker family* (wh_ν) *and satisfies $\xi \succeq -\rho$. Let $\alpha \in \Delta$.*

(a) *If $\xi(h_\alpha) - 1 \ge -\rho(h_\alpha)$ then for every $c \in [0, 1)$, the functional $\xi' := \xi - c\alpha$ dominates* (wh_ν) *and satisfies $\xi' \succeq -\rho$.*

(b) *If $\xi(h_\alpha) - 1 < -\rho(h_\alpha)$, then $i_\alpha(\xi)$ dominates* (wh_ν) *and satisfies $i_\alpha(\xi) \succeq -\rho$.*

The rest of this section will be dedicated to establishing this lemma. Before turning to the proof of the lemma we will show how Theorem 15.5 can be deduced from it.

Completion of the proof of Theorem 15.5. Let $\alpha \in \Delta$ and assume that $\xi \in \mathfrak{a}^*$ dominates (wh_ν) and satisfies $\xi \succeq -\rho$. Then $\xi(h_\alpha) \ge -\rho(h_\alpha)$. Let k be the smallest natural number such that $\xi(h_\alpha) - k < -\rho(h_\alpha)$. Then there exists a $c \in [0, 1)$ such that $\xi(h_\alpha) - kc = -\rho(h_\alpha)$. By applying (a) of the above lemma k-times successively, we find that $\xi'' := \xi - kc\alpha$ dominates (wh_ν), while $\xi'' \succeq -\rho$. Since $\xi''(h_\alpha) - 1 < -\rho(h_\alpha)$, we may apply (b) of the above lemma to conclude that $i_\alpha(\xi'')$ dominates (wh_ν) and satisfies $i_\alpha(\xi'') \succeq -\rho$ Since $i_\alpha(\xi'') = i_\sigma(\xi)$, we conclude that $i_\alpha(\xi)$ dominates (wh_ν) and satisfies $i_\alpha(\xi) \succeq -\rho$.

Let now $\alpha_1, \dots, \alpha_r$ be a numbering of the simple roots from Δ. Then by the above reasoning it follows that $\xi''' := i_{\alpha_r} \circ \cdots \circ i_{\alpha_1}(\xi)$ dominates (wh_ν) while $\xi''' \succeq -\rho$. Since $(h_\alpha)_{\alpha \in \Delta}$ is the basis of $\mathfrak{a}$ dual to Δ, it is readily verified that $\xi''' = -\rho$. □

Start of proof of Lemma 15.9. We assume that $(\mathrm{wh}_\nu)_{\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)}$ is a Whittaker family associated with (P, σ) which is dominated by $\xi \in \mathfrak{a}^*$. Moreover we assume that $\xi \succeq -\rho$. We fix a simple root $\alpha \in \Delta$ and put $\Phi := \Delta \setminus \{\alpha\}$. Our goal is to establish the two assertions (a) and (b) of Lemma 15.9. For this we will need a proper exploitation of the differential equations satisfied by the given Whittaker family.

For $X \in \mathfrak{g}_\mathbb{C}$ we denote by $\bar{X}$ the complex conjugate of X relative to the real form $\mathfrak{g}$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. The map $X \mapsto \bar{X}$ has a unique extension to a conjugate linear algebra isomorphism $U(\mathfrak{g}) \to U(\mathfrak{g})$, which is denoted by $u \mapsto \bar{u}$. In particular, this means that $\overline{uv} = \bar{u}\bar{v}$ for $u, v \in U(\mathfrak{g})$.

For $X \in \mathfrak{g}$ we have

$$R_X[\mathrm{wh}_\nu(f)](x) = \langle \pi(x^{-1})f\,,\,\pi_{\bar{P},\sigma,\bar{\nu}}(\bar{X}) j_{\bar{\nu}} \rangle, \tag{15.10}$$

for $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and $x \in G$. By complex linearity (15.10) is valid for all $X \in \mathfrak{g}_\mathbb{C}$, which leads to the similar formula with X replaced by a general element of $U(\mathfrak{g})$. Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{m}_1$; then

$$\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$$

is a Cartan subalgebra of $\mathfrak{g}$. We put $\mathfrak{h}^*_\mathbb{R} := i\mathfrak{t}^* \oplus \mathfrak{a}^*$. Then $\mathfrak{h}^*_\mathbb{R}$ is the real span of the roots of $\mathfrak{h}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Let $\gamma : \mathfrak{Z} \to P(\mathfrak{h})^{W(\mathfrak{h})}$ be the Harish-Chandra isomorphism for $(G, \mathfrak{h})$ and let $\gamma_{M_{1P}} : \mathfrak{Z}(M_{1P}) \to S(\mathfrak{h})^{W_P(\mathfrak{h})}$ be the similar isomorphism for $(M_{1P}, \mathfrak{h})$. Let $\Lambda_\sigma \in \mathfrak{h}^*_\mathbb{C} \cap \mathfrak{m}^*_{P\mathbb{C}}$ be an infinitesimal character for the representation σ of the discrete series of M_P, by which we mean that $\gamma_{1P}(\,\cdot\,, \Lambda_\sigma)$ is the character of $\mathfrak{Z}(M_P)$ by which it acts on H^∞_σ. We note that Λ_σ belongs to the real span of the roots of $(\mathfrak{m}_{1P}, \mathfrak{h})$, hence to $i\mathfrak{t}^* \oplus {}^*\mathfrak{a}^*_P$. Applying formula (15.10) with $Z \in \mathfrak{Z}$ in place of X, we find that

$$R_Z[\mathrm{wh}_\nu(f)] = \overline{\gamma(\bar{Z}, \Lambda_\sigma + \bar{\nu})}\, \mathrm{wh}_\nu(f) = \gamma(Z, -\Lambda_\sigma + \nu)\, \mathrm{wh}_\nu(f), \tag{15.11}$$

for all $Z \in \mathfrak{Z}$. Following an idea similar to the one in Section 4, but with dependence on parameters, we will exploit this system to establish the improvement step of Lemma 15.9.

The standard parabolic subgroup P_Φ has split component $A_\Phi := \exp \mathfrak{a}_\Phi = \exp \mathbb{R} h_\alpha$. We agree to write ${}^\backprime\mathfrak{a}_\Phi$ for the real linear span of the elements h_β with $\beta \in \Phi$ and, accordingly, ${}^\backprime A_\Phi = \exp({}^\backprime\mathfrak{a}_\Phi)$. We note that $\mathfrak{a} = {}^\backprime\mathfrak{a}_\Phi \oplus \mathfrak{a}_\Phi$ and $A = {}^\backprime A_\Phi A_\Phi$, see also (2.10).

We denote by $\mathfrak{Z}_{1\Phi} = \mathfrak{Z}_{\mathfrak{m}_{1\Phi}}$ the center of $U(\mathfrak{m}_{1\Phi})$. In view of the PBW theorem we have $U(\mathfrak{g}) = U(\mathfrak{m}_{1\Phi}) \oplus (\bar{\mathfrak{n}}_\Phi U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_\Phi)$. The associated projection $U(\mathfrak{g}) \to U(\mathfrak{m}_{1\Phi})$, restricted to $\mathfrak{Z}$, defines an algebra homomorphism

$$p : \mathfrak{Z} \to \mathfrak{Z}_{1\Phi}.$$

It is well known that p is injective and that $\mathfrak{Z}_{1\Phi}$ is a free $p(\mathfrak{Z})$-module of finite rank. Let $u_1, \ldots, u_\ell$ be a free basis of this module, and let E_Φ the complex linear span of this basis. Then it follows that

$$\mathfrak{Z}_{1\Phi} = E_\Phi p(\mathfrak{Z}).$$

Moreover, the map $(u, Z) \mapsto up(Z)$ induces a linear isomorphism $E_\Phi \otimes \mathfrak{Z} \simeq \mathfrak{Z}_{1\Phi}$. In the formulation of the following lemma, $\mathfrak{h}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Thus, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ with $\mathfrak{t}$ a maximal torus in $\mathfrak{m}$.

Lemma 15.10 *If I is a cofinite ideal of $\mathfrak{Z}$, then $I_\Phi := E_\Phi p(I)$ is a cofinite ideal in $\mathfrak{Z}_{1\Phi}$. Furthermore, if $\lambda_\Phi \in \mathfrak{h}_{\mathbb{C}}^*$ is an infinitesimal character for $\mathfrak{Z}_{1\Phi}$ appearing in the quotient module $\mathfrak{Z}_{1\Phi}/I_\Phi$, then there exists an infinitesimal character $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ for the $\mathfrak{Z}$-module $\mathfrak{Z}/I$ such that*

$$\lambda_\Phi \in W(\mathfrak{h})\lambda - \rho_\Phi. \tag{15.12}$$

Proof. Let $u \in E_\Phi$ and $W \in \mathfrak{Z}_{1\Phi}$ then $Wu = \sum_j u_j p(Z_j)$ with $Z_j \in \mathfrak{Z}$. It follows that $Wup(I) \subset \sum_j u_j p(Z_j)p(I) = \sum_j u_j p(Z_j I) \subset E_\Phi p(I)$. This shows that $E_\Phi p(I)$ is an ideal. In view of the linear isomorphism $E_\Phi \otimes \mathfrak{Z} \to \mathfrak{Z}_{1\Phi}$ we have $\mathfrak{Z}_{1\Phi}/E_\Phi p(I) = E_\Phi p(Z)/p(I)$ as complex vector spaces. Since $p(Z)/p(I)$ is finite dimensional, the cofiniteness of I_Φ follows.

Let $\xi \in \widehat{\mathfrak{Z}_{1\Phi}}$ be a character appearing in $\mathfrak{Z}_{1\Phi}/I_\Phi$. Then there exists a $k \in \mathbb{N}$ and an element $v \in \mathfrak{Z}_{1\Phi} \setminus I_\Phi$ such that $(W - \xi(W))^k v \in I_\Phi$ for all $W \in \mathfrak{Z}_{1\Phi}$. In particular the latter is valid for $W = p(Z)$, $Z \in \mathfrak{Z}$. Decompose $v = \sum u_i p(Z_i)$ with $Z_i \in \mathfrak{Z}$. Then it follows that $[p(Z) - \xi(p(Z))]^k p(Z_i) \in p(I)$ for all $Z \in \mathfrak{Z}$ and all $1 \leq i \leq \ell$. By injectivity of p this implies that

$$[Z - \xi(p(Z))]^k Z_i \in I, \tag{15.13}$$

for all $Z \in \mathfrak{Z}$ and all i. On the other hand, $v \notin I_\Phi$ implies that $Z_i \notin I$ for at least one i. Combining this with (15.13) we infer that the character $\xi \circ p \in \widehat{\mathfrak{Z}}$ appears in $\mathfrak{Z}/I$.

Let $\gamma : \mathfrak{Z} \to P(\mathfrak{h}_{\mathbb{C}}^*)^{W(\mathfrak{h})}$ and $\gamma_{1\Phi} : \mathfrak{Z}_{1\phi} \to P(\mathfrak{h}_{\mathbb{C}}^*)^{W_{1\Phi}(\mathfrak{h})}$ denote the canonical isomorphisms. Then it is well known that

$$\gamma_{1\Phi} \circ p = T_{\rho_\Phi} \circ \gamma,$$

where $T_{\rho_\Phi} \in \mathrm{Aut}(P(\mathfrak{h}_{\mathbb{C}}^*))$ is the translation $p \mapsto p(\,\cdot\, + \rho_\Phi)$.

Let λ_Φ be as stated. Then $\xi = \gamma_{1\Phi}(\,\cdot\,, \lambda_\Phi)$ is a character of $\mathfrak{Z}_{1\Phi}$ which appears in $\mathfrak{Z}_{1\Phi}/I_\Phi$. It follows that $\xi \circ p$ is a character of $\mathfrak{Z}$ appearing in $\mathfrak{Z}/I$ hence of the form $\gamma(\,\cdot\,, \lambda)$, with $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. We now conclude that for all $Z \in \mathfrak{Z}$ we have

$$\gamma(Z, \lambda_\Phi + \rho_\Phi) = \gamma(Z, \lambda).$$

This in turn implies that $\lambda_\Phi + \rho_\Phi \in W(\mathfrak{h})\lambda$. □

If $I \lhd \mathfrak{Z}$ is cofinite, we denote by $\mathrm{spec}(\mathfrak{Z}/I)$ the (finite) collection of infinitesimal characters appearing in $\mathfrak{Z}/I$. Since $U(\mathfrak{a}_\Phi)$ is a submodule of $\mathfrak{Z}_{1\Phi}$, the $\mathfrak{Z}_{1\Phi}$-module $\mathfrak{Z}_{1\Phi}/E_\Phi p(I)$ is a $U(\mathfrak{a}_\Phi)$-module as well.

Corollary 15.11 *Let $I \lhd \mathfrak{Z}$ be cofinite. Then the set of $\mathfrak{a}_\Phi$-weights in $\mathfrak{Z}_{1\Phi}/E_\Phi p(I)$ is contained in*

$$\cup_{\lambda \in \mathrm{spec}(\mathfrak{Z}/I)} W(\mathfrak{h})\lambda|_{\mathfrak{a}_\Phi} - \rho_\Phi.$$

Proof. Apply Lemma 15.10. □

We now specialize to the ideal $I_\nu = \ker \gamma(\,\cdot\,, -\Lambda_\sigma + \nu)$ for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$. Then

$$\mathrm{spec}(\mathfrak{Z}/I_\nu) = W(\mathfrak{h})(-\Lambda_\sigma + \nu).$$

Corollary 15.12 *The inclusion map induces a linear isomorphism*

$$E_\Phi \xrightarrow{\simeq} \mathfrak{Z}_{1\Phi}/\mathfrak{Z}_{1\Phi}p(I_\nu). \tag{15.14}$$

The set $\mathrm{wt}(\nu)$ *of (generalized)* $\mathfrak{a}_\Phi$*-weights in the displayed quotient equals*

$$\mathrm{wt}(\nu) = W(\mathfrak{h})(-\Lambda_\sigma + \nu)|_{\mathfrak{a}_\Phi} - \rho_\Phi.$$

We write $E_{\Phi,\nu}$ for the space E_Φ equipped with the $\mathfrak{a}_\Phi$ action for which the map (15.14) becomes an isomorphism of $\mathfrak{a}_\Phi$ modules. We agree to write $B_1(\nu)$ for the linear map by which h_α acts on $E_{\Phi,\nu}$. There exist unique $Z^k_j \in \mathfrak{Z}$ such that

$$h_\alpha u_j = \sum_{k=1}^{\ell} u_k p(Z^k_j), \qquad (1 \le j \le \ell). \tag{15.15}$$

Since $Z^k_j - \gamma(Z^k_j, -\Lambda_\sigma + \nu) \in I_\nu$ it follows that, for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$B_1(\nu)u_j \in \sum_k \gamma(Z^k_j, -\Lambda_\sigma + \nu)u_k + \mathfrak{Z}_{1\Phi}p(I_\nu).$$

Therefore, the matrix of $B_1(\nu)$ relative to the basis $u_1, \ldots, u_\ell$ of E_Φ is given by

$$B_1(\nu)^k_j = \gamma(Z^k_j, -\Lambda_\sigma + \nu). \tag{15.16}$$

In particular, it follows that $\nu \mapsto B_1(\nu)$ is polynomial $\mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{End}(E_\Phi)$.

Let I_χ denote the left ideal in $U(\mathfrak{g})$ generated by the elements $Y - \chi_*(Y)$ for $Y \in \mathfrak{n}_0$. Let $I_{\sigma,\nu}$ denote the left ideal in $U(\mathfrak{g})$ generated by the ideal I_ν.

Lemma 15.13 *There exist elements* $v_j \in \bar{\mathfrak{n}}_\Phi U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)$ *such that*

$$h_\alpha u_j - B_1(\nu)u_j - v_j \in I_\chi + I_{\sigma,\nu}, \qquad (1 \le j \le \ell).$$

Proof. Let $Z^k_j \in \mathfrak{Z}$ be as in (15.15). Now $p(Z^k_j) - Z^k_j \in \bar{\mathfrak{n}}_\Phi U(\mathfrak{g})$, and since $u_j \in U(\mathfrak{m}_{1\Phi})$ it follows that also

$$u_k p(Z^k_j) - u_k Z^k_j \in \bar{\mathfrak{n}}_\Phi U(\mathfrak{g}) = \bar{\mathfrak{n}}_\Phi U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)U(\mathfrak{n}_0).$$

We note that any $W \in U(\mathfrak{n}_0)$ equals $\chi_*(W)$ modulo I_χ, hence

$$u_k p(Z^k_j) - u_k Z^k_j \in v^k_j + I_\chi,$$

with $v^k_j \in \bar{\mathfrak{n}}_\Phi U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)$. It follows that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$u_k p(Z^k_j) - u_k \gamma(Z^k_j, -\Lambda_\sigma + \nu) - v^k_j \in I_\chi + I_{\sigma,\nu}.$$

Summing the above over k, putting $v_j = \sum_k v^k_j$ and using (15.16) we obtain the desired assertion. □

We proceed with the proof of the improvement step of Lemma 15.9. For $f \in I^\infty_{P,\sigma}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ we define the function $F(f,\nu) : A \to \mathbb{C}^\ell$ by

$$F(f,\nu,a)_j := \langle \pi_{-\nu}(a)^{-1} f \,,\, \pi_{\bar\nu}(u_j) j_{\bar\nu} \rangle,$$

where we briefly wrote π_ν for $\pi_{\bar P,\sigma,\nu}$. We define the function $R(f,\nu) : A \to \mathrm{End}(\mathbb{C}^\ell)$ by

$$R(f,\nu,a)_j := \langle \pi_{-\nu}(a)^{-1} f \,,\, \pi_{\bar\nu}(v_j) j_{\bar\nu} \rangle.$$

Furthermore, let $B(\nu) \in \mathrm{End}(\mathbb{C}^\ell)$ be the endomorphism with matrix equal to $\overline{B_1(\bar\nu)}^{\mathrm{T}}$.

For the following result we recall the decomposition $A = {}^{\backprime}A_\Phi A_\Phi$. It allows us to decompose any element $a \in A$ in a unique way as

$$a = {}^{\backprime}a a_t,$$

with ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and with $a_t = \exp t h_\alpha$, $(t \in \mathbb{R})$.

Lemma 15.14 *The function F introduced above satisfies the equation*

$$\frac{d}{dt} F(f,\nu,{}^{\backprime}a a_t) = B(\nu) F(f,\nu,{}^{\backprime}a a_t) + R(f,\nu,{}^{\backprime}a a_t),$$

for every $f \in I^\infty_{P,\sigma}$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)$, ${}^{\backprime}a \in A_\Phi$ and $t \in \mathbb{R}$.

Here B is a polynomial function $\mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{End}(\mathbb{C}^\ell)$. For every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the spectrum of $B(\nu)$ satsfies

$$\mathrm{spec}(B(\nu)) \subset \{ w(-\Lambda_\sigma + \nu)(h_\alpha) - \rho(h_\alpha) \mid w \in W(\mathfrak{h}) \}, \qquad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}).$$

Proof. Noting that $I_{\sigma,\bar\nu}$ and I_χ vanish on $j_{\bar\nu}$, we obtain that the functions F_j introduced above satisfy the equations

$$\begin{aligned} \frac{d}{dt} F_j(f,\nu,{}^{\backprime}a a_t) &= \langle \pi_{-\nu}(a)^{-1} f \,,\, \pi_{\bar\nu}(h_\alpha u_j) j_{\bar\nu} \rangle \\ &= \langle \pi_{-\nu}(a)^{-1} f \,,\, \pi_{\bar\nu}(B_1(\bar\nu) u_j + v_j) j_{\bar\nu} \rangle, \end{aligned}$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)$, ${}^{\backprime}a \in {}^{\backprime}A_\Phi$ and $t \in \mathbb{R}$. This gives the required equation, with $B(\nu)$ as asserted. The spectrum of $B(\nu)$ equals that of $\overline{B_1(\bar\nu)}$ hence consists of the elements

$$\overline{w(-\Lambda_\sigma + \bar\nu)(h_\alpha)} - \rho_\Phi(h_\alpha), \tag{15.17}$$

for $w \in W(\mathfrak{h})$. Since $\Lambda_\sigma \in \mathfrak{h}^*_{\mathbb{R}} = i\mathfrak{t}^* \oplus \mathfrak{a}^*$ whereas $W(\mathfrak{h})$ leaves $\mathfrak{h}_{\mathbb{R}}$ invariant, it follows that for each $w \in W(\mathfrak{h})$ the value $w(-\Lambda_\sigma)(h_\alpha) = -\Lambda_\sigma(w^{-1} h_\alpha)$ is real. Likewise, it follows that $\overline{w\bar\nu(h_\alpha)} = w\nu(h_\alpha)$. Finally, since $\rho(h_\alpha) = \rho_\Phi(h_\alpha)$, it follows that the element (15.17) equals $w(-\Lambda_\sigma + \nu)(h_\alpha) - \rho(h_\alpha)$. □

By integration the equation of Lemma 15.14 leads to the equality

$$F(f,\nu,{}^{\backprime}a a_t) = e^{tB(\nu)}F({}^{\backprime}a) + e^{tB(\nu)}\int_0^t e^{-\tau B(\nu)}R(f,\nu,a)\,d\tau, \tag{15.18}$$

for $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_0)$, ${}^{\backprime}a \in A_\Phi$ and $t \in \mathbb{R}$. We will first derive estimates for F and R, following from the information that $\xi \in \mathfrak{a}^*$ dominates $\mathrm{wh} = (\mathrm{wh}_\nu)$. This means that there exist $\varepsilon > 0, s > 0, N > 0$ and a continuous seminorm n on $I^\infty_{P,\sigma}$ such that (15.5) is valid for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and $a \in A$.

Lemma 15.15 *There exist $\varepsilon > 0, s > 0, N > 0$ and a continuous seminorm n on $I^\infty_{P,\sigma}$ such that*

$$\|F(f,\nu,a)\| \le |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^\xi n(f) \tag{15.19}$$

for all $f \in I^\infty_\sigma$, $\nu \in \mathfrak{a}^*(\varepsilon)$ and $a \in A$.

Proof. We take $\varepsilon > 0$ sufficiently small such that the estimate (15.5) is valid. Since

$$F(f,\nu,a)_j = \langle \pi_{-\nu}(a)^{-1}\pi_{-\nu}(\bar{u}_j^\vee)f\,,\, j_{\bar{\nu}}\rangle = \mathrm{wh}_\nu(\pi_{-\nu}(\bar{u}_j^\vee)f)(a)$$

the estimate (15.19) follows from (15.5) with the same s and possibly enlarged constant $N > 0$ and enlarged seminorm n. □

Remark 15.16 (structure of proof). Throughout the proof of Lemma 15.9 we will prove assertions of the form

$$\exists(\varepsilon,s,N,n) : A(\varepsilon,s,N,n) \tag{15.20}$$

where $A(\varepsilon,s,N,n)$ is an assertion (usually containing an estimate) depending on positive constants ε, s, N and a continuous seminorm n on $I^\infty_{P,\sigma}$. Moreover, the assertion has the property that for any (ε',s',N',n') with $\varepsilon' \le \varepsilon, s' \ge s, N' \ge N$ and $n' \ge n$,

$$A(\varepsilon,s,N,n) \Rightarrow A(\varepsilon',s',N',n'). \tag{15.21}$$

A typical assertion of this type is the assertion

$$\forall(f \in I^\infty_{P,\sigma}, \nu \in \mathfrak{a}^*_{P\mathbb{C}}, a \in A) : \|F(f,\nu,a)\| \le |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^\xi n(f)$$

of Lemma 15.15. The proofs of assertions of this type will make use of finitely many valid similar assertions $\exists(\varepsilon,s,N,n) : A_i(\varepsilon,s,N,n)$, for $i \in I$, with I a finite index set. If all A_i have the property (15.21) then it follows that also $\exists(\varepsilon,s,N,n) : \wedge_{i\in I}A_i(\varepsilon,s,N,n)$ is valid. Indeed, if $A_i(\varepsilon_i,s_i,N_i,n_i)$ is true for every $i \in I$, then $A_i(\varepsilon,s,N,n)$, for $i \in I$, are simultaneously valid as soon as $\varepsilon \le \min_i \varepsilon_i$, $s \ge \max_i s_i, N \ge \max_i N_i$ and $n \ge \max_i n_i$. In the proof we shall indicate this informally by saying that $A_i(\varepsilon,s,N,n)$ are valid for sufficiently small ε and sufficiently large s, N and n. A logical reasoning will then give the validity of $A(\varepsilon',s',N',n')$ for suitably chosen ε', s', N', n', which finally allows the conclusion that (15.20) is valid.

Lemma 15.17 *There exist $\varepsilon > 0, s > 0, N > 0$ and a continuous seminorm n on $I^\infty_{P,\sigma}$ such that*

$$\|R(f,\nu,a)\| \leq |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^{\xi-\alpha} n(f) \tag{15.22}$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and $a \in A$.

Proof. We recall that $R(f,\nu)$ is the $\mathbb{C}^\ell$-valued function defined by

$$R(f,\nu,a)_j := \langle \pi_{-\nu}(a)^{-1} f\,,\, \pi_{\bar\nu}(v_j) j_{\bar\nu} \rangle.$$

We will now derive the estimate for R with the required properties of uniformity. For this we note that $\pi_{\bar\nu}(v_j) j_{\bar\nu}$ may be written as a finite sum of terms of the form $\pi_{\bar\nu}(U) j_{\bar\nu}$, with $U \in U(\bar{\mathfrak{n}}_\Phi)U(\bar{\mathfrak{n}}_0 + \mathfrak{m}_1)$ of $\mathfrak{a}$-weight $-\mu \in -\sum_{\beta\in\Delta}\mathbb{N}\beta$ such that $\mu(h_\alpha) \geq 1$. Each corresponding term $r(f,\nu,a)$ in $R(f,\nu,a)_j$ may be rewritten as

$$\begin{aligned} r(f,\nu,a) &= \langle \pi_{-\nu}(U^*)\pi_{-\nu}({}^{`}aa_t)^{-1} f\,,\, j_{\bar\nu}\rangle \\ &= ({}^{`}aa_t)^{-\mu}\langle \pi_{-\nu}({}^{`}aa_t)^{-1}\pi_{-\nu}(U^*) f\,,\, j_{\bar\nu}\rangle. \end{aligned}$$

We note that the restriction of μ to ${}^{`}\mathfrak{a}_\Phi$ equals the restriction of $\mu_\Phi := \sum_{\beta\in\Phi}\mu_\beta\beta$ to this space. For each $\beta \in \Phi$ we may select a simple root vector $X_\beta \in \mathfrak{g}_\beta$ such that $\chi_*(X_\beta) = 1$. The product $X := \prod_{\beta\in\Phi} X_\beta^{\mu_\beta}$ belongs to $U(\mathfrak{n}_0)$, satisfies $\chi_*(X) = 1$ and has $\mathfrak{a}$-weight μ_Φ. Therefore,

$$\mathrm{Ad}({}^{`}aa_t)X = ({}^{`}a)^\mu X$$

and it follows that

$$\begin{aligned} r(f,\nu,a) &= ({}^{`}aa_t)^{-\mu}\langle \pi_{-\nu}(X^*)\pi_{-\nu}({}^{`}aa_t)^{-1}\pi_{-\nu}(U^*) f\,,\, j_{\bar\nu}\rangle \\ &= (a_t)^{-\mu}\langle \pi_{-\nu}({}^{`}aa_t)^{-1}\pi_{-\nu}(X^*U^*) f\,,\, j_{\bar\nu}\rangle \\ &= (a_t)^{-\mu}\mathrm{wh}_\nu(\pi_{-\nu}(X^*U^*)f)({}^{`}aa_t). \end{aligned}$$

We now select $\varepsilon, s, N > 0$ and n a continuous seminorm on $I^\infty_{P,\sigma}$ which make (15.5) valid for f, ν, a in the indicated sets. Then it follows that there exists a constant $C_\mu > 0$, only depending on μ, such that

$$\begin{aligned} |r(f,\nu,a)| &\leq C_\mu (a_t)^\mu ({}^{`}aa_t)^\xi |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log({}^{`}aa_t)|} n(\pi_{-\nu}(X^*U^*)f) \\ &\leq ({}^{`}aa_t)^{\xi-\alpha}|(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log({}^{`}aa_t)|} n'(f) \end{aligned}$$

with n' a seminorm on $I^\infty_{P,\sigma}$, independent of f. Combining the above estimates and enlarging n, N if necessary, we find that

$$\|R(f,\nu,a)\| \leq |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^{\xi-\alpha} n(f),$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and $a \in A$. □

Our next goal is to show how this stronger estimate on the remainder term leads to an improved estimate of F, hence of $\mathrm{wh}_\nu(f) = F_1(f, \nu)$. For this we need to decompose the formula (15.18) into parts corresponding to the spectrum of $B(\nu)$. In the course of the argument we will impose finitely many conditions on the constant $\varepsilon > 0$, ensuring it is sufficiently small.

Let us first analyze the spectrum of $B(\nu)$, in particular its dependence on ν. For $w \in W(\mathfrak{h})$ we write $x_w = -w(\Lambda_\sigma)(h_\alpha) - \rho(h_\alpha)$ which is a real number as shown in the proof of Lemma 15.14. For every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ the spectrum of $B(\nu)$ consists of complex numbers of the form

$$x_w + w(\nu)(h_\alpha),$$

for $w \in W(\mathfrak{h})$. Put

$$X := \{x_w \mid w \in W(\mathfrak{h})\}.$$

Let $\gamma > 0$ be a positive real number such that all distinct elements x_1 and x_2 of X satisfy $|x_1 - x_2| > 2\gamma$. Fix $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \frac{1}{2}\gamma |h_\alpha|^{-1}.$$

In the course of this section we will always assume that $0 < \varepsilon \le \varepsilon_1$.

We note that $W(\mathfrak{h})$ preserves the subspace $\mathfrak{h}_\mathbb{R} = i\mathfrak{t} + \mathfrak{a}$ of $\mathfrak{h}_\mathbb{C}$. Let $p_\mathfrak{a}$ denote the projection $\mathfrak{h}_\mathbb{R} \to \mathfrak{a}$ along $i\mathfrak{t}$. Then for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and $w \in W(\mathfrak{h})$ we have

$$\begin{aligned} |\mathrm{Re}\,(w\nu(h_\alpha))| &= |\mathrm{Re}\,[\nu(w^{-1}(h_\alpha))]| \\ &= |\mathrm{Re}\,[\nu(p_\mathfrak{a}(w^{-1}(h_\alpha)))]| \\ &\le |\mathrm{Re}\,\nu||p_\mathfrak{a}(w^{-1}(h_\alpha))| \le |\mathrm{Re}\,\nu||h_\alpha|. \end{aligned}$$

Likewise, for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we have

$$|\mathrm{Im}\,(w\nu(h_\alpha))| \le |\mathrm{Im}\,\nu||h_\alpha|.$$

Given $x \in X$ and $w \in W(\mathfrak{h})$ such that $x_w = x$ we thus see that for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ we have

$$|\mathrm{Re}\,(x_w + w\nu(h_\alpha)) - x| \le |\mathrm{Re}\,\nu||h_\alpha| < \gamma/2, \tag{15.23}$$

and

$$|\mathrm{Im}\,(x_w + w\nu(h_\alpha))| \le |\mathrm{Im}\,(\nu)||h_\alpha|. \tag{15.24}$$

For $t \in \mathbb{R}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ we define

$$C_{t,\nu} := \frac{\gamma}{2}(1 + |t|)^{-1}(1 + |\nu|)^{-1}.$$

and the rectangle $R(t, \nu) \subset \mathbb{C}$ to be the set of points $z \in \mathbb{C}$ such that

$$|\mathrm{Re}\,z| \le |\mathrm{Re}\,\nu||h_\alpha)| + C_{t,\nu}, \quad |\mathrm{Im}\,z| \le |\mathrm{Im}\,\nu||h_\alpha| + C_{t,\nu}.$$

Then it is readily seen that, for $t \in \mathbb{R}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, $R(t,\nu) \subset [-\gamma,\gamma] + i\mathbb{R}$, from which it readily follows that the translated rectangles $x + R(t,\nu)$ are mutually disjoint. For $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ we define $S(\nu)$ to be the rectangle of points with

$$|\mathrm{Re}\, z| \leq |\mathrm{Re}\, \nu||h_\alpha)|, \quad |\mathrm{Im}\, z| \leq |\mathrm{Im}\, \nu||h_\alpha|.$$

Then it is clear that $S(\nu)$ is contained in the interior of $R(t,\nu)$ so that the translated rectangles $x + S(\nu)$, $(x \in X)$, are mutually disjoint as well. Furthermore, it follows from the estimates (15.23) and (15.24) that

$$\mathrm{spec}\, B(\nu) = \bigcup_{x \in X} \mathrm{spec}\, B(\nu) \cap [x + S(\nu)]. \tag{15.25}$$

For $x \in X$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ we denote by $P_x(\nu)$ the spectral projection of $B(\nu)$ onto the sum of the generalized eigenspaces corresponding to the eigenvalues from $\mathrm{spec}\, B(\nu) \cap [x + S(\nu)]$. Then P_x is a holomorphic function on $\mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ with values in $\mathrm{End}(\mathbb{C}^\ell)$. We note that

$$I = \sum_{x \in X} P_x(\nu).$$

Lemma 15.18 *There exists a $C > 0$ such that for every $x \in X$, all $t \in \mathbb{R}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_1)$,*

$$\|e^{tB(\nu)} P_x(\nu)\| \leq C(1+|t|)^p (1+|\nu|)^p e^{xt + |h_\alpha||\mathrm{Re}\,\nu||t|}. \tag{15.26}$$

Proof. Since for all $t \in \mathbb{R}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_1)$ we have that

$$\mathrm{spec}\, B(\nu) \cap (x + R(t,\nu)) \subset x + S(\nu) \subset \mathrm{int}(x + R(t,\nu)), \tag{15.27}$$

it follows that

$$P_x(\nu) e^{tB(\nu)} = \frac{1}{2\pi i} \int_{x + \partial R(t,\nu)} e^{tz} (zI - B(\nu))^{-1}\, dz.$$

We will complete the proof by estimation of the integral. First of all, the length of the boundary of $x + R(t,\nu)$ is estimated by

$$\mathrm{length}(\partial R(t,\nu)) \leq 2\varepsilon |h_\alpha| + 2|\nu||h_\alpha| + 4C_{t,\nu} \leq 3\gamma\varepsilon^{-1}(1+|\nu|). \tag{15.28}$$

For $z \in x + \partial R_{t,\nu}$ we have

$$|e^{-tx} e^{tz}| \leq e^{|t||\mathrm{Re}\, z - x|} \leq e^{|t||\mathrm{Re}\,\nu||h_\alpha| + |t| C_{t,\nu}},$$

so that

$$|e^{tz}| \leq e^{tx} e^{|t||\mathrm{Re}\,\nu||h_\alpha|} e^{\gamma/2}. \tag{15.29}$$

If $x, x' \in X$ then the distance from $x + \partial R(t,\nu)$ to $x' + S(\nu)$ is at least $C_{t,\nu}$. From (15.25) and (15.27) we now see that for $z \in x + \partial R_{t,\nu}$ the distance of z to the spectrum of $B(\nu)$ is at least $C_{t,\nu}$ so that

$$|\det(zI - B(\nu))|^{-1} \leq C_{t,\nu}^{-\ell} \leq (2/\gamma)^{\ell}(1+|t|)^{\ell}(1+|\nu|)^{\ell}.$$

In view of Cramer's rule, there exists a constant $C_\ell > 0$ such that, for all $A \in \mathrm{GL}(\ell, \mathbb{C})$,

$$\|A^{-1}\| \leq C_\ell |\det A|^{-1}(1+\|A\|)^{\ell-1}.$$

Applying this with $A = (zI - B(\nu))$ we see that for $z \in x + \partial R(t,\nu)$,

$$\|(zI - B(\nu))^{-1}\| \leq C_\ell (2/\gamma)^{\ell}(1+|t|)^{\ell}(1+|\nu|)^{\ell}(|z| + \|B(\nu)\|)^{\ell-1}.$$

As $\nu \mapsto B(\nu)$ is polynomial in ν, there exist constants $N \in \mathbb{N}$ and $C' > 0$ such that for all $t \in \mathbb{R}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and $z \in x + \partial R(t,\nu)$,

$$\|(zI - B(\nu))^{-1}\| \leq C'(1+|t|)^{\ell}(1+|\nu|)^{N}. \tag{15.30}$$

Combining the estimates (15.28), (15.29) and (15.30) we infer the existence of a constant $C > 0$ such that for all $t \in \mathbb{R}$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon_1)$ the estimate (15.26) is valid with $p = \max(\ell, N+1)$. □

We now decompose $F(f,\nu,a)$ and $R(f,\nu,a)$ into components

$$F_x(f,\nu,a) := P_x(\nu)F(f,\nu,a), \qquad R_x(f,\nu,a) = P_x(\nu)R(f,\nu,a).$$

Then using (15.26) with $t = 0$ we obtain, after decreasing $\varepsilon > 0$ and increasing N and n suitably,

$$\|F_x(f,\nu,a)\| \leq |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^{\xi} n(f)$$

and

$$\|R_x(f,\nu,a)\| \leq |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|} a^{\xi-\alpha} n(f)$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, $a \in A$.

We will now obtain sharper estimates for F_x, for each $x \in X$. Our main tool will be the following identity which follows from (15.18) by application of $P_x(\nu)$;

$$F_x(f,\nu,`aa_t) = e^{tB(\nu)}F_x(f,\nu,`a) + e^{tB(\nu)}\int_0^t e^{-\tau B(\nu)} R_x(f,\nu,a)\, d\tau. \tag{15.31}$$

In the course of this proof we will need to distinguish two cases, depending on which of the following sets x belongs to:

$$X_+ := X \cap\,]\xi(h_\alpha) - 1, \infty[, \quad \text{and} \quad X_- := X \setminus X_+. \tag{15.32}$$

There exists a constant $c_0 \in [0,1[$ such that $(\xi(h_\alpha) - [c_0, 1[\,) \cap X = \emptyset$. Furthermore, we fix an arbitrary $c \in [c_0, 1[$. Then for all $x \in X$ the following assertion is valid:

$$x > \xi(h_\alpha) - 1 \iff x > \xi(h_\alpha) - c. \tag{15.33}$$

We put $\xi' := \xi - c\alpha$; then by (15.33) we have, for $x \in X$,

$$x \in X_\pm \iff \pm(x - \xi'(h_\alpha)) > 0.$$

We fix $\varepsilon_2 > 0$ such that

$$\xi'(h_\alpha) + [-\varepsilon_2|h_\alpha|, \varepsilon_2|h_\alpha|] \cap X = \emptyset.$$

In the course of this section we will always assume that $0 < \varepsilon \leq \varepsilon_2$. Then for every $x \in X$ and all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ the real part of the spectrum $\operatorname{spec}[B(\nu)|_{\operatorname{im}(P_x(\nu))}]$ is contained in $[\, x - \varepsilon|h_\alpha|, x + \varepsilon|h_\alpha|]$, which is contained in the interval $]\xi'(h_\alpha), \infty[$ if $x \in X_+$ and in the interval $] - \infty, \xi'(h_\alpha)[$ if $x \in X_-$.

Lemma 15.19 *Assume that $\xi \in \mathfrak{a}^*$ dominates* (wh_ν) *and let $\alpha > 0$ be simple. Then there exist $\varepsilon > 0$, $N > 0$, $s > 0$ and a continuous seminorm n on I^∞_σ such that for every $f \in I^\infty_\sigma$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$, and $t \geq 0$ the following estimates are valid.*

(a) *If $x \in X_-$, then*

$$\| \int_0^t e^{(t-\tau)B(\nu)} R_x(f, \nu, {}^\backprime a a_\tau)\, d\tau \,\| \leq |(\nu, {}^\backprime a a_t)|^N ({}^\backprime a a_t)^{\xi'} e^{s|\operatorname{Re}\nu||\log({}^\backprime a a_t)|} n(f).$$

(b) *If $x \in X_+$, then*

$$\| \int_t^\infty e^{(t-\tau)B(\nu)} R_x(f, \nu, {}^\backprime a a_\tau)\, d\tau \,\| \leq |(\nu, {}^\backprime a a_t)|^N ({}^\backprime a a_t)^{\xi'} e^{s|\operatorname{Re}\nu||\log({}^\backprime a a_t)|} n(f),$$

with absolutely converging integral.

Proof. After decreasing $\varepsilon > 0$ and increasing s, N and n if necessary, we obtain, for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ ${}^\backprime a \in {}^\backprime A_\Phi$ and $t, \tau \in \mathbb{R}$,

$$\|e^{(t-\tau)B(\nu)} R_x(f, \nu, {}^\backprime a a_\tau)\| \leq C_N(\nu, {}^\backprime a)\, D_N(\nu, t, \tau)\, n(f), \tag{15.34}$$

where

$$C_N(\nu, {}^\backprime a) = |(\nu, {}^\backprime a)|^N ({}^\backprime a)^\xi e^{s|\operatorname{Re}\nu||\log {}^\backprime a|}$$

and

$$\begin{aligned} D_N(\nu, t, \tau) &= |(t,\tau)|^N e^{(t-\tau)x} e^{(s+1)|\operatorname{Re}\nu||h_\alpha||\tau|} (a_\tau)^{\xi-\alpha} \\ &= |(t,\tau)|^N e^{tx} e^{(s+1)|\operatorname{Re}\nu||h_\alpha||\tau|} e^{[-x+\xi'(h_\alpha)]\tau} e^{(c-1)\tau}. \end{aligned}$$

Here we have used the notation $|(t,\tau)|^N = (1+|\tau|)^N(1+|t|)^N$ for $\tau, t \in \mathbb{R}$.

In order to prove (a), assume that $x < \xi'(h_\alpha)$. Then we may fix $\varepsilon_s > 0$ so that

$$\varepsilon_s(s+1)|h_\alpha| - x + \xi'(h_\alpha) > 0.$$

By decreasing ε further if necessary, we may asssume that $0 < \varepsilon \leq \varepsilon_s$. Then for $t \geq 0$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*(\varepsilon)$ we have that the function

$$\tau \mapsto e^{tx} e^{(s+1)|\mathrm{Re}\,\nu||h_\alpha||\tau|} e^{[-x+\xi'(h_\alpha)]\tau}$$

is increasing on $[0,t]$, hence dominated by its value at t, so that

$$\begin{aligned} \int_0^t D_N(\nu,\tau,t)d\tau &\leq e^{[(s+1)|\mathrm{Re}\,\nu||h_\alpha|+\xi'(h_\alpha)]t} \int_0^t (1+|(t,\tau)|)^N e^{(c-1)\tau}\, d\tau \\ &\leq C' e^{(s+1)|\mathrm{Re}\,\nu||\log a_t|} (a_t)^{\xi'} (1+|\log a_t|)^N, \end{aligned}$$

where

$$C' = \int_0^\infty (1+\tau)^N e^{(c-1)\tau}\, d\tau. \tag{15.35}$$

We observe that $({}^{\backprime}a)^{\xi} = ({}^{\backprime}a)^{\xi'}$. Accordingly, it now follows that

$$C_N(\nu, {}^{\backprime}a) \int_0^t D_N(\nu,\tau,t)d\tau \leq C'|(\nu, {}^{\backprime}a a_t)|^{2N} e^{2(s+1)|\mathrm{Re}\,\nu||\log({}^{\backprime}a a_t)|} ({}^{\backprime}a a_t)^{\xi'}.$$

In view of (15.34) we finally obtain the desired estimate of (a), with s, N and n chosen large enough.

We now turn to (b), and assume $x > \xi'(h_\alpha)$. Then there exists $\varepsilon_s' > 0$ such that

$$\varepsilon_s'(s+1)|h_\alpha| - x + \xi'(h_\alpha) < 0.$$

Decreasing ε further if necessary, we may assume that $0 < \varepsilon < \varepsilon_s'$. Then for $t \geq 0$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*(\varepsilon)$ the function

$$\tau \mapsto e^{tx} e^{(s+1)|\mathrm{Re}\,\nu||h_\alpha||\tau|} e^{[-x+\xi'(h_\alpha)]\tau}$$

is decreasing on $[t,\infty)$ hence dominated by its value at t, so that

$$\begin{aligned} \int_t^\infty D_N(\nu,\tau,t)d\tau &\leq e^{[(s+1)|\mathrm{Re}\,\nu||h_\alpha|+\xi'(h_\alpha)]t} \int_t^\infty (1+|(t,\tau)|)^N e^{(c-1)\tau}\, d\tau \\ &\leq C'(1+|\log a_t|)^N e^{(s+1)|\mathrm{Re}\,\nu||\log a_t|} (a_t)^{\xi'}, \end{aligned}$$

with C' given by (15.35). As in the first part of the proof, we now infer that

$$C_N(\nu, {}^{\backprime}a) \int_t^\infty D_N(\nu,\tau,t)d\tau \leq C'|(\nu, {}^{\backprime}a a_t)|^{2N} e^{2(s+1)|\mathrm{Re}\,\nu||\log({}^{\backprime}a a_t)|} ({}^{\backprime}a a_t)^{\xi'}.$$

Using (15.34) and further enlarging s, N and n if necessary, we obtain the desired estimate of (b). □

Proposition 15.20 *Let the Whittaker family* (wh_ν) *be dominated by* $\xi \in \mathfrak{a}^*$. *Assume that* $\xi \geq -\rho$. *Let* $x \in X$.

(a) *If* $x \leq \xi(h_\alpha) - 1$, *then* F_x *is dominated by* $\xi' = \xi - c\alpha$ *for all* $c \in [0, 1)$.

(b) *If* $x > \xi(h_\alpha) - 1 \geq -\rho(h_\alpha)$, *then* F_x *is dominated by* $\xi' = \xi - c\alpha$ *for all* $c \in [0, 1)$.

(c) *If* $x > \xi(h_\alpha) - 1$ *and* $\xi(h_\alpha) - 1 < -\rho(h_\alpha)$, *then* F_x *is dominated by* $i_\alpha(\xi)$.

Before we continue with the proof of Proposition 15.20, we will first argue that the proposition is sufficient for the proof of the improvement step asserted in Lemma 15.9. We first observe that F is dominated by a functional $\theta \in \mathfrak{a}^*$ if and only if every component F_x, for $x \in X$ is dominated by θ, where the obvious extension of the notion of domination is assumed.

Proof of Lemma 15.9. We begin by observing that the hypothesis on ξ in Lemma 15.9 (a) guarantees that $\xi' \geq -\rho$. In (b) of the lemma, $i_\alpha(\xi) \geq -\rho$ by virtue of Lemma 15.8. Thus it is sufficient to establish the asserted dominations.

To establish Lemma 15.9 (a), assume that $\xi(h_\alpha) - 1 \geq -\rho(h_\alpha)$. Let $x \in X$ and assume that $x > \xi(h_\alpha) - 1$. Then by (b) of the above proposition, it follows F_x is dominated by $\xi' := \xi - c\alpha$ for each $c \in [0, 1)$. By (a) of the above proposition, the same is true for all remaining $x \in X$. This establishes Lemma 15.9 (a).

To prove (b) of Lemma 15.9, assume that $\xi(h_\alpha) - 1 < -\rho(h_\alpha)$. Since $\xi(h_\alpha) \geq -\rho(h_\alpha)$, it follows that $i_\alpha(\xi) = \xi - d\alpha$ for a unique $d \in [0, 1)$.

If $x \in X$ satisfies $x \leq \xi(h_\alpha) - 1$, then according to (a) of the above proposition, F_x is dominated by $\xi' = \xi - d\alpha = i_\alpha(\xi)$.

On the other hand, if $x > \xi(h_\alpha) - 1$, then it follows from (c) of the above proposition that F_x is dominated by $i_\alpha(\xi)$.

We conclude that every F_x is dominated by $i_\alpha(\xi)$, hence so is F. This establishes Lemma 15.9 (b). □

Proof of Proposition 15.20. Before we proceed with the proof, we note that in the cases (a) and (b) it suffices to show that F_x is dominated by $\xi' = \xi - c\alpha$ for all $c \in [c_0, 1[$. Indeed assume this to be the case and let $c' \in [0, c_0]$ and put $\xi'' = \xi - c_\alpha$. Then $\xi'' \leq \xi'$ on ${}^\backprime\mathfrak{a}_\Phi + \mathbb{R}_+ h_\alpha$ whereas $\xi'' \leq \xi$ on ${}^\backprime\mathfrak{a}_\Phi + \mathbb{R}_- h_\alpha$. It therefore follows from the domination of F_x by both ξ and ξ' that F_x is dominated by ξ''.

For the actual proof, let $x \in X$. To establish (a) assume $x \leq \xi(h_\alpha) - 1$. Let $c \in [c_0, 1[$. Then obviously $x < \xi'(h_\alpha)$, where $\xi' = \xi - c\alpha$. We now have an estimate of the type of Lemma 15.19 (a). On the other hand, we also have the identity (15.31). From the domination assumption on F and the estimate of Lemma 15.19 (a) it follows that there exists $\varepsilon > 0$ such that for N, s and the continuous seminorm n all chosen large enough, we have, for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$, and $t \geq 0$,

$$e^{tB(\nu)} F_x(f, \nu, {}^\backprime a) \leq (1 + |t|)^N e^{t(x + |\mathrm{Re}\,\nu|)} |(\nu, {}^\backprime a)|^N a^{s|\mathrm{Re}\,\nu||\log {}^\backprime a|} ({}^\backprime a)^\xi n(f).$$

Using that $(1+|t|)^N = (1+|\log a_t|)^N \leq (1+|\log a|)^N$, that $({}^\backprime a)^\xi = ({}^\backprime a)^{\xi'}$, and that

$$e^{t(x+|\mathrm{Re}\,\nu|)} \leq e^{t\xi'(h_\alpha)}e^{t|\mathrm{Re}\,\nu||h_\alpha|} = e^{|\mathrm{Re}\,\nu||\log a_t|}(a_t)^{\xi'}$$

we obtain the estimate

$$e^{tB(\nu)}F_x(f,\nu,{}^\backprime a) \leq |(\nu,a)|^{2N}a^{(s+1)|\mathrm{Re}\,\nu||\log a|}a^{\xi'}n(f),$$

where $a = {}^\backprime a a_t$, $t \geq 0$. Combining this with the identity (15.31) and Lemma 15.19 (a) we see that we may enlarge N, s and the continuous seminorm n such that for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ all ${}^\backprime a \in {}^\backprime A_\Phi$ and all $t \geq 0$ we have the estimate

$$F_x(f,\nu,{}^\backprime a a_t) \leq |(\nu,a)|^N e^{s|\mathrm{Re}\,\nu||\log a|}a^{\xi'}n(f). \tag{15.36}$$

After decreasing ε, and increasing s, N, n, the same estimate becomes valid for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \leq 0$, provided ξ' is replaced by ξ. Since $\xi' \leq \xi$ on ${}^\backprime\mathfrak{a}_\Phi + (-\infty, 0]h_\alpha$, we see that the estimate (15.36) is in fact valid for all $t \in \mathbb{R}$. It follows that ξ' dominates F_x. This establishes (a).

We turn to (b) and (c) and assume that $x > \xi(h_\alpha) - 1$. Fix $c \in [c_0, 1[$. Then by (15.33) we have $x > \xi'(h_\alpha)$, where $\xi' = \xi - c\alpha$. From Lemma 15.19 (b) with $t = 0$ we now obtain that the integral

$$I_x(f,\nu,{}^\backprime a) := \int_0^\infty e^{-\tau B(\nu)}R_x(f,\nu,{}^\backprime a a_\tau)\,d\tau$$

converges for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ and satisfies the estimate

$$\|I_x(f,\nu,{}^\backprime a)\| \leq |(\nu,{}^\backprime a)|^N e^{s|\mathrm{Re}\,\nu||\log {}^\backprime a|}({}^\backprime a)^{\xi'}n(f), \tag{15.37}$$

provided that $\varepsilon > 0$ is taken sufficiently small, and s, N, n suffiently large. Put

$$F^\infty_x(f,\nu,{}^\backprime a) := F_x(f,\nu,{}^\backprime a) + I_x(f,\nu,{}^\backprime a). \tag{15.38}$$

From (15.18) we see that

$$F_x(f,\nu,{}^\backprime a a_t) = e^{tB(\nu)}F^\infty_x(f,\nu,{}^\backprime a) + R^\infty_x(f,\nu,{}^\backprime a a_t), \tag{15.39}$$

where the last term is given by the convergent integral

$$R^\infty_x(f,\nu,{}^\backprime a a_t) = -\int_t^\infty e^{(t-\tau)B(\nu)}R_x(f,\nu,{}^\backprime a a_\tau)\,d\tau. \tag{15.40}$$

From Lemma 15.19 (b) it follows that $\varepsilon > 0$ can be decreased, and s, N and n increased such that

$$\|R^\infty_x(f,\nu,{}^\backprime a a_t)\| \leq |(\nu,{}^\backprime a a_t)|^N({}^\backprime a a_t)^{\xi'}e^{s|\mathrm{Re}\,\nu||\log({}^\backprime a a_t)|}n(f), \tag{15.41}$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$ and $t \geq 0$.

From the domination assumption on F we obtain the estimate (15.37) for $F_x(f, \nu, {}^\backprime a)$ in place of $I_x(f, \nu, {}^\backprime a)$, provided we shrink ε and enlarge N, s, n if necessary. Here we need that $({}^\backprime a)^\xi = ({}^\backprime a)^{\xi'}$ for ${}^\backprime a \in {}^\backprime A_\Phi$. This observation leads to the estimate

$$\|F^\infty_x(f, \nu, {}^\backprime a)\| \leq |(\nu, {}^\backprime a)|^N a^{s|\mathrm{Re}\,\nu||\log {}^\backprime a|}({}^\backprime a)^{\xi'} n(f),$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, and ${}^\backprime a \in {}^\backprime A_\Phi$. Combining this estimate with (15.26) we see that we may increase N, n further to arrange that

$$\|e^{tB(\nu)} F^\infty_x(f, \nu, {}^\backprime a)\| \leq |(\nu, {}^\backprime a)|^N ({}^\backprime a)^\xi e^{s|\mathrm{Re}\,\nu||\log {}^\backprime a|} e^{xt+|\mathrm{Re}\,\nu||\log a_t|} n(f) \qquad (15.42)$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$, and $t \geq 0$.

We will first consider the case that $F^\infty_x = 0$. Then it follows from (15.39) and (15.41) that

$$\|F_x(f, \nu, {}^\backprime a a_t)\| \leq |(\nu, {}^\backprime a a_t)|^N ({}^\backprime a a_t)^{\xi'} e^{s|\mathrm{Re}\,\nu||\log {}^\backprime a a_t)|} n(f).$$

for $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$, $t \geq 0$. Here we used that $\xi' = \xi$ on ${}^\backprime\mathfrak{a}_\Phi$. In view of the assumed domination of (wh_ν) by ξ, we have the similar estimate for $t \leq 0$, with ξ in place of ξ'. Since $\xi' \geq \xi$ on ${}^\backprime\mathfrak{a}_\Phi + \mathfrak{a}^-_\Phi$, it follows that F_x is dominated by ξ'. In case (b) we still have that $\xi' \geq -\rho$. If the hypothesis of (c) is fulfilled this need not be the case. However, we may chose $c \in [c_0, 1[$ such that $\xi' \leq i_\alpha(\xi) \leq \xi$ on ${}^\backprime A_\Phi A^+_\Phi$ and by an argument similar to the previous argument, if follows that F_x is dominated by $i_\alpha(\xi)$. This establishes both (b) and (c) under the assumption $F^\infty_x = 0$.

To finish the proof, we assume that $F^\infty_x \neq 0$. Then it follows from the proposition below that $x \leq -\rho(h_\alpha)$ so that: $\xi(h_\alpha) - 1 < x \leq -\rho(h_\alpha)$. In particular, we are in (a subcase of) case (c). From (15.33) it follows that

$$\xi'(h_\alpha) < x \leq -\rho(h_\alpha) = i_\alpha(\xi)(h_\alpha).$$

It now follows from (15.39), (15.41) and (15.42) that, for a suitably enlarged seminorm n,

$$\|F_x(f, \nu, {}^\backprime a a_t)\| \leq |(\nu, {}^\backprime a a_t)|^N ({}^\backprime a a_t)^{i_\alpha(\xi)} e^{s|\mathrm{Re}\,\nu||\log {}^\backprime a a_t)|}$$

for $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, ${}^\backprime a \in {}^\backprime A_\Phi$, $t \geq 0$. Here we used that $i_\alpha(\xi) = \xi$ on ${}^\backprime\mathfrak{a}_\Phi$. In view of the assumed domination of (wh_ν) by ξ, we have the similar estimate for $t \leq 0$, with ξ in place of $i_\alpha(\xi)$. Since $i_\alpha(\xi) \geq \xi$ on ${}^\backprime\mathfrak{a}_\Phi + \mathfrak{a}^-_\Phi$, it follows that F_x is dominated by $i_\alpha(\xi)$. This completes the proof of Proposition 15.20. □

Proposition 15.21 *Let $\xi \in \mathfrak{a}^*$ satisfy $\xi \geq -\rho$ and dominate the Whittaker family (wh_ν). If $x \in X$ is such that $x > \max(\xi(h_\alpha) - 1, -\rho(h_\alpha))$, then, for $\varepsilon > 0$ sufficiently small,*

$$F^\infty_x(f, \nu, a) = 0$$

for all $f \in I^\infty_{P,\sigma}$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$, and $a \in A$.

To prove the proposition, we need some preparation.

Lemma 15.22 *For every left K-finite $f \in C^\infty(K/K_P : \sigma)$ and every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ there exist constants $m, C > 0$ such that*

$$|\mathrm{wh}_\nu(f)(a)| \leq C(1 + |\log {}^*a|)^m a^{\mathrm{Re}\,\nu - \rho}. \tag{15.43}$$

Proof. By K-finiteness, there exists a unitary representation (τ, V_τ) of K, a function $g \in C^\infty(\tau_P : K/K_P : \sigma)$ and a linear functional $\mu \in V_\tau^*$ such that $f = (I \otimes \mu) \circ g$. Define the function $\mathrm{wh}_\nu(g) : G \to V_\tau$ by

$$\mathrm{wh}_\nu(x) = \langle g\,,\, \pi_{\bar{P},\sigma,\bar{\nu}}(x) j_{\bar{\nu}}\rangle.$$

Then clearly, $\mathrm{wh}_\nu(f) = \mu \circ \mathrm{wh}_\nu(g)$. We now note that there exists a $\psi \in \mathcal{A}_{2,P}$ such that $\mathrm{wh}_\nu(g) = \mathrm{Wh}(P, \psi, \nu)$. From Lemma 9.13 it follows that there exist $m, C' > 0$ such that

$$\|\mathrm{wh}_\nu(g)(a)\| \leq C'(1 + |\log {}^*a|)^m a^{\mathrm{Re}\,\nu - \rho}.$$

This implies (15.43). □

Corollary 15.23 *For every left K-finite $f \in C^\infty(K/K_P : \sigma)$ and every $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$ there exist constants $m, C > 0$ such that*

$$\|F(f, \nu, a)\| \leq C(1 + |\log a|)^m a^{\mathrm{Re}\,\nu - \rho}. \tag{15.44}$$

Proof. The j-th component of F is given by $F_j(f, \nu, a) = \mathrm{wh}_\nu(\pi_{\bar{P},\sigma,-\nu}(u_j) f)$, with $u_j \in U(\mathfrak{g})$. The function $\pi_{\bar{P},\sigma,-\nu}(u_j) f$ is K-finite in $C^\infty(K/K_P, \sigma)$, hence F_j satisfies an estimate of the form (15.44). The proof is completed by the observation that $|\log {}^*a| \leq |\log a|$, for all $a \in A$, by orthogonality of the sum $\mathfrak{a} = {}^*\mathfrak{a}_\Phi + \mathfrak{a}_\Phi$. □

Proof of Proposition 15.21. Put $\Omega := \mathfrak{a}^*_{P\mathbb{C}}(P, 0)$. From (15.39) it follows that

$$e^{tB(\nu)} F_x^\infty(f, \nu, {}^\backprime a) = F_x(f, \nu, {}^\backprime a a_t) - R_x^\infty(f, \nu, {}^\backprime a a_t)$$

We first assume that $f \in I^\infty_{P,\sigma}$ is K-finite. Let $\varepsilon' < \varepsilon$. Then for $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon') \cap \Omega$ and ${}^\backprime a \in {}^\backprime A_\Phi$ it follows from Corollary 15.23 that there exist constants $m > 0$ and $C > 0$ such that for all $t \geq 0$,

$$\|F_x(f, \nu {}^\backprime a a_t)\| \leq C(1 + |t|)^m e^{t(\varepsilon'|h_\alpha| - \rho(h_\alpha))}. \tag{15.45}$$

From (15.40) and Lemma 15.19 (b) it follows, possible after adapting m and C, that also

$$\|R_x^\infty(f, \nu, {}^\backprime a a_t)\| \leq C(1 + |t|)^m e^{t(\varepsilon'|h_\alpha| + \xi'(h_\alpha))}), \tag{15.46}$$

again for $t \geq 0$. From the hypothesis on x combined with (15.33) we see that there exists $\delta > 0$ such that $x - \delta > \max(-\rho(h_\alpha), \xi'(h_\alpha))$. Keeping this in mind when combining the estimates (15.45) and (15.46) we obtain

$$\|e^{tB(\nu)} F_x^\infty(f, \nu, {}^\backprime a)\| \leq 2C(1 + |t|)^m e^{t(\varepsilon'|h_\alpha| + x - \delta)} \tag{15.47}$$

as $t \geq 0$. On the other hand, the expression inside the norm on the left-hand side is exponential polynomial in t with exponents whose real part is at least $x - \varepsilon'|h_\alpha|$. For $\varepsilon' > 0$ sufficiently small we have $x - \varepsilon'|h_\alpha| > \varepsilon'|h_\alpha| + x - \delta$ so that by uniqueness of asymptotics for $t \to \infty$ we find that

$$e^{tB(\nu)} F_x^\infty(f, \nu, {}^\backprime a) = 0$$

provided $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(\varepsilon') \cap \Omega$. Since the expression on the left is holomorphic in $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(\varepsilon)$, it follows by analytic continuation that the assertion of the lemma holds for K-finite f.

For every $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(\varepsilon)$ and ${}^\backprime a \in {}^\backprime A_\Phi$ it follows from the definitions given that $f \mapsto F_x^\infty(f, \nu, {}^\backprime a)$ is a linear map $C^\infty(K/K_P : \sigma) \to \mathbb{C}^\ell$. This linear map is continuous in view of (15.38) and the estimates (15.36) and (15.37) . As it vanishes on the dense subspace of K-finite functions, it follows that the given map is zero on the entire space $C^\infty(K/K_P : \sigma)$. This finishes the proof. □

16 Uniform temperedness of the Whittaker integral

Let P be a standard parabolic subgroup. We recall from (9.6) and (9.7) the definition of the space $\mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)$. For ψ in this space and for $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(P, 0)$ the Whittaker integral $\mathrm{Wh}(P, \psi, \nu)$, defined by (9.12), is a function in $C^\infty(\tau : G/N_0 : \chi)$.

Proposition 16.1 *Let ψ be as above. Then $\nu \mapsto \mathrm{Wh}(P, \psi, \nu)$ extends to a holomorphic function $\mathfrak{a}_{P\mathbb{C}}^* \to C^\infty(\tau : G/N_0 : \chi)$.*

Proof. As in the proof of Corollary 9.11 it suffices to prove this for $\psi = \psi_{f\otimes\xi}$ with $f \in C^\infty(\tau : K/K_P : \sigma)$ and $\xi \in \mathrm{Wh}_{\chi_P}(H_\sigma^\infty)$. In that case the result follows from Corollary 9.10 combined with Theorem 14.4. □

Theorem 16.2 (uniformly tempered estimate) *Let P be a standard parabolic subgroup. Then there exists an $\varepsilon > 0$, and for each $u \in U(\mathfrak{g})$ constants $s > 0$ and $C, N > 0$ such that*

$$\|L_u[\mathrm{Wh}(P, \psi, \nu)](x)\| \leq C\|\psi\|(1 + |\nu|)^N (1 + |H(x)|)^N e^{-\rho H(x) + s|\mathrm{Re}\,\nu|},$$

for all $\nu \in \mathfrak{a}_{P\mathbb{C}}^*(\varepsilon)$, all $x \in G$ and all $\psi \in \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)$.

Proof. We first assume that $G = {}^\circ G$. By finite dimensionality of $\mathcal{A}_{2,P}$, it suffices to prove the result for a fixed ψ of unit length. By linearity of the Whittaker integral in ψ, and using the decomposition (9.8) and the isomorphism (9.11) we may as well assume in addition that $\psi = \psi_{f\otimes\xi}$, with $\xi \in \mathrm{Wh}_{\chi_P}(H_\sigma^\infty)$ and

$$f \in C^\infty(\tau : K/K_P : \chi_P) \subset V_\tau \otimes I_{P,\sigma}^\infty.$$

By analytic continuation we then have that

$$\mathrm{Wh}(P,\psi,\nu)(x) = \langle f\,,\,\pi_{\bar P,\sigma,\bar\nu}(x)j(\bar P,\sigma,\bar\nu,\bar\xi)\rangle$$

for all $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ and $x \in G$. In view of Remark 15.3 the map $\mathrm{wh}_\nu : C^\infty(K/K_P : \sigma) \to C^\infty(G/N_0 : \chi)$ defined by

$$\mathrm{wh}_\nu({}^\backprime f)(x) = \langle {}^\backprime f\,,\,\pi_{\bar P,\sigma,\bar\nu}(x)j(\bar P,\bar\nu,\bar\xi)\rangle$$

defines a holomorphic family of Whittaker functions of moderate growth. By Cor. 15.6, there exists an $\varepsilon > 0$, and for each $u \in U(\mathfrak{g})$ constants $N, s > 0$ and a continuous seminorm ${}^\backprime n$ on $C^\infty(K/K_P : \sigma)$ such that

$$|L_u[\mathrm{wh}_\nu({}^\backprime f)](x)| \le (1+|\nu|)^N(1+|H(x)|)^N e^{s|\mathrm{Re}\,\nu||H(x)|}e^{-\rho H(x)}\,{}^\backprime n({}^\backprime f)$$

for all ${}^\backprime f \in I^\infty_{P,\sigma}$, $x \in G$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$. Applying this with ${}^\backprime f = (\zeta \otimes I)f$ for $\zeta \in V^*_\tau$ we find

$$|\zeta \circ L_u[\mathrm{Wh}(P,\psi,\nu)](x)| \le (1+|\nu|)^N(1+|H(x)|)^N e^{s|\mathrm{Re}\,\nu||H(x)|}e^{-\rho H(x)}\,{}^\backprime n((I\otimes\zeta)f).$$

Now there exists a constant $C > 0$ such that $|{}^\backprime n(I\otimes\zeta)f)| \le C|\zeta|$ for all $\zeta \in V^*_\tau$. We conclude that

$$\|L_u[\mathrm{Wh}(P,\psi,\nu)](x)\| \le C(1+|\nu|)^N(1+|H(x)|)^N e^{s|\mathrm{Re}\,\nu||H(x)|}e^{-\rho H(x)},$$

for all $x \in G$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}(\varepsilon)$ as required. This finishes the proof for the case $G = {}^\circ G$

In general, the group decomposes as $G = {}^\circ G \times A_\Delta$, where $A_\Delta = \exp \mathfrak{a}_\Delta$, with $\mathfrak{a}_\Delta := \cap_{\alpha\in\Delta}\ker\alpha$ central in $\mathfrak{g}$. From the definitions it then readily follows that the spaces $\mathcal{A}_{2,P}$ for G and ${}^\circ G$ coincide and that

$$\mathrm{Wh}(G,P,\psi,\nu)(xa_\Delta) = a_\Delta^\nu\,\mathrm{Wh}({}^\circ G, {}^\circ G\cap P, \psi, {}^\backprime\nu)(x),$$

for $\psi \in \mathcal{A}_{2,P}$, $x \in {}^\circ G$, $a_\Delta \in A_\Delta$ and $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with ${}^\backprime\nu$ the restriction of ν to ${}^\circ\mathfrak{g}\cap\mathfrak{a}$. All assertions now readily generalize from ${}^\circ G$ to G. □

Corollary 16.3 *Let $P \in \mathcal{P}_{\mathrm{st}}$. Then for all $u \in U(\mathfrak{g})$ and $v \in S(\mathfrak{a}^*_P)$ there exist constants $C > 0$ and $N > 0$ such that*

$$\|L_u[\mathrm{Wh}(P,\psi,\nu;v)](x)\| \le C\|\psi\|(1+|\nu|)^N(1+|H(x)|)^N e^{-\rho H(x)}$$

for all $\psi \in \mathcal{A}_{2,P}$, $\nu \in i\mathfrak{a}^*_P$ and $x \in G$.

Remark 16.4 In the displayed equation, we have used Harish-Chandra's convention to denote the action of a differential operator by putting it next to the variable relative to which it is applied, separated from the variable by a semi-colon. In the present context, if $v \in S(\mathfrak{a}^*_P)$, then v is viewed as a constant coefficient complex differential operator on $\mathfrak{a}^*_{P\mathbb{C}}$, and if $\varphi : \mathfrak{a}^*_{P\mathbb{C}} \to V$ is a holomorphic function with values in a locally convex space then $\varphi(\nu; v)$ stands for $v\varphi$ at the point $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$.

Proof. This follows from the estimates of Theorem 16.2 by using the Cauchy integral formula in the variable ν, with polydiscs of polyradius $\varepsilon(\dim\mathfrak{a}_P)^{-1}(1+|H(x)|)^{-1}$. □

We may now define a Fourier transform $\mathcal{F}_P : C(G/N_0 : \chi) \to C^0(i\mathfrak{a}_P^*, \mathcal{A}_{2,P})$ by the formula

$$\langle \mathcal{F}_P(f)(\nu)\,,\,\psi\rangle = \langle f\,,\, \mathrm{Wh}(P,\psi,\nu)\rangle_2 := \int_{G/N_0} f(x)\overline{\mathrm{Wh}(P,\psi,\nu)(x)}\;d\dot{x},$$

for $f \in C(G/N_0 : \chi)$, $\nu \in i\mathfrak{a}_P^*$ and $\psi \in \mathcal{A}_{2,P}$. Indeed, let $\ell = \dim A + 1$ then by Lemma 3.3 the function $x \mapsto (1 + |H(x)|)^{-\ell} e^{-2\rho H(x)}$ is absolutely integrable over G/N_0. By application of Theorem 16.2, we infer the existence of a constant $N > 0$ and a continuous seminorm n on $C(\tau : G/N_0 : \chi)$ such that for all $f \in C(\tau : G/N_0 : \chi)$ we have

$$|f(x)\overline{\mathrm{Wh}(P,\psi,\nu)(x)}| \leq (1+|\nu|)^N (1+|H(x)|)^{-\ell} e^{-2\rho H(x)} n(f), \qquad (x \in G). \tag{16.1}$$

It follows from this that the Fourier transform is defined by an absolutely converging integral, and defines a continuous linear operator $C(G/N_0 : \chi) \to C^0(i\mathfrak{a}_P^*, \mathcal{A}_{2,P})$. By application of Cor. 16.3 it follows that differentiation under the integral is allowed, and that $\mathcal{F}_P$ is continuous linear $C(G/N_0 : \chi) \to C^\infty(i\mathfrak{a}_P^*, \mathcal{A}_{2,P})$.

Lemma 16.5 *Let $P \in \mathcal{P}_{\mathrm{st}}$. Then for all $f \in C(\tau : G/N_0 : \chi)$ and every $Z \in \mathfrak{Z}$ we have*

$$\mathcal{F}_P(R_Z f)(\nu) = \underline{\mu}_P(Z,\nu)\mathcal{F}_P f(\nu), \qquad (\nu \in i\mathfrak{a}_P^*).$$

Proof. Let $\psi \in \mathcal{A}_{2,P}$. Since $\underline{\mu}(Z,\,\cdot\,)$ is polynomial with values in $\mathrm{End}(\mathcal{A}_{2,P})$ it follows from () by analytic continuation that

$$R_Z \mathrm{Wh}(P,\psi,\nu) = \mathrm{Wh}(P, \underline{\mu}_P(Z,\nu)\psi,\nu)$$

for all $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. Hence, by differentiation under the integral sign,

$$\begin{aligned}
\langle \mathcal{F}_P(R_Z f)(\nu)\,,\,\psi\rangle &= \langle L_{Z^\vee} f\,,\,\mathrm{Wh}(P,\psi,\nu)\rangle_2\\
&= \langle f\,,\,R_{\bar{Z}^\vee}\mathrm{Wh}(P,\psi,\nu)\rangle_2\\
&= \langle f\,,\,\mathrm{Wh}(P,\underline{\mu}_P(\bar{Z}^\vee,\nu)\psi,\nu)\rangle_2\\
&= \langle \mathcal{F}_P f\,,\,\underline{\mu}_P(\bar{Z}^\vee,\nu)\psi\rangle.
\end{aligned}$$

As this holds for arbitrary $\psi \in \mathcal{A}_{2,P}$ we conclude that

$$\mathcal{F}_P(R_Z f)(\nu) = \underline{\mu}_P(\bar{Z}^\vee,\nu)^*\,\mathcal{F}_P f(\nu), \quad (\nu \in i\mathfrak{a}_P^*).$$

where the star indicates that the adjoint is taken with respect to the L^2-Hilbert structure on $\mathcal{A}_{2,P}$. By a straightforward calculation it follows that

$$\underline{\mu}_P(\bar{Z}^\vee,\nu)^* = \overline{\underline{\mu}_P(\bar{Z}^\vee,\nu)^\vee} = \underline{\mu}_P(Z,-\bar{\nu}) = \underline{\mu}_P(Z,\nu)$$

for all $\nu \in i\mathfrak{a}_P^*$. □

Let $\mathcal{S}(i\mathfrak{a}_P^*)$ denote the usual space of Schwartz functions on the real vector space $i\mathfrak{a}_P^*$. Then the following result is valid.

Theorem 16.6 *$\mathcal{F}_P$ maps $C(\tau : G/N_0 : \chi)$ continuous linearly to $\mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$.*

The proof follows the usual strategy of applying partial integration, involving minus the Casimir operator associated with the invariant bilinear form B, see (2.1). The following lemma prepares for this.

Lemma 16.7 *Let $L \in \mathfrak{Z}$ be minus the Casimir operator. Then*

$$\lim_{\substack{\nu \in i\mathfrak{a}_P^* \\ |\nu| \to \infty}} (1 + |\nu|)^{-2} \, \underline{\mu}_P(L, \nu) = I$$

in $\mathrm{End}(\mathcal{A}_{2,P})$.

Proof. By finite dimensionality of $\mathcal{A}_{2,P}$ it suffices to prove the identity for the restriction of the endomorphisms to the subspace $\mathcal{A}_{2,P,\sigma} := \mathcal{A}_\sigma(\tau_P : K/K_P : \chi_P)$, with σ a representation of the discrete series of M_P. Let $\Lambda_\sigma \in \mathfrak{h}_\mathbb{C}^*$ be the infinitesimal character of σ and let δ be half of the sum of a choice of positive roots for the root system of $\mathfrak{h}$ in $\mathfrak{g}_\mathbb{C}$.

The restriction of $\underline{\mu}_P(L, \mu)$ to $\mathcal{A}_{2,P,\sigma}$ equals the restriction of $R_{\underline{\mu}_P(L,\mu)} = L_{\underline{\mu}_P(L,\mu)^\vee}$, which is given by multiplication by the scalar

$$\begin{aligned} \gamma_{M_P}(\underline{\mu}_P(L, \nu)^\vee, \Lambda_\sigma) &= \gamma_{M_P}(\underline{\mu}_P(L, \nu), -\Lambda_\sigma) \\ &= \gamma(L, \nu - \Lambda_\sigma) = |\nu|^2 + C_\sigma \end{aligned}$$

with $C_\sigma = -B^*(\Lambda_\sigma, \Lambda_\sigma) + B^*(\delta, \delta) \in \mathbb{R}$, where B^* is the dual of B. Accordingly, the restriction of the limit equals

$$\lim_{|\nu| \to \infty} (1 + |\nu|)^{-2}(|\nu|^2 + C_\sigma)I = I.$$

□

Proof of Theorem 16.6. In the above we already showed that $\mathcal{F}_P$ maps $C(\tau : G/N_0 : \chi)$ continuously to $C^\infty(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$. By Lemma 16.7 there exists a constant $R > 0$ such that for all $\nu \in i\mathfrak{a}_P^*$ with $|\nu| \geq R$ the endomorphism $\underline{\mu}_P(L, \nu)$ of $\mathcal{A}_{2,P}$ is invertible, whereas the operator norm of its inverse satisfies

$$\|\underline{\mu}_P(L, \nu)^{-1}\| \leq 2(1 + |\nu|)^{-2}.$$

We will finish the proof by showing that for every $u \in U(\mathfrak{a}_P^*)$ and all $k \in \mathbb{N}$ there exists a continuous seminorm $\mathbf{n}$ on $C(\tau : G/N_0 : \chi)$ such that for all $f \in C(\tau : G/N_0 : \chi)$ we have

$$\|\mathcal{F}_P f(\nu; u)\| \leq (1 + |\nu|)^{-k} \mathbf{n}(f), \qquad (|\nu| \geq R).$$

For this we proceed by induction on the order of u. Clearly, the result is true for u of order -1 since then $u = 0$. Thus, assume that u has order $m \geq 0$ and assume the result has been established for u of order strictly smaller than m.

It follows from (16.1) that for all $f \in C(\tau : G/N_0 : \chi)$ and $\nu \in i\mathfrak{a}_P^*$ we have

$$\|\mathcal{F}_P f(\nu; u)\| \leq (1 + |\nu|)^N \mathbf{n}(f). \tag{16.2}$$

Here $N \in \mathbb{N}$ and $\mathbf{n}$ is a continuous seminorm on $C(\tau : G/N_0 : \chi)$.

Fix $\ell \in N$ such that $N - 2\ell \leq -k$. Then it follows by application of Lemma 16.5 and the Leibniz rule that there exist a finite collection of polynomial functions $q_j \in P(\mathfrak{a}_P^*)$ and a finite collection of elements $u_j \in S(\mathfrak{a}_P^*)$ of order strictly smaller than the order of u such that

$$\underline{\mu}_P(L, \nu)^\ell \mathcal{F}_P f(\nu; u) = \mathcal{F}_P(L^\ell f)(\nu; u) + \sum_{j=1}^r q_j(\nu) \mathcal{F}_P f(\nu; u_j),$$

for all $f \in C(\tau : G/N_0 : \chi)$ and all $\nu \in i\mathfrak{a}_P^*$. By application of the initial estimate (16.2) and the inductive hypothesis there exists a continuous seminorm $\mathbf{n}'$ on $C(\tau : G/N_0 : \chi)$ such that for all f and all $|\nu| \geq R$, we have

$$\|\underline{\mu}_P(L, \nu)^\ell \mathcal{F}_P f(\nu; u)\| \leq (1 + |\nu|)^N \mathbf{n}'(f).$$

This implies that

$$\begin{aligned} \|\mathcal{F}_P f(\nu; u)\| &\leq \|\underline{\mu}_P(L, \nu)^{-1}\|^\ell \|\underline{\mu}_P(L, \nu)^\ell \mathcal{F}_P f(\nu; u)\| \\ &\leq 2^\ell (1 + |\nu|)^{N - 2\ell} \mathbf{n}'(f) \\ &\leq (1 + |\nu|)^{-k} \mathbf{n}'(2^\ell f). \end{aligned}$$

completing the induction. □

17 Appendix: factorization of polynomial functions

In this section we will prove the following result, which is needed in Section 12.

Proposition 17.1 *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function of degree $d \geq 1$ and assume that $\mathcal{H}$ is a locally finite collection of affine hyperplanes in $\mathbb{C}^n$ such that $f^{-1}(0) \subset \cup\mathcal{H}$. Then f can be expressed as a finite product $f = \ell_1 \cdots \ell_d$ with $\ell_j : \mathbb{C}^n \to \mathbb{C}$ a linear polynomial function whose zero set $\ell_j^{-1}(0)$ belongs to $\mathcal{H}$, for $1 \leq j \leq d$.*

The following lemma is a first step in the proof.

Lemma 17.2 *Let $f, \ell : \mathbb{C}^n \to \mathbb{C}$ be non-zero polynomial functions, with* $\deg \ell = 1$. *If f vanishes on the hyperplane $\ell^{-1}(0)$ then f/ℓ is polynomial.*

Proof. By application of a suitable affine coordinate transformation we may reduce to the case $\ell(z) = z_n$. In view of the hypothesis, $f(z', 0) = 0$ for all $z' \in \mathbb{C}^{n-1}$. This implies that all partial derivatives $\partial^\alpha f(0)$, with $\alpha \in \mathbb{N}^n$ and $\alpha_n = 0$ are zero. Hence,

$$f(z) = \sum_{\alpha \in F} c_\alpha z^\alpha$$

with $F \subset \mathbb{N}^{n-1} \times \mathbb{N}_+$. The result now follows. □

In the following we denote by $O = O_{\mathbb{C}^n}$ the sheaf of holomorphic functions on $\mathbb{C}^n$. For a point $a \in \mathbb{C}^n$ we denote by O_a the stalk at a, i.e., the ring of germs at a of locallly defined holomorphic functions. If no confusion is possible, we will switch between elements of O_a and local representatives for them without explicitly mentioning this.

Lemma 17.3 *Let $f \in O_0$ and let Ξ be a finite collection of non-zero linear functionals $\xi : \mathbb{C}^n \to \mathbb{C}$. Suppose that $f(0) = 0$ and that $f^{-1}(0) \subset \cup_{\xi \in \Xi} \ker \xi$ (in the sense of germs). Then there exists a functional $\xi \in \Xi$ such that $f = 0$ on $\ker \xi$ (in the sense of germs).*

Proof. By a suitable linear change of coordinates we may reduce to the case that f and each $\xi \in \Phi$ does not vanish identically on the coordinate axis $\mathbb{C}e_n$. Then by the Weierstrass preparation theorem, there exists an invertible element $q \in O_0$ such that $F = q^{-1} f$ is a Weierstrass polynomial given by

$$F(z', z_n) = z_n^N + \sum_{k<N} c_k(z') z_n^k, \tag{17.1}$$

for $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ sufficiently close to $(0, 0)$, with $c_k \in O_0' = O_0(\mathbb{C}^{n-1})$ such that $c_k(0) = 0$ for $0 \leq k < N$. Furthermore, for every $\xi \in \Phi$ there exists a linear functional $\eta_\xi : \mathbb{C}^{n-1} \to \mathbb{C}$ such that $\ker \xi \subset \mathbb{C}^{n-1} \times \mathbb{C}$ equals the graph of η_ξ.

Let $D \subset \mathbb{C}^n$ be a polydisk centered at 0 such that f, q and F admit representatives in $O(D)$. We decompose $D = D' \times D_n$ according to the decomposition $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$. Then $f^{-1}(0) \cap D = F^{-1}(0) \cap D$. Let Ω be the open dense subset of $\mathbb{C}^{n-1}$ consisting of $z' \in \mathbb{C}^{n-1}$ such that $\eta_{\xi_1}(z') \neq \eta_{\xi_2}(z')$ for all distinct $\xi_1, \xi_2 \in \Xi$.

By continuous dependence of the roots of (17.1) on the coefficients $c_k(z')$ we may shrink D' sufficiently so that for all $z' \in D'$ there exists an element $z_n \in D_n$ such that $F(z', z_n) = 0$. Accordingly, we may select $\alpha' \in \Omega \cap D'$ and $\alpha_n \in D_n$ such that $F(\alpha', \alpha_n) = 0$. Then $f(\alpha', \alpha_n) = 0$ so there exists a $\xi_0 \in \Xi$ such that $\xi_0(\alpha', \alpha_n) = 0$, or, equivalently,

$$\alpha_n = \eta_{\xi_0}(\alpha').$$

Since $\alpha' \in \Omega$, there exists $\varepsilon > 0$ such that for $z' = \alpha'$ we have

$$|\eta_\xi(z') - \alpha_n| > \varepsilon \qquad (\forall \xi \in \Xi \setminus \{\xi_0\}). \tag{17.2}$$

By continuity there exists an open neighborhood U of α' in $D' \cap \Omega$ such that the estimates (17.2) are still valid for all $z' \in U$.

By the continuity of roots mentioned above, we may shrink U so that in addition for every $z' \in U$ there exists a $z_n \in D(\alpha_n, \varepsilon)$ such that $f(z', z_n) = 0$. The latter implies that $z_n = \eta_\xi(z')$ for a $\xi \in \Xi$. Now $|\eta_\xi(z') - \alpha_n| < \varepsilon$ implies that $\xi = \xi_0$ and we see that

$$F(z', \eta_{\xi_0}(z')) = 0 \tag{17.3}$$

for all $z' \in U$. By analytic continuation, (17.3) is valid for all $z \in D'$. Therefore, $f = 0$ on $\{z \in D \mid z_n = \eta_{\xi_0}(z')\} = \ker \xi_0$. □

Corollary 17.4 *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function of positive degree and assume that $\mathcal{H}$ is a locally finite collection of affine hyperplanes in $\mathbb{C}^n$ such that $f^{-1}(0) \subset \cup\mathcal{H}$. Then there exists a hyperplane $H \in \mathcal{H}$ such that f vanishes on H.*

Proof. Since f has positive degree, $f^{-1}(0) \neq \emptyset$. By application of a suitable translation, we may reduce to the case that $f(0) = 0$. Let $\mathcal{H}_0$ be the finite collection of $H \in \mathcal{H}$ with $H \ni 0$. For each $H \in \mathcal{H}_0$ we fix $\xi_H \in (\mathbb{C}^n)^*$ such that $H = \ker \xi_H$. Put $\Xi = \{\xi_H \mid H \in \mathcal{H}_0\}$.

By application of Lemma 17.3 it follows that there exists a $\xi_0 \in \Xi$ and a polydisk $D \subset \mathbb{C}^n$ centered at 0 such that $f = 0$ on $\ker(\xi_0) \cap D$. By analytic continuation of $f|_{\ker \xi_0}$ this implies that f vanishes on $\ker \xi_0$. □

Proof of Proposition 17.1. In view of Corollary 17.4 and Lemma 17.2 the proof follows by a straightforward induction on d. □

18 Appendix: a Hartog type continuation result

In this paper we will need the following continuation result for holomorphic functions on a domain in $\mathbb{C}^n$ with values in a quasi-complete locally convex space.

Let $\Omega \subset \mathbb{C}^n$ be a connected open subset, $p : \Omega \to \mathbb{C}$ a non-zero holomorphic function, and $X \subset \Omega$ its zero locus $p^{-1}(0)$. We denote by X_r the set of points $z \in X$ such that X is a smooth complex hypersurface at z. By this we mean that there should exist an open neighborhood U of z in Ω such that $X \cap U$ is a complex submanifold of dimension $n - 1$. The complement of X_r in X is denoted by X_s. Clearly, X_s is closed in X hence in Ω. Since Ω is connected, the set X, hence also X_s, has empty interior.

Theorem 18.1 *Let V be a quasi-complete locally convex (Hausdorff) space. Then every holomorphic function $f : \Omega \backslash X_s \to V$ admits a unique extension to a holomorphic function $\Omega \to V$.*

Remark 18.2 For $V = \mathbb{C}$ the result is well known and can be obtained as a consequence of [8, Thm. 6.12], which asserts that the above result is valid for $V = \mathbb{C}$ and with X_s replaced by any analytic subset Y of Ω which is everywhere locally of codimension at least two. That result actually extends to V-valued holomorphic functions, but we have not found a decent reference to the literature for this. We have therefore chosen to present a self-contained proof of Theorem 18.1, following a strategy suggested in Exercise 4.26 in the set of lecture notes *Several Complex Variables*, by Jaap Korevaar and Jan Wiegerinck, version August 23, 2017. This turned out to be possible since Cauchy's integral formula is valid in the setting of V-valued holomorphic functions.

To prepare for the proof of Theorem 18.1 we make the following general observation. We assume that V is a quasi-complete locally convex space.

Lemma 18.3 *Let $\Omega \subset \mathbb{C}^n$ be an open subset and let Y be a closed subset of Ω which has empty interior. Let $f : \Omega \setminus Y \to V$ be holomorphic. Then the following assertions are equivalent.*

(a) *f extends to a holomorphic function $\Omega \to V$;*

(b) *for every $y \in Y$ there exists an open neighborhood $\omega \ni y$ in Ω such that $f|_{\omega \setminus Y}$ has a holomorphic extension to ω.*

Proof. That (a) implies (b) is obvious. Assume (b). Then one may cover Ω with open subsets ω_j for j in an index set I, such that for each $j \in I$ the function $f|_{\omega_j \setminus Y}$ has a holomorphic extension $f_j : \omega_j \to V$. Clearly, if $i, j \in I$ and $\omega_i \cap \omega_j \neq \emptyset$, then $f_i = f_j$ on $(\omega_i \cap \omega_j) \setminus Y$. By density this implies that $f_i = f_j$ on $\omega_i \cap \omega_j$. From this (a) readily follows. $\square$

Proof of Theorem 18.1. By Lemma 18.3 applied with $Y = X_s$ it suffices to show that for every $z \in X_s$ there exists an open neighborhood $\omega \ni z$ in Ω such that $f|_{\omega \setminus X_s}$ extends to a holomorphic function $\omega \to V$.

Let $z^0 \in X_s$. Then we may apply an affine coordinate transformation to arrange that $z^0 = 0$ and that locally at 0 the function p is z_n-regular, see [8, p. 109]. By the Weierstrass preparation theorem, see [8, Thm. III.2.7], locally at 0 the function p factors as a product of holomorphic functions $p_0 \cdot W$, with the germ at 0 of p_0 being a unit in the ring $\mathcal{O}_0 = \mathcal{O}_0(\mathbb{C}^n)$ of germs of holomorphic functions defined locally at 0 in $\mathbb{C}^n$ and with $W \in \mathcal{O}'_0[z_n]$ a Weierstrass polynomial of order d over the ring $\mathcal{O}'_0 = \mathcal{O}_0(\mathbb{C}^{n-1})$. Then for D a sufficiently small polydisk in $\mathbb{C}^n$ centered at 0 the germ W has a representative which is holomorphic on D and such that $D \cap X$ is contained in $W^{-1}(0)$. Note that $0 \in D \cap X$.

The ring $\mathcal{O}_0$ is a unique factorization domain, see [8, Thm. III.3.3]. Let $W = f_1^{m_1} \cdots f_r^{m_r}$ be a decomposition into irreducibles of $\mathcal{O}_0$ with the f_j mutually prime. Since W is z_n-regular, each f_j is z_n-regular as well. By the Weierstrass preparation theorem we may write $f_j = \varepsilon_j W_j$ with ε_j a unit in $\mathcal{O}_0$ and W_j a Weierstrass polynomial in $\mathcal{O}'_0[z_n]$. Then $W = \varepsilon W_1^{m_1} \cdots W_r^{m_r}$ with $\varepsilon = \varepsilon_1^{m_1} \cdots \varepsilon_r^{m_r}$ a unit in $\mathcal{O}_0$. Clearly the product $W_1^{m_1} \cdots W_r^{m_r}$ is a Weierstrass polynomial. By the uniqueness statement of the Weierstrass preparation theorem, it now follows that $W = W_1^{m_1} \cdots W_r^{m_r}$. The W_j are mutually distinct and irreducible in $\mathcal{O}_0$ hence in $\mathcal{O}'_0[z_n]$, see [9, Lemma II.5].

Let $`W = W_1 \cdots W_r$; then after sufficiently shrinking $D$, keeping it centered at 0, we obtain that $`W$ has a holomorphic representative on D and that $D \cap X$ equals the zero locus of $`W$ in D. We agree to write $D = D' \times D_n$ in accordance with the decomposition $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$.

If $z \in D$ satisfies $`W(z) = 0$ and $\partial_n `W(z) \neq 0$, then it follows by application of the submersion theorem that $`W^{-1}(0)$ is a complex differentiable submanifold of codimension 1 locally at $z$. This implies that $z \notin X_s$. It follows that $X_s \cap D$ is contained in the zero locus of both $`W$ and $\partial_n `W$.

Let now $z \in X_s \cap D$. Then it follows that the polynomial functions $`W(z', \cdot)$ and $\partial_n W(z', \cdot)$ have $z_n \in D_n$ as a common zero. Hence, $z_n$ is a root of higher multiplicity of $`W(z', \cdot)$ and it follows that the discriminant $\Delta(z')$ of $`W(z', \cdot)$ is zero. Since $\Delta(z')$ is a polynomial expression in the coefficients of $`W(z', \cdot)$, it follows that $\Delta \in \mathcal{O}(D')$. We view Δ as a polynomial function on $\mathbb{C}$ with coefficients in $\mathcal{O}(D')$ and conclude that

$$D \cap X_s \subset D \cap `W^{-1}(0) \cap \Delta^{-1}(0). \tag{18.1}$$

We will now establish the claim that Δ does not vanish identically on D. Let Q be the quotient field of $\mathcal{O}'_0 := \mathcal{O}_0(\mathbb{C}^{n-1})$. Then it follows from [8, Cor. 3.2 (2)] with $I = \mathcal{O}'_0$ that $W_1, \ldots, W_r$ are irreducible in the polynomial ring $Q[z_n]$. Furthermore, since $Q \setminus \{0\}$ is the set of units in $Q[z_n]$, no distinct W_i and W_j are related by a unit factor in $Q[z_n]$. Since $Q[z_n]$ is a unique factorization domain, it follows that the factors W_j are prime. Furthermore, the elements $`W$ and $\partial_n `W$ have a greatest common divisor $\gamma \in Q[z_n]$, which, up to a unit factor, may be written as the product of those factors W_j that divide $\partial_n `W$. By application of Leibniz's rule for differentiation one sees that such a factor W_j must divide $W_1 \cdots W_{j-1} \cdot \partial_n W_j \cdot W_{j+1} \cdots W_r$, hence W_j must divide $\partial_n W_j$ which is impossible since the latter has lower degree than the former. We conclude that γ is a unit hence belongs to $Q \setminus \{0\}$.

By the Euclidean division algorithm there exist $\lambda_1, \lambda_2 \in Q[z_n]$ such that $1 = \lambda_1 `W + \lambda_2 \partial_n `W$. Let $a \in \mathcal{O}'_0$ be a non-zero element such that $^\circ\lambda_j := a\lambda_j$ belong to $\mathcal{O}'_0[z_n]$, for $j = 1, 2$. Then

$$a = {}^\circ\lambda_1 `W + {}^\circ\lambda_2 \partial_n `W. \tag{18.2}$$

Shrinking the polydisk D sufficiently we may arrange that this equation is valid for all $z \in D$. Shrinking the polydisk D' sufficiently, we may also arrange that for every $z' \in D'$ the polynomial functions $`W(z', \cdot)$ and $\partial_n `W(z', \cdot)$ have all their roots contained in D_n. Suppose now that $z' \in D$ and $\Delta(z') = 0$. Then the polynomial functions $`W(z', \cdot)$ and $`\partial_n `W(z', \cdot)$ have a common root ζ, which must be contained in D_n. Evaluating (18.2) in (z', ζ) we find that $a(z') = 0$. We infer that $D' \cap \Delta^{-1}(0) \subset D' \cap \Delta^{-1}(0)$. Since a is non-zero, we infer that Δ is not vanishing on all of D'; the validity of the claim follows.

Since the germ of Δ at 0 is non-zero we may apply a linear transformation in the first $n-1$ coordinates of $\mathbb{C}^n$ to arrange that $\Delta = \delta W'$ in $\mathcal{O}'_0 := \mathcal{O}_0(\mathbb{C}^{n-1})$, with δ a unit in $\mathcal{O}'_0$ and with $W' \in \mathcal{O}''_0[z_{n-1}]$ a Weierstrass polynomial of degree d'; here $\mathcal{O}''_0 = \mathcal{O}_0(\mathbb{C}^{n-2})$. Shrinking D' to a polydisk centered at 0 with respect to the new coordinates, we may arrange to be in the situation that (18.1) is still true, and such that W' can be represented by a holomorphic function on D' whereas δ can be represented by a nowhere vanishing function on D'. Accordingly, $D \cap \Delta^{-1}(0) = D \cap (W')^{-1}(0)$.

We fix a sufficiently small $r > 0$ such that $\partial D(0, r) \subset D_n$. Furthermore, we shrink the polydisk D', keeping it centered at 0, such that ${}^{\backprime}W$ is nowhere zero on an open neighborhood $D' \times A$ of $D' \times \partial D(0, r)$ in D. It follows that $(D' \times A) \cap X_s = \emptyset$, so that f is well-defined and holomorphic on $D' \times A$. We write $D' = D'' \times D_{n-1}$ according to the decomposition $\mathbb{C}^{n-1} = \mathbb{C}^{n-2} \times \mathbb{C}$ and fix $r' > 0$ such that $\partial D(0, r') \subset D_{n-1}$. Then it follows by application of Cauchy's integral formula to the $(n-1)$-th coordinate that for $(z'', z_{n-1}, z_n) \in D'' \times D(0, r') \times A$,

$$f(z'', z_{n-1}, z_n) = \frac{1}{2\pi i} \int_{\partial D(0,r')} \frac{f(z'', \zeta_{n-1}, z_n)}{\zeta_{n-1} - z_{n-1}} \, d\zeta_{n-1}. \tag{18.3}$$

There exists a sufficiently small polydisk $D''_0 \subset D''$, centered at 0, such that W' does not vanish on an open neighborhood $D''_0 \times B$ of $D''_0 \times \partial D(0, r')$. Since W' is constant as a function of the n-th variable, it follows that X_s has empty intersection with $D''_0 \times B \times D_n$, so that f is holomorphic on the latter set as well. In particular, it follows that the integrand of (18.3) is a holomorphic function of $z_n \in D_n$ as long as $z'' \in D''_0$ and $\zeta_{n-1} \in \partial D(0, r')$. Applying Cauchy's integral formula to this holomorphic function of z_n, we obtain

$$f(z'', z_{n-1}, z_n) = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial D(0,r') \times \partial D(0,r)} \frac{f(z'', \zeta_{n-1}, \zeta_n)}{(\zeta_{n-1} - z_{n-1})(\zeta_n - z_n)} \, d\zeta_n d\zeta_{n-1},$$

for $z \in D''_0 \times D(0, r') \times A$. On the other hand, since f is holomorphic on $D''_0 \times B \times A$, the integral on the right defines a V-valued holomorphic function $F(z)$ of $z \in {}^{\backprime}D := D''_0 \times D(0, r') \times D(0, r)$. It follows from the last displayed equality that $F = f$ on the non-empty open subset $\mathcal{O} := D''_0 \times D(0, r') \times (A \cap D(0; r))$ of ${}^{\backprime}D$. By analytic continuation it follows that $F = f$ on the connected open set ${}^{\backprime}D \setminus X_s$. Thus, ${}^{\backprime}D$ is an open neigborhood of z^0 such that $f|_{{}^{\backprime}D \setminus X_s}$ extends to the holomorphic function $F : {}^{\backprime}D \to V$. □

References

[1] E.P. van den Ban. The principal series for a reductive symmetric space. II. Eisenstein integrals. *J. Funct. Anal.*, 109(2):331–441, 1992.

[2] E.P. van den Ban and J.J. Kuit. Cusp forms for reductive symmetric spaces of split rank one. *Represent. Theory*, 21:467–533, 2017.

[3] E.P. van den Ban and H. Schlichtkrull. Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces. *J. Reine Angew. Math.*, 380:108–165, 1987.

[4] E.P. van den Ban and H. Schlichtkrull. Polynomial estimates for c-functions on reductive symmetric spaces. *Int. Math. Res. Not. IMRN*, (6):1201–1229, 2012.

[5] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[6] D. Bălibanu and E.P. van den Ban. Convexity theorems for semisimple symmetric spaces. *Forum Math.*, 28(6):1167–1204, 2016.

[7] J. Carmona and P. Delorme. Transformation de Fourier sur l'espace de Schwartz d'un espace symétrique réductif. *Invent. Math.*, 134(1):59–99, 1998.

[8] K. Fritzsche and H. Grauert. *From holomorphic functions to complex manifolds*, volume 213 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[9] R.C. Gunning and H. Rossi. *Analytic functions of several complex variables*. AMS Chelsea Publishing, Providence, RI, 2009. Reprint of the 1965 original.

[10] Harish-Chandra. Harmonic analysis on real reductive groups. I. The theory of the constant term. *J. Functional Analysis*, 19:104–204, 1975.

[11] Harish-Chandra. Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula. *Ann. of Math. (2)*, 104(1):117–201, 1976.

[12] Harish-Chandra. *Collected papers. V (Posthumous). Harmonic analysis in real semisimple groups*. Springer, Cham, 2018. Edited by Ramesh Gangolli and V. S. Varadarajan, with assistance from Johan Kolk.

[13] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[14] S. Helgason. *Groups and geometric analysis*, volume 113 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Integral geometry, invariant differential operators, and spherical functions.

[15] A.W. Knapp and E.M. Stein. Intertwining operators for semisimple groups. II. *Invent. Math.*, 60(1):9–84, 1980.

[16] J.A.C. Kolk and V.S. Varadarajan. On the transverse symbol of vectorial distributions and some applications to harmonic analysis. *Indag. Math. (N.S.)*, 7(1):67–96, 1996.

[17] B. Kostant. On the tensor product of a finite and an infinite dimensional representation. *J. Functional Analysis*, 20(4):257–285, 1975.

[18] M. Scott Osborne. *Locally convex spaces*, volume 269 of *Graduate Texts in Mathematics*. Springer, Cham, 2014.

[19] V.S. Varadarajan. *Harmonic analysis on real reductive groups*. Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977.

[20] N.R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[21] N.R. Wallach. *Real reductive groups. II*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1992.

[22] N.R. Wallach. On the Whittaker Plancherel theorem for real reductive groups. *arXiv*, 1705.06787[math.RT]:1–56, 2017.

[23] G. Warner. *Harmonic analysis on semi-simple Lie groups. I.* Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972.

E. P. van den Ban
Mathematical Institute
Utrecht University
PO Box 80 010
3508 TA Utrecht
The Netherlands
E-mail: E.P.vandenBan@uu.nl