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The Principal Series for a Reductive Symmetric Space. II. Eisenstein Integrals

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Contents.

INTRODUCTION

In this paper we develop a theory of Eisenstein integrals related to the principal series for a reductive symmetric space G/H. Here G is a real reductive group of Harish–Chandra's class, σ an involution of G and H an open subgroup of the group G^σ of fixed points for σ. The group G itself is a symmetric space for the left $\times$ right action of $G \times G$: we refer to this

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setting as the group case. Up to a normalization, our Eisenstein integrals generalize those of Harish-Chandra [18] associated with a minimal parabolic subgroup in the group case.

In [4] we studied the principal series for G/H and their H-fixed generalized vectors, motivated by the expectation that they constitute the building blocks for an explicit Plancherel decomposition of $L^2_{\mathrm{mc}}(G/H)$, the most continuous part of $L^2(G/H)$. Let K be a σ-stable maximal compact subgroup of G. Then on the level of left K-finite functions the decomposition should be described in terms of matrix coefficients of K-finite and H-fixed vectors, i.e., in terms of Eisenstein integrals. In the present paper we concentrate on the Eisenstein integrals, and their asymptotic behaviour towards infinity. The main results are: (1) a unitarity result for c-functions, (2) uniform tempered estimates for the Eisenstein integral, and (related to this) (3) a functional equation for H-fixed generalized vectors. These results will be applied in a forthcoming joint paper with H. Schlichtkrull [8] where the decomposition of $L^2_{\mathrm{mc}}(G/H)$ will be given.

We shall now describe the results of this paper in more detail (for unspecified notations see Section 1). The principal series for G/H is a series of parabolically induced representations $\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$, with P a minimal $\sigma \circ \theta$-stable parabolic subgroup (here θ is the Cartan involution associated with K). Moreover, if $P = MAN$ is the Langlands decomposition of P, then $\xi \in \hat{M}_{\mathrm{ps}}$, an appropriate set of finite dimensional irreducible unitary representations of M, and $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$, where $\mathfrak{a}_{\mathfrak{q}}$ is the -1 eigenspace for σ in the Lie algebra $\mathfrak{a}$ of A. The main object of study in [4] was the space of H-fixed elements in the space $C^{-\infty}(P : \xi : \lambda)$ of generalized vectors of $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$. We established the existence of a fixed finite dimensional Hilbert space $V(\xi)$ and a linear map $j(P : \xi : \lambda) : V(\xi) \to C^{-\infty}(P : \xi : \lambda)^H$, depending meromorphically on $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$, and bijective for generic λ.

Eisenstein integrals, defined in Section 3 of the present paper, are essentially linear combinations of matrix coefficients of K-finite vectors with the H-fixed vectors $j(P : \xi : \lambda : \eta)$, $\eta \in V(\xi)$ (cf. Section 4). They depend meromorphically on the parameter $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$ and behave finitely and semisimply under the action of the algebra $\mathbf{D}(G/H)$ of invariant differential operators. Hence by [10, 2] they may be represented by converging series expansions describing their asymptotic behaviour towards infinity. In order to control the dependence of these expansions on λ we adopt a technique which was used in [5], see Sections 11–14.

Let us discuss what the expansions look like for the simplest case of left K-invariant Eisenstein integrals. These Eisenstein integrals occur as matrix coefficients for the induced representations with $\xi = 1$ and generalize the elementary spherical functions of a Riemannian symmetric space (cf. [14]) as well as the spherical functions introduced by Oshima and Sekiguchi [27] for the symmetric spaces of K_ε-type. They are parametrized as follows.

Consider the Weyl group $W = N_K(\mathfrak{a}_\mathfrak{q})/Z_K(\mathfrak{a}_\mathfrak{q})$ and its subgroup $W_{K \cap H}$, the canonical image of $N_{K \cap H}(\mathfrak{a}_\mathfrak{q})$. Let $\mathcal{W}$ be a fixed set of representatives for $W/W_{K \cap H}$ in $N_K(\mathfrak{a}_\mathfrak{q})$. Then $w \mapsto PwH$ is a bijective map from $\mathcal{W}$ onto the set of open H-orbits in $P\backslash G$. In our example we may identify $V(1)$ with $\mathbf{C}^{\mathcal{W}}$ provided with the standard inner product. If $\eta \in \mathbf{C}^{\mathcal{W}}$, then $j(P:1:\lambda:\eta) \in C^{-\infty}(P:1:\lambda)^H$ is completely determined by $j(P:1:\lambda:\eta)(w) = \eta_w$, $w \in \mathcal{W}$. Notice that in the Riemannian case (i.e., $H = K$) we have $\mathbf{C}^{\mathcal{W}} = \mathbf{C}$ and $j(P:1:\lambda:1)$ equals the function 1_λ defined by $1_\lambda(nak) = a^{\lambda + \rho_P}$ (we induce from the left).

The K-fixed Eisenstein integrals may be parametrized by $\mathbf{C}^{\mathcal{W}}$ as well (for general K-types the situation is more complicated). They are defined as matrix coefficients:

$$E(P:\eta:\lambda)(x) = \langle 1_{-\lambda}, \pi_{1,\lambda}(x) j(P:1:\bar{\lambda}:\eta) \rangle \qquad (\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*, x \in G).$$

Notice that in the Riemannian case $E(P:1:\lambda)$ equals the elementary spherical function $\varphi_{-i\lambda}$.

The asymptotic expansions may now be described as follows. Consider the Cartan decomposition $G = KA_\mathfrak{q}H$. Let Q be a second minimal $\sigma \circ \theta$-stable parabolic subgroup containing $A_\mathfrak{q}$. Then Q determines a positive system $\Sigma(Q)$ of roots for $\mathfrak{a}_\mathfrak{q}$ and an associated positive Weyl chamber $A_\mathfrak{q}^+(Q)$. The closure of the set $\bigcup_{w \in \mathcal{W}} w^{-1} A_\mathfrak{q}^+(Q) w$ is a fundamental domain for the Cartan decomposition. Along each set $KA_\mathfrak{q}^+(Q)wH$ the asymptotic behaviour of the K-fixed Eisenstein integral is described by an (actually converging) expansion of the form

$$E(P:\eta:\lambda)(aw) \sim a^{-\rho_Q} \sum_{\substack{s \in W \\ \mu \in \mathbf{N}\Sigma(Q)}} a^{s\lambda - \mu} \Gamma_{Q,w,\mu}(\lambda)\eta \qquad (a \xrightarrow{Q} \infty)$$

Here the $\Gamma_{Q,w,\mu}(\lambda)$ are linear functionals on $\mathbf{C}^{\mathcal{W}}$, meromorphically depending on $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. We define c-functions $C_{Q|P}(s:\lambda) \in \mathrm{End}(\mathbf{C}^{\mathcal{W}})$ by

$$\mathrm{pr}_w \circ C_{Q|P}(s:\lambda) = \Gamma_{Q,w,0}(\lambda),$$

where pr_w denotes projection onto the coordinate determined by w. For general K-types the situation is similar, but more involved (see Section 14). Thus c-functions are defined in terms of leading coefficients of expansions in (generally) more than one chamber, in contrast with the group case, where only one chamber is involved.

One of the main results of this paper is Theorem 16.3, which asserts that the Eisenstein integral allows a normalization so that the associated normalized c-functions are unitary endomorphisms for $\lambda \in i\mathfrak{a}_\mathfrak{q}^*$. For the K-fixed case treated above, this is equivalent to the existence of a meromorphic scalar function $\eta(\lambda)$, independent of P, Q, and s, such that

$$C_{Q|P}(s:-\bar{\lambda})^* C_{Q|P}(s:\lambda) = \eta(\lambda), \qquad \lambda \in \mathfrak{a}_{\mathfrak{q}c}^*.$$

This is analogous to a fundamental result of Harish-Chandra [18, Lemma 3, p. 153]. In the Riemannian case it comes down to $c(-s\lambda)\,c(s\lambda) = c(-\lambda)\,c(\lambda)$, cf., e.g., [20, p. 451, (16)]. In [8] it will be shown that the corresponding part of the Plancherel measure is essentially given by $\eta(\lambda)^{-1}$ times Lebesgue measure on $i\mathfrak{a}_{\mathfrak{q}}^*$, in analogy with the group case.

The second main result of this paper is that Eisenstein integrals satisfy uniform tempered estimates (Theorem 19.2). In the K-fixed case this comes down to estimates of the following form, with $u \in S(\mathfrak{a}_{\mathfrak{q}}^*)$, $X \in U(\mathfrak{g})$, and $C, N > 0$ constants depending on u, X:

$$\|\pi(\lambda)\, E(\eta:\lambda; u:X; aw)\| \leqslant C\, \|\eta\|\, (1+|\lambda|)^N (1+|\log a|)^N\, a^{-\rho_Q}$$

for $w \in \mathscr{W}$, $a \in cl\, A_{\mathfrak{q}}^+(Q)$, $\lambda \in i\mathfrak{a}_{\mathfrak{q}}^*$. Here π is a suitable polynomial function cancelling the singularities of the Eisenstein integral along $i\mathfrak{a}_{\mathfrak{q}}^*$. The estimates allow us, in the final section, to define a Fourier transform on a Schwartz space on G/H generalizing Harish-Chandra's Schwartz space for the group case (cf. Theorem 19.1).

From what has been said so far, it is clear that the results of this paper are deeply inspired by analogous results of Harish-Chandra. Indeed we owe much to the ideas of his papers [16–18]. Nevertheless there are fundamental differences. The first one, already referred to above, is that the c-functions are obtained from (generally) several asymptotic expansions: in [8] this will turn out to be intimately related with the occurrence of multiplicities in the most continuous part of the Plancherel formula. The second difference is the meromorphic dependence of the Eisenstein integral on λ. This is caused by the fact that (in [4]) the map $j(P:\xi:\lambda)$ was obtained by meromorphic continuation starting from a region in $\mathfrak{a}_{\mathfrak{q}c}^*$ which is quite apart from the imaginary points. This makes it hard to get estimates of the uniformly tempered type. Let us finish this introduction by indicating how we obtain them.

In Sections 8 and 9 we derive a functional equation for the map $j(P:\xi:\lambda)$. This result, Theorem 9.3, is the third main result of our paper. Its proof involves, among others, an argument inspired by Zuckerman's translation principle. The obtained functional equation is sufficiently explicit to give a priori estimates for the Eisenstein integral with uniformity in λ (see Proposition 10.3).

When this paper was almost finished I learned that our functional equation in the group case is related to recent work on intertwining operators by Vogan and Wallach [31] and by Zhu [36]. Indeed, in the group case, $j(P:\xi:\lambda)$ is essentially a distribution kernel of an intertwining operator (cf. [7]).

In Section 18 we use the differential equations satisfied by the Eisenstein integral to improve upon the initial estimates, and get estimates of uniformly tempered type. The proof is inspired by a technique of Wallach (cf. [33, Theorem 5.6, p. 328]), related to the theory of Jacquet modules: it allows one to improve initial estimates for matrix coefficients in a number of steps, each step involving the asymptotic behaviour along a maximal parabolic subgroup. We have to do this along maximal $\sigma\theta$-stable parabolic subgroups however, and with uniformity in the parameter λ (see Proposition 18.6 and Theorem 18.3).

1. Notations and Preliminaries

In this section we recall some notations and preliminaries from [4]. Lie groups will be denoted by italic capitals, their Lie algebras by the corresponding German lowercase letters (parabolic subalgebras will sometimes be denoted by German capitals). If $\mathfrak{m}$ is a real Lie algebra, we shall write $U(\mathfrak{m})$, resp. $S(\mathfrak{m})$, for the universal enveloping resp. symmetric algebra of the complexification $\mathfrak{m}_c$ of $\mathfrak{m}$. Let M be a Lie group with algebra $\mathfrak{m}$. Then we denote the left (resp. right) regular action of M on $C^\infty(M)$ by L (resp. R). The associated infinitesimal representations are denoted by the same symbols. Moreover, given $f\in C^\infty(G)$, we shall also use the notations $f(u;x):=L_u f(x)$, $f(x;u):=R_u f(x)$, and $uf:=L_u f$, for $u\in U(\mathfrak{m})$, $x\in M$.

Throughout the paper G will be a real reductive group of Harish-Chandra's class, σ an involution of G, and H an open subgroup of the group G^σ of its fixed points. Let θ be a Cartan involution which commutes with σ, and K the associated maximal compact subgroup of G. The derivative of σ (resp. θ) at e is denoted by the same symbol; let $\mathfrak{h}$ (resp. $\mathfrak{k}$) denote its $+1$ eigenspace, and $\mathfrak{q}$ (resp. $\mathfrak{p}$) its -1 eigenspace. The composition $\sigma\theta$ is an involution as well: the associated $+1$, -1 eigenspaces in $\mathfrak{g}$ are denoted by $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively. Thus

$$\mathfrak{g}_+=\mathfrak{k}\cap\mathfrak{h}\oplus\mathfrak{p}\cap\mathfrak{q} \qquad \mathfrak{g}_-=\mathfrak{k}\cap\mathfrak{q}\oplus\mathfrak{p}\cap\mathfrak{h} \tag{1}$$

and

$$\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_- \tag{2}$$

as direct sums of vector spaces.

We extend the Killing form on $\mathfrak{g}_1=[\mathfrak{g},\mathfrak{g}]$ to a non-degenerate G-invariant bilinear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$, and for which centre $(\mathfrak{g}) \cap \mathfrak{h}$ and centre$(\mathfrak{g}) \cap \mathfrak{q}$ are orthogonal. Moreover, we define a $\mathrm{Ad}(K)$-invariant positive definite inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X, \theta Y)$, and denote the associated norm by $|\cdot|$. All the above decompositions are orthogonal with respect to $\langle \cdot, \cdot \rangle$.

If $\mathfrak{j}$ is a commutative subalgebra of a Lie algebra $\mathfrak{l}$, consisting of semisimple elements, then we write $\Sigma(\mathfrak{l}, \mathfrak{j})$ for the set of non-zero $\mathfrak{j}_c$-weights in $\mathfrak{l}_c$. If $\Sigma(\mathfrak{l}, \mathfrak{j})$ is a (non-reduced) root system, then we denote the associated refection group by $W(\mathfrak{l}, \mathfrak{j})$.

We fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{q}$ of $\mathfrak{p} \cap \mathfrak{q}$ and extend it to a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}$. Given a linear subspace $\mathfrak{e} \subset \mathfrak{g}$, we agree to write $\mathfrak{e}_\mathfrak{h} = \mathfrak{e} \cap \mathfrak{h}$, $\mathfrak{e}_\mathfrak{p} = \mathfrak{e} \cap \mathfrak{p}$, $\mathfrak{e}_{\mathfrak{kq}} = \mathfrak{e} \cap \mathfrak{k} \cap \mathfrak{q}$, etc. Then $\mathfrak{a}_\mathfrak{q} = \mathfrak{a}_{0\mathfrak{q}}$. The root systems of $\mathfrak{a}_\mathfrak{q}$ and $\mathfrak{a}_0$ in $\mathfrak{g}$ are denoted by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_\mathfrak{q})$ and $\Sigma_0 = \Sigma(\mathfrak{g}, \mathfrak{a}_0)$ and we fix compatible positive systems Σ^+ and Σ_0^+, respectively. The set $\Sigma_+ = \Sigma(\mathfrak{g}_+, \mathfrak{a}_\mathfrak{q})$ is a subsystem of Σ. Let $\Sigma_+^+ = \Sigma_+ \cap \Sigma^+$ and let $A_\mathfrak{q}^+$ denote the associated open positive Weyl chamber in $A_\mathfrak{q} = \exp(\mathfrak{a}_\mathfrak{q})$. Then we have the Cartan decomposition

$$G = K\, cl(A_\mathfrak{q}^+) H. \tag{3}$$

Further down we will see that the middle part of the corresponding decomposition of an element need not be uniquely determined, if H is not connected.

By $\mathscr{P}_\sigma$ we denote the (finite) set of all $\sigma\theta$-stable parabolic subgroups of G containing $A_\mathfrak{q}$. Given $P \in \mathscr{P}_\sigma$ we write $P = M_P A_P N_P$ for its Langlands decomposition and put $M_{1P} = M_P A_P$, $A_{P\mathfrak{h}} = A_P \cap H$, $A_{P\mathfrak{q}} = \exp \mathfrak{a}_{P\mathfrak{q}}$, and $M_{\sigma P} = M_P A_{P\mathfrak{h}}$. Notice that $A_{P\mathfrak{q}} \subset A_\mathfrak{q}$. Hence if $\alpha \in \Sigma$, then either $\mathfrak{g}^\alpha = 0$ or $\mathfrak{g}^\alpha \subset \mathfrak{n}_P$. Put

$$\Sigma(P) = \{\alpha \in \Sigma : \mathfrak{g}^\alpha \subset \mathfrak{n}_P\}.$$

Then $\mathfrak{n}_P = \sum_{\alpha \in \Sigma(P)} \mathfrak{g}^\alpha$. Let $\bar{P} = \theta P$. Then $\bar{P} = \sigma P$ and $\Sigma(\bar{P}) = -\Sigma(P)$. Let M_1 denote the centralizer of $\mathfrak{a}_\mathfrak{q}$ in G, and define $\mathfrak{a} = \mathrm{centre}(\mathfrak{m}_1) \cap \mathfrak{p}$. Then $\mathfrak{a}_\mathfrak{q} = \mathfrak{a} \cap \mathfrak{q}$. The linear functional $\rho_P \in \mathfrak{a}^*$ defined by $\rho_P(X) = (1/2)\, \mathrm{tr}(\mathrm{ad}(X) \mid \mathfrak{n}_P)$ vanishes on $\mathfrak{a}_\mathfrak{h}$. Thus $\rho_P \in \mathfrak{a}_\mathfrak{q}^*$ and in fact $\rho_P \in \mathfrak{a}_{P\mathfrak{q}}^*$ if we embed $\mathfrak{a}_{P\mathfrak{q}}^* \subset \mathfrak{a}_\mathfrak{q}^* \subset \mathfrak{a}^*$ via the inner product $\langle \cdot, \cdot \rangle$.

If $P \in \mathscr{P}_\sigma$, then $A_{P\mathfrak{q}} \subset A_\mathfrak{q}$. Moreover, equality holds iff P belongs to the set $\mathscr{P}_\sigma(A_\mathfrak{q})$ of minimal $\sigma\theta$-stable parabolic subgroups containing $A_\mathfrak{q}$. Let $\mathfrak{m}$ denote the orthocomplement of $\mathfrak{a}$ in $\mathfrak{m}_1$, and set $A = \exp \mathfrak{a}$, $A_\mathfrak{h} = A \cap H$, $M = (M_1 \cap K) \exp(\mathfrak{m} \cap \mathfrak{p})$, and $M_\sigma = M A_\mathfrak{h}$. Then $M_1 = MA = M_\sigma A_\mathfrak{q}$ as direct products of groups. For every $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$ we have that $M_P = M$ and $A_P = A$.

The map $P \mapsto \Sigma(P)$ is a bijective correspondence from $\mathscr{P}_\sigma(A_\mathfrak{q})$ onto the set of positive systems for Σ (cf. [4, Sect. 2]. Writing $A_\mathfrak{q}^+(P)$ for the open

Weyl chamber in $A_{\mathfrak{q}}$ associated with the positive system $\Sigma(P)$, $P \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$, we have that

$$cl(A_{\mathfrak{q}}^+) = \bigcup_{\substack{P \in \mathscr{P}_\sigma(A_{\mathfrak{q}}) \\ \Sigma(P) \cap \Sigma_+ = \Sigma_+^+}} cl(A_{\mathfrak{q}}^+(P)).$$

If $P \in \mathscr{P}_\sigma$, then the group $W = N_K(\mathfrak{a}_{\mathfrak{q}})/Z_K(\mathfrak{a}_{\mathfrak{q}})$, the normalizer modulo the centralizer of $\mathfrak{a}_{\mathfrak{q}}$ in K is naturally isomorphic with the reflection group of the root system Σ. By conjugation it acts simply transitively on the set $\mathscr{P}_\sigma(A_{\mathfrak{q}})$. Let $W_{K \cap H}$ be the canonical image of $N_{K \cap H}(\mathfrak{a}_{\mathfrak{q}})$ in W. Throughout this paper $\mathscr{W}$ will be a fixed set of representatives for $W/W_{K \cap H}$ in $N_K(\mathfrak{a}_{\mathfrak{q}})$. If $P \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$ then $w \mapsto PwH$ establishes a one-to-one correspondence from $\mathscr{W}$ onto the set of open H-orbits on $P \backslash G$ (cf. [4, Sect. 3]).

At this point we discuss the decomposition (3) in more detail. The group $W_{K \cap H}$ acts naturally on Σ_+. Let $W^\circ_{K \cap H}$ denote the subgroup of elements leaving Σ_+^+ invariant, or, equivalently, leaving $A_{\mathfrak{q}}^+$ invariant.

LEMMA 1.1. $W_{K \cap H} \simeq W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}}) \rtimes W^\circ_{K \cap H}$.

Proof. We first observe that $W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}}) \simeq W_{K \cap H_e}$. (Here the index e indicates that the identity component of the group is taken.) The product map is bijective, since $W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}})$ acts simply transitively on the Σ_+-chambers in $\mathfrak{a}_{\mathfrak{q}}$. Moreover, since $K \cap H$ normalizes $K \cap H_e = (K \cap H)_e$, it follows that $W_{K \cap H}$ normalizes $W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}})$. ∎

Remark. Notice that it follows from the above that $W^\circ_{K \cap H}$ is trivial iff $W_{K \cap H} = W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}})$ which in turn is equivalent to

$$H = H_e Z_{K \cap H}(\mathfrak{a}_{\mathfrak{q}}), \tag{4}$$

i.e., H is essentially connected (cf. [4, Lemma 4.1]).

LEMMA 1.2. *Let* $X, Y \in cl(\mathfrak{a}_{\mathfrak{q}}^+)$. *Then* $\exp X \in K \exp YH \Leftrightarrow X \in W^\circ_{K \cap H} H$.

Proof. We have that $H = N_{K \cap H}(\mathfrak{a}_{\mathfrak{q}}) H_e$. Hence $\exp X \in K \exp(\tilde{w} Y) H_e$ for some $\tilde{w} \in N_{K \cap H}(\mathfrak{a}_{\mathfrak{q}})$. It now follows from the results in [12, Sect. 4] that $X = wY$ for some $w \in W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}}) W_{K \cap H} = W_{K \cap H}$. Write $w = uv$, with $u \in W(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}})$, $v \in W^\circ_{K \cap H}$. Then $vY \in cl(\mathfrak{a}_{\mathfrak{q}}^+)$, and $u(vY) \in cl(\mathfrak{a}_{\mathfrak{q}}^+)$. It is well known that this implies $v = 1$. Hence $X \in W^\circ_{K \cap H} Y$. The reversed implication is obvious. ∎

We recall that by $\hat{M}_{\mathrm{ps}}$ we denote the set of (equivalence classes of) irreducible finite dimensional unitary representations $(\xi, \mathscr{H}_\xi)$ of M which possess a $w(M \cap H)w^{-1}$-fixed vector for some $w \in \mathscr{W}$. A representation $\xi \in \hat{M}_{\mathrm{ps}}$ is trivial on $\mathfrak{m} \cap \mathfrak{p}$. By trivial extension we will sometimes view it as a representation of $M_1 = MA$. Given $\xi \in \hat{M}_{ps}$, $w \in \mathscr{W}$ we write $\mathscr{V}(\xi, w)$ for the set of $w(M \cap H)w^{-1}$-fixed vectors of ξ. We endow the spaces $\mathscr{V}(\xi, w)$ $(w \in \mathscr{W})$ with the unitary structure inherited from ξ and define a formal direct sum of Hilbert spaces $V(\xi) = \amalg_{w \in \mathscr{W}} \mathscr{V}(\xi, w)$. Let $V(\xi, w)$ denote the canonical image of $\mathscr{V}(\xi, w)$ in $V(\xi)$. Then

$$V(\xi) = \bigoplus_{w \in \mathscr{W}} V(\xi, w) \tag{5}$$

is an orthogonal direct sum decomposition.

Let $P \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$, $\xi \in \hat{M}_{ps}$, and $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. Later in this paper we will need different function spaces associated with the principal series representation $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$. We write

$$C^{-\infty}(P : \xi : \lambda) \tag{6}$$

for the space of generalized functions (i.e., the continuous linear functionals on the compactly supported C^∞-densities) $f : G \to \mathscr{H}_\xi$ transforming according to the rule

$$f(manx) = a^{\lambda + \rho_P} \xi(m) f(x) \qquad ((m, a, n) \in M \times A \times N_P). \tag{7}$$

The group G acts on (6) via the right regular representation R.

It will be useful to work with the compact picture of this induced representation. Restriction to K induces a bijective linear map from (6) onto

$$C^{-\infty}(K : \xi), \tag{8}$$

the space of generalized functions $\varphi : K \to \mathscr{H}_\xi$ transforming according to the rule

$$\varphi(mk) = \xi(m)\, \varphi(k) \qquad \text{for} \quad m \in K_M = K \cap M.$$

Via the restriction map we transfer the induced representation on (6) to a λ-dependent representation $\pi_{P, \xi, \lambda}$ of G on (8).

If $q \in \mathbf{N} \cup \{\infty\}$, then we shall write $C^q(K : \xi)$ for the subspace of (8) consisting of the q times continuously differentiable functions. We provide this space with the usual Fréchet topology. For q finite this is in fact a Banach topology, and we fix a norm $\|\cdot\|_q$ once and for all. Moreover, we let $C^{-q}(K : \xi)$ denote the subspace of (8) consisting of the generalized functions of order at most q. This space was denoted by $\mathscr{D}'_q(K : \xi)$ in [4].

Let dk be the normalized Haar measure on K. If $q \in \mathbf{N} \cup \{\infty\}$, then the map

$$(f, g) \mapsto \langle f, g \rangle = \int_K \langle f(k), g(k) \rangle_\xi \, dk \tag{9}$$

defines a non-degenerate pairing

$$C^{-q}(K:\xi)\times C^{q}(K:\xi)\to \mathbf{C} \tag{10}$$

which is anti-linear in its second variable. It defines a linear isomorphism of C^{-q} with the topological anti-linear dual of C^{q}. We provide $C^{-q}(K:\xi)$ with the associated strong dual topology. When q is finite this is a Banach topology with the dual (operator) norm $\|\cdot\|_{-q}$.

If $q\in \mathbf{Z}\cup\{-\infty,\infty\}$, we define $C^{q}(P:\xi:\lambda)$ to be the preimage of $C^{q}(K:\xi)$ for the (bijective) restriction map from (6) onto (8). The space is topologized by transference of structure. The pairing (10) induces a G-equivariant Hermitian pairing

$$C^{-q}(P:\xi:\lambda)\times C^{q}(P:\xi:-\bar{\lambda})\to \mathbf{C} \tag{11}$$

which establishes a G-equivariant identification of $C^{-q}(P:\xi:\lambda)$ with the strong topological anti-linear dual of $C^{q}(P:\xi:-\bar{\lambda})$.

For $w\in\mathcal{W}$ the evaluation map $\mathrm{ev}_w: f\mapsto f(w)$ is well defined on the space $C^{-\infty}(P:\xi:\lambda)^H$ of H-fixed generalized functions, with values in $\mathcal{V}(\xi,w)$. Let

$$\mathrm{ev}: C^{-\infty}(P:\xi:\lambda)^H\to V(\xi) \tag{12}$$

be the direct sum of the maps ev_w. Then for generic $\lambda\in\mathfrak{a}_{\mathfrak{q}c}^*$ (i.e., for λ in a Baire subset) the map (12) is bijective. Moreover there exists a unique meromorphic map

$$j(P:\xi:\lambda): V(\xi)\to C^{-\infty}(P:\xi:\lambda)^H$$

such that $\mathrm{ev}\circ j(P:\xi:\lambda)=I$ on $V(\xi)$ (cf. [4, Sect. 5]). Here meromorphy should be interpreted with respect to the compact picture of the induced representation: $j(P:\xi:\lambda)$ is meromorphic as a map $V(\xi)\to C^{-\infty}(K:\xi)$ in the sense of [4, p. 375].

If $P, Q\in\mathscr{P}_\sigma(A_\mathfrak{q})$, we recall from [4] that by the methods of [25] we have an intertwining operator $A(P:Q:\xi:\lambda): C^{-\infty}(Q:\xi:\lambda)\to C^{-\infty}(P:\xi:\lambda)$, depending meromorphically on λ. Its action on H-fixed generalized functions is described by

$$A(Q:P:\xi:\lambda)\circ j(P:\xi:\lambda)=j(Q:\xi:\lambda)\circ B(Q:P:\xi:\lambda).$$

Here $B(Q:P:\xi:\lambda)\in\mathrm{End}(V(\xi))$ depends meromorphically on $\lambda\in\mathfrak{a}_{\mathfrak{q}c}^*$ (cf. [4, Proposition 6.1]).

2. Invariant Differential Operators

In this section we gather some properties of the algebra $\mathbf{D}(G/H)$ of invariant differential operators on G/H needed in this paper, meanwhile fixing notations.

We recall that the right regular action of G on $C^\infty(G)$ induces a surjective algebra homomorphism $r: U(\mathfrak{g})^H \to \mathbf{D}(G/H)$ with kernel $\ker r = U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$ (cf. [19]). Thus r factorizes to an isomorphism of algebras

$$\bar{r}: U(\mathfrak{g})^H/(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}) \to \mathbf{D}(G/H). \tag{13}$$

Let $\lambda: S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map. Then we have the following direct sum of vector spaces:

$$U(\mathfrak{g})^H = (U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}) \oplus \lambda[S(\mathfrak{q})^H] \tag{14}$$

(cf. [19]). It follows from the above that r maps $\lambda[S(\mathfrak{q})^H]$ bijectively onto $\mathbf{D}(G/H)$. Set

$$\mathbf{D} := U(\mathfrak{g})^{\mathfrak{h}}/(U(\mathfrak{g})^{h} \cap U(\mathfrak{g})\mathfrak{h}). \tag{15}$$

Then by the above we have a natural isomorphism $\mathbf{D}(G/H_e) \simeq \mathbf{D}$. More generally the inclusion $U(\mathfrak{g})^H \subset U(\mathfrak{g})^{\mathfrak{h}}$ induces an embedding of algebras $\mathbf{D}(G/H) \hookrightarrow \mathbf{D}$. The following result was communicated to me by professor T. Oshima, several years ago.

Lemma 2.1. *The natural embedding* $\mathbf{D}(G/H) \hookrightarrow \mathbf{D}$ *is an isomorphism onto.*

Before proving this lemma we fix notations that will be useful elsewhere too. Let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{q}$, containing $\mathfrak{a}_\mathfrak{q}$. Then $\mathfrak{b} = \mathfrak{b}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{q}$. We recall the duality of [9]. Define a dual real form in $\mathfrak{g}_\mathfrak{c}$ by

$$\mathfrak{g}^d = \mathfrak{g}_+ \oplus i\mathfrak{g}_-. \tag{16}$$

Put $\mathfrak{k}^d = \mathfrak{h}_\mathfrak{c} \cap \mathfrak{g}^d$ and $\mathfrak{p}^d = \mathfrak{q}_\mathfrak{c} \cap \mathfrak{g}^d$. Then

$$\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$$

is a Cartan decomposition for the reductive algebra $\mathfrak{g}^d$, corresponding to the Cartan involution $\theta^d = \sigma_\mathfrak{c} \mid \mathfrak{g}^d$ (here $\sigma_\mathfrak{c}$ denotes the complex linear extension). Notice that $\mathfrak{a}_0^d = \mathfrak{b}_\mathfrak{c} \cap \mathfrak{g}^d$ is a maximal abelian subspace of $\mathfrak{p}^d$, containing $\mathfrak{a}_\mathfrak{q}$. Moreover, we clearly have

$$S(\mathfrak{q})^{\mathfrak{h}} = I(\mathfrak{p}^d), \tag{17}$$

the algebra of $\mathrm{ad}(k^d)$-invariants in $S(\mathfrak{p}^d)$.

Proof of Lemma 2.1. In view of (14) it suffices to show that $S(\mathfrak{q})^H = S(\mathfrak{q})^{\mathfrak{h}}$. Let K_c^d be the commutant of $\theta_c^d = \sigma_c$ in the complex adjoint group G_c. Then $\mathrm{Ad}_G(H) \subset K_c^d$(here we use that $\mathrm{Ad}_G(G) \subset G_c$). Hence it suffices to show that K_c^d acts trivially on (17). Now this is seen as follows. Let F be the (finite) group of elements of order 2 in $\exp \circ \mathrm{ad}(i\mathfrak{a}_0^d)$. Then $K_c^d = F(K_c^d)_e$, hence it suffices to show that F acts trivially on (17). Let B^d be an extension of the Killing form to a non-degenerate bilinear form on $\mathfrak{g}^d$ which is positive definite on $\mathfrak{p}^d$ and for which $[\mathfrak{g}^d, \mathfrak{g}^d]$ and centre($\mathfrak{g}^d$) are orthogonal. Then B^d is G_c-invariant, hence its restriction to $\mathfrak{p}^d$ is F-invariant. In particular the orthogonal projection $\mathfrak{p}^d \to \mathfrak{a}_0^d$ commutes with F. The induced map $I(\mathfrak{p}^d) \to S(\mathfrak{a}_0^d)$ being injective by Chevalley's theorem, it follows that F centralizes $I(\mathfrak{p}^d)$. ∎

Let $W(\mathfrak{b})$ denote the reflection group of the root system $\Sigma(\mathfrak{b}) = \Sigma(\mathfrak{g}, \mathfrak{b}) = \Sigma(\mathfrak{g}^d, \mathfrak{a}_0^d)$. Then the algebra $I(\mathfrak{b})$ of $W(\mathfrak{b})$-invariants in $S(\mathfrak{b})$ equals the algebra $I(\mathfrak{a}_0^d)$ of invariants in $S(\mathfrak{a}_0^d)$ for the reflection group W_0^d of $\Sigma(\mathfrak{g}^d, \mathfrak{a}_0^d)$. Since $\mathbf{D} = U(\mathfrak{g}^d)^{\mathfrak{k}^d}/(U(\mathfrak{g}^d)^{\mathfrak{k}^d} \cap U(\mathfrak{g}^d)\mathfrak{k}^d)$ we have a Harish-Chandra isomorphism $\gamma^d : \mathbf{D} \to I(\mathfrak{a}_0^d) = I(\mathfrak{b})$. Via the natural isomorphism $\mathbf{D}(G/H) \simeq \mathbf{D}$ we transfer γ^d to what we call the Harish-Chandra isomorphism

$$\gamma : \mathbf{D}(G/H) \to I(\mathfrak{b}). \tag{18}$$

If $Q \in \mathscr{P}_\sigma$, we put $H_{1Q} = M_{1Q} \cap H$, and $H_Q = M_Q \cap H$. The natural isomorphism $M_Q/H_Q \times A_{Q\mathfrak{q}} \simeq M_{1Q}/H_{1Q}$ induces an isomorphism

$$\mathbf{D}(M_{1Q}/H_{1Q}) \simeq \mathbf{D}(M_Q/H_Q) \otimes S(\mathfrak{a}_{Q\mathfrak{q}}). \tag{19}$$

Given $D \in \mathbf{D}(G/H)$ we define $'\mu_Q(D)$ to be the element of $\mathbf{D}(M_{1Q}/H_{1Q})$ satisfying

$$D - {}'\mu_Q(D) \in \mathfrak{n}_Q U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}. \tag{20}$$

Here we have slightly abused notations by not distinguishing between elements of $\mathbf{D}(G/H)$ (resp. $\mathbf{D}(M_{1Q}/H_{1Q})$) and their representatives in $U(\mathfrak{g})^H$ (resp. $U(\mathfrak{m}_{1Q})^{H_{1Q}}$). We will continue to do this, as it will not cause any ambiguity. One readily verifies that $D \mapsto {}'\mu_Q(D)$ is a homomorphism of algebras. In view of the decomposition (19) we may view $'\mu_Q(D)$ as a $\mathbf{D}(M_Q/H_Q)$-valued polynomial function on $\mathfrak{a}^*_{Q\mathfrak{q}c}$: we denote its value at λ by $'\mu_Q(D : \lambda)$.

Now consider the function $d_Q : M_{1Q} \to \mathbf{R}^+$ defined by

$$d_Q(m) = \sqrt{|\det \mathrm{Ad}(m)|_{\mathfrak{n}_Q}|} \qquad (m \in M_{1Q}).$$

Then $d_Q = 1$ on $M_{\sigma Q}$ and $d_Q(a) = a^{\rho_Q}$ for $a \in A_{Q\mathfrak{q}}$. Moreover, the function d_Q is right H_{1Q}-invariant.

We define the algebra automorphism T_Q of $\mathbf{D}(M_{1Q}/H_{1Q})$ by

$$T_Q(D) = d_Q^{-1} \circ D \circ d_Q.$$

Moreover, we put $\mu_Q = T_Q \circ {}'\mu_Q$ and $\mu'_Q = T_Q \circ \mu_Q$. Now $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{m}_{1Q} \cap \mathfrak{q}$ containing $\mathfrak{a}_\mathfrak{q}$. Let γ_Q be the Harish-Chandra isomorphism from $\mathbf{D}(M_{1Q}/H_{1Q})$ onto the algebra $I_Q(\mathfrak{b})$ of $W_Q(\mathfrak{b}) = W(\mathfrak{m}_{1Q}, \mathfrak{b})$-invariants in $S(\mathfrak{b})$. By rephrasing the above definitions in terms of $\mathbf{D}$ and subalgebras of the dual real form one sees that

$$\gamma_Q \circ \mu_Q = \gamma. \tag{21}$$

In particular, μ_Q is an embedding.

It is well known that $S(\mathfrak{b})$ is a free $I(\mathfrak{b})$-module of rank $\# W(\mathfrak{b})$. In fact, if E is the set of $W(\mathfrak{b})$-harmonic polynomials in $S(\mathfrak{b})$ then the natural multiplication map

$$I(\mathfrak{b}) \otimes E \to S(\mathfrak{b}) \tag{22}$$

is an isomorphism. Similarly we have an isomorphism

$$I_Q(\mathfrak{b}) \otimes E_Q \to S(\mathfrak{b}), \tag{23}$$

where E_Q denotes the space of $W_Q(\mathfrak{b})$-harmonic polynomials in $S(\mathfrak{b})$. Taking $W_Q(\mathfrak{b})$-invariants in (22) we see that

$$I_Q(\mathfrak{b}) \simeq I(\mathfrak{b}) \otimes E^Q, \tag{24}$$

where we have written E^Q for the set of $W_Q(\mathfrak{b})$-invariants in E. Combining these isomorphisms we see that

$$E \simeq E_Q \otimes E^Q. \tag{25}$$

Hence $\dim E^Q = [W(\mathfrak{b}) : W_Q(\mathfrak{b})]$ and we infer that $I_Q(\mathfrak{b})$ is a free $I(\mathfrak{b})$-module of rank $[W(\mathfrak{b}) : W_Q(\mathfrak{b})]$.

It now follows from (21) that $\mu_Q : \mathbf{D}(G/H) \to \mathbf{D}(M_{1Q}/H_{1Q})$ is an injective homomorphism of algebras. Moreover, $\mathbf{D}(M_{1Q}/H_{1Q})$ is a free $\mu_Q(\mathbf{D}(G/H))$-module of rank $[W(\mathfrak{b}) : W_Q(\mathfrak{b})]$. Let V be the linear subspace of $\mathbf{D}(M_{1Q}/H_{1Q})$ defined by

$$V = T_Q^{-1} \gamma_Q^{-1}(E^Q). \tag{26}$$

Then by (21) and (24) we have a natural isomorphism

$$\mathbf{D}(M_{1Q}/H_{1Q}) \simeq V \otimes {}'\mu_Q(\mathbf{D}(G/H)). \tag{27}$$

Moreover, notice that $1 \in V$. For $\nu \in \mathfrak{b}_c^*$ we define the following ideal of codimension 1 in $\mathbf{D}(G/H)$:

$$\mathscr{I}_\nu = \ker \gamma(. : \nu).$$

LEMMA 2.2. *Let $\nu \in \mathfrak{a}_{0c}^*$. Then $V \otimes {}'\mu_Q(\mathscr{I}_\nu)$ naturally embeds onto an ideal $\mathscr{J}_\nu$ of $\mathbf{D}(M_{1Q}/H_{1Q})$.*

Proof. By (27) the natural map is a linear embedding. Since $\mathscr{I}_\nu$ is an ideal, whereas μ_Q is a homomorphism of algebras, we have that

$${}'\mu_Q(\mathbf{D}(G/H))\, {}'\mu_Q(\mathscr{I}_\nu) \subset {}'\mu_Q(\mathscr{I}_\nu).$$

Combining this with (27) we infer that $\mathbf{D}(M_{1Q}/H_{1Q})\, {}'\mu_Q(\mathscr{I}_\nu) \subset V\, {}'\mu_Q(\mathscr{I}_\nu)$. The reversed inclusion is obviously valid. ∎

LEMMA 2.3. *The inclusion $V \hookrightarrow \mathbf{D}(M_{1Q}/H_{1Q})$ induces a bijection from V onto $\mathbf{D}(M_{1Q}/H_{1Q})/\mathscr{J}_\nu$.*

Proof. In view of (27) and the previous lemma we have natural isomorphisms

$$\begin{aligned} \mathbf{D}(M_{1Q}/H_{1Q})/\mathscr{J}_\nu &\simeq V \otimes {}'\mu_Q(\mathbf{D}(G/H))/{}'\mu_Q(\mathscr{I}_\nu) \\ &\simeq V \otimes {}'\mu_Q(\mathbf{D}(G/H)/\mathscr{I}_\nu) \\ &\simeq V \otimes \mathbf{C} \end{aligned}$$

since ${}'\mu_Q$ is injective. Via these indentifications the induced map corresponds to the map $V \to V \otimes \mathbf{C}$, $x \mapsto x \otimes 1$. ∎

Via the isomorphism $V \simeq \mathbf{D}(M_{1Q}/H_{1Q})/\mathscr{J}_\nu$ described in the above lemma, the space V carries a ν-dependent structure of $\mathbf{D}(M_{1Q}/H_{1Q})$-module which we denote by τ_ν. We shall write V_ν for the space V endowed with the structure τ_ν of $\mathbf{D}(M_{1Q}/H_{1Q})$-module.

LEMMA 2.4. *Let $\nu \in \mathfrak{b}_c^*$. Then the set of $\mathfrak{a}_{Q\mathfrak{q}}$-weights of V_ν equals $(W(\mathfrak{b})\nu + \rho_Q) \mid \mathfrak{a}_{Q\mathfrak{q}}$.*

Proof. Equivalently we must show that $W(\mathfrak{b})\nu \mid \mathfrak{a}_{Q\mathfrak{q}}$ is the set of $\mathfrak{a}_{Q\mathfrak{q}}$-weights of $\mathbf{D}(M_{1Q}/H_{1Q})/T_Q(\mathscr{J}_\nu)$. Let I_ν be the ideal of $p \in I(\mathfrak{b})$ with $p(\nu) = 0$. Then γ_Q induces an isomorphism of $\mathbf{D}(M_{1Q}/H_{1Q})/T_Q(\mathscr{J}_\nu)$ onto $I_Q(\mathfrak{b})/J_\nu$, where $J_\nu = E^Q I_\nu$ is the ideal in $I_Q(\mathfrak{b})$ generated by I_ν (use (24)). Since γ_Q is $\mathfrak{a}_{Q\mathfrak{q}}$-equivariant, we must show that $W(\mathfrak{b})\nu \mid \mathfrak{a}_{Q\mathfrak{q}}$ equals the set of $\mathfrak{a}_{Q\mathfrak{q}}$-weights of $I_Q(\mathfrak{b})/J_\nu$. Let $\mathfrak{b}_c^Q$ denote the space of $W_Q(\mathfrak{b})$-invariants in $\mathfrak{b}_c$, then $\mathfrak{a}_{Q\mathfrak{q}} \subset \mathfrak{b}_c^Q$. Thus the assertion will follow from our claim that the set of weights of the $\mathfrak{b}_c^Q$-module $I_Q(\mathfrak{b})/J_\nu$ equals $W(\mathfrak{b})\nu \mid \mathfrak{b}_c^Q$.

To see the validity of the claim, notice that EI_ν is the ideal in $S(\mathfrak{b})$ generated by I_ν. The $\mathfrak{b}$-module $S(\mathfrak{b})/EI_\nu$ has $W(\mathfrak{b})\nu$ as its set of weights (apply duality and use [5, Proposition 4.1]). From the decompositions (22), (23), (24), (25) we see that the multiplication map $E^Q \otimes I_Q(\mathfrak{b}) \to S(\mathfrak{b})$ induces a linear isomorphism

$$E^Q \otimes (I_Q(\mathfrak{b})/J_\nu) \to S(\mathfrak{b})/EI_\nu. \tag{28}$$

The above map is equivariant for the $\mathfrak{b}_c^Q$-action if we let $\mathfrak{b}_c^Q$ act on the second component in the tensor product. Since the set of weights of the $\mathfrak{b}_c^Q$-module on the right equals $W(\mathfrak{b})\nu \mid \mathfrak{b}_c^Q$, this proves the claim. ∎

3. Definition of the Eisenstein Integral

Throughout the paper F will be a finite subset of the set $\hat{K}$ of (equivalence classes) of finite dimensional irreduible representations of K. Moreover, we write

$$\mathbf{V} = C(K)_{F^\vee}$$

for the space of right K-finite functions whose isotopy types for the right regular representation R are contained in $F^\vee$. It inherits the unitary inner product from $L^2(K, dk)$. Let τ denote the restriction of R to $\mathbf{V}$. We put

$$H_{\mathbf{M}} = H \cap M, \qquad K_{\mathbf{M}} = K \cap M, \qquad \text{and} \qquad \tau_{\mathbf{M}} = \tau \mid K_{\mathbf{M}}.$$

Given $w \in N_K(\mathfrak{a}_\mathfrak{q})$, we denote the space of $\tau_{\mathbf{M}}$-spherical functions from $M/wH_{\mathbf{M}}w^{-1}$ into $\mathbf{V}$ by

$$C(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}}). \tag{29}$$

This space is finite dimensional because the inclusion $K_{\mathbf{M}} \subset M$ induces a diffeomorphism from $K_{\mathbf{M}}/w(K \cap H_{\mathbf{M}})w^{-1}$ onto $M/wH_{\mathbf{M}}w^{-1}$ (cf. [4, Lemma 3.5]). We fix a M-invariant measure dm on $M/wH_{\mathbf{M}}w^{-1}$ of total measure one and provide (29) with the unitary inner product induced by those of $\mathbf{V}$ and $L^2(M/wH_{\mathbf{M}}w^{-1}, dm)$. If ξ is an irreducible finite dimensional unitary representation of M, we write $C_\xi(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}})$ for the subspace of (29) consisting of the functions all of whose components are of left isotypy type ξ. Then clearly we have an orthogonal decomposition

$$C(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}}) = \bigoplus_{\xi \in X} C_\xi(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}}),$$

where X is the finite set of $\xi \in \hat{M}_{ps}$ which have a $K_{\mathbf{M}}$-type in common with $\tau_{\mathbf{M}}^{\vee}$.

Recall that $\mathscr{W} \subset N_K(\mathfrak{a}_{\mathfrak{q}})$ is a finite set of representatives for $W/W_{K \cap H}$ and consider the formal direct sum of Hilbert spaces

$$^\circ\mathscr{C} = \coprod_{w \in \mathscr{W}} C(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}}). \tag{30}$$

The image of $C(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}})$ in $^\circ\mathscr{C}$ is denoted by $^\circ\mathscr{C}_w$. Thus $^\circ\mathscr{C} = \bigoplus_{w \in \mathscr{W}} {^\circ\mathscr{C}_w}$. Given $\psi \in {^\circ\mathscr{C}}$ we write ψ_w for its component in $^\circ\mathscr{C}_w$ (often we shall identify this component with a function in (29)). The left regular representation induces a unitary action of M on $^\circ\mathscr{C}$ in a natural way. Given $\xi \in \hat{M}_{ps}$ we write

$$^\circ\mathscr{C}(\xi) = \coprod_{w \in \mathscr{W}} C_\xi(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}})$$

and we see that the following result holds.

LEMMA 3.1. *We have the orthogonal decomposition* $^\circ\mathscr{C} = \bigoplus_{\xi \in X} {^\circ\mathscr{C}(\xi)}$, *where X is the finite set of $\xi \in \hat{M}_{ps}$ which have a $K_{\mathbf{M}}$-type in common with $\tau_{\mathbf{M}}^{\vee}$, and where each space $^\circ\mathscr{C}(\xi)$ is finite dimensional.*

Fix $P = MAN \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$, $w \in \mathscr{W}$, and $\psi_m \in {^\circ\mathscr{C}_w}$. For $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ with $\operatorname{Re} \lambda + \rho_P$ strictly $\bar{P}$-dominant (i.e., strictly dominant with respect to $\Sigma(\bar{P}) = -\Sigma(P)$), we define the function $\tilde{\psi}_w(P : \lambda) : G \to \mathbf{V}$ by

$$\tilde{\psi}_w(P : \lambda)(namwh) = a^{\lambda + \rho_P}\psi_w(m), \tag{31}$$

for $n \in N$, $a \in A$, $m \in M$, $h \in H$, and by

$$\tilde{\psi}_w(P : \lambda) = 0 \text{ outside } PwH. \tag{32}$$

In view of [4, Proposition 5.6] the function $\tilde{\psi}_w(P : \lambda)$ is continuous on G. It is easily seen to be right H-invariant. We now define the function $\tilde{\psi}(P : \lambda) : G \to \mathbf{V}$ by

$$\tilde{\psi}(P : \lambda) = \sum_{w \in \mathscr{W}} \tilde{\psi}_w(P : \lambda).$$

Finally we define the Eisenstein integral by

$$E(P : \psi : \lambda)(x) = \int_K \tau(k)^{-1}\tilde{\psi}(P : \lambda)(kx)\, dk, \tag{33}$$

for $x \in G$. Let $C(G/H : \tau)$ denote the space of continuous τ-spherical functions from G/H into $\mathbf{V}$. Then $\psi \mapsto E(P : \psi : \lambda)$ defines a linear map from ${}^{\circ}\mathscr{C}$ into $C(G/H : \tau)$.

4. Relation with the Principal Series

In this section we study the relation of the Eisenstein integral $E(P : \psi : \lambda)$ with matrix coefficients of the principal series representation $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$. This relation is then used to extend the Eisenstein integral meromorphically in λ, and to compute the action of $\mathbf{D}(G/H)$ on it.

Let $\mathscr{H}_\xi$ be a Hilbert space model for ξ, and write

$$\mathscr{H}_{\xi,F} := C(K : \xi)_F, \tag{34}$$

where K-types with respect to the right regular representation are taken. We endow the above space with the unitary inner product induced by the unitary structures of ξ and $L^2(K, dk)$.

If V is a complex linear space, we denote the conjugate complex linear space by $\bar{V}$. If V' is a second complex linear space, then we define

$$V' \bar{\otimes} V := V' \otimes_{\mathbf{C}} \bar{V}.$$

Recall the definition of the finite dimensional Hilbert space $V(\xi)$ from Section 1. In a natural fashion the space $\bar{V}(\xi)$ inherits a unitary inner product from $V(\xi)$: if $(.,.)$ denotes the inner product of $V(\xi)$ then the inner product $\langle .,. \rangle$ of $\bar{V}(\xi)$ is defined by $\langle v, w \rangle = (w, v)$. We provide

$$\mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$$

with the induced structure of Hilbert space. Given an element $T = f \bar{\otimes} \eta$ of $\mathscr{H}_{\xi,F} \bar{\otimes} V(\xi, w)$ (where $w \in \mathscr{W}$) we define a function $\psi_T : M/wH_{\mathbf{M}}w^{-1} \to C(K)$ by

$$\psi_T(m)(k) = \langle f(k^{-1}), \xi(m)\eta \rangle_\xi.$$

One easily checks that $\psi_T \in C_\xi(M/wH_{\mathbf{M}}w^{-1} : \tau_{\mathbf{M}})$. By linearity $T \mapsto \psi_T$ is extended to a complex linear map from $\mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$ into ${}^{\circ}\mathscr{C}(\xi)$. Set $d(\xi) = \dim \xi$, then we have:

Lemma 4.1. *The map $T \mapsto d(\xi)^{1/2}\psi_T$ is a bijective isometry from $\mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$ onto ${}^{\circ}\mathscr{C}(\xi)$.*

Proof. Fix $w \in \mathscr{W}$. Then it suffices to prove that the map is an isometry from $\mathscr{H}_{\xi,F} \bar{\otimes} V(\xi, w)$ onto ${}^{\circ}\mathscr{C}_w(\xi)$.

Let $C_\xi(M/wH_{\mathbf{M}}w^{-1})$ denote the space of complex valued functions on $M/wH_{\mathbf{M}}w^{-1}$ which are of left isotopy type ξ. Then the linear map $m_w: \mathscr{H}_\xi \bar{\otimes} V(\xi, w) \to C_\xi(M/wH_{\mathbf{M}}w^{-1})$, determined by

$$m_w(v \bar{\otimes} \eta)(m) = \langle v, \xi(m)\eta \rangle_\xi$$

is bijective. The representation $\xi | K_{\mathbf{M}}$ is irreducible (cf. [4, Lemma 5.3]), hence by the Schur orthogonality relations the map $\mathbf{m}_w = d(\xi)^{1/2} m_w$ is an isometry.

Let S be the endomorphism of $C(K)$ defined by $Sf(k) = f(k^{-1})$. Then S is an isometry from $C(K)_F$ onto $\mathbf{V}$ (where K-types with respect to R are being considered). Hence $S \otimes \mathbf{m}_w$ is an isometry from $E_1 = C(K)_F \otimes [\mathscr{H}_\xi \bar{\otimes} V(\xi, w)]$ onto $E_2 = \mathbf{V} \otimes C_\xi(M/wH_{\mathbf{M}}w^{-1})$. Let π_1 be the representation $L \otimes \xi \otimes 1$ of $K_{\mathbf{M}}$ in E_1, and let π_2 be the representation $\tau_{\mathbf{M}} \otimes L$ of $K_{\mathbf{M}}$ in E_2. Then one readily verifies that $S \otimes \mathbf{m}_w$ intertwines π_1 with π_2, hence maps $(E_1)^{K_{\mathbf{M}}} \simeq \mathscr{H}_{\xi, F} \bar{\otimes} V(\xi, w)$ isometrically onto $(E_2)^{K_{\mathbf{M}}} \simeq {}^\circ\mathscr{C}_w(\xi)$. Now observe that $(S \otimes \mathbf{m}_w)(T) = d(\xi)^{1/2}\, \psi_T$ for $T \in \mathscr{H}_{\xi, F} \bar{\otimes} V(\xi, w)$. ∎

We can now relate the Eisenstein integral to matrix coefficients of principal series representations.

LEMMA 4.2. *If* $T = f \bar{\otimes} \eta \in \mathscr{H}_{\xi, F} \bar{\otimes} V(\xi)$, *then for* $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$ *with* $\operatorname{Re} \lambda + \rho_P$ *strictly* $\bar{P}$*-dominant we have*

$$E(P : \psi_T : \lambda)(x)(k) = \langle f, \pi_{P, \xi, \bar{\lambda}}(kx)\, j(P : \xi : \bar{\lambda})\eta \rangle, \tag{35}$$

for $x \in G, k \in K$.

Proof. It suffices to prove this for $\eta = \eta_w \in V(\xi, w)$, $w \in \mathscr{W}$. From the definition of ψ_T we deduce that

$$\psi_T(m)(k) = \langle f(k^{-1}), j(P : \xi : \bar{\lambda} : \eta_w)(mw) \rangle_\xi$$

for $m \in M$, $k \in K$. From the transformation properties under the left action by $N_P A$ and the right action by H it follows that

$$\tilde{\psi}_T(x)(k) = \langle f(k^{-1}), j(P : \xi : \bar{\lambda} : \eta_w)(x) \rangle_\xi$$

for $x \in PwH$. Both the left and the right hand side of the above equation are zero outside PwH so that it actually holds for all $x \in G$. Now use (33) and the definition (9) of the equivariant pairing (11). ∎

Let $\mathscr{F}$ be a Fréchet space. Then a $\mathscr{F}$-valued function f on a complex analytic manifold Ω will be called meromorphic if locally at every point $z \in \Omega$ there exists a homomorphic function φ such that φf is holomorphic in a neighbourhood of z.

Let $C^\infty(G/H:\tau)$ denote the space of τ-spherical C^∞-functions $G/H \to \mathbf{V}$.

COROLLARY 4.3. *Let $\psi \in {}^\circ\mathscr{C}$. If $\operatorname{Re}\lambda + \rho_P$ is strictly $\bar{P}$-dominant, then $E(P:\psi:\lambda)$ belongs to $C^\infty(G/H:\tau)$, depending holomorphically on λ. Moreover, $\lambda \mapsto E(P:\psi:\lambda)$ extends to a meromorphic $C^\infty(G/H:\tau)$-valued function on $\mathfrak{a}^*_{\mathfrak{q}c}$.*

Proof. By Lemmas 3.1 and 4.1 it suffices to prove this for $\psi = \psi_T$, with $T \in \mathscr{H}_{\xi,F} \bar{\otimes} V(\xi, w)$, $w \in \mathscr{W}$. The result is then an immediate consequence of Lemma 4.2 and the meromorphy of $j(P:\xi:\lambda)$, cf. [4, Lemma 5.7 and Theorem 5.10]. ∎

In the rest of this section we will discuss the action of the algebra of invariant differential operators on Eisenstein integrals.

Recall the definition of $\mu_P\colon \mathbf{D}(G/H) \to \mathbf{D}(M_1/M_1 \cap H)$ from Section 2. Given $w \in \mathscr{W}$ we define $\mu_P^w\colon \mathbf{D}(G/wHw^{-1}) \to \mathbf{D}(M_1/M_1 \cap wHw^{-1})$ similarly but with H replaced by wHw^{-1}. Now $\operatorname{Ad}(w)$ maps $U(\mathfrak{g})^H$ into $U(\mathfrak{g})^{wHw^{-1}}$ and induces an isomorphism of algebras $\overline{\operatorname{Ad}(w)}: \mathbf{D}(G/H) \to \mathbf{D}(G/wHw^{-1})$. We define $\mu_{P,w}: \mathbf{D}(G/H) \to \mathbf{D}(M_1/M_1 \cap wHw^{-1})$ by

$$\mu_{P,w} = \mu_P^w \circ \overline{\operatorname{Ad}(w)}.$$

Given $X \in U(\mathfrak{g})^H$ let $\mu_{P,w}(X:\xi:\lambda)$ denote the endomorphism by which $\mu_{P,w}(X:\lambda)$ acts on $\mathscr{V}(\xi, w) \subset \mathscr{H}_\xi$, and define $\underline{\mu}_P(X:\xi:\lambda): V(\xi) \to V(\xi)$ to be the direct sum of these maps.

LEMMA 4.4. *Let $X \in U(\mathfrak{g})^H$. Then*

$$R_X j(P:\xi:\lambda) = j(P:\xi:\lambda) \circ \underline{\mu}_P(X:\xi:\lambda).$$

Proof. Since R_X preserves the subspace of H-invariant functions in (6) it suffices to establish the identity which results if we apply ev_w on the left (use [4, Theorem 5.10]). For $w=1$ this identity is a straightforward consequence of the equivariance properties of j locally at e, and the definition of $\mu_{P,1} = \mu_P$. The identity now follows for arbitrary w if we observe that

$$\mathrm{ev}_w \circ R_X \circ j(P:\xi:\lambda) = \mathrm{ev}_1 \circ R_{\operatorname{Ad}(w)X} \circ j'(P:\xi:\lambda),$$

where $j'(P:\xi:\lambda)$ is the map $V(\xi) \to C^{-\infty}(P:\xi:\lambda)^{wHw^{-1}}$ associated with wHw^{-1} and the set $\mathscr{W}' = \mathscr{W}w^{-1}$ of representatives for $W/W_{K \cap wHw^{-1}}$. ∎

Given $D \in \mathbf{D}(G/H)$ we define an endomorphism of ${}^\circ\mathscr{C}$ by

$$\underline{\mu}_P(D:\lambda) = \bigoplus_{w \in \mathscr{W}} R(\mu_{P,w}(D:\lambda)).$$

LEMMA 4.5. *Let* $D \in \mathbf{D}(G/H)$. *Then*

$$DE(P : \psi : \lambda) = E(P : \underline{\mu}_P(D : \lambda)\psi : \lambda).$$

Proof. By linearity it suffices to prove this for a D with real coefficients and for $\psi = \psi_T$ with $T = f_w \bar{\otimes} \eta_\omega \in \mathscr{H}_{\xi, F} \bar{\otimes} V(\xi, w)$ for some $\xi \in \hat{M}_{\mathrm{ps}}$, $w \in \mathscr{W}$. Let X be a real representative of D in $U(\mathfrak{g})^H$. Then from the definition of ψ_T it follows straightforwardly that

$$\underline{\mu}_P(D : \lambda)\psi_T = \psi_{f_w \otimes \mu_{P,w}(X : \xi : \lambda)\eta_w}.$$

Now use Lemmas 4.2 and 4.4 to complete the proof. ∎

We finish this section with a description of the eigenvalues of the endomorphisms $\underline{\mu}_P(D : \lambda)$. The following lemma will be needed at a later stage as well. Let $\mathfrak{j}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$.

LEMMA 4.6. *Let* $w \in N_K(\mathfrak{a}_\mathfrak{q})$. *Then there exists a* $s \in W(\mathfrak{g}, \mathfrak{j})$ *normalizing* $\mathfrak{b}$ *and* $\mathfrak{a}_\mathfrak{q}$, *and such that* $s \mid \mathfrak{a}_\mathfrak{q} = \mathrm{Ad}(w) \mid \mathfrak{a}_\mathfrak{q}$. *Moreover, if* $\xi \in \hat{M}_{\mathrm{ps}}$ *has infinitesimal character* $\Lambda \in \mathfrak{j}_\mathfrak{c}^*$, *then* $w\xi$ *has infinitesimal charater* $s\Lambda$.

Proof. Using the duality of Section 2, notice that $W(\mathfrak{g}^d, \mathfrak{a}_0^d) = W(\mathfrak{g}, \mathfrak{b})$. Let

$$W_{0\sigma}^d = \{s \in W(\mathfrak{g}^d, \mathfrak{a}_0^d);\ \sigma^d \circ s = s \circ \sigma^d\}.$$

Then according to [28, Proposition 7.17] (see also [4, Lemma 1.1]), restriction induces a surjective map $W_{0\sigma}^d \to W$. Now $\mathrm{Ad}(w) \mid \mathfrak{a}_\mathfrak{q} \in W$, hence $\mathrm{Ad}(w) \mid \mathfrak{a}_\mathfrak{q} = s_1 \mid \mathfrak{a}_\mathfrak{q}$ for some $s_1 \in W(\mathfrak{g}, \mathfrak{b})$. Now $\mathfrak{j}^d = \mathfrak{j}_\mathfrak{c} \cap \mathfrak{g}^d$ is a θ^d-stable Cartan subalgebra of $\mathfrak{g}^d$ containing $\mathfrak{a}_0^d$. Hence the normalizer of $\mathfrak{a}_0^d$ in $W(\mathfrak{g}^d, \mathfrak{j}^d) = W(\mathfrak{g}, \mathfrak{j})$ maps onto $W(\mathfrak{g}^d, \mathfrak{a}_0^d) = W(\mathfrak{g}, \mathfrak{b})$ and we see that $s_1 = s \mid \mathfrak{b}$ for some $s \in W(\mathfrak{g}, \mathfrak{j})$.

Since $\mathrm{Ad}(w^{-1})\mathfrak{j}_\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{m}_{1\mathfrak{c}}$, there exists a $\varphi_1 \in \mathrm{Aut}(\mathfrak{m}_{1\mathfrak{c}})^\circ$ such that $\mathrm{Ad}(w^{-1})\,\mathfrak{j}_\mathfrak{c} = \varphi_1(\mathfrak{j}_\mathfrak{c})$. Now $\mathrm{Ad}(w) \circ \varphi_1 \in \mathrm{Aut}(\mathfrak{g}_\mathfrak{c})^\circ$ and normalizes $\mathfrak{j}_\mathfrak{c}$, hence defines an element $t \in W(\mathfrak{g}, \mathfrak{j})$. Moreover, $t \mid \mathfrak{a}_\mathfrak{q} = \mathrm{Ad}(w) \mid \mathfrak{a}_\mathfrak{q} = s \mid \mathfrak{a}_\mathfrak{q}$, hence $t^{-1}s \in W(\mathbf{m}_1, \mathfrak{j})$. Hence $t^{-1}s = \varphi_2 \mid \mathfrak{j}_\mathfrak{c}$ for some $\varphi_2 \in \mathrm{Aut}(\mathfrak{m}_{1\mathfrak{c}})^\circ$. Put $\varphi = \varphi_1 \circ \varphi_2$. Then $\varphi \in \mathrm{Aut}(\mathfrak{m}_{1\mathfrak{c}})^\circ$ and $\psi := \mathrm{Ad}(w) \circ \varphi$ normalizes $\mathfrak{j}_\mathfrak{c}$ and satisfies $\psi \mid \mathfrak{j}_\mathfrak{c} = s \mid \mathfrak{j}_\mathfrak{c}$.

Given any automorphism φ of $\mathfrak{m}_{1\mathfrak{c}}$ we write ξ^φ for the infinitesimal representation $\xi \circ \varphi^{-1}$ of $\mathfrak{m}_{1\mathfrak{c}}$. In particular, $\xi^{\mathrm{Ad}(w)}$ denotes the differential of $w\xi$. If φ is any element of the identity component of $\mathrm{Aut}(\mathfrak{m}_{1\mathfrak{c}})$, then it is readily verified that ξ^φ is equivalent to ξ. Hence $w\xi$ has the same infinitesimal character as ξ^ψ. Now ψ is an automorphism of $\mathfrak{m}_{1\mathfrak{c}}$ which normalizes $\mathfrak{j}_\mathfrak{c}$. This implies that ξ^ψ has infinitesimal character $(\psi^{-1})^*\Lambda = s\Lambda$. ∎

The space $\mathfrak{b}_{\mathfrak{k}}$ is a Cartan subspace of $\mathfrak{m} \cap \mathfrak{q}$. Let Σ_{M}^{+} be a system of positive roots for $\Sigma_{\mathrm{M}} = \Sigma(\mathfrak{m}, \mathfrak{b}_{\mathfrak{k}})$, and let ρ_{M} be half the sum of the positive roots, counting multiplicities. Let W_{M} be the associated reflection group, and write $I_{\mathrm{M}}(\mathfrak{b}_{\mathfrak{k}})$ for the algebra of W_{M}-invariants in $S(\mathfrak{b}_{\mathfrak{k}})$. Then we have a Harish-Chandra isomorphism $\gamma_{\mathrm{M}} : \mathbf{D}(M/H_{\mathrm{M}}) \to I_{\mathrm{M}}(\mathfrak{b}_{\mathfrak{k}})$. Notice that for any $Q \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$ we have

$$\gamma_Q = \gamma_{\mathrm{M}} \otimes \mathrm{id}_{S(\mathfrak{a}_{\mathfrak{q}})}$$

with respect to the decomposition (19). Now let L be the set of $\Lambda \in i\mathfrak{b}_{\mathfrak{k}}^*$ which lift to a character of the torus $B_{\mathfrak{k}} = \exp \mathfrak{b}_{\mathfrak{k}}$.

PROPOSITION 4.7. *For every $D \in \mathbf{D}(G/H)$, $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ the endomorphism $\mu_P(D : \lambda)$ of ${}^\circ\mathscr{C}$ is semisimple and respects the decomposition ${}^\circ\mathscr{C} = \bigoplus {}^\circ\mathscr{C}_w(\xi)$ ($\xi \in X$, $w \in \mathscr{W}$). Moreover, let $w \in \mathscr{W}$, and let s be as in Lemma 4.6. Then the eigenvalues of $\mu_P(D : \lambda) \mid {}^\circ\mathscr{C}_w$ are of the form $\gamma(D : s\Lambda + \rho_{\mathrm{M}} + \lambda)$, with $\Lambda \in L$.*

We begin by studying the action of $\mathbf{D}(M/H_{\mathrm{M}})$ on the space $C^\infty(M/H_{\mathrm{M}})_{K_{\mathrm{M}}}$ of left K_{M}-finite smooth functions on M/H_{M}. The following result will be needed at a later stage as well.

LEMMA 4.8. *The algebra $\mathbf{D}(M/H_{\mathrm{M}})$ acts finitely and semisimply on $C^\infty(M/H_{\mathrm{M}})_{K_{\mathrm{M}}}$. The simultaneous eigenvalues of the action are all of the form $D \mapsto \gamma_{\mathrm{M}}(D : \Lambda + \rho_{\mathrm{M}})$, with $\Lambda \in L$.*

Proof. We first notice that $\mathfrak{b}_{\mathfrak{k}}$ is also a Cartan subspace of $\mathfrak{k}_{\mathrm{M}} \cap \mathfrak{q}$. Moreover, since $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{h}$, it follows that $[\mathfrak{b}_{\mathfrak{k}}, \mathfrak{m} \cap \mathfrak{p}] \subset \mathfrak{m} \cap \mathfrak{p} \cap \mathfrak{q} = 0$. Hence $\Sigma(\mathfrak{k}_{\mathrm{M}}, \mathfrak{b}_{\mathfrak{k}}) = \Sigma_{\mathrm{M}}$, including multiplicities. Set $H_0 = K_{\mathrm{M}} \cap H$, and

$$\mathbf{D}_0 = U(\mathfrak{k}_{\mathrm{M}})^{\mathfrak{h}_0} / U(\mathfrak{k}_{\mathrm{M}})^{\mathfrak{h}_0} \cap U(\mathfrak{k}_{\mathrm{M}})\mathfrak{h}_0.$$

Then we also have a Harish-Chandra isomorphism $\gamma_{K_{\mathrm{M}}} : \mathbf{D}_0 \to I_{\mathrm{M}}(\mathfrak{b}_{\mathfrak{k}})$. It is related to γ_{M} as follows. From $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{h}$ it follows that $U(\mathfrak{m}) = U(\mathfrak{k}_{\mathrm{M}}) + U(\mathfrak{m})(\mathfrak{h} \cap \mathfrak{m})$. Let $p_0 : U(\mathfrak{m}) \to U(\mathfrak{k}_{\mathrm{M}})\mathfrak{h}_0$ be the associated linear surjective map. The induced map $p_1 : U(\mathfrak{m})^{H_{\mathrm{M}}} \to (U(\mathfrak{k}_{\mathrm{M}})^{H_0} \cap U(\mathfrak{k}_{\mathrm{M}})\, \mathfrak{h}_0)$ is easily seen to be an algebra homomorphism with kernel $\ker p_1 = U(\mathfrak{m})^{H_{\mathrm{M}}} \cap U(\mathfrak{m})(\mathfrak{m} \cap \mathfrak{h})$. In view of the fact that $[\mathfrak{m} \cap \mathfrak{h} \cap \mathfrak{p}, \mathfrak{k}_{\mathrm{M}}] \subset \mathfrak{m} \cap \mathfrak{h}$, it follows that p_1 is actually surjective, hence induces an isomorphism of algebras

$$p : \mathbf{D}(M/H_{\mathrm{M}}) \to \mathbf{D}(K_{\mathrm{M}}/H_0).$$

The second algebra allows a natural embedding in $\mathbf{D}_0$ (cf. Section 2). Moreover, from the above definition of p it is clear that

$$\gamma_{K_{\mathrm{M}}} \circ p = \gamma_{\mathrm{M}} \tag{36}$$

(use that the definitions of the two Harish-Chandra isomorphisms involve the same rho-shift). In particular we see that $\mathbf{D}(K_{\mathbf{M}}/H_0) \simeq \mathbf{D}_0$. The natural map $i : K_{\mathbf{M}}/H_0 \hookrightarrow M/H_{\mathbf{M}}$ is a diffeomorphism (cf. [4, Lemma 3.5]). The associated pull-back $i^* : C^\infty(M/H_{\mathbf{M}}) \to C^\infty(K_{\mathbf{M}}/H_0)$ is a bijective $K_{\mathbf{M}}$-equivariant topological linear isomorphism and from the above definition of p one readily checks that $i^* \circ p(D) = D \circ i^*$ for all $D \in \mathbf{D}(M/H_{\mathbf{M}})$. Therefore it suffices to study the right action of $U(\mathfrak{k}_{\mathbf{M}})^{H_0}$ on $C^\infty(K_{\mathbf{M}}/H_0)$.

Let $\mathscr{L}_1$ be the set of equivalence classes of finite dimensional irreducible representations of $K_{\mathbf{M}}$ possessing a H_0-fixed vector. Then by the Peter–Weyl theorem we have the following isomorphism of $K_{\mathbf{M}}$, $U(\mathfrak{k}_{\mathbf{M}})^{H_0}$ modules:

$$C^\infty(K_{\mathbf{M}}/H_0)_{K_{\mathbf{M}}} \simeq \bigoplus_{\xi \in \mathscr{L}_1} V_\xi^* \otimes V_\xi^{H_0}. \tag{37}$$

Hence it suffices to consider the action of $U(\mathfrak{k}_{\mathbf{M}})^{H_0}$ on $V_\xi^{H_0}$. We consider the action on the possibly bigger space $V_\xi^{\mathfrak{h}_0}$. Let $V_\xi = V_1 \oplus \cdots \oplus V_m$ be a decomposition of V_ξ into irreducible $(K_{\mathbf{M}})^\circ$ modules. Then

$$V_\xi^{\mathfrak{h}_0} = V_1^{\mathfrak{h}_0} \oplus \cdots \oplus V_m^{\mathfrak{h}_0}$$

and this decomposition is preserved by $U(\mathfrak{k}_{\mathbf{M}})^{H_0}$. It suffices to consider the action of $U(\mathfrak{k}_{\mathbf{M}})^{H_0}$, on $V^{\mathfrak{h}_0}$, with V an irreducible $(K_{\mathbf{M}})^\circ$ module. If $V^{\mathfrak{h}_0} = 0$ then there is nothing to prove. In the remaining case we have $\dim V^{\mathfrak{h}_0} = 1$, and it is well known that V has a highest weight $\Lambda \in i\mathfrak{b}_{\mathfrak{k}}^*$: clearly $\Lambda \in L$. It is also standard that $X \in U(\mathfrak{k}_{\mathbf{M}})^{H_0}$ acts on $V^{\mathfrak{h}_0}$ by the scalar $\gamma_{K_{\mathbf{M}}}(X : \Lambda + \rho_{\mathbf{M}})$. It follows that $D \in \mathbf{D}(M/H_{\mathbf{M}})$ acts semisimply on $C^\infty(M/H_{\mathbf{M}})_{K_{\mathbf{M}}}$, and with eigenvalues $\gamma_{K_{\mathbf{M}}}(p(D) : \Lambda + \rho_{\mathbf{M}})$. Now use (36). ∎

Proof of Proposition 4.7. From the definition of μ_P^w one readily deduces that

$$\mu_P^w \circ \overline{\mathrm{Ad}(w)} = \overline{\mathrm{Ad}(w)} \circ \mu_{w^{-1}Pw},$$

where in the right hand side of the equation $\overline{\mathrm{Ad}(w)}$ denotes the isomorphism $\mathbf{D}(M/H_{\mathbf{M}}) \to \mathbf{D}(M/wH_{\mathbf{M}}w^{-1})$ induced by $\mathrm{Ad}(w) : U(\mathfrak{m})^{H_{\mathbf{M}}} \to U(\mathfrak{m})^{wH_{\mathbf{M}}w^{-1}}$. Hence for $D \in \mathbf{D}(G/H)$ we have

$$\mu_{P,w}(D : \lambda) = \overline{\mathrm{Ad}(w)}\, \mu_{w^{-1}Pw}(D : w^{-1}\lambda).$$

Now consider the bijective intertwining map $R_w : C^\infty(M/H_{\mathbf{M}}) \to C^\infty(M/wH_{\mathbf{M}}w^{-1})$ defined by $R_w f(m) = f(mw)$. Then $R_w \circ \mu = [\overline{\mathrm{Ad}(w)}\mu] \circ R_w$ for $\mu \in \mathbf{D}(M/H_{\mathbf{M}})$. It follows that the eigenvalues of $\mu_{P,w}(D : \lambda)$ are the same as those of $\mu_{w^{-1}Pw}(D : w^{-1}\lambda)$. In view of Lemma 4.8 they are all of the following form, with $Q = w^{-1}Pw$, $\Lambda_1 \in L$:

$$\begin{aligned}\gamma_{\mathbf{M}}(\mu_Q(D : w^{-1}\lambda))(\Lambda_1 + \rho_{\mathbf{M}}) &= \gamma_Q(\mu_Q(D : s^{-1}\lambda))(\Lambda_1 + \rho_{\mathbf{M}}) \\ &= \gamma(D : \Lambda_1 + \rho_{\mathbf{M}} + s^{-1}\lambda) \\ &= \gamma(D : s\Lambda + \rho_{\mathbf{M}} + \lambda),\end{aligned}$$

where $\Lambda = \Lambda_1 + \rho_{\mathbf{M}} - s^{-1}\rho_{\mathbf{M}}$. Now s normalizes $\mathfrak{a}_{\mathfrak{q}}$, hence $\mathfrak{m}$, $\mathfrak{b}_{\mathfrak{k}}$, and $\Sigma_{\mathbf{M}}$. Therefore $\rho_{\mathbf{M}} - s^{-1}\rho_{\mathbf{M}}$ is an integral linear combination of roots in $\Sigma_{\mathbf{M}}$, hence belongs to L. ∎

5. Finite Dimensional Class (1, 1) Representations

The purpose of this section is to describe the finite dimensional irreducible representations of G possessing both a H- and a K-fixed vector. These representations will be needed in the translation arguments of Sections 8 and 9. Most of the results of this section are essentially due to [21].

A continuous representation π of the group G in a finite dimensional complex linear space V is said to be of class 1 if there exists a non-trivial vector $v \in V$ which is K-fixed. If in addition there exists a non-trivial vector $w \in V$ which is H-fixed, then we shall say that π is of class (1, 1). Let us first recall the Cartan–Helgason description of finite dimensional irreducible representations of class 1, meanwhile fixing notations. With notations as in Section 1 let $\mathfrak{j}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$. Let $\Sigma^+(\mathfrak{j})$ be a system of positive roots for $\Sigma(\mathfrak{j}) = \Sigma(\mathfrak{g}, \mathfrak{j})$ which is compatible with Σ_0^+.

Let $\Lambda(\mathfrak{j})$ denote the set of integral weights in $\mathfrak{j}_c^*$, and let $\Lambda(\mathfrak{a}_0)$ denote the set of $\nu \in \mathfrak{a}_{0c}^*$ such that

$$\frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \qquad \text{for each} \quad \alpha \in \Sigma_0.$$

Via the decomposition $\mathfrak{j} = \mathfrak{j}_{\mathfrak{k}} \oplus \mathfrak{a}_0$ we identify $\mathfrak{a}_{0c}^*$ with a subspace of $\mathfrak{j}_c^*$. Then $\Lambda(\mathfrak{a}_0) \subset \Lambda(\mathfrak{j})$.

If π is an irreducible class 1 representation of G in a finite dimensional complex vector space V, then it is well known that dim $V^K = $ dim $V^{\mathfrak{t}} = 1$, and that V is an irreducible $\mathfrak{g}_c$-module. Let $\nu(\pi) \in \Lambda(\mathfrak{j})$ be its $\Sigma^+(\mathfrak{j})$-highest weight. Then $\nu(\pi)$ belongs to

$$\Lambda^+(\mathfrak{a}_0) = \{\mu \in \Lambda(\mathfrak{a}_0);\ \ \langle \nu, \alpha \rangle \geqslant 0 \qquad \text{for} \quad \alpha \in \Sigma_0^+\},$$

Conversely, if $\nu \in \Lambda^+(\mathfrak{a}_0)$, then $\nu = \nu(\pi)$ for a unique finite dimensional irreducible class 1 representation π of G (up to equivalence). We shall call π the class 1 representation of highest weight ν. For G connected, semisimple and with finite centre these results can be found, e.g., in [35, Sect. 3.3]. They are easily extended to groups of Harish-Chandra's class.

If $\mathfrak{l}$ is a real abelian Lie algebra, and V a complex vector space on which $\mathfrak{l}$ acts finitely, then by $V_\lambda(\mathfrak{l})$ we denote the generalized weight space of weight $\lambda \in \mathfrak{l}_c^*$ in the $\mathfrak{l}$-module V. For future use we list some facts which are easy to prove.

LEMMA 5.1. *Let* $\nu \in \Lambda^+(\mathfrak{a}_0)$, *and let* (π, V) *be the associated class* 1 *representation of* G *of highest weight* ν. *Then*: (1) $V_\nu(\mathfrak{a}_0) = V_\nu(\mathfrak{j})$; (2) *if* $v \in V_\nu(\mathfrak{a}_0)\backslash\{0\}$ *and* $\varepsilon \in (V^*)^K\backslash\{0\}$, *then* $\varepsilon(v) \neq 0$; *and* (3) $Z_K(\mathfrak{a}_0)$ *acts trivially on* $V_\nu(\mathfrak{a}_0)$.

We now recall some results due to [21].

LEMMA 5.2. *Let* $X \in \mathfrak{p}$, $Y \in \mathfrak{q}$, *and assume that both* X *and* Y *centralize* $\mathfrak{a}_\mathfrak{q}$. *Then* $[X, Y] = 0$.

Proof. It suffices to prove this for the case that $\mathfrak{g}$ is semisimple. Moreover, by maximality of $\mathfrak{a}_\mathfrak{q}$ in $\mathfrak{p} \cap \mathfrak{q}$ we may as well assume that $X \in \mathfrak{p} \cap \mathfrak{h}$ and $Y \in \mathfrak{q} \cap \mathfrak{k}$. Then $Z = [X, Y]$ belongs to $[\mathfrak{h}, \mathfrak{q}] \cap [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{q} \cap \mathfrak{p}$. Clearly Z centralizes $\mathfrak{a}_\mathfrak{q}$ and we infer that $Z \in \mathfrak{a}_\mathfrak{q}$. But using the invariance of the Killing form one readly chacks that Z is Killing perpendicular to $\mathfrak{a}_\mathfrak{q}$: hence $Z = 0$. ∎

Recall that $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{q}$, containing $\mathfrak{a}_\mathfrak{q}$.

COROLLARY 5.3. $[\mathfrak{a}_0, \mathfrak{b}] = 0$.

By the above result the subspace $\mathfrak{a}_0 + \mathfrak{b}$ is an abelian subalgebra of $\mathfrak{g}$ which consists of semisimple elements. We may therefore choose an abelian subspace $\mathfrak{j}_{\mathfrak{kh}} \subset \mathfrak{k} \cap \mathfrak{h}$ such that $\mathfrak{j} = \mathfrak{j}_{\mathfrak{kh}} \oplus (\mathfrak{a}_0 + \mathfrak{b})$ is a Cartan subalgebra of $\mathfrak{g}$. Notice that $\mathfrak{j}$ is both σ- and θ-invariant. Via the decomposition of $\mathfrak{j}$ induced by (1), (2) we identify $\mathfrak{a}_{\mathfrak{q}c}^*, \mathfrak{a}_{0c}^*$, and $\mathfrak{b}_c^*$ with subspaces of $\mathfrak{j}_c^*$. Let $\Sigma(\mathfrak{b}) = \Sigma(\mathfrak{g}, \mathfrak{b})$. The following result (cf. [21, Lemma 1.5]) will allow us to fix suitable choices of positive roots.

LEMMA 5.4. *Let* $\alpha \in \Sigma(\mathfrak{j})$ *be a root whose restriction to* $\mathfrak{a}_\mathfrak{q}$ *is zero. Then either* $\alpha \mid \mathfrak{a}_0 = 0$ *or* $\alpha \mid \mathfrak{b} = 0$.

Proof. Let X_α be any element in $\mathfrak{g}_c^\alpha$. Then $\mathfrak{a}_\mathfrak{q}$ centralizes the element $Y = X_\alpha + \theta X_\alpha - \sigma(X_\alpha + \theta X_\alpha)$. Now $Y \in \mathfrak{q} \cap \mathfrak{k}$ so in view of Lemma 5.2 we infer that $\mathfrak{a}_0$ centralizes Y. This is only possible in one of the following two cases.

(1) $\alpha \mid \mathfrak{a}_0 = 0$. There is nothing left to prove.

(2) At least one of the roots $\theta\alpha$, $\sigma\alpha$, $\sigma\theta\alpha$ equals α. If $\theta\alpha = \alpha$, then $\alpha \mid \mathfrak{a}_0 = 0$ and if $\sigma\alpha = \alpha$ then $\alpha \mid \mathfrak{b} = 0$. Finally if $\sigma\theta\alpha = 0$ then $\alpha = 0$ on $\mathfrak{j} \cap \mathfrak{g}_- \supset \mathfrak{j}_{\mathfrak{ph}} \oplus \mathfrak{j}_{\mathfrak{kq}}$ hence on $\mathfrak{a}_0 + \mathfrak{b}$. ∎

In view of the above we may fix compatible systems of positive roots for Σ, Σ_0, $\Sigma(\mathfrak{b})$, and $\Sigma(\mathfrak{j})$. We indicate these choices by the superscript $+$.

Let $\Lambda(\mathfrak{b})$ denote the set of $v \in \mathfrak{b}_{\mathfrak{c}}^*$ such that $\langle \alpha, \alpha \rangle^{-1} \langle v, \alpha \rangle \in \mathbf{Z}$ for each $a \in \Sigma(\mathfrak{b})$, and define

$$\Lambda(\mathfrak{a}_{\mathfrak{q}}) = \Lambda(\mathfrak{a}_0) \cap \Lambda(\mathfrak{b}). \tag{38}$$

Then the following result describes the finite dimensional class (1, 1) representations. Recall that H is said to be essentially connected iff (4).

PROPOSITION 5.5. *Let $v \in \Lambda^+(\mathfrak{a}_0)$, and let (π, V) be the associated finite dimensional class* 1 *representation of highest weight v. Then V possesses a non-trivial $\mathfrak{h}$-fixed vector iff $v \in \Lambda(\mathfrak{a}_{\mathfrak{q}})$. Let $v \in \Lambda(\mathfrak{a}_{\mathfrak{q}})$. Then*:

(1) dim $V^{\mathfrak{h}} = 1$. *If H is essentially connected then $V^{\mathfrak{h}} = V^H$.*

(2) *Assume $v \in V_v(\mathfrak{j}) \setminus \{0\}$. If $\varepsilon \in (V^*)^{\mathfrak{h}} \setminus \{0\} \cup (V^*)^{\mathfrak{k}} \setminus \{0\}$ then $\varepsilon(v) \neq 0$.*

(3) $V_v(\mathfrak{a}_{\mathfrak{q}}) = V_v(\mathfrak{j})$.

(4) *M_σ acts trivially on $V_v(\mathfrak{a}_{\mathfrak{q}})$.*

Proof. In view of the results described earlier in this section, V is an irreducible $\mathfrak{g}_{\mathfrak{c}}$-module of highest weight v.

Recall the duality of Section 2. Then obviously

$$V^{\mathfrak{h}} = V^{\mathfrak{k}^d}. \tag{39}$$

It follows from the Cartan–Helgason description that (39) is non-trivial iff $v \in \Lambda(\mathfrak{a}_0^d) = \Lambda(\mathfrak{b})$. The latter conditition is equivalent to $v \in \Lambda(\mathfrak{a}_{\mathfrak{q}})$. Moreover, if that condition is fulfilled, then the space (39) has dimension 1. Now assume that $v \in \Lambda(\mathfrak{a}_{\mathfrak{q}})$.

For (1) it remains to be shown that $Z_{H \cap K}(\mathfrak{a}_{\mathfrak{q}})$ acts trivially on $V^{\mathfrak{h}}$, in view of (4). Observe that

$$K \exp(\mathfrak{a}_{\mathfrak{q}}) H_e = K \exp(\mathfrak{a}_{\mathfrak{q}}) H = G \tag{40}$$

(this holds always, regardless of whether H is essentially connected or not). Now fix $e_{\mathfrak{h}} \in V^{\mathfrak{h}} \setminus \{0\}$, and $\varepsilon \in (V^*)^K \setminus \{0\}$. Since π is irreducible, it follows from (40) that the real analytic function $x \mapsto \varepsilon(\pi(x) e_{\mathfrak{h}})$, $A_{\mathfrak{q}} \to \mathbf{C}$ is not identically zero. Hence there exists a $X \in \mathfrak{a}_{\mathfrak{q}}$ such that $\varepsilon(\pi(X) e_{\mathfrak{h}}) \neq 0$. We can now finish the proof of (1). Let $m \in Z_{H \cap K}(\mathfrak{a}_{\mathfrak{q}})$. Since Ad$(m)$ normalizes $\mathfrak{h}$, $\pi(m)$ normalizes the one dimensional space $V^{\mathfrak{h}}$ hence acts by a scalar $c \in \mathbf{C}$

on it. It follows that $c\langle \varepsilon, \pi(X)e_{\mathfrak{h}}\rangle = \langle \varepsilon, \pi(X)\,\pi(m)e_{\mathfrak{h}}\rangle = \langle \pi^{\vee}(m^{-1})\varepsilon, \pi(X)e_{\mathfrak{h}}\rangle = \langle \varepsilon, \pi(X)e_{\mathfrak{h}}\rangle$, hence $c=1$.

For (2), notice that by Lemma 5.1(1) and duality we have $V_{\nu}(\mathfrak{j}) = V_{\nu}(\mathfrak{a}_0) = V_{\nu}(\mathfrak{b})$. Now apply Lemma 5.1(2) and duality.

To prove (3) notice that M_1 leaves the space $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$ invariant. We claim that in fact $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$ is an irreducible $\mathfrak{m}_1$-module. Indeed let V_0 be a non-trivial $\mathfrak{m}_1$-invariant subspace of $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$. Then $\mathfrak{n}$ annihilates V_0 and from $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m}_1 \oplus \mathfrak{n}$ we see that $V = U(\mathfrak{g})V_0 = U(\bar{\mathfrak{n}})V_0$, hence $V_{\nu}(\mathfrak{a}_{\mathfrak{q}}) = V_{\nu}(\mathfrak{a}_{\mathfrak{q}}) \cap U(\bar{\mathfrak{n}})V_0 = V_0$. This proves the claim. Now $\mathfrak{m}_1 = \mathfrak{m}_{\sigma} \oplus \mathfrak{a}_{\mathfrak{q}}$, and since $\mathfrak{a}_{\mathfrak{q}}$ acts by scalars it follows that $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$ is an irreducible $\mathfrak{m}_{\sigma}$-module as well. Now fix $\varepsilon^{\mathfrak{h}} \in (V^*)^{\mathfrak{h}} \setminus \{0\}$ and $\varepsilon^{\mathfrak{k}} \in (V^*)^{\mathfrak{k}} \setminus \{0\}$. Since $V_{\nu}(\mathfrak{a}_{\mathfrak{q}}) \supset V_{\nu}(\mathfrak{j})$ we have that $\varepsilon^{\mathfrak{h}}$ and $\varepsilon^{\mathfrak{k}}$ are not identially zero on $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$. This implies in particular that $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$ has a non-zero K_M-fixed vector w (use that $\varepsilon^{\mathfrak{k}}$ is K-fixed). From $\mathfrak{m}_{\sigma} \cap \mathfrak{p} \subset \mathfrak{m}_1 \subset \mathfrak{h}$ it follows that $M_{\sigma} = \exp(\mathfrak{m}_1 \cap \mathfrak{h})K_M$. We infer that for all $x \in M_{\sigma}$ we have that $\varepsilon^{\mathfrak{h}}(\pi(x)w) = \varepsilon^{\mathfrak{h}}(w)$. Hence $\varepsilon^{\mathfrak{h}}(\pi(x)\,\pi(y)w) = \varepsilon^{\mathfrak{h}}(\pi(x)w)$ for all $x, y \in M_{\sigma}$, and since $\varepsilon^{\mathfrak{h}} \mid V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$ is a cyclic vector for the contragredient $\mathfrak{m}_{\sigma}$-module $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})^*$ it follows that $\pi(y)w = w$ for all $y \in M_{\sigma}$. Hence $V_{\nu}(\mathfrak{a}_{\mathfrak{q}})$ is the (one-dimensional) trivial M_{σ}-module. ∎

LEMMA 5.6. *For $\alpha \in \Sigma_0 \cup \Sigma(\mathfrak{b})$, write $\hat{\alpha} = \alpha \mid \mathfrak{a}_{\mathfrak{q}}$. Then*

$$4\,\frac{\langle \hat{\alpha}, \hat{\alpha}\rangle}{\langle \alpha, \alpha\rangle} \in \mathbf{Z}. \tag{41}$$

Proof. We restrict to the case that $\alpha \in \Sigma(\mathfrak{b})$, the other case being similar. Then $2\hat{\alpha} = \alpha - \theta\alpha$, hence the right hand side of (41) equals $2 - 2\langle \alpha, \theta\alpha\rangle \langle \alpha, \alpha\rangle^{-1}$ and the result follows. ∎

Remark. In [21, Lemma 2.3], it is actually shown that (41) belongs to $\{1, 2, 4\}$, but we shall not need this.

The following is now obvious.

COROLLARY 5.7. *Let $\nu \in \mathfrak{a}_{\mathfrak{q}c}^*$. Then*

$$\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in 4\mathbf{Z} \text{ for each } \alpha \in \Sigma \quad \Rightarrow \quad \nu \in \Lambda(\mathfrak{a}_{\mathfrak{q}}).$$

6. FUNCTIONS OF S-POLYNOMIAL GROWTH

In Sections 8, 9, 10, and 16 we will be dealing with meromorphic functions of $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ whose singular and growth behaviour are of a specific type. The purpose of this section is to describe this type of behaviour, meanwhile developing some useful terminology.

Let S be a finite subset of $\mathfrak{a}_{\mathfrak{q}c}^* \setminus \{0\}$. Then we denote by $\Pi_S(\mathfrak{a}_\mathfrak{q})$ the subset of $S(\mathfrak{a}_\mathfrak{q})$ consisting of 1 and all products of linear functions $\mathfrak{a}_{\mathfrak{q}c}^* \to \mathbf{C}$ of the form

$$l(\lambda) = \langle \lambda, \xi \rangle - c, \tag{42}$$

with $\xi \in S$ and $c \in \mathbf{C}$. Here $\langle .,. \rangle$ denotes the Hermitian extension of the dual of the given inner product on $\mathfrak{a}_\mathfrak{q}$. Of course the decomposition of an element of $\Pi_S(\mathfrak{a}_\mathfrak{q})$ as a product of linear factors is unique up to the order of the factors. We endow $\Pi_S(\mathfrak{a}_\mathfrak{q})$ with the partial ordering $\preccurlyeq$ defined by $p \preccurlyeq q$ iff p divides q. Then clearly every subset T of $\Pi_S(\mathfrak{a}_\mathfrak{q})$ has a greatest lower bound $\inf T$ in $\Pi_S(\mathfrak{a}_\mathfrak{q})$.

Let V be a Fréchet space. We will say that a holomorphic V-valued function f, defined on an open set $\Omega \subset \mathfrak{a}_{\mathfrak{q}c}^*$ has exponential growth on Ω if there exists a constant $r \geqslant 0$ and for every continuous seminorm s on V constants $N \in \mathbf{N}$ and $C > 0$ such that

$$s(f(\lambda)) \leqslant C(1 + |\lambda|)^N e^{r|\mathrm{Re}\,\lambda|} \tag{43}$$

for all $\lambda \in \Omega$. The function f is said to have polynomial growth on Ω if the above holds with $r = 0$.

We will say that a meromorphic function $f: \Omega \to V$ has S-exponential (resp. S-polynomial) growth if there exists a polynomial $q \in \Pi_S(\mathfrak{a}_\mathfrak{q})$ such that qf is holomorphic and of exponential (resp. polynomial) growth on Ω.

In particular we will be interested in functions of S-exponential growth on open sets of the form

$$\mathfrak{a}_\mathfrak{q}^*(P, R) := \{\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*;\ \langle \lambda, \alpha \rangle < R \qquad \text{for} \quad \alpha \in \Sigma(P)\}; \tag{44}$$

here $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$ and $R \in \mathbf{R}$. The following result will enable us to reduce on the polynomial q in the definition of S-exponential growth.

Lemma 6.1. *There exists a constant $a > 0$ such that for every $R \in \mathbf{R}$ and every holomorphic function f on $\mathfrak{a}_\mathfrak{q}^*(P, R)$ with values in a Fréchet space V the following holds. Let $p \in \Pi_S(\mathfrak{a}_\mathfrak{q})$ be of degree d and suppose we have an estimate*

$$s(p(\lambda) f(\lambda)) \leqslant C(1 + |\lambda|)^N e^{r|\mathrm{Re}\,\lambda|} \qquad (\lambda \in \mathfrak{a}_\mathfrak{q}^*(P, R)),$$

with s a seminorm, $r \geqslant 0$, $N \in \mathbf{N}$, and $C > 0$. Then for every $0 < \varepsilon < 1$ we have the estimate

$$s(f(\lambda)) \leqslant C(2^N a d)^d \left(\frac{1 + r}{\varepsilon}\right)^d (1 + |\lambda|)^N e^{r|\mathrm{Re}\,\lambda|} \tag{45}$$

for all $\lambda \in \mathfrak{a}_\mathfrak{q}^*(P, R - \varepsilon)$.

Proof. It suffices to prove the result for $d = 1$. The above estimate will then follow if we apply this result d times with $d^{-1}\varepsilon$ instead of ε. Thus we assume that $d = 1$ and that p has the form (42).

Let

$$m = \min_{v \in S} |v|, \qquad M = \max_{v \in \Sigma \cup S} |v|,$$

and write $\eta = \tau\xi$, with

$$\tau = \frac{\varepsilon}{(1+r)(1+M)^2}.$$

Let $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(P, R-\varepsilon)$. If $|p(\lambda)| \geqslant (1/2)\tau\,|\xi|^2$, then

$$|p(\lambda)|^{-1} \leqslant 2\left(\frac{1+M}{m}\right)^2\left(\frac{1+r}{\varepsilon}\right),$$

and (45) follows with $a = a_1 := 2m^{-2}(1+M)^2$. We therefore assume that $|p(\lambda)| < (1/2)\,\tau\,|\xi|^2$. For every $\alpha \in \Sigma$ we have $|\langle \eta, \alpha\rangle| \leqslant \tau M^2 \leqslant \varepsilon$. Hence if $z \in \mathbf{C}$, $|z| \leqslant 1$ then $\lambda + z\eta \in \mathfrak{a}_{\mathfrak{q}}^*(P, R)$. On the other hand, if $|z| = 1$, then

$$|p(\lambda + z\eta)| \geqslant |\langle \eta, \xi\rangle| - |p(\lambda)| > \tfrac{1}{2}\tau\,|\xi|^2.$$

Hence

$$s(f(\lambda + z\eta)) \leqslant DC(1+|\lambda|)^N\, e^{r|\mathrm{Re}\,\lambda|},$$

with

$$D = \frac{2}{\tau\,|\xi|^2}(1+\tau\,|\xi|)^N\, e^{r\tau|\xi|} \leqslant 2^N a_1 e\left(\frac{1+r}{\varepsilon}\right).$$

The required estimate now follows with $a = a_1 e$ if we apply the above to estimate the integrand in Cauchy's integral formula for the function $z \mapsto f(\lambda + z\eta)$ over the unit circle in $\mathbf{C}$. ∎

7. S-Genericity

In this section we define a notion of genericity which will be used in Sections 8 and 9.

Let a finite subset $S \subset \mathfrak{a}_{\mathfrak{q}c}^* \backslash \{0\}$ be given. Then by a S-hyperplane we will mean a hyperplane in $\mathfrak{a}_{\mathfrak{q}c}^*$ of the form $l^{-1}(0)$ with $l \in \Pi_S(\mathfrak{a}_{\mathfrak{q}})$, $\deg l = 1$. Moreover, we will say that a λ-dependent statement ($\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$) holds for S-generic λ if the statement holds for λ in the complement in $\mathfrak{a}_{\mathfrak{q}c}^*$ of a locally finite union of S-hyperplanes.

Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$ as defined below Corollary 5.3. For future use we fix a particular finite and W-invariant subset $\mathrm{S} \subset \mathfrak{a}_{\mathfrak{q}c}^* \setminus \{0\}$ such that the following conditions are satisfied.

(1) $\Sigma \subset \mathrm{S}$.

(2) If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{j})$, $w \in W(\mathfrak{g}, \mathfrak{j})$ then $(\alpha - w\alpha) \mid \mathfrak{a}_\mathfrak{q} \in \mathrm{S} \cup \{0\}$.

Remark 7.1. The first of the above conditions guarantees that the map $j(P : \xi : \lambda)$ is well defined as a map from $V(\xi)$ into $C^{-\infty}(P : \xi : \lambda)^H$ for S-generic $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, by [4, Lemma 9.5]. Moreover, ev being a left inverse (cf. [4, Theorem 5.10]) the map $j(P : \xi : \lambda)$ is injective as soon as it is well defined.

The second of the above conditions guarantees that the following lemma is valid. Note that $W(\mathfrak{m}_1, \mathfrak{j})$ is the centralizer of $\mathfrak{a}_\mathfrak{q}$ in $W(\mathfrak{g}, \mathfrak{j})$.

LEMMA 7.2. *Let $\eta_1, \eta_2 \in \mathfrak{j}_c^*$ be such that $\eta_1 \notin W(\mathfrak{m}_1, \mathfrak{j})\eta_2$. Then there exists a polynomial $q \in \Pi_\mathrm{S}(\mathfrak{a}_\mathfrak{q})$ such that for $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ with $q(\lambda) \neq 0$ we have*

$$\lambda + \eta_1 \neq \omega(\lambda + \eta_2) \qquad \textit{for all} \quad w \in W(\mathfrak{g}, \mathfrak{j}).$$

Proof. If $w \in W(\mathfrak{m}_1, \mathfrak{j})$, then the required assertion holds for any $\lambda \in \mathfrak{a}_\mathfrak{q}$, in view of the assumption on η_1, η_2.

For each $w \in W(\mathfrak{g}, \mathfrak{j}) \setminus W(\mathfrak{m}_1, \mathfrak{j})$ thee exists a root $\beta_w \in \Sigma(\mathfrak{g}, \mathfrak{j})$ such that the restriction $v_w = (\beta_w - w^{-1}\beta_w) \mid \mathfrak{a}_\mathfrak{q}$ is non-zero. The second of the above conditions guarantees that $v_w \in \mathrm{S}$. Set $l_w(\lambda) = \langle \lambda, v_w \rangle - \langle w\eta_2 - \eta_1, \beta_w \rangle$. Then $\lambda + \eta_1 = w(\lambda + \eta_2)$ implies $l_w(\lambda) = 0$. Hence $q(\lambda) = \prod_{w \notin W(\mathfrak{m}_1, \mathfrak{j})} l_w(\lambda)$ satisfies our requirements. ∎

8. PROJECTION ALONG INFINITESIMAL CHARACTERS

In this section we will study projection along an infinitesimal character in the tensor product of a principal series representation with a finite dimensional class (1, 1) representation, inspired by an idea of Zuckerman (cf. [37]). The results will be used in the derivation of the functional equation for j in the next section.

Let $\mathfrak{j}$ be the Cartan subalgebra of $\mathfrak{g}$ introduced above Lemma 5.4. If V is a Harish-Chandra module and $\eta \in \mathfrak{j}_c^*$ an infinitesimal character, then we denote the projection in V onto the generalized weight space for $\mathscr{Z}(\mathfrak{g})$ corresponding to η by p_η^V or just p_η.

Let $\mu \in \Lambda(\mathfrak{a}_\mathfrak{q})$ (cf. (38)), and assume that (π, F) is the finite dimensional irreducible class 1 representation of extremal weight μ.

Let $\xi \in \hat{M}_{\mathrm{ps}}$, and let $\Lambda \in (\mathfrak{m}_{\sigma c} \cap \mathfrak{j}_c)^* \subset \mathfrak{j}_c^*$ be its infinitesimal character. If $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ then $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes 1)$ has infinitesimal charater $\Lambda + \lambda$.

PROPOSITION 8.1. *Let* $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$ *and* $\mu \in \Lambda(\mathfrak{a}_\mathfrak{q})$. *Then for* S-*generic* $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ *we have that*

$$p_{\Lambda+\lambda+\mu}[C(Q:\xi:\lambda)_K \otimes F] \simeq C(Q:\xi:\lambda+\mu)_K. \tag{46}$$

Proof. Let $\mathscr{H}_{\xi\lambda}$ denote the space $\mathscr{H}_\xi$ provided with the Q-module structure $\xi \otimes \lambda \otimes 1$. We consider the G-equivariant map

$$\varphi_\lambda : C^{-\infty}(Q:\xi:\lambda) \otimes F \to C^{-\infty}\, \mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda} \otimes F|_Q)$$

determined by

$$\varphi_\lambda(f \otimes v)(x) = f(x) \otimes \pi(x) v.$$

Then on the level of K-finite vectors, φ_λ is an isomorphism of $(\mathfrak{g}, K)$-modules (the proof of this statement goes exactly as the proof suggested by [24, p. 384, Exercise 6]). In particular this implies that φ_λ is injective on the space of generalized functions.

We shall first deal with the case that μ is Q-dominant. Then by Proposition 5.5 the $\mathfrak{a}_\mathfrak{q}$-weight space $F_\mu = F_\mu(\mathfrak{a}_\mathfrak{q})$ is a one dimensional subrepresentation of $F|_Q$, on which M_σ acts trivially. Consider the short exact sequence of Q-modules

$$0 \to F_\mu \to F|_Q \to F/F_\mu \to 0.$$

Let $\Lambda(F)$ be the set of $\mathfrak{a}_\mathfrak{q}$-weights of F. Then the composition factors of the Q-module F/F_μ are all of the form $\tau \otimes \nu \otimes 1$, with τ a finite dimensional irreducible representation of M_σ and $\nu \in \Lambda(F) \backslash \{\mu\}$. Let $\mathscr{C}$ be the set of composition factors of the M_σ-modules occurring in $\xi \otimes \tau$, with τ as above. One easily verifies that $\omega \mapsto \mathrm{Ind}_Q^G(\omega)_K$ is an exact functor from the category of finite dimensional Q-modules to the category of admissible $(\mathfrak{g}, K)$-modules. Hence every composition factor of the $(\mathfrak{g}, K)$-module

$$\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda} \otimes (F/F_\mu))_K \tag{47}$$

is a composition factor of an induced module of the form $\mathrm{Ind}_Q^G(\delta \otimes (\lambda + \nu) \otimes 1)_K$, with $\delta \in \mathscr{C}$, $\nu \in \Lambda(F) \backslash \{\mu\}$. Therefore every generalized infinitesimal character of (47) is of the form $\Lambda_\delta + \lambda + \nu$ with $\Lambda_\delta \in \mathfrak{j}_c^*$ the infinitesimal character of $\delta \in \mathscr{C}$, and $\nu \in \Lambda(F) \backslash \{\mu\}$. Now suppose that

$$\Lambda + \lambda + \mu \neq w(\Lambda_\delta + \lambda + \nu), \tag{48}$$

for all $\delta \in \mathscr{C}$, $\nu \in \Lambda(F) \backslash \{\mu\}$, and $w \in W(\mathfrak{g}_c, \mathfrak{j}_c)$. (According to Lemma 7.2 this condition is fulfilled for λ in the complement of a finite union of S-hyperplanes.) Then $p_{\Lambda+\lambda+\mu}$ annihilates (47). On the other hand it is the identity on $\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda}\otimes F_\mu)_K = C(Q:\xi:\lambda+\mu)_K$. Using exactness of induction once more we infer that $p_{\Lambda+\lambda+\mu}$ maps $\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda}\otimes F|_Q)_K$ onto $\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda}\otimes F_\mu)_K$. Applying the isomorphism φ_λ we infer that (46) holds for λ in the complement of a finite union of S-hyperplanes, when μ is Q-dominant.

Finally, let $Q'\in\mathscr{P}_\sigma(A_{\mathfrak{q}})$. Then the intertwining operators $A(Q':Q:\xi:\lambda)\otimes I$ and $A(Q:\xi:\lambda+\mu)$ are isomorphisms for λ S-generic. Hence (46) remains valid if we replace Q by Q'. ∎

We will now investigate the extension of $p_{\Lambda+\lambda+\mu}$ from the K-finite level to the space of generalized functions

$$C^{-\infty}(Q:\xi:\lambda)\otimes F \tag{49}$$

and its dependence on λ. First we need a lemma. Recall the definition of the λ-dependent representation $\pi_\lambda=\pi_{Q,\xi,\lambda}$ of G on (8).

LEMMA 8.2. *Let $X\in U(\mathfrak{g})$ be of order at most d, and let $r\in\mathbf{R}$. Then $\lambda\mapsto\pi_\lambda(X)$ is polynomial (of degree at most d) as a function on $\mathfrak{a}_{\mathfrak{q}c}^*$ with values in the Banach space of bounded linear maps $C^r(K:\xi)\to C^{r-d}(K:\xi)$.*

Proof. Clearly it suffices to prove this for $d=1$, and then we may as well assume that $X\in\mathfrak{p}$. Let $\varphi\in C^{-\infty}(K:\xi)$. We define $\varphi_\lambda\in C^{-\infty}(Q:\xi:\lambda)$ by $\varphi_\lambda\mid K=\varphi$. Then

$$\begin{aligned}\pi_\lambda(X)\,\varphi(k)&=\varphi_\lambda(k;X)\\&=\varphi_\lambda(\mathrm{Ad}(k)X^\vee;k).\end{aligned}\tag{50}$$

Now modulo $\mathfrak{n}$, $\mathrm{Ad}(k)X^\vee$ can be written as a finite sum of terms $c(k)(U+V+W)$, where $c\in C^\infty(K)$, and $U\in\mathfrak{a}_{\mathfrak{q}}$, $V\in\mathfrak{m}_\sigma$, $W\in\mathfrak{k}$. Hence (50) can be written as a finite sum of terms

$$c(k)\,L_{(U+V+W)}\varphi_\lambda=c(k)[\langle U,\lambda+\rho_P\rangle+\xi(V)+L_W]\,\varphi(k).\tag{51}$$

From this the assertion easily follows. ∎

PROPOSITION 8.3. *There exist a polynomial $q\in\Pi_{\mathbf{S}}(\mathfrak{a}_{\mathfrak{q}})$ and a meromorphic family $p_\mu(Q:\xi:\lambda)$ $(\lambda\in\mathfrak{a}_{\mathfrak{q}c}^*)$ of equivariant continuous linear endomorphisms of (49) with the following properties.*

(1) *For S-generic λ we have*

$$p_\mu(Q:\xi:\lambda)=p_{\Lambda+\lambda+\mu}\qquad on\quad C(Q:\xi:\lambda)_K\otimes F.\tag{52}$$

(2) *There exists a $d\in\mathbf{N}$ such that for every $r\in\mathbf{Z}$ the map*

$\lambda \mapsto q(\lambda)\, p_\mu(Q : \xi : \lambda)$ is polynomial as a function on $\mathfrak{a}_{\mathfrak{q}c}^*$ with values in the Banach space of bounded linear maps

$$C^r(K : \xi) \otimes F \to C^{r-d}(K : \xi) \otimes F. \tag{53}$$

Proof. Let $\mu_1 = \mu$, $\mu_2, ..., \mu_m$ be the collection of distint $\mathfrak{j}$-weights of π. Then it follows from [23, Theorem 5.1] that (49) is admissible and of finite length, and that

$$\prod_{j=1}^{m} [Z - \gamma(Z, \Lambda + \lambda + \mu_j)] \tag{54}$$

acts by zero on (49).

We may assume $\mu_1, ..., \mu_m$ to be ordered so that for a suitable $1 \leqslant k \leqslant m$ we have $1 \leqslant j \leqslant k$ iff $\Lambda + \mu_j \in W(\mathfrak{m}_1, \mathfrak{j})(\Lambda + \mu)$. Then by Lemma 7.2 there exists a polynomial $\tilde{q} \in \Pi_S(\mathfrak{a}_\mathfrak{q})$ such that for $j > k$ and for every λ with $\tilde{q}(\lambda) \neq 0$ we have that $\Lambda + \lambda + \mu_j$ is not $W(\mathfrak{g}, \mathfrak{j})$-conjugate to $\Lambda + \lambda + \mu$. Given an element $Z \in \mathscr{Z}(\mathfrak{g})$ we define

$$b(Z, \lambda) = \prod_{j=k+1}^{m} [\gamma(Z, \Lambda + \lambda + \mu) - \gamma(Z, \Lambda + \lambda + \mu_j)].$$

Let I be the ideal generated by the polynomials $b(Z)$, $Z \in \mathscr{Z}(\mathfrak{g})$, and let V_I be its zero set. We claim that $\tilde{q} = 0$ on V_I.

To see this, let $E \subset \mathscr{Z}(\mathfrak{g})$ be a finite dimensional linear subspace which generates the algebra $\mathscr{Z}(\mathfrak{g})$. If $\lambda \in V_I$, then the polynomial function $E \to \mathbf{C}$, $Z \mapsto b(Z, \lambda)$ is identically zero, hence for some $k+1 \leqslant j \leqslant m$ we have that $\gamma(\cdot, \Lambda + \lambda + \mu) = \gamma(\cdot, \Lambda + \lambda + \mu_j)$ on E. Since γ is an algebra homomorphism, this identity actually holds on all of $\mathscr{Z}(\mathfrak{g})$, and it follows that $\Lambda + \lambda + \mu_j$ is $W(\mathfrak{g}, \mathfrak{j})$-conjugate to $\Lambda + \lambda + \mu$, hence $\tilde{q}(\lambda) = 0$. This proves the claim.

In particular we see that there exists a $Z \in \mathscr{Z}(\mathfrak{g})$ such that $b(Z)$ is not identically zero. For $Z \in \mathscr{Z}(\mathfrak{g})$ we write

$$D(Z, \lambda) = \prod_{j=k+1}^{m} [Z - \gamma(Z, \Lambda + \lambda + \mu_j)].$$

Since $\gamma(Z, \Lambda + \lambda + \mu_j) = \gamma(Z, \Lambda + \lambda + \mu)$ for all $1 \leqslant j \leqslant k$, $Z \in \mathscr{Z}(\mathfrak{g})$, and $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, we have that

$$[Z - \gamma(Z, \Lambda + \lambda + \mu)]^k\, D(Z, \lambda) \tag{55}$$

equals (54) hence acts by 0 on (49). To complete the proof we need the following.

LEMMA 8.4. *Let $Z \in \mathscr{Z}(\mathfrak{g})$. Then for* S*-generic λ we have*

$$[\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes 1) \otimes \pi](D(Z, \lambda)) = b(Z, \lambda)\, p_{\Lambda + \lambda + \mu}$$

on the K-finite level.

Proof. The space $\ker p_{\Lambda+\lambda+\mu}$ equals the sum of the generalized weight spaces corresponding to infinitesimal characters not contained in $W(\mathfrak{g}, \mathfrak{j})(\Lambda + \lambda + \mu)$. Hence the power at the left in (55) acts invertibly on $\ker p_{\Lambda+\lambda+\mu}$. The whole of (55) acts by zero, hence $D(Z, \lambda) = 0$ on $\ker p_{\Lambda+\lambda+\mu}$.

On the other hand $\mathscr{Z}(\mathfrak{g})$ acts semisimply by the infinitesimal character $\Lambda + \lambda + \mu$ on $\mathrm{im}\, p_{\Lambda+\lambda+\mu}$ for S-generic λ, in view of Proposition 8.1. From this we see that the equation holds on $\mathrm{im}\, p_{\Lambda+\lambda+\mu}$ as well. ∎

Completion of the Proof of Proposition 8.3. Let $Z \in \mathscr{Z}(\mathfrak{g})$ be such that $b(Z) \neq 0$. Then by the above lemma the meromorphic family

$$p_\mu(Q : \xi : \lambda) := b(Z, \lambda)^{-1} [\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes 1) \otimes \pi](D(Z, \lambda))$$

of equivariant continuous linear maps does not depend on the particular choice of Z. Set $d(Z) = (m-1)\deg(Z)$. Then in view of Lemma 8.2 it follows from the above definitions that $\lambda \mapsto b(Z, \lambda)\, p_\mu(Q : \xi : \lambda)$ is polynomial as a function with values in the Banach space of bounded linear maps from $C^r(K : \xi) \otimes F$ into $C^{r - d(Z)}(K : \xi) \otimes F$.

By the Nullstellen Satz there exists a constant $\nu \in \mathbf{N}$ such that $q = \tilde{q}^\nu$ belongs to I. Hence we may write

$$q(\lambda) = \sum_{k=1}^{n} a_k(\lambda)\, b(Z_k, \lambda)$$

with $Z_k \in \mathscr{Z}(\mathfrak{g})$ such that $b(Z_k) \neq 0$, and with $a_k \in S(\mathfrak{a}_\mathfrak{q})$. Let $d = \max_{1 \leqslant k \leqslant n} d(Z_k)$. Then we infer that $q(\lambda)\, p_\mu(Q : \xi : \lambda)$ is a polynomial function of λ with values in the Banach space of bounded linear maps (53).

Finally let Ω_k be the complement of $b(Z_k)^{-1}(0)$ in $\mathfrak{a}_{\mathfrak{q}c}^*$. By Lemma 8.4 there exists a locally finite union $\mathscr{H}_k$ of S-hyperplanes such that for $\lambda \in \Omega_k \backslash \mathscr{H}_k$ we have $p_\mu(Q : \xi : \lambda) = p_{\Lambda+\lambda+\mu}$. Put $\mathscr{H} = \bigcup_{k=1}^n \mathscr{H}_k$. If $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^* \backslash \mathscr{H}$, $q(\lambda) \neq 0$, then $\lambda \in \Omega_k \backslash \mathscr{H}_k$ for some k, and (52) follows. ∎

In the following two lemmas we list transformation properties which will be useful at a later stage.

LEMMA 8.5. *Let $Q_1, Q_2 \in \mathscr{P}_\sigma(A_\mathfrak{q})$ and consider the intertwining operator*

$A(Q_2 : Q_1 : \xi : \lambda) \otimes I$ from $C^{-\infty}(Q_1 : \xi : \lambda) \otimes F$ into $C^{-\infty}(Q_2 : \xi : \lambda) \otimes F$. We have

$$\begin{aligned} p_\mu(Q_2 : \xi : \lambda) &\circ [A(Q_2 : Q_1 : \xi : \lambda) \otimes I] \\ &= [A(Q_2 : Q_1 : \xi : \lambda) \otimes I] \circ p_\mu(Q_1 : \xi : \lambda). \end{aligned}$$

Proof. By equivariance we have that

$$p_{\Lambda + \lambda + \mu} \circ [A(Q_2 : Q_1 : \xi : \lambda) \otimes I] = [A(Q_2 : Q_1 : \xi : \lambda) \otimes I] \circ p_{\Lambda + \lambda + \mu}$$

on the K-finite level. Now apply (52) and a density argument. ∎

LEMMA 8.6. *Let $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $w \in N_K(\mathfrak{a}_\mathfrak{q})$, and consider the intertwining operator $L(w) \otimes I$ from $C^{-\infty}(Q : \xi : \lambda) \otimes F$ into $C^{-\infty}(wQw^{-1} : w\xi : w\lambda) \otimes F$. We have*

$$[L(w) \otimes I] \circ p_\mu(Q : \xi : \lambda) = p_{w\mu}(wQw^{-1} : \omega\xi : w\lambda) \circ [L(w) \otimes I].$$

Proof. By equivariance we have that

$$[L(w) \otimes I] \circ p_{\Lambda + \lambda + \mu} = p_{\Lambda + \lambda + \mu} \circ [L(w) \otimes I] \tag{56}$$

on the K-finite level.

According to Lemma 4.6, there exists a $s \in W(\mathfrak{g}, \mathfrak{j})$ which normalizes $\mathfrak{a}_\mathfrak{q}$, and such that $s \mid \mathfrak{a}_\mathfrak{q} = \mathrm{Ad}(w) \mid \mathfrak{a}_\mathfrak{q}$. Moreover, $w\xi$ has infinitesimal character $s\Lambda$ (we view ξ as a representation of M_1, cf. Section 1). Finally, $w\mu$ is an extremal $\mathfrak{a}_\mathfrak{q}$-weight for F, so ti follows that on $C(wQw^{-1} : w\xi : w\lambda)_K \otimes F$ we have (for S-generic λ)

$$\begin{aligned} p_{\Lambda + \lambda + \mu} = p_{s(\Lambda + \lambda + \mu)} &= p_{s\Lambda + w\lambda + w\mu} \\ &= p_{w\mu}(wQw^{-1} : w\xi : w\lambda). \end{aligned}$$

Here we have used Proposition 8.3 to obtain the third equality. Substituting the above relation into the right hand side of (56), and substituting $p_{\Lambda + \lambda + \mu} = p_\mu(Q : \xi : \lambda)$ into its left hand side we obtain the desired equality. ∎

9. ESTIMATES FOR j

This section is devoted to the proof of the following result; in the next section it will provide us with an initial estimate for Eisenstein integrals. Recall the terminology of Section 6.

THEOREM 9.1. *Let $\xi \in M_{ps}^{\wedge}$, $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, and $R > 0$. Then there exists a constant $s \in \mathbf{R}$ such that for each $\eta \in V(\xi)$*

$$\lambda \mapsto j(P : \xi : \lambda)\eta \tag{57}$$

defines a meromorphic $C^s(K : \xi)$-valued function of Σ-polynomial growth on $\mathfrak{a}_\mathfrak{q}^*(P, R)$.

This result will be proved by means of a functional equation for $j(P : \xi : \lambda)$, see Theorem 9.3.

It suffices to prove Theorem 9.1 for H essentially connected (see also the argument in [4, Remark on p. 381]). We therefore assume condition (4) to be fulfilled.

Let $\mu \in \Lambda(\mathfrak{a}_\mathfrak{q})$ and let (π, F) be the finite dimensional irreducible class 1 representation of G with extremal weight μ. Then F is of class (1, 1), i.e., it possesses a non-trivial H-fixed vector (cf. Proposition 5.5). The contragredient representation $(\pi^\vee, F^*)$ is also of class (1, 1) and has extremal weight $-\mu \in \Lambda(\mathfrak{a}_\mathfrak{q})$.

Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, and assume that μ is $\bar{P}$-dominant. Then we may use the equivariant pairing $F^* \times F \to \mathbf{C}$ to define an equivariant embedding ε_μ of F into $C(P : 1 : \mu - \rho_P)_K$ as follows. Fix a non-zero vector $e^{-\mu}$ of weight $-\mu$ in F^*. Then $e^{-\mu}$ is N_P and M_σ-fixed (cf. Proposition 5.5), and we may define the map ε_μ by

$$\varepsilon_\mu(v)(x) = \langle e^{-\mu}, \pi(x)v\rangle \qquad (v \in F,\ x \in G).$$

Let $e_K \in F$ be a K-fixed vector satisfying $\langle e^{-\mu}, e_K\rangle = 1$. Then the right K-invariant function $\varepsilon_\mu(e_K)$ vanishes nowhere. We define a continuous linear map

$$\mathscr{M}_\mu : C^{-\infty}(P : \xi : \lambda + \mu) \to C^{-\infty}(P : \xi : \lambda) \otimes F$$

by

$$f \mapsto \varepsilon_\mu(e_K)^{-1} f \otimes e_K.$$

Thus, as a map from $C^{-\infty}(K : \xi)$ into $C^{-\infty}(K : \xi) \otimes F$, $\mathscr{M}_\mu$ is given by $f \mapsto f \otimes e_K$. Fix H-fixed vectors $e_H \in F$ and $e^H \in F^*$ such that $\langle e^H, e_H\rangle = 1$. Given $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$ we define the linear map ε^H from $C^{-\infty}(Q : \xi : \lambda) \otimes F$ into $C^{-\infty}(Q : \xi : \lambda)$ by

$$\varepsilon^H\left(\sum \varphi_j \otimes v_j\right) = \sum \langle e^H, v_j\rangle \varphi_j. \tag{58}$$

Finally recall the definition of $p_\mu(P:\xi:\lambda)$ in the previous section, and define the differential operator

$$D_\mu(\xi:\lambda): C^{-\infty}(P:\xi:\lambda+\mu) \to C^{-\infty}(P:\xi:\lambda)$$

by

$$D_\mu(\xi:\lambda) = \varepsilon^H \circ p_\mu(P:\xi:\lambda) \circ \mathcal{M}_\mu.$$

LEMMA 9.2. *There exists a polynomial $q \in \Pi_{\mathrm{S}}(\mathfrak{a}_{\mathfrak{q}})$ and a constant $d \in \mathbf{N}$ such that for every $r \in \mathbf{Z}$ the map $\lambda \mapsto q(\lambda)\, D_\mu(\xi:\lambda)$ is polynomial as a function on $\mathfrak{a}_{\mathfrak{q}c}^*$ with values in the Banach space of bounded linear maps $C^r(K:\xi) \to C^{r-d}(K:\xi)$.*

Proof. This is a straightforward consequence of Proposition 8.3. ∎

We can now formulate the functional equation for j.

THEOREM 9.3. *Let μ be $\bar{P}$-dominant. Then there exists a rational* $\mathrm{End}(V(\xi))$*-valued function $\lambda \mapsto R_\mu(\xi:\lambda)$ on $\mathfrak{a}_{\mathfrak{q}c}^*$ such that*

$$j(P:\xi:\lambda) = D_\mu(\xi:\lambda) \circ j(P:\xi:\lambda+\mu) \circ R_\mu(\xi:\lambda). \tag{59}$$

Moreover, the function $\lambda \mapsto R_\mu(\xi:\lambda)$ is of S*-polynomial growth on $\mathfrak{a}_{\mathfrak{q}c}^*$.*

Before turning to the proof of this theorem we shall use it to establish Theorem 9.1.

Proof of Theorem 9.1. Let Ω denote the set of $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ such that

$$\langle \operatorname{Re}\lambda + \rho_P, \alpha \rangle < -1 \qquad \text{for all} \quad \alpha \in \Sigma(P).$$

Then $\lambda \mapsto j(P:\xi:\lambda)\eta$ is holomorphic $C^0(P:\xi:\lambda)$-valued, and of polynomial growth on Ω (cf. [4, Proof of Proposition 5.6]). In view of Corollary 5.7 we may select $\mu \in \Lambda(\mathfrak{a}_{\mathfrak{q}})$ such that $\langle \mu, \alpha \rangle < 0$ for all $\alpha \in \Sigma(P)$ and such that in addition $\mathfrak{a}_{\mathfrak{q}}^*(P, R+1/2) + \mu \subset \Omega$. Let F be the finite dimensional irreducible class $(1,1)$ representation of G of P-lowest weight μ. Then in view of Lemma 9.2 and Theorem 9.3 the right hand side of (59) is meromorphic and of S-polynomial growth on $\mathfrak{a}_{\mathfrak{q}}^*(P, R+1/2)$ as a $V(\xi)^* \otimes C^{-d}(K:\xi)$-valued function. Hence $\lambda \mapsto j(P:\xi:\lambda)\eta$ is of S-polynomial growth on $\mathfrak{a}_{\mathfrak{q}}^*(P, R+1/2)$. On the other hand, by [4, Lemma 9.5] we know already that for some $q \in \Pi_\Sigma(\mathfrak{a}_{\mathfrak{q}})$ the map $\lambda \mapsto q(\lambda)\, j(P:\xi:\lambda)\eta$ is holomorphic on $\mathfrak{a}_{\mathfrak{q}}^*(P, R+1/2)$. According to Lemma 6.1 the latter map is therefore of polynomial growth on $\mathfrak{a}_{\mathfrak{q}}^*(P, R)$. ∎

The remaining part of this section will be devoted to the proof of Theorem 9.3. As before we assume that μ is $\bar{P}$-dominant. Define the equivariant map

$$\Phi_\mu(P:\xi:\lambda): C^{-\infty}(P:\xi:\lambda)\otimes F \to C^{-\infty}(P:\xi:\lambda+\mu)$$

by

$$f\otimes v \mapsto \varepsilon_\mu(v) f.$$

Then the following result is a straightforward consequence of the definitions.

LEMMA 9.4. *For every $p\in\mathbf{Z}$ the map $\Phi_\mu(P:\xi:\lambda)$ restricts to a bounded linear map from $C^p(K:\xi)\otimes F$ into $C^p(K:\xi)$ which is independent of λ. Moreover,*

$$\Phi_\mu(P:\xi:\lambda)\circ \mathscr{M}_\mu = I. \tag{60}$$

In particular, $\Phi_\mu(P:\xi:\lambda)$ is surjective.

Notice that $\mathscr{M}_\mu$ is not equivariant. Our next objective is to find an equivariant right inverse for $\Phi_\mu(P:\xi:\lambda)$, still assuming that μ is $\bar{P}$-dominant.

LEMMA 9.5. *Let μ be $\bar{P}$-dominant. Then*

$$\Phi_\mu(P:\xi:\lambda)\circ p_\mu(P:\xi:\lambda) = \Phi_\mu(P:\xi:\lambda). \tag{61}$$

Proof. By equivariance we have

$$\begin{aligned}\Phi_\mu(P:\xi:\lambda)\circ p_{\Lambda+\lambda+\mu} &= p_{\Lambda+\lambda+\mu}\circ\Phi_\mu(P:\xi:\lambda)\\ &= \Phi_\mu(P:\xi:\lambda),\end{aligned} \tag{62}$$

on the level of K-finite vectors. Now use (52) and meromorphic continuation to complete the proof. ∎

We now define

$$\Psi_\mu(P:\xi:\lambda): C^{-\infty}(P:\xi:\lambda+\mu)\to C^{-\infty}(P:\xi:\lambda)\otimes F$$

by

$$\Psi_\mu(P:\xi:\lambda) = p_\mu(P:\xi:\lambda)\circ\mathscr{M}_\mu.$$

Notice that

$$D_\mu(\xi:\lambda) = \varepsilon^H\circ\Psi_\mu(P:\xi:\lambda). \tag{63}$$

Now let $q\in\Pi_S(\mathfrak{a}_\mathfrak{q})$ and $d\in\mathrm{N}$ be as in Proposition 8.3 with $Q=P$.

LEMMA 9.6. *For every* $r \in \mathbf{R}$ *the function* $\lambda \mapsto q(\lambda)\, \Psi_\mu(P : \xi : \lambda)$ *is polynomial as a function on* $\mathfrak{a}_{\mathfrak{q}c}^*$ *with values in the Banach space of bounded linear maps from* $C^r(K : \xi) \otimes F$ *into* $C^{r-d}(M : \xi)$. *If* $q(\lambda) \neq 0$, *then the map* $\Psi_\mu(P : \xi : \lambda)$ *is equivariant and we have*

$$\Phi_\mu(P : \xi : \lambda) \circ \Psi_\mu(P : \xi : \lambda) = I; \tag{64}$$

$$\Psi_\mu(P : \xi : \lambda) \circ \Phi_\mu(P : \xi : \lambda) = p_\mu(P : \xi : \lambda). \tag{65}$$

Proof. The assertion about the polynomial dependence is a straightforward consequence of Proposition 8.3. By meromorphy it suffices to prove the identities (64) and (65) for generic $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. We suppress P and ξ in the notations. Using (61) we obtain that

$$\Phi_\mu(\lambda) \circ \Psi_\mu(\lambda) = \Phi_\mu(\lambda) \circ p_\mu(\lambda) \circ \mathcal{M}_\mu \tag{66}$$

$$= \Phi_\mu(\lambda) \circ \mathcal{M}_\mu = I. \tag{67}$$

To prove the second identity, we first notice that $\Phi_\mu(\lambda)$ maps $(\text{im}\, p_\mu(\lambda))_K$ equivariantly onto $C(P : \xi : \lambda + \mu)_K$. A surjective endomorphism of an admissible $(\mathfrak{g}, K)$-module is automatically bijective. Thus from (46) and Proposition 8.3 we infer that for S-generic $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ the map $\Phi_\mu(\lambda)$ is injective on im $p_\mu(\lambda)$. Next we observe that (64) implies that

$$\Phi_\mu(\lambda) \circ [\Psi_\mu(\lambda) \circ \Phi_\mu(\lambda)] = I \circ \Phi_\mu(\lambda)$$
$$= \Phi_\mu(\lambda) \circ p_\mu(\lambda).$$

Using the injectivity of $\Phi_\mu(\lambda)$ we may now conclude that (65) holds for S-generic λ.

Finally it follows from (64) and (65) that $\Phi_\mu(\lambda)$ is a bijection from $p_\mu(C^{-\infty}(P : \xi : \lambda) \otimes F)$ onto $C^{-\infty}(P : \xi : \lambda + \mu)$ with inverse $\Psi_\mu(\lambda)$. Thus the equivariance of $\Psi_\mu(\lambda)$ follows from the equivariance of $\Phi_\mu(\lambda)$. ∎

Our interest in $\Phi_\mu(P : \xi : \lambda)$ originates from the following observations. Let m_μ be the endomorphism of $V(\xi)$ defined by

$$m_\mu = \langle e^{-\mu}, \pi(w)\, e_H \rangle I \qquad \text{on } V(\xi, w),$$

for $w \in \mathcal{W}$.

LEMMA 9.7. *The endomorphism* m_μ *of* $V(\xi)$ *is invertible.*

Proof. Assume not. Then $\langle e^{-\mu}, \pi(w) e_H \rangle = 0$ for some $w \in \mathcal{W}$. But then the function $\varepsilon_\mu(e_H)(x) = \langle e^{-\mu}, \pi(x) e_H \rangle$ vanishes on the open set PwH by its transformation properties, and hence on the whole of G, because it is real analytic. On the other hand it is the matrix coefficient of two non-trivial vectors of an irreducible representation so it cannot be identically zero. ∎

LEMMA 9.8. *For every $\eta \in V(\xi)$ we have*

$$\Phi_\mu(P:\xi:\lambda)[j(P:\xi:\lambda)\eta \otimes e_H] = j(P:\xi:\lambda+\mu)m_\mu\eta. \tag{68}$$

Proof. By meromorphy it suffices to prove the equation for generic $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ (i.e., for λ in a Baire subset). The left hand side of (68) belongs to $C^{-\infty}(P:\xi:\lambda+\mu)^H$. Application of ev_w to the left hand side of (68) yields

$$\begin{aligned}\varepsilon_\mu(e_H)(w)\,\mathrm{ev}_w(j(P:\xi:\lambda)\eta) &= \langle e^{-\mu}, \pi(w)\,e_H\rangle\,\mathrm{pr}_w\eta \\ &= \mathrm{pr}_w(m_\mu\eta),\end{aligned}$$

for $w \in \mathcal{W}$. Since $\mathrm{ev}: C^{-\infty}(P:\xi:\lambda+\mu)^H \to V(\xi)$ is bijective for generic λ with inverse $j(P:\lambda+\mu)$ (cf. [4, Lemma 5.7]), this implies the result. ∎

If Q is any parabolic subgroup in $\mathscr{P}_\sigma(A_\mathfrak{q})$, then the map ε^H defined by (58) maps $[C^{-\infty}(Q:\xi:\lambda)\otimes F]^H$ into $[C^{-\infty}(Q:\xi:\lambda+\mu)]^H$. We define the linear endomorphism $M_\mu(Q:\xi:\lambda)$ of $V(\xi)$ by

$$M_\mu(Q:\xi:\lambda)\eta = \mathrm{ev}\circ\varepsilon^H\circ p_\mu(Q:\xi:\lambda)[j(Q:\xi:\lambda)\eta\otimes e_H]. \tag{69}$$

LEMMA 9.9. *Let $q \in \Pi_S(\mathfrak{a}_\mathfrak{q})$ be as in Proposition 8.3. Then $\lambda \mapsto q(\lambda)\,M_\mu(Q:\xi:\lambda)$ is a polynomial map from $\mathfrak{a}_{\mathfrak{q}c}^*$ into* $\mathrm{End}(V(\xi))$.

Proof. If $X \in U(\mathfrak{g})$ then one readily verifies that

$$\mathrm{ev}\circ\varepsilon^H\circ(R\otimes\pi)(X)[j(Q:\xi:\lambda)\eta\otimes e_H]$$

depends polynomially on λ. Hence $M_\mu(Q:\xi:\lambda)$ depends rationally on $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. On the other hand, since the restriction of $j(Q:\xi:\lambda)\eta$ to the open H-orbits on $P\backslash G$ depends holomorphically on λ, it follows that $q(\lambda)\,M_\mu(Q:\xi:\lambda)$ depends holomorphically and hence polynomially on λ. ∎

LEMMA 9.10. *If $Q, Q' \in \mathscr{P}_\sigma(A_\mathfrak{q})$, then*

$$M_\mu(Q':\xi:\lambda)\circ B(Q':Q:\xi:\lambda) = B(Q':Q:\xi:\lambda)\circ M_\mu(Q:\xi:\lambda).$$

Proof. Since $\mathrm{ev}: C^{-\infty}(Q:\xi:\lambda)^H \to V(\xi)$ is bijective for generic λ, with inverse $j(Q:\xi:\lambda)$ (cf. [4, Lemma 5.7]), it follows that

$$\varepsilon^H\circ p_\mu(Q:\xi:\lambda)[j(Q:\xi:\lambda)\,\eta\otimes e_H] = j(Q:\xi:\lambda)\,M_\mu(Q:\xi:\lambda)\eta. \tag{70}$$

The operator $A(Q':Q:\xi:\lambda)\otimes I$ from $C^{-\infty}(Q:\xi:\lambda)\otimes F$ into

$C^{-\infty}(Q' : \xi : \lambda) \otimes F$ is equivariant hence commutes with $p_{\Lambda+\lambda+\mu}$. Moreover,

$$A(Q' : Q : \xi : \lambda) \circ \varepsilon^H = \varepsilon^H \circ [A(Q' : Q : \xi : \lambda) \otimes I].$$

Hence application of $A(Q' : Q : \xi : \lambda)$ to (70) yields

$$\begin{aligned} \varepsilon^H \circ p_\mu(Q' : \xi : \lambda)[j(Q' : \xi : \lambda) \circ B(Q' : Q : \xi : \lambda)\, \eta \otimes e_H] \\ = j(Q' : \xi : \lambda) \circ B(Q' : Q : \xi : \lambda) \circ M_\mu(Q : \xi : \lambda)\eta. \end{aligned}$$

Application of the evaluation map ev completes the proof. ∎

PROPOSITION 9.11. *There exists a non-zero constant $c \in \mathbf{C}$ and two polynomials $q_1, q_2 \in \Pi_{\mathbf{S}}(\mathfrak{a}_\mathfrak{q})$ (all independent of Q) such that*

$$\det M_\mu(Q : \xi : \lambda) = c\, \frac{q_1(\lambda)}{q_2(\lambda)}. \tag{71}$$

Before turning to the proof of this proposition we shall use it to establish Theorem 9.3.

Proof of Theorem 9.3. Applying $\Psi_\mu(P : \xi : \lambda)$ to both sides of (68) and using (65), we find that

$$p_\mu(P : \xi : \lambda)[j(P : \xi : \lambda)\eta \otimes e_H] = \Psi_\mu(P : \xi : \lambda) \circ j(P : \xi : \lambda + \mu)\, m_\mu \eta.$$

From (70) we now obtain

$$j(P : \xi : \lambda)\eta = \varepsilon^H \circ \Psi_\mu(P : \xi : \lambda) \circ j(P : \xi : \lambda + \mu)[m_\mu \circ M_\mu(P : \xi : \lambda)^{-1}\eta].$$

Since $D_\mu(\xi : \lambda) = \varepsilon^H \circ \Psi(P : \xi : \lambda)$, this proves the functional equation with

$$R_\mu(\xi : \lambda) = m_\mu \circ M_\mu(P : \xi : \lambda)^{-1}. \quad ∎$$

The rest of this section will be devoted to the proof of Proposition 9.11. In view of Lemma 9.10 the determinant (71) is independent of Q. This will be crucial for the proof.

LEMMA 9.12. *Let $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$. Then for* S*-generic $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$ the map*

$$\eta \mapsto p_\mu(Q : \xi : \lambda)(j(Q : \xi : \lambda)\, \eta \otimes e_H) \tag{72}$$

is injective from $V(\xi)$ into $(C^{-\infty}(Q : \xi : \lambda) \otimes F)^H$.

Proof. In view of Lemma 8.5 we may as well assume that μ is $\bar{Q}$-dominant. Using (61) we then infer that

$$\begin{aligned}\Phi_\mu(Q:\xi:\lambda)\circ p_\mu(Q:\xi:\lambda)(j(Q:\xi:\lambda)\,\eta\otimes e_H)\\ =\Phi_\mu(Q:\xi:\lambda)(j(Q:\xi:\lambda)\,\eta\otimes e_H).\end{aligned} \tag{73}$$

Evaluation of (73) at w yields

$$\varepsilon_\mu(e_H)(w)\,\mathrm{ev}_w\circ j(Q:\xi:\lambda)\eta=\mathrm{pr}_w(m_\mu\eta).$$

This proves that (72) is injective as soon as it is well defined (i.e., λ is not a pole). Now this is true for S-generic λ. ∎

LEMMA 9.13. *Let $Q\in\mathscr{P}_\sigma(A_{\mathfrak{q}})$, and assume that $\mu\in\Lambda(\mathfrak{a}_{\mathfrak{q}})$ is Q-dominant. Then there exists a unique rational function $\psi_\mu(Q:\xi:\lambda):\mathfrak{a}^*_{\mathfrak{q}c}\to$ End$(V(\xi,1))$ such that for $\eta\in V(\xi,1)$ we have*

$$(\mathrm{ev}_1\otimes I)\circ p_\mu(Q:\xi:\lambda)[j(Q:\xi:\lambda)\eta\otimes e_H]=\psi_\mu(Q:\xi:\lambda:\eta)\otimes e_\mu. \tag{74}$$

Moreover if q is as in Proposition 8.3 *then $q(\lambda)\,\psi_\mu(Q:\xi:\lambda)$ is polynomial in λ and invertible for* S*-generic λ.*

Proof. We use the notations of the proof of Proposition 8.1. As in the proof of Lemma 9.9 it follows that $q(\lambda)$ times the left hand side of (74) defines an element of $V(\xi)\otimes F$ which depends polynomially on λ. We will first show that in fact it belongs to $V(\xi,1)\otimes F_\mu$.

From the definition of φ_λ in the proof of Proposition 8.1 it follows that

$$\mathrm{ev}_1\circ\varphi_\lambda=\mathrm{ev}_1\otimes I\qquad\text{on}\quad[C^{-\infty}(Q:\xi:\lambda)\otimes F]^H.$$

Therefore the left hand side of (74) may be rewritten as

$$\mathrm{ev}_1\circ\varphi_\lambda\circ p_\mu(Q:\xi:\lambda)[j(Q:\xi:\lambda)\eta\otimes e_H]. \tag{75}$$

In the proof of Proposition 8.1 it was shown (under the assumption that μ is Q-dominant) that for S-generic λ the projection $p_{\Lambda+\lambda+\mu}$ maps $C^{-\infty}\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda}\otimes F|_Q)$ into its subspace $C^{-\infty}\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda}\otimes F_\mu)$. By equivariance we have $\varphi_\lambda\circ p_{\Lambda+\lambda+\mu}=p_{\Lambda+\lambda+\mu}\circ\varphi_\lambda$. Hence

$$\mathrm{im}(\varphi_\lambda\circ p_\mu(Q:\xi:\lambda))\subset C^{-\infty}\,\mathrm{Ind}_Q^G(\mathscr{H}_{\xi\lambda}\otimes F_\mu)$$

for S-generic λ (use Proposition 8.3). We conclude that (75) may be rewritten as $\psi(\lambda:\eta)\otimes e_\mu$ with $q(\lambda)\,\psi(\lambda:\eta)\in\mathscr{H}_\xi$ depending polynomially on λ and linearly on $\eta\in V(\xi,1)$. Moreover from the $H\cap M$-invariance of (75) it follows that $\psi(\lambda:\eta)\in V(\xi,1)$.

Observe that $F_\mu=\mathbf{C}e_\mu$, by Proposition 5.5. Hence $v\otimes e_\mu\mapsto v$ defines a linear isomorphism $\mathscr{H}_\xi \otimes F_\mu \xrightarrow{\simeq} \mathscr{H}_\xi$. This map in turn induces an isomorphism of Q-modules $\mathscr{H}_{\xi\lambda} \otimes F_\mu \xrightarrow{\simeq} \mathscr{H}_{\xi(\lambda+\mu)}$, hence an isomorphism

$$\nu : C^{-\infty}\operatorname{Ind}_Q^G(\mathscr{H}_{\xi\lambda} \otimes F_\mu) \xrightarrow{\simeq} C^{-\infty}(Q : \xi : \lambda + \mu).$$

Put

$$u(\lambda : \eta) := \nu \circ \varphi_\lambda \circ p_\mu(Q : \xi : \lambda)[j(Q : \xi : \lambda)\eta \otimes e_H].$$

Then from the above it follows that $u(\lambda : \eta)$ is H-invariant and that $\mathrm{ev}_1 u(\lambda : \eta) = \psi(\lambda : \eta)$, for $\eta \in V(\xi, 1)$. The support of $u(\lambda : \eta)$ is obviously contained in the closure of QH; hence

$$u(\lambda : \eta) = j(Q : \xi : \lambda)\, \psi(\lambda : \eta), \qquad \eta \in V(\xi, 1),$$

as meromorphic functions of λ (use [4, Theorem 5.1 and Lemma 5.7]). In view of Lemma 9.12 the map $\eta \mapsto u(\lambda : \eta)$ is injective from $V(\xi, 1)$ into $[C^{-\infty}(Q : \xi : \lambda + \mu)]^H$, for S-generic λ. This implies that $\psi_\mu(Q : \xi : \lambda) = \psi(\lambda)$ is injective for S-generic λ. ∎

Now assume that μ is Q-dominant. For every $w \in \mathscr{W}$, let $e_{w^{-1}\mu}$ be a non-zero $\mathfrak{a}_\mathfrak{q}$-weight vector in F of weight $w^{-1}\mu$, and define the endomorphism $\psi_w(\lambda)$ of $V(\xi, w)$ by

$$\psi_w(\lambda)) = L(\xi, w^{-1})^{-1} \circ \psi_{w^{-1}\mu}(w^{-1}Qw : w^{-1}\xi : w^{-1}\lambda) \circ L(\xi, w^{-1}).$$

Here $L(\xi, w^{-1})$ is the map $V(\xi) \to V(w^{-1}\xi)$ defined in [4, Lemma 6.10].

Corollary 9.14. *For every $\eta \in V(\xi)$ we have*

$$(\mathrm{ev}_w \otimes I) \circ p_\mu(Q : \xi : \lambda)[j(Q : \xi : \lambda)\eta \otimes e_H]$$
$$= \psi_w(\lambda : \mathrm{pr}_w(\eta)) \otimes e_{w^{-1}\mu}. \tag{76}$$

Proof. Since the map $p_\mu(Q : \xi : \lambda)$ is support preserving, nothing changes if we replace η in the left hand side of (76) by its $V(\xi, w)$-component $\mathrm{pr}_w \eta$. Hence we may as well assume that $\eta \in V(\xi, w)$ already.

We have that

$$L(\xi, w^{-1}) \circ (\mathrm{ev}_w \otimes I) = (\mathrm{ev}_1 \otimes I) \circ [L(w^{-1}) \otimes I].$$

Using Lemma 8.6 we may rewrite the left hand side of (76) as

$$L(\xi, w^{-1})^{-1} \circ (\mathrm{ev}_1 \otimes I) \circ p_{w^{-1}\mu}(w^{-1}Qw : w^{-1}\xi : w^{-1}\lambda)$$
$$[j(w^{-1}Qw : w^{-1}\xi : w^{-1}\lambda)\, L(\xi, w^{-1})\eta \otimes e_H]. \tag{77}$$

Now $w^{-1}\mu$ is an extremal weight for F which is $w^{-1}Qw$-dominant. Applying Lemma 9.13 we now infer that (77) equals

$$\begin{aligned} L(\xi, w^{-1})^{-1}\psi_{w^{-1}\mu}(w^{-1}Qw : w^{-1}\xi : w^{-1}\lambda)[L(\xi, w^{-1})\eta] \otimes e_{w^{-1}\mu} \\ = \psi_w(\lambda : \eta) \otimes e_{w^{-1}\mu} = \psi_w(\lambda : \mathrm{pr}_w\, \eta) \otimes e_{w^{-1}\mu}. \quad \blacksquare \end{aligned}$$

Proof of Proposition 9.11. In view of Lemma 9.10 it suffices to prove the assertion when μ is Q-dominant. But then it follows from Corollary 9.14 that

$$\begin{aligned} \mathrm{pr}_w M_\mu(Q : \xi : \lambda)\eta &= \mathrm{ev}_w \circ \varepsilon^H \circ p_\mu(Q : \xi : \lambda)[j(Q : \xi : \lambda)\eta \otimes e_H] \\ &= \varepsilon^H \circ (\mathrm{ev}_w \otimes I) \circ p_\mu(Q : \xi : \lambda)[j(Q : \xi : \lambda)\eta \otimes e_H] \\ &= \langle e^H, e_{w^{-1}\mu} \rangle\, \psi_w(\lambda : \mathrm{pr}_w \eta). \qquad (78) \end{aligned}$$

This proves that $M_\mu(\lambda) = M_\mu(Q : \xi : \lambda)$ preserves the decomposition (5), and that its determinant is given by the formula

$$\det M_\mu(\lambda) = \prod_{\substack{w \in \mathcal{W} \\ V(\xi, w) \neq 0}} \langle e^H, e_{w^{-1}\mu} \rangle \det \psi_w(\lambda).$$

Since $\langle e^H, e_{w^{-1}\mu} \rangle \neq 0$ (cf. the proof of Lemma 9.7), it now follows by application of Lemma 9.13 that there exists a $q_1 \in \Pi_S(\mathfrak{a}_\mathfrak{q})$ such that $q_1(\lambda) \det M_\mu(\lambda)$ is a polynomial which is non-zero for S-generic λ. Any such polynomial is of the form cq_1, with $q_1 \in \Pi_S(\mathfrak{a}_\mathfrak{q})$ and c a non-zero scalar. $\blacksquare$

10. Initial Estimates for Eisenstein Integrals

In this section we will derive an initial estimate for the Eisenstein integral. Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\xi \in \hat{M}_{\mathrm{ps}}$, and write $\pi_\lambda = \pi_{P, \xi, \lambda}$. In addition to Lemma 8.2 we need the following result.

Lemma 10.1. *Let $s \in \mathbf{N}$. Then there exist constants $C > 0$, $r > 0$ such that for every $a \in A_\mathfrak{q}$ the operator $\pi_\lambda(a)$ maps $C^s(K : \xi)$ into itself with operator norm*

$$\|\pi_\lambda(a)\| \leqslant C(1 + |\lambda|)^q\, e^{(r + |\mathrm{Re}\, \lambda|)\, |\log a|}.$$

Proof. Let $\varphi \in C^{-\infty}(K : \xi)$ and define $\varphi_\lambda \in C^{-\infty}(P : \xi : \lambda)$ by $\varphi_\lambda \mid K = \varphi$.

Define the maps $H_P: G \to \mathfrak{a}$, $\mu_P: G \to \exp(\mathfrak{m} \cap \mathfrak{p})$ and $\kappa_P: G \to K$ by $x \in N_P \exp H_P(x)\, \mu_P(x)\, \kappa_P(x)$. Then

$$\begin{aligned}\pi_\lambda(a)\, \varphi(k) &= \varphi_\lambda(ka)\\ &= e^{(\lambda + \rho_P) H_P(ka)}\, \xi(\mu_P(ka))\, \varphi(\kappa_P(ka)).\end{aligned}$$

Using that ξ is unitary and that

$$|H_P(ka)| \leqslant |\log a|$$

for all $k \in K$, $a \in A_\mathfrak{q}$ one obtains the desired estimate for $s = 0$.

Now let s be arbitrary, $\varphi \in C^s(K)$, and suppose that $Y \in U_s(\mathfrak{k})$. Then

$$\begin{aligned}R_Y \pi_\lambda(a)\, \varphi(k) &= \pi_\lambda(a)\, \pi_\lambda(\mathrm{Ad}(a^{-1})\, Y)\, \varphi(k)\\ &= \sum_i c_i(a)\, \pi_\lambda(a)\, \pi_\lambda(Y_i)\, \varphi(k),\end{aligned}$$

for finitely many $Y_i \in U_s(\mathfrak{g})$ and finitely many smooth functions c_i on $A_\mathfrak{q}$ satisfying bounds of the form $|c_i(a)| \leqslant \exp(r\, |\log a|)$. The result now follows by applying Lemma 8.2 and the first part of this proof. ∎

COROLLARY 10.2. *Let $\xi \in \hat{M}_{\mathrm{ps}}$, $R \in \mathbf{R}$. Then there exists a polynomial function $p \in \Pi_\Sigma(\mathfrak{a}_\mathfrak{q})$ and a constant $s \in \mathbf{N}$, such that*

(1) *for every $\eta \in V(\xi)$ the function $\lambda \mapsto p(\lambda)\, j(P : \xi : \lambda)\eta$ is holomorphic $C^{-s}(P : \xi : \lambda)$-valued on $\mathfrak{a}_\mathfrak{q}^*(P, R)$, and*

(2) *there exist constants $N \in \mathbf{N}$, $C > 0$, $r > 0$ such that*

$$\|\pi_\lambda(a)\, p(\lambda)\, j(P : \xi : \lambda)\eta\|_{-s} \leqslant C(1 + |\lambda|)^N e^{(r + |\mathrm{Re}\,\lambda|)\, |\log a|}\, \|\eta\|,$$

for all $\eta \in V(\xi)$, $\lambda \in \mathfrak{a}_\mathfrak{q}^*(P, R)$, and $a \in A_\mathfrak{q}$.

Proof. The first assertion is a reformulation of Theorem 9.1. The second one follows immediately by application of the previous lemma. ∎

PROPOSITION 10.3. *Let $R \in \mathbf{R}$. Then there exists a polynomial function $p \in \Pi_\Sigma(\mathfrak{a}_\mathfrak{q})$ such that for each $\psi \in {}^\circ\mathscr{C}$ the mapping $(\lambda, x) \mapsto p(\lambda)\, E(P : \psi : \lambda)(x)$ is a C^∞-function on $\mathfrak{a}_\mathfrak{q}^*(P, R) \times G/H$, which is in addition holomorphic in its first variable. Moreover, if $p \in \Pi_\Sigma(\mathfrak{a}_\mathfrak{q})$ is any polynomial with this property, then there exist a constant $r > 0$ and for every $X \in U(\mathfrak{g})$ constants $N \in \mathbf{N}$ and $C > 0$, such that*

$$\|p(\lambda)\, E(P : \psi : \lambda)(X; a)\| \leqslant C(1 + |\lambda|)^N e^{(r + |\mathrm{Re}\,\lambda|)\, |\log a|}\, \|\psi\| \qquad (79)$$

for all $\psi \in {}^\circ\mathscr{C}$, $\lambda \in \mathfrak{a}_\mathfrak{q}^*(P, R)$, and $a \in A_\mathfrak{q}$.

Proof. It suffices to prove the proposition for a fixed ψ, and we may as well assume that $\psi = \psi_T$, with $T = f \bar{\otimes} \eta \in \mathscr{H}_{\xi, F} \bar{\otimes} V(\xi)$ as in the proof of Lemma 4.2. Let $p_0(\lambda)$ be the polynomial corresponding to $j(P:\xi:\lambda)\eta$ as in Corollary 10.2 and let $p(\lambda)$ be the polynomial defined by $\overline{p(\lambda)} = p_0(\bar{\lambda})$. Then $p \in \Pi_\Sigma(\mathfrak{a}_\mathfrak{q})$ because Σ is invariant under complex conjugation. Moreover,

$$p(\lambda)\, E(P : \psi : \lambda)(X; a)(k) = \langle \pi_\lambda(X)\, R_{k^{-1}} f_w, \pi_\lambda(a)\, p_0(\bar{\lambda}) j(P : \xi : \bar{\lambda})\eta_w \rangle.$$

The last expression may be suitably estimated when we apply Corollary 10.2 and Lemma 8.2.

11. Families of Sperical Modules

In this section we will investigate the structure of certain families of spherical $(\mathfrak{g}, K)$-modules, related to algebraic models of the spherical principal series. Our interest in them originates from the following. Given $\nu \in \mathfrak{b}_c^*$, let $f \in C^\infty(G/H)$ satisfy the system of differential equations

$$Df = \gamma(D : \nu) f, \qquad D \in \mathbf{D}(G/H)$$

(notations of Section 2). Then f generates a $(\mathfrak{g}, H)$-module from the right. Via duality this module corresponds to a quotient of a spherical principal series $(\mathfrak{g}, K)$-module Y_ν. With a similar motivation this module has been studied by [5]. We need stronger results concerning the dependence on the parameter ν however. The main results of this section, Proposition 11.7 resp. Corollary 11.15, and their dual companions, Proposition 12.4 resp. Proposition 18.8 will be applied in the study of the asymptotic behaviour of eigenfunctions in Sections 12 and 18.

We start by fixing notations. Let $W_0 = W(\mathfrak{g}, \mathfrak{a}_0)$ and let Δ denote the set of simple roots in Σ_0^+ (cf. Section 1). Given a subset $F \subset \Delta$ we shall write P_F for the associated standard parabolic subgroup, $P_F = M_F A_F N_F$ for its Langlands decomposition, and $M_{1F} = M_F A_F$. Moreover, we put $\bar{N}_F = \theta N_F$. If F is the empty set, then we shall also use the subscript 0 instead of $\varnothing$. Thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_0 \oplus \bar{\mathfrak{n}}_0$ is an Iwasawa decomposition for $\mathfrak{g}$. We also adopt the notations of Section 2 for the special case $\sigma = \theta$. A sub- or superscript P_F will then be replaced by F. In particular γ_0 denotes the isomorphism from $\mathbf{D}(G/K)$ onto $I(\mathfrak{a}_0)$.

Let X be a complex linear space, and suppose that for every value of a parameter ω ranging in a connected open subset Ω of a finite dimensional complex linear space, a $(\mathfrak{g}, K)$-representation π_ω in X is given. We shall write X_ω for X together with the structure π_ω of $(\mathfrak{g}, K)$-module. Moreover, if $\delta \in \hat{K}$ then we shall write $X(\omega, \delta)$ for the isotypical component of type δ for $\pi_\omega \mid K$. If $\vartheta \subset \hat{K}$, we put

$$X(\omega, \vartheta) = \bigoplus_{\delta \in \vartheta} X(\omega, \delta).$$

DEFINITION 11.1. We will say that $(\pi_\omega; \omega \in \Omega)$ is a holomorphic (resp. polynomial) family of Harish-Chandra modules in X if the following conditions are fulfilled.

(1) for every $\omega \in \Omega$ the $(\mathfrak{g}, K)$-module X_ω is finitely generated and admissible;

(2) for every $u \in U(\mathfrak{g})$ and $x \in X$ there exists a finite dimensional subspace $S \subset X$ such that for all $\omega \in \Omega$ one has $\pi_\omega(u)x \in S$ and $\pi_\omega(K)x \subset S$ and moreover

(a) the map $\omega \mapsto \pi_\omega(u)x$, $\Omega \to S$ is holomorphic (resp. polynomial), and

(b) the map $(\omega, k) \mapsto \pi_\omega(k)x, \Omega \times K \to S$ is continuous and in addition holomorphic (resp. polynomial) in its first variable.

LEMMA 11.2. *Let $(\pi_\omega; \omega \in \Omega)$ be a holomorphic family of Harish-Chandra modules in X. Then for every finite dimensional subspace $S \subset X$ there exists a finite subset $\vartheta \subset \hat{K}$ such that*

$$S \subset X(\omega, \vartheta)$$

for all $\omega \in \Omega$. Conversely, if ϑ' is a finite subset of $\hat{K}$, then there exists a finite dimensional subspace $S' \subset X$ such that

$$X(\omega, \vartheta') \subset S'$$

for all $\omega \in \Omega$.

Proof. Let T be the linear span of the vectors $\pi_\omega(k)x$, $x \in S$, $k \in K$, $\omega \in \Omega$. Then by (2)(b), T is finite dimensional. If $\delta \in \hat{K}$, let $P_{\omega,\delta}: X \to X$ denote the projection onto the isotypical component of type δ for $\pi_\omega \mid K$. Then

$$P_{\omega,\delta} = \int_K \dim(\delta)\, \chi_\delta(k^{-1})\, \pi_\omega(k)\, dk, \tag{80}$$

where dk is the normalized Haar measure of K, and χ_δ the character of δ. The operators $P_{\omega,\delta}$ map S into the finite dimensional space T, and from

(80) we infer that the map $\Omega \to \mathrm{Hom}(S, T)$, $\omega \mapsto P_{\omega,\delta} \mid S$ is holomorphic. Hence for $\delta \in \hat{K}$ the subset

$$\Omega(\delta) = \{\omega \in \Omega;\, P_{\omega,\delta} \mid S \neq 0\}$$

is either empty or open dense in Ω. Now let ϑ be the subset of $\delta \in \hat{K}$ for which $\Omega(\delta) \neq \emptyset$. Then obviously $S \subset X(\omega, \vartheta)$ for every $\omega \in \Omega$. We will show that ϑ is a finite set. Indeed, if ϑ_0 is any finite ubset of ϑ, then $\Omega(\vartheta_0) = \bigcap_{\delta \in \vartheta_0} \Omega(\delta)$ is open dense. Fix $\omega_0 \in \Omega(\vartheta_0)$. Then for every $\delta \in \vartheta_0$ the space $P_{\omega_0.\delta}(S)$ is a non-trivial subspace in T. Since X is the direct sum of the spaces $P_{\omega_0,\delta}(X)$ $(\delta \in \hat{K})$ it follows that $|\vartheta_0| \leqslant \dim T$. Hence ϑ is a finite set, and the first assertion follows.

To prove the second assertion, we may as well assume that $\vartheta' = \{\delta\}$. We first show that the function $d(\omega) = \dim X(\omega, \delta)$ is uniformly bounded. Indeed assume this were not so, and let $\Omega_j = \{\omega \in \Omega;\ \dim X(\omega, \delta) > j\}$. Then $\emptyset \neq \Omega_{j+1} \subset \Omega_j$ for all $j \geqslant 1$. If $\omega_0 \in \Omega_j$, put $S = X(\omega_0, \delta)$, and let T be as in the first part of the proof. Then the map $\omega \mapsto P_{\omega,\delta} \mid S$, $\Omega \mapsto \mathrm{Hom}(S, T)$ is holomorphic. Since $P_{\omega_0,\delta}$ is the identity on S it follows that the set of $\omega \in \Omega$ for which $P_{\omega,\delta} \mid S$ is injective, is open and dense in Ω. But Ω_j contains this set, hence is open and dense in Ω as well. By the Baire category theorem it now follows that $\Omega_\infty = \bigcap_{j \geqslant 1} \Omega_j$ is non-empty. Fix $\omega_\infty \in \Omega_\infty$. Then $X(\omega_\infty, \delta)$ is infinite dimensional, contradicting the admissibility of X_{ω_∞}.

Let m be the maximal value of the function $d = \dim X(\cdot, \delta)$, and let $\Omega_{\max}$ be the set of $\omega \in \Omega$ for which $d(\omega) = m$. Then $\Omega_{\max} = \Omega_{m-1}$, hence open and dense. Fix $\omega_1 \in \Omega_{\max}$, let $S = X(\omega_1, \delta)$, and define T as in the first line of the proof. Then the rank of $P(\omega, \delta) \mid S \in \mathrm{Hom}(S, T)$ is at most m. Moreover, it is m for $\omega = \omega_1$, hence for ω in an open dense subset $\Omega' \subset \Omega$. The set $P(\omega, \delta)S$ is contained in $X(\omega, \delta)$ for any $\omega \in \Omega$; hence for dimensional reasons we have that $P(\omega, \delta)S = X(\omega, \delta)$ for $\omega \in \Omega_{\max} \cap \Omega'$. It follows that $X(\omega, \delta)$ is contained in T for $\omega \in \Omega_{\max} \cap \Omega'$. We complete the proof by showing that in fact this holds for all $\omega \in \Omega$. Indeed let $x \in X$ be arbitrary, and let T' be the linear space spanned by T and $\pi_\omega(k)x$ $(\omega \in \Omega$, $k \in K)$. Then T' is finite dimensional, and $\varphi: \omega \mapsto P_{\omega,\delta}(x)$ is a holomorphic function with values in T'. But in the above we showed that $\varphi(\omega) \in T$ for all $\omega \in \Omega_{\max} \cap \Omega'$. By continuity and density this holds for all $\omega \in \Omega$. Hence $X(\omega, \delta) = P_{\omega,\delta}(X) \subset T$ for all ω. ∎

Holomorphic families of Harish-Chandra modules may be obtained by using coinduction. We first discuss the induction procedure without parameter dependence.

Via the isomorphism (13) (in the special case $\sigma = \theta$) we shall view the space $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}$ as a right $\mathbf{D}(G/K)$-module. Of course it is also a $(\mathfrak{g}, K)$-

module for the left action by $\mathfrak{g}$ and the adjoint action by K. If χ is a representation of $\mathbf{D}(D/K)$ in a finite dimensional complex vector space W, then we define the $(\mathfrak{g}, K)$-module Y_χ by

$$Y_\chi := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} \otimes_{\mathbf{D}(G/K)} W.$$

It is a finitely generated admissible $(\mathfrak{g}, K)$-module (use [34, Corollary 3.4.7]).

Let E denote the space of W_0-harmonic polynomials in $S(\mathfrak{a}_0)$, and define

$$\mathscr{U} = U(\bar{\mathfrak{n}}_0) \otimes E.$$

We shall view $\mathscr{U}$ as a left $U(\bar{\mathfrak{n}}_0)$-module. The following result is contained in [5, Proposition 5.1] (notice that $E = T_{\rho_0} E$).

Lemma 11.3. *The map $\Gamma: \mathscr{U} \otimes \mathbf{D}(G/K) \to U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}$ induced by $u \otimes e \otimes D \mapsto ueD$ is an isomorphism of left $U(\mathfrak{n}_0)$- and right $\mathbf{D}(G/K)$-modules.*

Corollary 11.4. *The linear map $\mathscr{U} \otimes W \to Y_\chi$ induced by $x \otimes e \otimes w \mapsto xe \otimes w$ is an isomorphism of left $U(\bar{\mathfrak{n}}_0)$-modules.*

Proof. Write $\mathbf{D} = \mathbf{D}(G/K)$. Then we have

$$\begin{aligned} Y_\chi = [U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}] \otimes_{\mathbf{D}} W &\simeq [U(\bar{\mathfrak{n}}_0) \otimes E \otimes \mathbf{D}] \otimes_{\mathbf{D}} W \\ &\simeq U(\bar{\mathfrak{n}}_0) \otimes E \otimes [\mathbf{D} \otimes_{\mathbf{D}} W]. \end{aligned}$$

Now use that $\mathbf{D} \otimes_{\mathbf{D}} W \simeq W$. ∎

We shall consider the above construction for a representation χ_ω of $\mathbf{D}(G/K)$ in W depending on a parameter $\omega \in \Omega$. The family $(\chi_\omega;\, \omega \in \Omega)$ will be called holomorphic (resp. polynomial) if for every $D \in \mathbf{D}(G/K)$ the map $\omega \mapsto \chi_\omega(D)$ is holomorphic (resp. polynomial) from Ω into $\mathrm{End}(W)$. Let W_ω denote W provided with the structure of $\mathbf{D}(G/K)$-module induced by χ_ω. Writing Y_ω for Y_{χ_ω} we have

$$Y_\omega = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} \otimes_{\mathbf{D}(G/K)} W_\omega.$$

Moreover, let $\mathscr{Y} = \mathscr{U} \otimes W$. Then by Corollary 11.4 the linear map $\varphi_\omega: \mathscr{Y} \to Y_\omega$ induced by $x \otimes e \otimes w \mapsto xe \otimes w$ is an isomorphism of left $U(\bar{\mathfrak{n}}_0)$-modules. We shall write π_ω for the ($(\mathfrak{g}, K)$-representation which $\mathscr{Y}$ inherits via pull back by φ_ω.

Proposition 11.5. *Let $(\chi_\omega;\, \omega \in \Omega)$ be a holomorphic (resp. polynomial) family of $\mathbf{D}(G/K)$-representations in W. Then π_ω is a holomorphic (resp. polynomial) family of Harish-Chandra modules in $\mathscr{Y}$.*

Proof. Since we observed already that each $\mathscr{Y}_\omega$ is a finitely generated admissible $(\mathfrak{g}, K)$-module it remains to verify condition (2) of Definition 11.1, and it suffices to do this for $x = y \otimes e \otimes w$, with $y \in U(\bar{\mathfrak{n}}_0)$, $e \in E$, $w \in W$. Let $u \in U(\mathfrak{g})$. Then $uye \equiv \sum_i y_i e_i D_i$ modulo $U(\mathfrak{g})\mathfrak{k}$ with finitely many $y_i \in U(\bar{\mathfrak{n}}_0)$, $e_i \in E$, $D_i \in \mathbf{D}(G/K)$. Hence

$$\pi_\omega(u)(y \otimes e \otimes w) = \sum_i y_i \otimes e_i \otimes \chi_\omega(D_i) w.$$

We conclude that $\omega \mapsto \pi_\omega(u)x$ is a holomorphic (resp. polynomial) map into a finite dimensional subspace of $\mathscr{Y}$.

Finally, let $x = y \otimes e \otimes w$ be as above. Then $k \mapsto \mathrm{Ad}(k)(ye)$ is a continuous map from K into a finite dimensional linear subspace of $U(\mathfrak{g})$. In view of Lemma 11.3 we may write $\mathrm{Ad}(k)(ye) = \sum_i m_i(k) y_i e_i D_i$ modulo $U(\mathfrak{g})\mathfrak{k}$, with finitely many $y_i \in U(\bar{\mathfrak{n}}_0)$, $e_i \in E$, $D_i \in \mathbf{D}(G/K)$, and finitely many continuous functions $m_i \colon K \to \mathbf{C}$. Now

$$\pi_\omega(k)(y \otimes e \otimes w) = \sum_i m_i(k) y_i \otimes e_i \otimes \chi_\omega(D_i) w,$$

and one sees that condition (2)(b) holds. ∎

Since M_{1F} normalizes the algebra $\bar{\mathfrak{n}}_F$, the quotient spaces $Y_\omega^j = Y_\omega / \bar{\mathfrak{n}}_F^j Y_\omega$ $(j \geqslant 1)$ are $(\mathfrak{m}_{1F}, K_F)$-modules. In fact they are finitely generated and admissible, cf. [34, Sect. 4.3].

Let $\mathscr{Y}^j = \mathscr{Y} / \bar{\mathfrak{n}}_F^j \mathscr{Y}$, and let π_ω^j be the $(\mathfrak{m}_{1F}, K_F)$-module structure inherited from π_ω. Then clearly φ_ω factorizes to an isomorphism of $(\mathfrak{m}_{1F}, K_F)$-modules $\varphi_\omega^j \colon (\mathscr{Y}^j, \pi_\omega^j) \to Y_\omega^j$.

The proof of the following result amounts to a straightforward verification of condition (2) of Definition 11.1.

Proposition 11.6. *Assume that χ_ω is a holomorphic (resp. polynomial) family and let $j \geqslant 1$. Then $(\pi_\omega^j;\ \omega \in \Omega)$ is a holomorphic (resp. polynomial) family of Harish-Chandra $(\mathfrak{m}_{1F}, K_F)$-modules in $\mathscr{Y}^j$.*

We now apply all the above to a specific situation. Let $\Omega = \mathfrak{a}_{0\mathrm{c}}^*$, $W = \mathbf{C}$, and for $\nu \in \mathfrak{a}_{0\mathrm{c}}^*$ define the character χ_ν of $\mathbf{D}(G/K)$ by $\chi_\nu(D) = \gamma_0(D : \nu)$. Then for $j \geqslant 1$ the family π_ν^j is polynomial, hence by Lemma 11.2 there exists a finite dimensional subspace $\bar{\mathscr{V}}_j\ \mathscr{Y}^j$ such that

$$\mathscr{Y}^j(\nu, 1) \subset \bar{\mathscr{V}}_j \qquad \text{for all} \quad \nu \in \mathfrak{a}_{0\mathrm{c}}^*.$$

Let $\tilde{\mathscr{V}}_j$ be a finite dimensional subspace of $\mathscr{Y}$ which is mapped bijectively onto $\bar{\mathscr{V}}_j$ under the canonical projection $p_j: \mathscr{Y} \to \mathscr{Y}^j$, and which contains $1 \otimes 1 \otimes 1$. Moreover, let $\mathscr{V}_j$ denote the image of $\tilde{\mathscr{V}}_j$ under the map

$$\mathbf{m}: \mathscr{Y} = U(\bar{\mathfrak{n}}_0) \otimes E \otimes \mathbf{C} \to U(\mathfrak{g}), \qquad u \otimes e \otimes z \mapsto zue. \tag{81}$$

Then $\mathscr{V}_j$ is a finite dimensional subspace of $U(\bar{\mathfrak{n}}_0 + \mathfrak{a}_0)$ containing 1.

PROPOSITION 11.7. *Let $j \geqslant 1$. Then there exist*

(1) *an endomorphism $x_\nu \in \mathrm{End}(\mathscr{V}_j)$, depending polynomially on $\nu \in \mathfrak{a}_{0c}^*$, and such that $x_\nu(1) = 1$ for all $\nu \in \mathfrak{a}_{0c}^*$;*

(2) *an algebra homomorphism $b_j(\nu, \cdot)$ from $U(\mathfrak{m}_{1F})^{K_F}$ into $\mathrm{End}(\mathscr{V}_j)$, depending polynomially on $\nu \in \mathfrak{a}_{0c}^*$; and*

(3) *a bilinear map $y_\nu: U(\mathfrak{m}_{1F})^{K_F} \times \mathscr{V}_j \to \bar{\mathfrak{n}}_F^j\, U(\bar{\mathfrak{n}}_0 + \mathfrak{a}_0)$, depending polynomially on $\nu \in \mathfrak{a}_{0c}^*$,*

such that for all $\nu \in \mathfrak{a}_{0c}^*$, $D \in U(\mathfrak{m}_{1F})^{K_F}$, and $v \in \mathscr{V}_j$ we have

$$Dx_\nu(v) \equiv x_\nu(b_j(\nu, D)v) + y_\nu(D, v) \qquad \mathrm{mod}\ J_\nu.$$

Here J_ν denotes the left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{k}$ and

$$\{D - \gamma_0(D : \nu);\ D \in U(\mathfrak{g})^K\}.$$

Proof. Let P_ν denote the projection in $\mathscr{Y}^j$ onto the isotypical component of type 1 for $\pi_\nu \mid K_F$. Then P_ν maps the space $\mathscr{Y}^j$ into $\bar{\mathscr{V}}_j$. Put $\bar{x}_\nu = P_\nu \mid \bar{\mathscr{V}}_j$. Then as in the proof of Lemma 11.2 one verifies that the map $\nu \mapsto \bar{x}_\nu$ maps $\mathfrak{a}_{0c}^*$ polynomially into $\mathrm{End}(\bar{\mathscr{V}}_j)$.

Define the algebra homomorphism $\bar{b}_j(\nu, \cdot): U(\mathfrak{m}_{1F})^{K_F} \to \mathrm{End}(\bar{\mathscr{V}}_j)$ by

$$\bar{b}_j(\nu, D) = P_\nu \circ \pi_\nu^j(D) \circ P_\nu \mid \bar{\mathscr{V}}_j. \tag{82}$$

Then $\bar{b}_j(\nu, D)$ depends polynomially on ν. Using that P_ν commutes with $\pi_\nu^j(D)$ for every $D \in U(\mathfrak{m}_{1F})^{K_F}$, we see that

$$\begin{aligned}\pi_\nu^j(D) \circ \bar{x}_\nu &= P_\nu \circ \pi_\nu^j(D) \circ P_\nu \mid \bar{\mathscr{V}}_j \\ &= (P_\nu \mid \bar{\mathscr{V}}_j) \circ P_\nu \circ \pi_\nu^j(D) \circ P_\nu \mid \bar{\mathscr{V}}_j \\ &= \bar{x}_\nu \circ \bar{b}_j(\nu, D).\end{aligned}$$

The next step is to transport this structure from $\bar{\mathscr{V}}_j$ to $\mathscr{V}_j$. Let $\eta: \mathscr{V}_j \to \tilde{\mathscr{V}}_j$ be the inverse of the bijective map $\mathbf{m} \mid \tilde{\mathscr{V}}_j : \tilde{\mathscr{V}}_j \to \mathscr{V}_j$ (cf. (81)) and define $\xi = p_j \circ \eta$, where p_j is the canonical projection $\mathscr{Y} \to \mathscr{Y}^j$. Then ξ is a linear isomorphism from $\mathscr{V}_j$ onto $\bar{\mathscr{V}}_j$. For $\nu \in \mathfrak{a}_{0c}^*$ and $D \in U(\mathfrak{m}_{1F})^{K_F}$ we define $x_\nu, b_j(\nu, D) \in \operatorname{End}(\mathscr{V}_j)$ by

$$x_\nu = \xi^{-1} \circ \bar{x}_\nu \circ \xi,$$
$$b_j(\nu, D) = \xi^{-1} \circ \bar{b}_j(\nu, D) \circ \xi.$$

Let $1_{\mathscr{Y}}$ denote the element $1 \otimes 1 \otimes 1 \in \mathscr{Y}$. Then $1_{\mathscr{Y}}$ is a cyclic vector for the $U(\mathfrak{g})$-module $\mathscr{Y}_\nu$ $(\nu \in \mathfrak{a}_{0c}^*)$. Let $p_\nu \colon U(\mathfrak{g}) \to \mathscr{Y}$, $u \mapsto \pi_\nu(u) 1_{\mathscr{Y}}$ be the corresponding epimorphism, and define

$$\tilde{y}_\nu(D, v) = p_\nu(D x_\nu(v) - x_\nu(b_j(\nu, D) v)),$$

for $\nu \in \mathfrak{a}_{0c}^*$, $D \in U(\mathfrak{m}_{1F})^{K_F}$, $v \in \mathscr{V}_j$. Then

$$\tilde{y}_\nu(D, v) = \pi_\nu(D)[\eta \circ x_\nu(v)] - \eta \circ x_\nu(b_j(\nu, D) v)$$

which is easily seen to have canonical image zero in $\mathscr{Y}^j$. Hence $\tilde{y}_\nu(D, v) \in \bar{\mathfrak{n}}_F^j \mathscr{Y}$ and it follows that

$$y_\nu(D, v) := \mathbf{m}(\tilde{y}_\nu(D, v))$$

belongs to $\bar{\mathfrak{n}}_F^j\, U(\bar{\mathfrak{n}}_0) E$. Moreover, using that $p_\nu \circ \mathbf{m} = I$ on $\mathscr{Y}$ we see that

$$D x_\nu(v) - x_\nu(b_j(\nu, D) v) - y_\nu(D, v) \tag{83}$$

belongs to $\ker p_\nu$. One readily checks that $\ker p_\nu = J_\nu$. ∎

Let $\mathfrak{a}$ be a real abelian Lie algebra, and suppose that X is a complex vector space in which $U(\mathfrak{a})$ has a locally finite representation π, i.e., $\dim \pi(U(\mathfrak{a}))x < \infty$ for all $x \in X$. If $\lambda \in \mathfrak{a}_c^*$ then we shall write $X(\pi, \lambda)$ for the associated generalized $\mathfrak{a}$-weight space. Let $\Lambda(\pi)$ denote the set of $\mathfrak{a}$-weights of π, i.e., the set of $\lambda \in \mathfrak{a}_c^*$ such that $X(\pi, \lambda) \neq 0$. Then of course

$$X = \bigoplus_{\lambda \in \Lambda(\pi)} X(\pi, \lambda).$$

We say that a weight $\lambda \in \Lambda(\pi)$ has finite order if there exists a positive integer m such that for all $H \in \mathfrak{a}$ we have that $(\pi(H) - \lambda(H))^m$ vanishes on $X(\pi, \lambda)$. The smallest m having this property is said to be the order of λ in π, notation $o(\pi, \lambda)$. If $\lambda \in \Lambda(\pi)$ is not of finite order we define $o(\pi, \lambda) = \infty$, and if $\lambda \in \mathfrak{a}_c^* \backslash \Lambda(\pi)$ we set $o(\pi, \lambda) = 0$.

PROPOSITION 11.8. *Let $j \geqslant 1$. Then Proposition* 11.7 *holds with the additional properties*

(1) $\Lambda(b_j(\nu, \cdot) \mid \mathfrak{a}_F) \subset \Lambda(\pi_\nu^j \mid \mathfrak{a}_F) \cup \{0\}$;

(2) *if* $\lambda \in \Lambda(b_j(\nu, \cdot) \mid \mathfrak{a}_F)$ *then* $o(b_j(\nu, \cdot), \lambda) \leqslant \max\{o(\pi_\nu^j, \lambda), 1\}$.

Proof. We use the notations of the proof of Proposition 11.7. By (83) it suffices to prove the assertions with $\bar{b}_j$ instead of b_j. Write $\bar{\mathscr{V}}_j = \mathrm{im}(P_\nu) \oplus \bar{\mathscr{V}}_{j,\nu}$, where $\bar{\mathscr{V}}_{j,\nu} = \bar{\mathscr{V}}_j \cap \ker P_\nu$. If $D \in U(\mathfrak{m}_{1F})^{K_F}$, then $\bar{b}_j(\nu, D)$ acts by zero on $\bar{\mathscr{V}}_{j,\nu}$. Moreover, $\pi_\nu^j(D)$ leaves $\mathrm{im}(P_\nu)$ invariant, and by (82), $\bar{b}_j(\nu, D) - \pi_\nu^j(D)$ acts by zero on $\mathrm{im}(P_\nu)$. From this all assertions follow. ∎

Our next goal is to investigate the weights of π_ν^j.

LEMMA 11.9. *There exists a positive integer m such that for every $\nu \in \mathfrak{a}_{0c}^*$ and every $\lambda \in \Lambda(\pi_\nu^1 \mid \mathfrak{a}_F)$ we have $o(\pi_\nu^1, \lambda) \leqslant m$.*

Proof. Let E' be the image of $\mathbf{C} \otimes E \otimes \mathbf{C}$ in $\mathscr{Y}^1$. According to Lemma 11.2 there exists a finite subset $\vartheta \subset \hat{K}_F$ and a finite dimensional subspace $E'' \subset \mathscr{Y}^1$ such that $E' \subset \mathscr{Y}^1(\nu, \vartheta) \subset E''$. One readily verifies that $\pi_\nu^1(U(\mathfrak{m}_{1F}))\, E' = \mathscr{Y}^1$ for every $\nu \in \mathfrak{a}_{0c}^*$. Hence

$$\pi_\nu^1(U(\mathfrak{m}_{1F}))\, X(\nu, \vartheta) = X \qquad \text{for every} \quad \nu \in \mathfrak{a}_{0c}^*. \tag{84}$$

Since $\mathfrak{a}_F$ is centralized by M_{1F}, $\pi_\nu^1(\mathfrak{a}_F)$ leaves the space $X(\nu, \vartheta)$ invariant and by (84) it suffices to majorize the orders of the weights of $\pi_\nu^1 \mid \mathfrak{a}_F$ restricted to $X(\nu, \vartheta)$. Thus the result is valid with $m = \dim E''$. ∎

PROPOSITION 11.10. *If $k \geqslant 1$ then the weights of $\pi_\nu^k \mid \mathfrak{a}_F$ are all of the form $(w\nu - \rho_0) \mid \mathfrak{a}_F - \xi$, where $w \in W_0$, and where ξ can be written as a sum $\xi = \alpha_1 + \cdots + \alpha_l$ $(0 \leqslant l < k)$ of roots $\alpha_i \in \Sigma(\mathfrak{n}_F, \mathfrak{a}_F)$.*

Let $\mathscr{A}$ be a subset of $\mathfrak{a}_{0c}^*$ such that $\mathrm{Re}\, \mathscr{A}$ *is bounded. Then for every $\xi \in \mathbf{N}\Sigma(\mathfrak{n}_F, \mathfrak{a}_F)$ there exists a $d_\xi \geqslant 1$ such that for every $k \geqslant 1$, $w \in W_0$ one has*

$$o(\pi_\nu^k \mid \mathfrak{a}_F, (w\nu - \rho_0) \mid \mathfrak{a}_F - \xi) \leqslant d_\xi \qquad \textit{for all} \quad \nu \in \mathscr{A}.$$

Proof. The assertion about the set of weights is proved in [6, Lemma 1.2]. To get a bound on the order we shall inspect the argument given there. First we need some notations.

The adjoint representation induces a finite dimensional representation μ_k of M_{1F} in $\mathscr{M}_k := \bar{\mathfrak{n}}_F^k\, U(\bar{\mathfrak{n}}_F) / \bar{\mathfrak{n}}_F^{k+1} U(\bar{\mathfrak{n}}_F)$ $(k \geqslant 1)$. The set $\Lambda_k = \Lambda(\mu_k \mid \mathfrak{a}_F)$ of $\mathfrak{a}_F$-weights of this module equals

$$\Lambda_k = \{\alpha_1 + \cdots + \alpha_k;\ \alpha_i \in -\Sigma(\mathfrak{n}_F, \mathfrak{a}_F)\}.$$

Consider the natural exact sequences of $U(\bar{\mathfrak{n}}_F)$-modules

$$\mathscr{M}_k \otimes \mathscr{Y}^1 \xrightarrow{a_k} \mathscr{Y}^{k+1} \xrightarrow{b_k} \mathscr{Y}^k \longrightarrow 0,$$

as defined in [6]. They induce exact sequences of $(\mathfrak{m}_{1F}, K_F)$-modules

$$\mu_k \otimes \pi_\nu^1 \xrightarrow{a_k} \pi_\nu^{k+1} \xrightarrow{b_k} \pi_\nu^k \longrightarrow 0.$$

Since $\mathfrak{a}_F \subset \mathrm{centre}(\mathfrak{m}_{1F})$, these are also exact sequences of locally finite $\mathfrak{a}_F$-modules. Thus for any $\lambda \in \mathfrak{a}_{Fc}^*$ we have

$$o_\nu^k(\lambda) \leqslant o_\nu^{k+1}(\lambda) \leqslant o_\nu^k(\lambda) + o(\mu_k \otimes \pi_\nu^1 \mid \mathfrak{a}_F, \lambda);$$

here we have written $o_\nu^k(\lambda) = o(\pi_\nu^k \mid \mathfrak{a}_F, \lambda)$.

The action of $\mathfrak{a}_F$ on $\mathscr{M}_k$ is semisimple, so in view of Lemma 11.9 it follows that $o(\mu_k \otimes \pi_\nu^1 \mid \mathfrak{a}_F, \lambda) \leqslant m$. Hence

$$o_\nu^k(\lambda) \leqslant km \qquad \text{for all} \quad \nu \in \mathfrak{a}_{0c}^*, k \geqslant 1. \tag{85}$$

However, there is a better estimate since the sequence $o_\nu^k(\lambda)$ becomes stationary. Indeed let $\mathscr{A}$ be a subset of $\mathfrak{a}_{0c}^*$ such that $\mathrm{Re}\,\mathscr{A}$ is bounded, fix $w \in W_0$, $\xi \in N\Sigma(\mathfrak{n}_F, \mathfrak{a}_F)$, and write $\lambda_\nu = (w\nu - \rho_0) \mid \mathfrak{a}_F - \xi$. Then there exists a bounded subset $\mathscr{A}'$ of $\mathfrak{a}_F^*$ such that for all $\nu \in \mathscr{A}$ one has $\mathrm{Re}\,\lambda_\nu + (-\Lambda(\pi_\nu^1 \mid \mathfrak{a}_F)) \subset \mathscr{A}'$. Now fix k_0 such that $k \geqslant k_0 \Rightarrow \mathscr{A}' \cap \Lambda_k = \varnothing$. Then

$$o(\mu_k \otimes \pi_\nu^1 \mid \mathfrak{a}_F, \lambda_\nu) = 0 \qquad \text{for all} \quad k \geqslant k_0, \nu \in \mathscr{A}.$$

Hence $o_\nu^k(\lambda_\nu) = o_\nu^{k_0}(\lambda_\nu)$ for $k \geqslant k_0$, and combining this with (85) we conclude that $o_\nu^k(\lambda_\nu) \leqslant mk_0$ for all $\nu \in \mathscr{A}$, $k \geqslant 1$. Notice that $d_{\xi,w} = mk_0$ only depends on $\mathscr{A}$, w, and ξ. This proves the result with $d_\xi = \max_{w \in W_0} d_{\xi,w}$. ∎

In the rest of this section we shall investigate the structure of the family π_ν^1 of Harish-Chandra $(\mathfrak{m}_{1F}, K_F)$-modules in $\mathscr{Y}^1$ in more detail.

Let τ_ν be the representation of $\mathbf{D}(M_{1F}/K_F)$ in $V \subset \mathbf{D}(M_{1F}/K_F)$ defined above Lemma 2.4 in the case $\sigma = \theta$, $Q = \bar{P}_F$. (Notice the bar!) In particular the set of $\mathfrak{a}_F$-weights of τ_ν equals

$$\Lambda(\tau_\nu \mid \mathfrak{a}_F) = (W_0\nu - \rho_F) \mid \mathfrak{a}_F. \tag{86}$$

The family $(\tau_\nu; \nu \in \mathfrak{a}_{0c}^*)$ is polynomial. Hence we may apply the construction of a family of Harish-Chandra modules discussed in the first part of this section to the pair (M_{1F}, K_F) and the data $\Omega = \mathfrak{a}_{0c}^*$, $W = V$, $\chi_\nu = \tau_\nu$. Then $Z_\nu := Y_{\tau_\nu}$ is the $(\mathfrak{m}_{1F}, K_F)$-module given by

$$Z_\nu = U(\mathfrak{m}_{1F})/U(\mathfrak{m}_{1F})\mathfrak{k}_F \otimes_{\mathbf{D}(M_{1F}/K_F)} V_\nu.$$

Let ${}^*\bar{\mathfrak{n}}_F = \bar{\mathfrak{n}}_0 \cap \mathfrak{m}_{1F}$, and define

$$\mathscr{X} = U({}^*\bar{\mathfrak{n}}_F) \otimes E_F \otimes V.$$

Then the linear map $\psi_\nu : \mathscr{X} \to Z_\nu$ induced by $x \otimes e \otimes v \mapsto xe \otimes v$ is an isomorphism of $U({}^*\bar{\mathfrak{n}}_F)$-modules. By pull-back under ψ_ν we obtain a representation π_ν^F of $(\mathfrak{m}_{1F}, K_F)$ on $\mathscr{X}$. According to Proposition 11.5, $(\pi_\nu^F; \nu \in \mathfrak{a}_{0c}^*)$ is a polynomial family of Harish-Chandra $(\mathfrak{m}_{1F}, K_F)$-modules in $\mathscr{X}$.

Consider the linear map $\tilde{\beta}_\nu : U(\mathfrak{m}_{1F}) \otimes V \to U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} \otimes \mathbf{C}$ defined by $\tilde{\beta}_\nu(x \otimes v) = xv \otimes 1$ (here we view V as a subspace of $U(\mathfrak{m}_{1F})/U(\mathfrak{m}_{1F})\mathfrak{k}$).

LEMMA 11.11. *The map $\tilde{\beta}_\nu$ factorizes to a surjective homomorphism $\beta_\nu : Z_\nu \to Y_\nu^1$ of $(\mathfrak{m}_{1F}, K_F)$-modules.*

Proof. From the fact that K_F centralizes V viewed as a subspace of $U(\mathfrak{m}_{1F})/U(\mathfrak{m}_{1F})\mathfrak{k}_F$, it follows that $\tilde{\beta}_\nu$ is a homomorphism of $(\mathfrak{m}_{1F}, K_F)$-modules. Hence the induced map $\beta_\nu : U(\mathfrak{m}_{1F}) \otimes V \to Y_\nu^1$ is. From the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{m}_{1F}) \oplus (\bar{\mathfrak{n}}_F\, U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k})$$

we infer that β_ν maps $U(\mathfrak{m}_{1F})/U(\mathfrak{m}_{1F})\mathfrak{k}_F \otimes 1$ onto Y_ν^1, hence is an epimorphism. Using once more that K_F centralizes V, we see that β_ν maps $U(\mathfrak{m}_{1F})\mathfrak{k}_F \otimes V$ onto 0, so it remains to be shown that

$$\beta_\nu(D \otimes v) = \beta_\nu(1 \otimes \tau_\nu(D)v), \tag{87}$$

for $D \in \mathbf{D}(M_{1F}/K_F)$, $v \in V$. By (27) we may express Dv as a finite sum

$$Dv = \sum_i v_i\, {}'\mu(X_i), \tag{88}$$

with $v_i \in V$, $X_i \in \mathbf{D}(G/K)$. Here we have written μ for μ_{P_F}. On the other hand, $v_i\, {}'\mu(X_i) \cong v_i X_i$ modulo $\bar{\mathfrak{n}}_F(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k})$, hence

$$\begin{aligned}
[v_i\, {}'\mu(X_i) \otimes 1] &= [v_i X_i \otimes 1] \\
&= [v_i \otimes \chi_\nu(X_i)] \\
&= [\gamma_0(X_i : \nu) v_i \otimes 1],
\end{aligned} \tag{89}$$

where the brackets indicate that the images in Y_ν^1 are taken. By definition we have

$$\tau_\nu(D)v = \sum_i \gamma_0(X_i : \nu) v_i \tag{90}$$

(use (27) and Lemmas 2.2, 2.3). Combining (88), (89), and (90) we obtain $[Dv \otimes 1] = [\tau_\nu(D)v \otimes 1]$, hence (87). ∎

LEMMA 11.12. *For every $\nu \in \mathfrak{a}_{0c}^*$, the map $\beta_\nu : Z_\nu \to Y_\nu^1$ is an isomorphism of $(\mathfrak{m}_{1F}, K_F)$-modules.*

Proof. Let $\alpha_\nu : \mathscr{X} \to \mathscr{Y}^1$ be the map which makes the following diagram commutative:

$$\begin{array}{ccc} Z_\nu & \xrightarrow{\beta_\nu} & Y_\nu^1 \\ {\scriptstyle \psi_\nu}\uparrow & & \uparrow{\scriptstyle \varphi_\nu^1} \\ \mathscr{X} & \xrightarrow{\alpha_\nu} & \mathscr{Y}^1 \end{array}$$

Then α_ν is an epimorphism of $U(\bar{\mathfrak{n}}_F)$-modules, and it suffices to show that α_ν is injective. If $(z_i;\ 1 \leqslant i \leqslant m)$ is a linear basis for the finite dimensional complex linear space $E_F \otimes V$, then $(1 \otimes z_i;\ 1 \leqslant i \leqslant m)$ is a free basis for the free $U(\bar{\mathfrak{n}}_F)$-module $\mathscr{X}$. Therefore it suffices to show that $\alpha_\nu \mid \mathbf{C} \otimes E_F \otimes V$ is injective.

If $e \in E_F$, $v \in V$, then $\psi_\nu(1 \otimes e \otimes v) = [e \otimes v]$ (brackets denote canonical images in the appropriate quotients). Given $\eta \in \mathfrak{a}_{0c}^*$, define $T_\eta \in \mathrm{Aut}(S(\mathfrak{a}_0))$ by $T_\eta(X) = X + \eta(X)$ $(X \in \mathfrak{a}_0)$. Define ${}^*\rho_F \in \mathfrak{a}_0^*$ by ${}^*\rho_F(X) = (1/2)\,\mathrm{tr}(\mathrm{ad}(X) \mid \mathfrak{n}_F)$, $(X \in \mathfrak{a}_0)$. Moreover, write $\gamma_F = \gamma_{P_F}$, and $'\gamma_F = T_{{}^*\rho_F} \circ \gamma_F$. Then $ev \equiv e\,'\gamma_F(v)$ modulo $\bar{\mathfrak{n}}_F\, U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$. Hence

$$\beta_\nu \psi_\nu(1 \otimes e \otimes v) = [ev \otimes 1] = [e\,'\gamma_F(v) \otimes 1].$$

In view of Lemma 11.13 below we have

$$[e\,'\gamma_F(v) \otimes 1] = \varphi_\nu([1 \otimes e\,'\gamma_F(v) \otimes 1]).$$

Hence

$$\alpha_\nu(1 \otimes e \otimes v) = [1 \otimes e\,'\gamma_F(v) \otimes 1].$$

The injectivity of $\alpha_\nu \mid \mathbf{C} \otimes E_F \otimes V$ now follows by application of Lemma 11.13 combined with the observation that $E \to \mathscr{Y}^1$, $e \mapsto [1 \otimes e \otimes 1]$ is an injective linear map. ∎

LEMMA 11.13. *The linear map $E_F \otimes V \to S(\mathfrak{a}_0)$ determined by $e \otimes v \mapsto e\,'\gamma_F(v)$ is a bijection onto E.*

Proof. For $v \in V$ we have

$$'\gamma_F(v) = T_{{}^*\rho_F}(\gamma_F(v)) = T_{\rho_0}(\gamma_F(T_{\rho_F} v)).$$

Using (26) we see that $'\gamma_F$ is a bijection from V onto $T_{\rho_0}(E^F)$, and it suffices to prove that the multiplication map $E_F \otimes T_{\rho_0}(E^F) \to S(\mathfrak{a}_0)$ is a bijection onto E. Now this follows from (25) in view of the invariance of E_F and E under the automorphism T_{ρ_0}. ∎

COROLLARY 11.14. *The map* $D \mapsto [D \otimes 1]$, $V_\nu \to Y_\nu^1$ *is an injective morphism of* $U(\mathfrak{m}_{1F})^{K_F}$*-modules.*

Proof. Use that $[D \otimes 1] = \beta_\nu \psi_\nu(1 \otimes 1 \otimes D) = \varphi_\nu \alpha_\nu(1 \otimes 1 \otimes D)$. ∎

Let J_ν be the left ideal of $U(\mathfrak{g})$ generated by $U(\mathfrak{g})\mathfrak{k}$ and $D - \gamma_0(D : \nu)$, $D \in \mathbf{D}(G/K)$.

COROLLARY 11.15. *There exists a bilinear map* $y_\nu : \mathbf{D}(M_{1F}/K_F) \times V \to \bar{\mathfrak{n}}_F U(\bar{\mathfrak{n}}_0 \oplus \mathfrak{a}_0)$ *depending polynomially on* $\nu \in \mathfrak{a}_{0c}^*$, *such that*

$$Dv - \tau_\nu(D)v - y_\nu(D, v) \in J_\nu, \tag{91}$$

for all $D \in \mathbf{D}(M_{1F}/K_F)$, $v \in V$, *and* $\nu \in \mathfrak{a}_{0c}^*$.

Proof. Recall the definitions of $\mathbf{m} : \mathscr{Y} \to U(\mathfrak{g})$, $1_{\mathscr{Y}}$, and $p_\nu : U(\mathfrak{g}) \to \mathscr{Y}$ from the proof of Proposition 11.7. Then p_ν is zero on $U(\mathfrak{g})\mathfrak{k}$, hence it makes sense to define

$$\tilde{y}_\nu(D, v) = p_\nu(Dv - \tau_\nu(D)v).$$

The canonical image in $\mathscr{Y}^1$ equals

$$\begin{aligned}
[\tilde{y}_\nu(D, v)] &= \pi_\nu^1(Dv - \tau_\nu(D)v][1_{\mathscr{Y}}] \\
&= \varphi_\nu^1([(Dv - \tau_\nu(D)v) \otimes 1]) \\
&= \varphi_\nu^1 \circ \beta_\nu([D \otimes v - 1 \otimes \tau_\nu(D)v]) \\
&= 0.
\end{aligned}$$

Hence $\tilde{y}_\nu(D, v) \in \bar{\mathfrak{n}}_F \mathscr{Y}$, and it follows that

$$y_\nu(D, v) := \mathbf{m}(\tilde{y}_\nu(D, v))$$

belongs to $\bar{\mathfrak{n}}_F U(\bar{\mathfrak{n}}_0)E$. The assertion (91) now follows as in the proof of Proposition 11.7. ∎

12. ASYMPTOTICS OF EIGENFUNCTIONS

In this section we will analyze the asymptotic behaviour of joint eigenfunctions for $\mathbf{D}(G/H)$, using the methods of [33, 5, 6].

Let $\|\cdot\| : G \to [1, \infty[$ be the distance function defined in [6, p. 643] (see also [5, p. 112]). (Notice that in these papers $\mathfrak{a}_0$ is denoted by $\mathfrak{a}$.) As in [6] we define

$$\|f\|_r = \sup_{x \in G} \|x\|^{-r} |f(x)|,$$

for $r \in \mathbf{R}$ and any function $f: G \to \mathbf{C}$. The Banach space of continuous functions $f: G \to \mathbf{C}$ satisfying $\|f\|_r < \infty$ is denoted by $C_r(G)$. It is invariant under both the left regular representation L and the right regular representation R (cf. [5, (2.4-5)]). The Banach space of C^q-vectors for L in $C_r(G)$ is denoted by $C_r^q(G)$ and the Fréchet space of C^∞-vectors is denoted by $C_r^\infty(G)$. The norm on $C_r^q(G)$ is denoted by $\|\cdot\|_{q,r}$. In [6, p. 643] it is observed that the estimates (2.2-7) of [5] are valid.

The above function spaces are of importance for analysis on G/H for reasons to be explained shortly. Let $\|\cdot\|_\sigma$ be the distance function $G \to [1, \infty[$ defined by $\|x\|_\sigma^2 = \|x\sigma(x)^{-1}\|$. Then $\|\cdot\|_\sigma$ is right H-invariant and left K-invariant (use [5, Lemma 2.1]). Moreover, since $\|a^2\| = \|a\|^2$ for $a \in A_\mathfrak{q}$ we deduce that

$$\|kah\|_\sigma = \|a\| \qquad (k \in K, a \in A_\mathfrak{q}, h \in H). \tag{92}$$

LEMMA 12.1. *For every $x \in G$ we have*

$$\|x\| \geqslant \|x\|_\sigma.$$

Proof. Since $\|\cdot\|$ and $\|\cdot\|_\sigma$ are left K-invariant, we may factor out $K \cap \mathrm{centre}(G)$ and reduce to the case that $G \simeq G_1 \times \exp \mathfrak{a}_{0\Sigma}$, where G_1 is connected and semisimple and where

$$\mathfrak{a}_{0\Sigma} = \{X \in \mathfrak{a}_0;\ \alpha(X) = 0 \text{ for all } \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_0)\}$$

is contained in the centre of G. Let $X \in \mathfrak{a}_{0\Sigma}$. Then we may write $X = X_\mathfrak{q} + X_\mathfrak{h}$, where $X_\mathfrak{q} \in \mathfrak{a}_{0\Sigma} \cap \mathfrak{q}$ and $X_\mathfrak{h} \in \mathfrak{a}_{0\Sigma} \cap \mathfrak{h}$. Since $X_\mathfrak{q}$ and $X_\mathfrak{h}$ are orthogonal, we have that $|X_\mathfrak{q}| \leqslant |X|$. But for every $x \in G$ one has that

$$\|x \exp X\| = \|x\|\, e^{|X|} \qquad \text{and} \qquad \|x \exp X\|_\sigma = \|x\|_\sigma\, e^{|X_q|}.$$

Hence it suffices to prove the assertion for the case that G is connected and semisimple.

In view of the decomposition $G = KA_\mathfrak{q}H$ and the left K-invariance of both distance functions we may assume that $x = ah$ ($a \in A_\mathfrak{q}$, $h \in H$) and then we must show that

$$\|ah\| \geqslant \|a\|,$$

by (92). Now use [5, Lemma 14.4]. ∎

COROLLARY 12.2. *Let $r \geqslant 0$. Then for every $f \in C(G/H)$ we have that*

$$\|f\|_r = \sup_{x \in G} \|x\|_\sigma^{-r} |f(x)|.$$

Proof. In view of Lemma 12.1 we have that

$$\|f\|_r \leqslant \sup_{x \in G} \|x\|_\sigma^{-r} |f(x)|$$

for every $f \in C(G)$. If in addition f is right H-invariant, then for $x = kah$ with $k \in K$, $a \in A_{\mathfrak{q}}$, and $h \in H$ we have

$$\|x\|_\sigma^{-r} |f(x)| = \|ka\|^{-r} |f(ka)| \leqslant \|f\|_r$$

and the asserted equality follows. ∎

From Lemma 4.5, Proposition 10.3, and the above corollary we see that the components of Eisenstein integrals are $\mathbf{D}(G/H)$-finite functions in $C_r^\infty(G)$, for suitable r.

Let $\nu \in \mathfrak{b}_{\mathfrak{c}}^*$. Then we denote the space of functions $f \in C^\infty(G/H)$ satisfying the system of differential equations

$$Df = \gamma(D : \nu) f \qquad (D \in \mathbf{D}(G/H))$$

by $\mathscr{E}_\nu^\infty(G/H)$. If $r \in \mathbf{R}$, then the space

$$\mathscr{E}_{\nu,r}^\infty(G/H) = \mathscr{E}_\nu^\infty(G/H) \cap C_r^\infty(G)$$

is a closed subspace of $C_r^\infty(G)$, hence a Fréchet space.

The following lemma will be useful at a later stage.

LEMMA 12.3. *Let $f \in \mathscr{E}_\nu^\infty(G/H)$ be left K-finite. Then there exists a $r > 0$ such that $f \in \mathscr{E}_{\nu,r}^\infty(G/H)$.*

Proof. We use the techniques of [10, 2]. Define the μ-spherical function $F: G/H \to E$ as in [2, p. 248, Proof of Theorem 7.3]. Then F behaves finitely under the action of centre $U(\mathfrak{g})$. Moreover, $f = \eta \circ F$ for some $\eta \in E^*$. For every $P \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$ let $\mathscr{L}_P$ denote the (finite) set of P-leading exponents of F as defined in [2]. Moreover fix $\xi_P \in \mathfrak{a}_{\mathfrak{q}}^*$ such that

$$\nu \in \mathscr{L}_P \Rightarrow \operatorname{Re} \nu \leqslant \xi_P \qquad \text{on} \quad cl\,\mathfrak{a}_{\mathfrak{q}}^+(P).$$

Then according to [2, Theorem 6.1], thee exist constants $C > 0$, $m \in \mathbf{N}$ such that for each $P \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$ we have

$$\|F(a)\| \leqslant C a^{\xi_P} (1 + |\log a|)^m \qquad (a \in A_q^+(P)).$$

Let $u \in U(\mathfrak{g})$. Then using [2, Lemma 7.6] we infer that the same estimate (with C, m depending on u) holds for $L_u F$.

There exists a constant $r > 0$ such that for each $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$ and all $m \in \mathbf{N}$ the function $a \mapsto \|a\|^{-r} a^{\xi_P}(1 + |\log a|)^m$ is bounded on $A_\mathfrak{q}^+(P)$. It follows that for every $u \in U(g)$ there exists a constant $C > 0$ such that $\|L_u F(a)\| \leqslant C\, \|a\|^r$ for all $a \in A_\mathfrak{q}$. Using the decomposition $G = KA_\mathfrak{q}H$ and the fact that F is left K-spherical we finally conclude that for every $u \in U(\mathfrak{g})$ we have an estimate

$$\|L_u F(x)\| \leqslant C_u\, \|x\|_\sigma^r \qquad (x \in G),$$

with $C_u > 0$ a constant depending on u. In view of Corollary 12.2 this implies that $f \in C_r^\infty(G)$. ∎

Let $\Lambda \in \mathfrak{b}_{\mathfrak{k}\mathrm{c}}^*$ be fixed from now on, and let λ denote a variable in $\mathfrak{a}_{\mathfrak{q}\mathrm{c}}^*$. Let $Q \in \mathscr{P}_\sigma$ be fixed (cf. Section 1), and write

$$\mathfrak{a}_{Q\mathfrak{q}}^+ = \{X \in \mathfrak{a}_{Q\mathfrak{q}};\ \alpha(X) > 0 \text{ for all } \alpha \in \Sigma(Q)\}.$$

We shall investigate the asymptotic behaviour of a function $f \in \mathscr{E}_{\Lambda+\lambda, r}^\infty(G/H)$ along $\mathfrak{a}_{Q\mathfrak{q}}^+$.

Without loss of generality we may assume that $\Sigma(Q)$ is compatible with Σ^+. We recall the duality of Section 2 and select a system Σ_0^{d+} of positive roots for $\Sigma_0^d = \Sigma(\mathfrak{g}^d, \mathfrak{a}_0^d) = \Sigma(\mathfrak{b})$. Let Δ_0^d denote the set of simple roots in Σ_0^{d+}. Denoting parabolic subalgebras with German capitals we have that $\mathfrak{Q}_\mathrm{c} \cap \mathfrak{g}^d = \mathfrak{P}_F^d$ for the finite subset $F \subset \Delta_0^d$ of roots α with $\alpha \mid \mathfrak{a}_{Q\mathfrak{q}} = 0$ (cf. also [4, Sect. 2]). Let G^d be any connected real reductive group of Harish-Chandra's class with Lie algebra $\mathfrak{g}^d$, let K^d be the analytic subgroup with Lie algebra $\mathfrak{k}^d$, and let P_F^d be the normalizer of $\mathfrak{P}_F^d$ in G^d. Put

$$X_Q(\Lambda, \lambda) = \{0\} \cup \{\nu \mid \mathfrak{a}_{Q\mathfrak{q}};\ \nu \in W(\mathfrak{b})(\Lambda + \lambda) - \rho_Q + [-\mathbf{N}\Sigma(Q)]\},$$

and fix $k \geqslant 1$. Then applying Propositions 11.7, 11.8, and 11.10 to $\mathfrak{g}^d$, K^d, P_F^d, and the parameter $\nu = \Lambda + \lambda \in \mathfrak{b}_\mathrm{c}^* = \mathfrak{a}_{0\mathrm{c}}^{d*}$ we infer the existence of a finite dimensional linear subspace $\mathscr{V}_k \subset U(\bar{\mathfrak{n}}_F^d \oplus \mathfrak{m}_{1F}^d) = U(\bar{\mathfrak{n}}_Q \oplus \mathfrak{m}_{1Q})$, containing 1, and such that the following holds.

PROPOSITION 12.4. *There exist*

(1) *an endomorphism* $x_\lambda \in \mathrm{End}(\mathscr{V}_k)$, *depending polynomially on* $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathrm{c}}^*$, *and such that* $x_\lambda(1) = 1$ *for all* $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathrm{c}}^*$;

(2) *an algebra homomorphism* $b_k(\lambda, \cdot)$ *from* $U(\mathfrak{m}_{1Q})^{\mathfrak{b}_Q}$ *into* $\mathrm{End}(\mathscr{V}_k)$, *depending polynomially on* $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathrm{c}}^*$; *and*

(3) *a bilinear map* $y_\lambda : U(\mathfrak{m}_{1Q})^{\mathfrak{h}_Q} \times \mathscr{V}_k \to \bar{\mathfrak{n}}_Q^k U(\bar{\mathfrak{n}})\, U(\mathfrak{m}_1)$, *depending polynomially on* $\lambda \in \mathfrak{a}_{\mathfrak{qc}}^*$,

such that for all $\lambda \in \mathfrak{a}_{\mathfrak{qc}}^*$, $D \in U(\mathfrak{m}_{1Q})^{\mathfrak{h}_Q}$, *and* $v \in \mathscr{V}_k$ *we have*

$$Dx_\lambda(v) \equiv x_\lambda(b_k(\lambda, D)v) + y_\lambda(D, v) \qquad \mathrm{mod}\ J_{\Lambda+\lambda},$$

where $J_{\Lambda+\lambda}$ *denotes the left ideal in* $U(\mathfrak{g})$ *generated by* $\mathfrak{h}$ *and*

$$\{D - \gamma(D : \Lambda + \lambda);\, D \in U(\mathfrak{g})^{\mathfrak{h}}\}.$$

Moreover,

$$\Lambda(b_k(\lambda, \cdot) \mid \mathfrak{a}_{Q\mathfrak{q}}) \subset X_Q(\Lambda, \lambda),$$

and there exists a locally bounded function $d : [0, \infty[\to \mathbf{N}$ *such that for all* $\lambda \in \mathfrak{a}_{\mathfrak{qc}}^*$, $\xi \in \Lambda(b_k(\lambda, \cdot) \mid \mathfrak{a}_{Q\mathfrak{q}})$ *we have*

$$o(b_k(\lambda, \cdot), \xi) \leqslant d(|\mathrm{Re}\,\lambda| + |\mathrm{Re}\,\xi|).$$

Define the function $\beta_Q : \mathfrak{a}_{Q\mathfrak{q}} \to \mathbf{R}$ by

$$\beta_Q(X) = \min\{\alpha(X);\, \alpha \in \Sigma(Q)\},$$

and fix $r \in \mathbf{R}$. Then the following lemma is proved in the same fashion as Lemma 6.2 in [5].

LEMMA 12.5. *Let* $k \in \mathbf{N}$, *and put*

$$\gamma(X) = |r|\, c_2\, |X| - k\beta_Q(X), \tag{93}$$

for $X \in \mathfrak{a}_{Q\mathfrak{q}}$, *where* c_2 *is the constant of* [5, *Lemma* 2.1(iv)].

For each $y \in \bar{\mathfrak{n}}_Q^k U(\bar{\mathfrak{n}}_Q + \mathfrak{m}_{1Q})$ *there exist constants* $q \in \mathbf{N}$, $r' \geqslant r$, *and* $C > 0$ *such that for all* $X \in \mathfrak{a}_{Q\mathfrak{q}}^+$ *we have*

$$\|R_{\exp X} R_y f\|_{r'} \leqslant C\, \|f\|_{q,r}\, e^{\gamma(X)}$$

for $f \in C_r^q(G)$.

We now have the following version of [5, Proposition 6.1], but along $\mathfrak{a}_{Q\mathfrak{q}}$. Fix $\lambda_0 \in \mathfrak{a}_{\mathfrak{qc}}^*$, $X_0 \in \mathfrak{a}_{Q\mathfrak{q}}^+$, and $r \in \mathbf{R}$. If A_1 and A_2 are Banach spaces, we write $B(A_1, A_2)$ for the space of bounded linear maps from A_1 into A_2.

PROPOSITION 12.6. *There exist, for each* $N \in \mathbf{R}$

(a) *open neighbourhoods* Ω *of* λ_0 *in* $\mathfrak{a}_{\mathfrak{qc}}^*$ *and* U *of* X_0 *in* $\mathfrak{a}_{Q\mathfrak{q}}^+$;

(b) *constants* $k, q \in \mathbf{N}$, $r' \geqslant r$, *and* $C, \varepsilon > 0$;

(c) *a continuous map* $\Psi\colon \Omega \times U \to B(C_r^q(G), \mathscr{V}_k^* \otimes C_{r'}(G))$, *holomorphic in its first variable*; *and*

(d) *an element* $\eta \in \mathscr{V}_k^{**}$

such that

(i) $\Psi(\lambda, X)$ *intertwines the left actions of G on* $C_r^q(G)$ *and* $C_{r'}(G)$, *for all* $(\lambda, X) \in \Omega \times U$, *and*

(ii) *for every* $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ *and every* $f \in \mathscr{E}_{\Lambda+\lambda}^\infty(G/H) \cap C_r^q(G)$ *we have that*

$$\|R_{\exp tX} f - (\eta \circ \exp[b_k(\lambda, X)^*] \otimes 1)\, \Psi(\lambda, X) f\|_{r'} \leqslant C\, \|f\|_{q,r}\, e^{(N-\varepsilon)t}$$

for all $X \in U$ *and* $t \geqslant 0$.

Remark 12.7. It should be noted that the formulation of Proposition 6.1 in [5] is not entirely correct. It becomes correct if one replaces $\mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y}$ by its dual in (c) and (d), and $\tau_\lambda^k(tH)$ by its adjoint in (ii). The erroneous formulation has no consequences for the applications in the paper because the eigenvalues of $\tau_\lambda^k(tH)$ are the same as those of its adjoint (counting multiplicities). A similar error has been made in the formulation of Proposition 1.3 in [6], but again this has no consequences for the other results in the paper.

Proof of Proposition 12.6. Fix $N \in \mathbf{R}$, and select $k \in \mathbf{N}$ such that $\gamma(X_0) < N$; here γ is given by (93). Let $S(\lambda)$ denote the set of weights of the representation $\tau_\lambda^k = b_k(\lambda, \cdot) \mid \mathfrak{a}_{Q\mathfrak{q}}$ of $\mathfrak{a}_{Q\mathfrak{q}}$ in $\mathscr{V}_k$. Then $S(\lambda) \subset X_Q(\Lambda, \lambda)$. Following [5] we split the set $S(\lambda)$ into two parts. Fix $\varepsilon > 0$ such that $\gamma(X_0) + \varepsilon < N$ and such that for $\xi \in S(\lambda_0)$ we have

$$\mathrm{Re}\, \xi(X_0) \notin [N - 2\varepsilon, N[.$$

Next fix a relatively compact connected open neighbourhood U of X_0 in $\mathfrak{a}_{Q\mathfrak{q}}^+$ such that

$$\gamma(X) + \varepsilon < N$$

and

$$\mathrm{Re}\, \xi(X) \notin [N - 2\varepsilon, N - \tfrac{1}{2}\varepsilon] \tag{94}$$

for $X \in U$ and $\xi \in S(\lambda_0)$. Finally fix a connected bounded open neighbourhood Ω of λ in $\mathfrak{a}_{\mathfrak{q}c}^*$ such that (94) holds for $\lambda \in \Omega$, $\xi \in S(\lambda)$, and $X \in U$. Then for $\lambda \in \Omega$, the set $S(\lambda)$ is a disjoint union of the subsets $S_\pm(\lambda)$ defined by

$$\xi \in S_+(\lambda) \Leftrightarrow \qquad \mathrm{Re}\, \xi(X) > N - \tfrac{1}{2}\varepsilon,\ \forall X \in U,$$

$$\xi \in S_-(\lambda) \Leftrightarrow \qquad \mathrm{Re}\, \xi(X) < N - 2\varepsilon,\ \forall X \in U.$$

Still following [5] we let $V_{\pm}(\lambda)$ denote the sums of the corresponding generalized weight spaces for τ_λ^k, and $E_{\pm}(\lambda)$ the projection onto $V_{\pm}(\lambda)$ along $V_{\mp}(\lambda)$ (in [5] the analogous projection operators are denoted by $Q_{\pm}(\lambda)$). Then $E_{\pm}(\lambda)$ depend holomorphically on λ (use [5, Proposition 5.8] or Lemma 20.1 of the present paper). If necessary we shrink Ω such that the operator norms of $E_{\pm}(\lambda)$ are uniformly bounded for $\lambda \in \Omega$.

From Lemma 12.5 we now infer that there exist numbers $q \in \mathbf{N}$ and $r' \geqslant r$, and constants $C, c > 0$ such that

$$\|R_{\exp tX} R(x_\lambda(v)) f\|_{r'} \leqslant C\,|v|\,\|f\|_{q,r}\, e^{ct}, \tag{95}$$

$$\|R_{\exp tX} R(y_\lambda(X, v)) f\|_{r'} \leqslant C\,|v|\,\|f\|_{q,r}\, e^{\gamma(X)t}, \tag{96}$$

for all $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$, $X \in U$, $t \geqslant 0$, and $v \in \mathscr{V}_k$. The first of the above inequalities follows from Lemma 12.5 with $k = 0$, since $\mathscr{V}_k$ is a subset of $U(\bar{\mathfrak{n}}_Q)\, U(\mathfrak{m}_{1Q})$.

We now define bounded linear maps $F_\lambda(X, t)$ and $G_\lambda(X, t)$ from $C_r^q(G)$ into $C_{r'}(G) \otimes \mathscr{V}_k^*$ by

$$\langle F_\lambda(X, t) f, v\rangle = R_{\exp tX} R(x_\lambda(v)) f,$$

$$\langle G_\lambda(X, t)\, d, v\rangle = R_{\exp tX} R(y_\lambda(X, v)) f.$$

The main difference with [5] is that we have not introduced a basis, and that F depends on the parameter λ. The operator norms of F and G satisfy the following estimates, analogous to (6.5-6) in [5]:

$$\|F_\lambda(X, t)\| \leqslant Ce^{ct}$$

and

$$\|G_\lambda(X, t)\| \leqslant Ce^{\gamma(X)t}$$

for all $\lambda \in \Omega$, $X \in U$, and $t \geqslant 0$.

As in [5] the reason for these definitions is that if $f \in \mathscr{E}^\infty_{\Lambda+\lambda}(G/H)$ then by Proposition 12.4 we have that

$$R_X R(x_\lambda(v)) f = R(x_\lambda(b_\lambda(X)) v)) f + R(y_\lambda(X, v)) f,$$

for $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$, $X \in \mathfrak{a}_{Q\mathrm{q}}$, and $v \in \mathscr{V}_k$. Now put

$$B(\lambda, X) = b_k(\lambda, X)^*.$$

(In [5] the matrix $B(\lambda, H)$ should have been defined as the transpose of the matrix of $\tau_\lambda^k(H)$, in order that (6.7) be valid.)

We obtain the $C_{r'}(G)$-valued differential equation

$$\frac{d}{dt} F_\lambda(X, t) f = [B(\lambda, X)\, F_\lambda(X, t) + G_\lambda(X, t)] f$$

for every $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, $f \in \mathscr{E}^\infty_{\Lambda+\lambda}(G/H) \cap C^q_r(G)$, and all $X \in \mathfrak{a}_{Q\mathfrak{q}}$, $t \in \mathbf{R}$. The proof is now completed in the same manner as the proof of Proposition 6.1 in [5]. Here the map Ψ is given by

$$\Psi(\lambda, X) = E_+(\lambda)\, F_\lambda(X, 0) + \int_0^\infty E_+(\lambda)\, e^{-sB(\lambda, X)} G_\lambda(X, s)\, ds,$$

for $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ and $X \in U$. Moreover, η is the image of $1 \in \mathscr{V}_k$ under the canonical isomorphism $\mathscr{V}_k \simeq \mathscr{V}_k^{**}$. ∎

Let

$$\mathscr{E}^\infty_{\Lambda+\lambda,*}(G/H) = \bigcup_{r \in \mathbf{R}} \mathscr{E}^\infty_{\Lambda+\lambda,r}(G/H).$$

Then we have the following generalization of [5, Theorem 3.5] (see also [6, Theorem 1.5]). If V is a finite dimensional real vector space, and $m \in \mathbf{N}$, then we denote by $P_m(V)$ the space of polynomial functions $V \to \mathbf{C}$ of degree at most m. Let $d: [0, \infty[\to \mathbf{N}$ be the locally bounded function of Proposition 12.4.

THEOREM 12.8. *Let $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$.*

(i) *Let $f \in \mathscr{E}^\infty_{\Lambda+\lambda,*}(G/H)$, $x \in G$. Then there exist unique polynomials $p_{\lambda,\xi}(Q \mid f, x)$ on $\mathfrak{a}_{Q\mathfrak{q}}$ of degree at most $d(|\mathrm{Re}\,\lambda| + |\mathrm{Re}\,\xi|)$, for $\xi \in X_Q(\Lambda, \lambda)$, such that*

$$f(x \exp tX) \sim \sum_{\xi \in X_Q(\Lambda, \lambda)} p_{\lambda,\xi}(Q \mid f, x, tX)\, e^{t\xi(X)} \qquad (t \to \infty) \tag{97}$$

at every $X_0 \in \mathfrak{a}_{Q\mathfrak{q}}^+$.

(ii) *Let $r \in \mathbf{R}$, $\xi \in X_Q(\Lambda, \lambda)$, and put $d = d(|\mathrm{Re}\,\lambda| + |\mathrm{Re}\,\xi|)$. Then there exists $r' \in \mathbf{R}$ such that $f \mapsto p_{\lambda,\xi}(Q \mid f)$ is a continuous linear map from $\mathscr{E}^\infty_{\Lambda+\lambda,r}(G/H)$ into $C^\infty_{r'}(G) \otimes P_d(\mathfrak{a}_{Q\mathfrak{q}})$, equivariant for the left regular actions of G on $\mathscr{E}^\infty_{\Lambda+\lambda,r}(G/H)$ and $C^\infty_{r'}(G)$.*

Proof. By the same arguments as in [5, p. 129], it follows from Proposition 12.6 that for each $\xi \in X_Q(\Lambda, \lambda)$ there exists a unique continuous function $p_{\lambda,\xi}(Q \mid f, x)$ on $\mathfrak{a}_{Q\mathfrak{q}}^+$ which is radially polynomial of degree $\leqslant d$, such that (97) holds, at every $X_0 \in \mathfrak{a}_{Q\mathfrak{q}}^+$. Let $r \in \mathbf{R}$ and $\xi \in X_Q(\Lambda, \lambda)$. Then given $X_0 \in \mathfrak{a}_{Q\mathfrak{q}}^+$ there exists a relatively compact open neighbourhood U of X_0 in $\mathfrak{a}_{Q\mathfrak{q}}^+$ such that

$$(f, X) \mapsto p_{\lambda,\xi}(Q \mid f, \cdot, X) \tag{98}$$

is a continuous map from $\mathscr{E}^\infty_{\Lambda+\lambda,r} \times U$ into $C^\infty_{r'}(G)$, which is linear in its first variable, and equivariant for the left regular actions. It remains to be shown that (98) is polynomial of degree $\leqslant d$ in its second variable X. Via restriction we identify $P_d(\mathfrak{a}_{Q\mathfrak{q}})$ with a finite dimensional hence closed subspace of the Fréchet space $C(U)$. Then by equivariance it suffices to show that the function

$$q(f) = p_{\lambda,\xi}(Q \mid f, e, \cdot) \in C(U)$$

belongs to $P_d(\mathfrak{a}_{Q\mathfrak{q}})$, for every $f \in \mathscr{E}^{\infty}_{\Lambda+\lambda,r}(G/H)$. By density and continuity it suffices to prove this for left K-finite $f \in \mathscr{E}^{\infty}_{\Lambda+\lambda,r}(G/H)$. But for such f it follows from the (converging) asymptotic expansions in [2] and by uniqueness of asymptotics that each function $p_{\lambda,\eta}(Q \mid f, e, \cdot)$ is a polynomial, hence $q(f) \in P(\mathfrak{a}_{Q\mathfrak{q}})$. From the already established fact that $t \mapsto q(f)(tX)$ is polynomial of degree $\leqslant d$ for every $X \in U$ it finally follows that $q(f) \in P_d(\mathfrak{a}_{Q\mathfrak{q}})$. ∎

We also have a generalization of [5, Theorem 3.6], for holomorphic families of eigenfunctions.

Following [5] we say that a map φ from an open subset Ω of $\mathbf{C}^n$ into $C^{\infty}_r(G)$ is holomorphic if for each $q \in \mathbf{N}$ it maps Ω holomorphically into the Banach space $C^q_r(G)$. Equivalently, this means that for every $u \in U(\mathfrak{g})$ the map $L_u \circ \varphi$ maps Ω holomorphically into $C_r(G)$.

Let Ω_0 be an open subset of $\mathfrak{a}^*_{\mathfrak{q}c}$. If f is a function $\Omega_0 \times G/H \to \mathbf{C}$, then given $\lambda \in \Omega_0$ we shall write f_λ for the function $G/H \to \mathbf{C}$, $x \mapsto f(\lambda, x)$. We define

$$\mathscr{E}_*(G/H, \Lambda, \Omega_0)$$

to be the space of C^{∞}-functions $f: \Omega_0 \times G/H \to \mathbf{C}$ such that

(1) for every $\lambda \in \Omega_0$ the function f_λ belongs to $\mathscr{E}^{\infty}_{\Lambda+\lambda,*}(G/H)$, and

(2) for every $\lambda_0 \in \Omega_0$ there exists a constant $r \in \mathbf{R}$ such that $\lambda \mapsto f_\lambda$ maps a neighbourhood of λ_0 holomorphically into $C^{\infty}_r(G)$.

We now have the following generalization of [5, Theorem 3.6].

THEOREM 12.9. *Let* $f \in \mathscr{E}_*(G/H, \Lambda, \Omega_0)$, *and fix* $\lambda_0 \in \Omega_0$ *and* $\xi_0 \in X_Q(\Lambda, \lambda)$. *Let* $\Xi(\lambda)$ *be the union of the set* $\{0\} \cap \{\xi_0\}$ *with the set of*

$$w(\Lambda + \lambda) \mid \mathfrak{a}_{Q\mathfrak{q}} - \rho_Q - \mu \qquad (w \in W(\mathfrak{b}), \mu \in \mathbf{N}\Sigma(Q))$$

such that

$$w(\Lambda + \lambda_0) \mid \mathfrak{a}_{Q\mathfrak{q}} - \rho_Q - \mu = \xi_0.$$

Then there exists an open neighbourhood $\Omega \subset \Omega_0$ of λ_0 and a constant $r' \in \mathbf{R}$ such that the map

$$(\lambda, X) \mapsto \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(Q \mid f_\lambda, \cdot, X)\, e^{\xi(X)}$$

is continuous from $\Omega \times \mathfrak{a}_{Q\mathfrak{q}}$ into $C_{r'}^\infty(G)$, and in addition holomorphic in λ.

Proof. The proof is essentially the same as the proof of Theorem 3.6 in [5] at the bottom of p. 129. ∎

13. Properties of the Coefficients

The purpose of this section is to investigate properties of the coefficients $p_{\lambda,\xi}(Q \mid f)$ in the asymptotic expansion (97). Here $Q \in \mathscr{P}_\sigma$. We will show that the coefficients satisfy certain differential equations. When $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$ these will allow us to limit the set of exponents.

We start with some simple transformation properties.

Lemma 13.1. *Let $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, $f \in \mathscr{E}_{\Lambda+\lambda,*}^\infty(G/H)$, and $\xi \in X_Q(\Lambda, \lambda)$. Then*

$$p_{\lambda,\xi}(Q \mid f, xma, X) = p_{\lambda,\xi}(Q \mid f, x, X + \log a)\, a^\xi$$

for all $x \in G$, $m \in M_{1Q} \cap H$, and $a \in A_{Q\mathfrak{q}}$.

Proof. The proof is essentially the same as the proof of [5, Lemma 8.5]. ∎

Next we will show that the coefficients are related by recurrence relations. Recall from Section 2 the definition of the algebra homomorphism $\mu'_Q: \mathbf{D}(G/H) \to \mathbf{D}(M_{1Q}/H_{1Q})$. It is well known that $\mu'_Q = {}'\mu_Q$. Let $D \in \mathbf{D}(G/H)$, and let u be an element of $U(\mathfrak{g})^H$ whose canonical image in $\mathbf{D}(G/H)$ equals D. Then there exists a $w \in \bar{\mathfrak{n}}_Q U(\bar{\mathfrak{n}}_Q + \mathfrak{m}_{1Q})$ such that

$$u - \mu'_Q(D) \in w + U(\mathfrak{g})\mathfrak{h}.$$

The element w can be written as a finite sum $w = \sum_i w_i$, with $w_i \in U(\bar{\mathfrak{n}}_Q + \mathfrak{m}_{1Q})$ such that $\operatorname{ad}(\mathfrak{a}_{Q\mathfrak{q}})$ acts on w_i by a non-zero weight $-\mu_i$, with $\mu_i \in \mathbf{N}\Sigma(Q)$.

Proposition 13.2. *Let $D \in \mathbf{D}(G/H)$, and u, w_i as above. Then*

$$[\mu'_Q(D) - \gamma(D : \Lambda + \lambda)]\, p_{\lambda,\xi}(Q \mid f, \cdot, X) = \sum_i w_i\, p_{\lambda,\xi+\mu_i}(Q \mid f, \cdot, X),$$

for all $f \in \mathscr{E}^{\infty}_{\Lambda+\lambda,*}(G/H)$, $\xi \in X_Q(\Lambda, \lambda)$, *and* $X \in \mathfrak{a}_{Q\mathfrak{q}}$. *Here we have adopted the convention that* $p_{\lambda,\eta} = 0$ *if* $\eta \notin X_Q(\Lambda, \lambda)$.

Proof. Proceed as in the proof of [6, Proposition 2.1]. ∎

We define the partial order $\preccurlyeq_Q$ on $\mathfrak{a}^*_{Q\mathfrak{q}c}$ by

$$\eta_1 \preccurlyeq_Q \eta_2 \Leftrightarrow \eta_2 - \eta_1 \in \mathbf{N}\Sigma(Q).$$

Let $f \in \mathscr{E}^{\infty}_{\Lambda+\lambda,*}(G/H)$. Then an element $\eta \in X_Q(\Lambda, \lambda)$ will be called an exponent of f along Q if $p_{\lambda,\eta}(Q \mid f, \cdot)$ is not identically zero. The set of exponents of f along Q is denoted by $\mathscr{E}(Q \mid f)$. The $\preccurlyeq_Q$-minimal elements of $\mathscr{E}(Q \mid f)$ are called the leading exponents of f along Q; the set of these leading exponents is denoted by $\mathscr{E}_L(Q \mid f)$. The following is now obvious.

COROLLARY 13.3. *Let* $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$, *and* $f \in \mathscr{E}^{\infty}_{\Lambda+\lambda,*}(G/H)$. *If* ξ *is a leading exponent of* f *along* Q, *then the function* $\varphi \in C^{\infty}(M_{1Q})$ *defined by*

$$\varphi(m) = p_{\lambda,\xi}(Q \mid f, m, 0)$$

is right H_{1Q}*-invariant and satisfies the system of differential equations*

$$\mu'_Q(D)\varphi = \gamma(D : \Lambda + \lambda)\varphi \qquad (D \in \mathbf{D}(G/H)).$$

Our next objective is to solve the above system when Q is a minimal $\sigma\theta$-stable parabolic subgroup, for generic values of λ. Thus from now on we assume that $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$. Then $M_{1Q} = M_1$, and

$$\mathfrak{a}^+_{Q\mathfrak{q}} = \mathfrak{a}^+_\mathfrak{q}(Q)$$

is a Weyl chamber in $\mathfrak{a}_\mathfrak{q}$ for the root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_\mathfrak{q})$. The set $\Sigma(Q)$ is the associated system of positive roots for Σ. Fix a system $\Sigma^+_\mathbf{M}$ of positive roots for $\Sigma_\mathbf{M} = \Sigma(\mathfrak{m}_1, \mathfrak{b})$, and let $\rho_\mathbf{M}$ be half the sum of the positive roots, counting multiplicities. Recall the definition of the set $L \subset i\mathfrak{b}^*_\mathfrak{k}$ above Proposition 4.7. This set being a lattice, we may fix a basis $\mathscr{B}$ for the real linear space $i\mathfrak{b}^*_\mathfrak{k}$ such that $\langle \mu, \beta \rangle \in \mathbf{Z}$ for all $\mu \in L$, $\beta \in \mathscr{B}$.

Let $\Lambda \in i\mathfrak{b}^*_\mathfrak{k}$. Then for every $p = (w, \beta) \in W(\mathfrak{b}) \times \mathscr{B}$ and $\mu \in L + \rho_\mathbf{M}$ we define

$$\mathscr{H}_{p,\mu} := \{\lambda \in \mathfrak{a}^*_{\mathfrak{q}c};\ \langle \lambda, w^{-1}\beta \rangle = \langle \mu + \rho_Q - w\Lambda, \beta \rangle\}.$$

If p belongs to the set Π of pairs $(w, \beta) \in W(\mathfrak{b}) \times \mathscr{B}$ such that $w^{-1}\beta \mid \mathfrak{a}_\mathfrak{q} \neq 0$, then $\mathscr{H}_{p,\mu}$ is a hyperplane in $\mathfrak{a}^*_{\mathfrak{q}c}$. Let $\mathscr{H}^1_\Lambda$ denote the (locally finite) union of the hyperplanes $\mathscr{H}_{p,\mu}$, $p \in \Pi$, $\mu \in L + \rho_\mathbf{M}$.

If $\alpha \in \Sigma(\mathfrak{b})$ and $\alpha \mid \mathfrak{a}_\mathfrak{q} \neq 0$, define the hyperplane

$$\mathscr{H}^2_\alpha := \{\lambda \in \mathfrak{a}^*_{\mathfrak{q}c};\ \langle \Lambda + \lambda, \alpha \rangle = 0\}$$

in $\mathfrak{a}_{\mathfrak{qc}}^*$. Let $\mathscr{H}_\Lambda^2$ be the finite union of these hyperplanes, and let $\mathfrak{a}_{\mathfrak{qc}}^{*\prime}$ denote the set of regular points in $\mathfrak{a}_{\mathfrak{qc}}^*$. Then

$$\mathfrak{a}_{\mathfrak{qc}}^{*\prime}(\Lambda) = \mathfrak{a}_{\mathfrak{qc}}^{*\prime} \backslash (\mathscr{H}_\Lambda^1 \cup \mathscr{H}_\Lambda^2) \tag{99}$$

is the complement of a locally finite union of hyperplanes.

LEMMA 13.4. *Let $\lambda \in \mathfrak{a}_{\mathfrak{qc}}^{*\prime}(\Lambda)$, $\mu \in L + \rho_{\mathbf{M}}$, and $w \in W(\mathfrak{b})$.*

(1) *If $w(\Lambda + \lambda) - \rho_Q - \mu \in \mathfrak{a}_{\mathfrak{qc}}^*$ then w normalizes $\mathfrak{a}_{\mathfrak{q}}$.*

(2) *If w centralizes $\Lambda + \lambda$ then w centralizes $\mathfrak{a}_{\mathfrak{q}}$.*

Proof. The hypothesis of (1) implies that $\lambda \in \mathscr{H}_{(\beta, w)\mu}$ for all $\beta \in \mathscr{B}$. In view of the condition on λ this can only be true if $w^{-1}\beta \mid \mathfrak{a}_{\mathfrak{q}} = 0$ for every $\beta \in \mathscr{B}$, or equivalently if w normalizes $\mathfrak{b}_{\mathfrak{k}}$. Hence w normalizes $\mathfrak{b}_{\mathfrak{k}}$'s Killing orthocomplement $\mathfrak{a}_{\mathfrak{q}} \cap \mathfrak{g}_1$ in $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. It follows that w normalizes $\mathfrak{a}_{\mathfrak{q}}$.

Now assume that $w \in W(\mathfrak{b})$ centralizes $\Lambda + \lambda$. Then w is a product of reflections s_α, with $\langle \alpha, \Lambda + \lambda \rangle = 0$. Since $\lambda \notin \mathscr{H}_\Lambda^2$ the latter condition implies that $\alpha \mid \mathfrak{a}_{\mathfrak{q}} = 0$, hence each reflection s_α centralizes $\mathfrak{a}_{\mathfrak{q}}$. ∎

PROPOSITION 13.5. *Let $\Lambda \in \mathfrak{b}_{\mathfrak{kc}}^*$, $\lambda \in \mathfrak{a}_{\mathfrak{qc}}^{*\prime}(\Delta)$. Then for every solution $\varphi \in C^\infty(M_1/H_{\mathbf{M}_1})$ of the system of differential equations*

$$\mu_Q'(D)\varphi = \gamma(D : \Lambda + \lambda)\varphi \qquad (D \in \mathbf{D}(G/H)) \tag{100}$$

there exist unique functions $\varphi_w \in C^\infty(M/H_{\mathbf{M}})$, $w \in W = W(\mathfrak{g}, \mathfrak{a}_{\mathfrak{q}})$, such that

$$\varphi(m \exp X) = \sum_{w \in W} \varphi_w(m)\, e^{(w\lambda - \rho_Q)(X)}, \tag{101}$$

for $m \in M$, $X \in \mathfrak{a}_{\mathfrak{q}}$. Moreover, if $w \in W$ and $s \in W(\mathfrak{g}, \mathfrak{j})$ is as in Lemma 4.6, then

$$D\varphi_w = \gamma_{\mathbf{M}}(D : s\Lambda)\varphi_w \qquad (D \in \mathbf{D}(M/H_{\mathbf{M}})). \tag{102}$$

Proof. Let $\mathscr{E}(M_1, \Lambda)$ denote the space of functions $\varphi \in C^\infty(M_1/H_{\mathbf{M}_1})$ satisfying the system (100). The map $M \times \mathfrak{a}_{\mathfrak{q}} \to M_1$, $(m, X) \mapsto m \exp X$ induces a diffeomorphism $t : M/H_{\mathbf{M}} \times \mathfrak{a}_{\mathfrak{q}} \to M_1/H_{\mathbf{M}_1}$. By pull-back under t we identify $C^\infty(M_1/H_{\mathbf{M}_1})$ with $C^\infty(M/H_{\mathbf{M}} \times \mathfrak{a}_{\mathfrak{q}})$, and $\mathbf{D}(M_1/H_{\mathbf{M}_1})$ with $\mathbf{D}(M/H_{\mathbf{M}}) \otimes S(\mathfrak{a}_{\mathfrak{q}})$. Since $\mathbf{D}(M_1/H_{\mathbf{M}_1})$ is a finite $\mu_Q'(\mathbf{D}(G/H))$-module, every $\varphi \in \mathscr{E}(M_1, \Lambda)$ behaves finitely under the action of $\mathbf{D}(M/H_{\mathbf{M}})$, and in view of Lemma 4.8 it suffices to consider the case that φ is an eigenfunction for $\mathbf{D}(M/H_{\mathbf{M}})$. Let $\gamma_{\mathbf{M}}(\cdot : \Lambda_0)$ be the associated eigenvalue ($\Lambda_0 \in L + \rho_{\mathbf{M}}$). We define the map $C_{\Lambda_0} : S(\mathfrak{b}) \to S(\mathfrak{a}_{\mathfrak{q}})$ by $C_{\Lambda_0}(X \otimes Y) = X(\Lambda_0 - \rho_{\mathbf{M}})Y$, for

$X \in S(\mathfrak{b}_{\mathfrak{k}})$, $Y \in S(\mathfrak{a}_{\mathfrak{q}})$. Then the action of $D \in \mathbf{D}(M_1/H_{\mathbf{M}_1})$ on φ is described by

$$(D\varphi) \mid \mathfrak{a}_{\mathfrak{q}} = C_{\Lambda_0}[\gamma'_{\mathbf{M}_1}(D)](\varphi \mid \mathfrak{a}_{\mathfrak{q}}). \tag{103}$$

Here $\gamma'_{\mathbf{M}_1} = T_{\rho_{\mathbf{M}}} \circ \gamma_{\mathbf{M}_1}$, and we have identified $\mathfrak{a}_{\mathfrak{q}}$ with the subspace $\{e\} \times \mathfrak{a}_{\mathfrak{q}}$ of $M/H_{\mathbf{M}} \times \mathfrak{a}_{\mathfrak{q}}$. Let $\rho_1 = \rho_{\mathbf{M}} + \rho_Q$ (this is a rho for $\Sigma(\mathfrak{b})$), and define $\gamma' = T_{\rho_1} \circ \gamma$. Then $\gamma' = \gamma'_{\mathbf{M}_1} \circ \mu'_Q$. Hence applying (103) to the system (100) we infer that $\varphi \mid \mathfrak{a}_{\mathfrak{q}}$ satisfies the system

$$C_{\Lambda_0}[\gamma'(D)](\varphi \mid \mathfrak{a}_{\mathfrak{q}}) = \gamma(D : \Lambda + \lambda)\, \varphi \mid \mathfrak{a}_{\mathfrak{q}} \qquad (D \in \mathbf{D}(G/H)).$$

Now define $\psi : \mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{q}} \to \mathbf{C}$ by $\psi(X+Y) = e^{(\Lambda_0 - \rho_{\mathbf{M}})(X)}\varphi(Y)$. Then it follows that

$$[e^{-\rho_1} \circ u \circ e^{\rho_1}]\psi = u(\Lambda + \lambda)\psi \qquad (u \in S(\mathfrak{b})^{W(\mathfrak{b})}).$$

Using [20, Chap. III] we now infer that we have a unique expression

$$\psi = \sum_{w \in W(\mathfrak{b})/W_0} q_w\, e^{w(\Lambda + \lambda) - \rho}, \tag{104}$$

where W_c denotes the centralizer of $\Lambda + \lambda$ in $W(\mathfrak{b})$, and where each q_w is a W_c-harmonic polynomial on $\mathfrak{b}$. By Lemma 13.4(2), the group W_c is contained in the centralizer of $\mathfrak{a}_{\mathfrak{q}}$ in $W(\mathfrak{b})$ which in turn may be identified with $W(\mathfrak{m}_1, \mathfrak{b})$.

In view of the definition of ψ we must have that

$$w(\Lambda + \lambda) - \rho_Q - \Lambda_0 \in \mathfrak{a}^*_{\mathfrak{qc}} \tag{105}$$

for every $w \in W(\mathfrak{b})$ with $q_{w+W_0} \neq 0$. In view of Lemma 13.4(1), the above condition (105) implies that w belongs to the normalizer $W_{\mathfrak{q}}$ of $\mathfrak{a}_{\mathfrak{q}}$ in $W(\mathfrak{b})$, and also that $w\Lambda = \Lambda_0$. It follows that

$$\psi = e^{\Lambda_0 - \rho_{\mathbf{M}}} \sum_{w \in W_{\mathfrak{q}}/W_c} p_w\, e^{w\lambda - \rho_Q},$$

where each p_w is a W_c-harmonic polynomial on $\mathfrak{b}$. Since $W_c \subset W(\mathfrak{m}_{1\mathfrak{c}}, \mathfrak{b})$ it follows that p_w is annihilated by differentiations from $\mathfrak{a}_{\mathfrak{q}}$ hence belongs to $S(\mathfrak{b}^*_{\mathfrak{k}})$. We conclude that for each $\varphi \in \mathscr{E}(M_1, \Lambda)$ we have

$$\varphi \mid \mathfrak{a}_{\mathfrak{q}} = \sum_{w \in W} c_w(\varphi)\, e^{w\lambda - \rho_Q}, \tag{106}$$

with $c_w(\varphi) \in \mathbf{C}$. Since λ is a regular element of $\mathfrak{a}_{\mathfrak{q}}$, the functions $e^{w\lambda}$, $w \in W$ are linearly independent. Therefore we may fix points $X_v \in \mathfrak{a}_{\mathfrak{q}}$, $v \in W$ such that the $c_w(\varphi)$ can be solved uniquely from the equations obtained by evaluating (106) in the points X_v, $v \in W$. It follows that each c_w is a continuous linear functional of order 0 on $\mathscr{E}(M_1, \Lambda)$. We define continuous linear maps C_w from $\mathscr{E}(M_1, \Lambda)$ into $C^\infty(M/H_{\mathbf{M}})$ by $C_w(\varphi)(m) = c_w(L_{m^{-1}}\varphi)$. Then (101) holds with $\varphi_w = C_w(\varphi)$ and it is clear that the φ_w are uniquely determined. Moreover, the maps C_w are left M-equivariant by uniqueness. Finally Eqs. (102) have been checked along $\mathfrak{a}_\mathfrak{q}$ in the course of the proof. This is sufficient in view of the equivariance of the C_w. ∎

COROLLARY 13.6. *Let* $\Lambda \in \mathfrak{b}_{\mathfrak{k}c}^*$, $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^{*\prime}(\Lambda)$. *If* $\mathscr{E}^\infty_{\Lambda+\lambda,*}(G/H) \neq 0$, *then* $\Lambda \in sL + \rho_{\mathbf{M}}$ *for some* $s \in W(\mathfrak{b})$, *normalizing* $\mathfrak{a}_\mathfrak{q}$.

Proof. Let the above hypotheses be fulfilled. If $f \in \mathscr{E}^\infty_{\Lambda+\lambda,*}(G/H)$ is non-trivial, then its asymptotic expansion does not vanish identically (use reduction to K-finite f as in the proof of Theorem 12.8). Hence there exists a leading exponent $\xi \in \mathscr{E}_L(Q \mid f)$. Replacing f by a left translate if necessary, and using equivariance, we may assume that the function $\varphi \in C^\infty(M_1)$ defined by $\varphi(m) = p_{\lambda,\xi}(f, m, 0)$ is non-trivial. Moreover, it satisfies the system of differential equations of Corollary 13.3. By Proposition 13.5 there exists a $w \in W$ such that the system (102) has a non-trivial solution. In view of Lemma 4.8 this implies that $s\Lambda \in W(\mathfrak{m}_1, \mathfrak{b})(L + \rho_{\mathbf{M}}) = L + \rho_{\mathbf{M}}$. Hence $\Lambda \in s^{-1}(L + \rho_{\mathbf{M}}) = s^{-1}L + \rho_{\mathbf{M}}$. ∎

For holomorphic families of eigenfunctions we can obtain a severe restriction on the exponents along the parabolic subgroup $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$.

If $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ we define

$$X(Q, \lambda) = \{w\lambda - \rho_Q - \mu;\ \ w \in W, \mu \in \mathbf{N}\Sigma(Q)\}.$$

THEOREM 13.7. *Let* $\Lambda \in \mathfrak{b}_{\mathfrak{k}c}^*$, Ω_0 *an open subset of* $\mathfrak{a}_{\mathfrak{q}c}^*$, *and assume that* $f \in \mathscr{E}_*(G/H, \Lambda, \Omega_0)$. *Then for every* $\lambda \in \Omega_0 \cap \mathfrak{a}_{\mathfrak{q}c}^{*\prime}$ *we have that*

$$f_\lambda(x \exp tX) \sim \sum_{\xi \in X(Q,\lambda)} p_{\lambda,\xi}(Q \mid f_\lambda, x, tX)\, e^{t\xi(X)} \qquad (t \to \infty) \qquad (107)$$

for $x \in G$, $X \in \mathfrak{a}_\mathfrak{q}^+(Q)$. *Moreover, if* $\lambda_0 \in \Omega_0$, $\xi_0 \in X(Q, \lambda_0)$, *put*

$$\Xi(\lambda) = \{w\lambda - \rho_Q - \mu;\ \ w \in W, \mu \in \mathbf{N}\Sigma(Q) \text{ with } w\lambda_0 - \rho_Q - \mu = \xi_0\}.$$

Then there exists an open neighbourhood Ω *of* λ_0 *in* Ω_0 *and a constant* $r' \in \mathbf{R}$, *such that the map*

$$(\lambda, X) \mapsto \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(Q \mid f_\lambda, \cdot, X)\, e^{\xi(X)}$$

is continuous from $\Omega \times \mathfrak{a}_\mathfrak{q}$ *into* $C^\infty_{r'}(G)$ *and in addition holomorphic in* λ.

Proof. In view of Theorems 12.8 and 12.9 it suffices to show that

$$\mathscr{E}(Q \mid f_\lambda) \subset X(Q, \lambda) \qquad \text{for every} \quad \lambda \in \Omega_0. \tag{108}$$

We first assume that $\Omega_0 \subset \mathfrak{a}_{\mathfrak{qc}}^{*\prime}(\Lambda)$. Let $\lambda_0 \in \Omega_0$ be fixed, and let ξ be a leading exponent of f_{λ_0} along Q. Then from Corollary 13.3 and Proposition 13.5 it follows that there exist unique $\varphi_w \in C^\infty(M/H_{\mathrm{M}})$ for $w \in W$, such that

$$p_{\lambda_0,\xi}(Q \mid f_{\lambda_0}, m \exp X, 0) = \sum_{w \in W} \varphi_w(m)\, e^{(w\lambda - \rho_Q)(X)}$$

for $m \in M$ and $X \in \mathfrak{a}_{\mathfrak{q}}$. On the other hand, from Lemma 13.1 we infer that

$$p_{\lambda_0,\xi}(Q \mid f_{\lambda_0}, m \exp X, 0) = p_{\lambda_0,\xi}(Q \mid f_{\lambda_0}, m, X)\, e^{\xi(X)}.$$

It follows that $\xi \in W\lambda_0 - \rho_Q$ for every leading exponent of f_{λ_0}, whence (108).

For a general open set Ω_0, fix $\lambda_0 \in \Omega_0$ and assume that $\xi_0 \in X_Q(\Lambda, \lambda_0)$, but $\xi_0 \notin X(Q, \lambda_0)$. Let $\Xi(\lambda)$ and Ω be as in Theorem 12.9. Notice that $\Xi(\lambda_0) = \{\xi_0\}$, hence $\Xi(\lambda_0) \cap X(Q, \lambda_0) = \emptyset$. Shrinking Ω if necessary we may assume that

$$\Xi(\lambda) \cap X(Q, \lambda) = \emptyset \qquad \text{for every} \quad \lambda \in \Omega. \tag{109}$$

If $x \in G$, $X \in \mathfrak{a}_{\mathfrak{q}}$, then the function

$$\psi(\lambda) = \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(Q \mid f_\lambda, x, X)\, e^{\xi(X)}$$

is holomorphic in the open neighbourhood Ω of λ_0. By the first part of the proof it follows that $\psi = 0$ on the open dense subset $\Omega \cap \mathfrak{a}_{\mathfrak{qc}}^{*\prime}(\Lambda)$ of Ω. Hence ψ vanishes identically on Ω. In particular we have that

$$p_{\lambda_0,\xi_0}(Q \mid f_{\lambda_0}, x, X) = \psi(\lambda_0) = 0,$$

and we infer that $\xi_0 \notin \mathscr{E}(Q \mid f_{\lambda_0})$. This implies (108). ∎

Remark 13.8. Combining Theorem 13.7 with Corollary 13.6 we see that $\mathscr{E}_*(G/H, \Lambda, \Omega_0) \neq 0$ implies that $\Lambda \in sL + \rho_{\mathrm{M}}$ for some $s \in W(\mathfrak{b})$, normalizing $\mathfrak{a}_{\mathfrak{q}}$.

We will conclude this section by showing that for generic λ the polynomial functions $X \mapsto p_{\lambda,\xi}(Q \mid f_\lambda, \cdot, X)$ are constant.

Recall that $\alpha^\vee = 2\langle \alpha, \alpha \rangle^{-1}\alpha$ for $\alpha \in \Sigma$, and let

$${}'\mathfrak{a}_{\mathfrak{qc}}^* = \{\lambda \in \mathfrak{a}_{\mathfrak{qc}}^*;\, \forall \alpha \in \Sigma\colon \langle \lambda, \alpha^\vee \rangle \notin \mathbf{Z}\}.$$

LEMMA 13.9. *The set* $'\mathfrak{a}_{qc}^*$ *is the complement of a locally finite union of hyperplanes. Moreover if* $\lambda \in {}'\mathfrak{a}_{qc}^*$, *and* $s\lambda - t\lambda \in \mathbf{Z}\Sigma$ *for* $s, t \in W$, *then* $s = t$.

Proof. The first assertion is obvious. As for the second, write

$$\mathscr{A}' = \{\lambda \in \mathfrak{a}_{qc}^*;\ \ \forall \alpha \in \Sigma^+ : \langle \lambda, \alpha^\vee \rangle \notin -\mathbf{N}\}.$$

Then $v'\mathfrak{a}_{qc}^* \subset \mathscr{A}'$, for every $v \in W$. If $\mu \in \mathscr{A}'$, then it follows from [22, Appendix II, Proposition 2(2)] that $w\mu - \mu \in \mathbf{N}\Sigma^+$ implies $w = 1$, for $w \in W$. Now let $\lambda \in {}'\mathfrak{a}_{qc}^*$, and suppose that $s\lambda - t\lambda \in \mathbf{Z}\Sigma$. Then there exists a $v \in W$ such that $vt^{-1}s\lambda - v\lambda \in \mathbf{N}\Sigma^+$. But $v\lambda \in \mathscr{A}'$, hence $vt^{-1}sv^{-1} = 1$, and it follows that $s = t$. ∎

THEOREM 13.10. *Under the assumptions of Theorem* 13.7, *let* $s \in W$, $v \in \mathbf{N}\Sigma(Q)$. *Then for* $\lambda \in {}'\mathfrak{a}_{qc}^* \cap \Omega$ *the* $C^\infty(G)$-*valued polynomial* $X \mapsto p_{\lambda, s\lambda - \rho_Q - v}(Q \mid f_\lambda, \cdot, X)$ *is constant. Its value*

$$p_{Q,v}(f : s : \lambda) := p_{\lambda, s\lambda - \rho_Q - v}(Q \mid f_\lambda, \cdot, 0) \tag{110}$$

is holomorphic as a $C^\infty(G)$-*valued function of* $\lambda \in {}'\mathfrak{a}_{qc}^* \cap \Omega$ *and allows a meromorphic extension to* Ω. *If* $\lambda_0 \in \Omega$, *then there exist an open neighbourhood* Ω_0 *of* λ_0 *in* Ω *and a constant* $r' \in \mathbf{R}$ *such that* (110) *defines a meromorphic* $C_{r'}^\infty(G)$-*valued function of* $\lambda \in \Omega_0$.

Proof. Write $'\Omega = {}'\mathfrak{a}_{qc}^* \cap \Omega$. If $\lambda \in {}'\Omega$, $s_1, s_2 \in W$, and $\mu_1, \mu_2 \in \mathbf{N}\Sigma(Q)$, then from Lemma 13.9 we see that

$$s_1\lambda - \rho_Q - \mu_1 = s_2\lambda - \rho_Q - \mu_2 \Rightarrow s_1 = s_2,\ \mu_1 = \mu_2.$$

Hence from Theorem 13.7 it follows that for each $s \in W$, $v \in \mathbf{N}\Sigma(Q)$ the function

$$p_{\lambda, s, v} = p_{\lambda, s\lambda - \rho_Q - v}(Q \mid f_\lambda),$$

depends holomorphically on $\lambda \in {}'\Omega$. Thus, in order to show that these functions are of degree zero in their second variable, we may restrict λ to the set $\Omega' = {}'\Omega \cap \mathfrak{a}_{qc}^{*\prime}(\Lambda)$. This will be understood from now on. We proceed by induction on v with respect to the partial ordering $\preccurlyeq\, = \,\preccurlyeq_Q$.

For $\lambda \in \Omega'$, $s \in W$, the exponent $s\lambda - \rho_Q$ is a leading exponent. Applying Corollary 13.3 we infer that the function $\varphi : M_1 \to \mathbf{C}$, $m \mapsto p_{\lambda, s, 0}(m, 0)$ satisfies the system (100). By Proposition 13.5 we infer that φ allows an expression of the form (101). Comparing this with Lemma 13.1 we conclude that all φ_w, $w \neq s$, in the expression (101) are zero. It also follows from the comparison that the polynomials $X \mapsto p_{\lambda, s, 0}(m, X)$ are constant. By equivariance we have that $p_{\lambda, s, 0}(x, X) = p_{\lambda, s\lambda - \rho_Q}(Q \mid L(x^{-1}) f_\lambda, e, X)$. Thus the assertion about zero degree holds for $v = 0$.

Next, let $\nu \in \mathbf{N}\Sigma(Q)$, $\nu \neq 0$, and suppose that the assertion has been established for $\mu \prec \nu$ (where $\prec$ stands for the strict ordering). Fix $\lambda \in \Omega'$ and write $\xi = s\lambda - \rho_Q - \nu$. Then for $\eta \succ \xi$ we have that $R_Y - \eta(Y)$ annihilates $p_\eta = p_{\lambda,\eta}(Q \mid f_\lambda, \cdot, 0)$ for every $Y \in \mathfrak{a}_\mathfrak{q}$. Hence if $\mathrm{ad}(\mathfrak{a}_\mathfrak{q})$ acts on $w \in U(\bar{\mathfrak{n}}_Q + \mathfrak{m}_1)$ by a non-zero weight $-\mu$, $\mu \in \mathbf{N}\Sigma(Q)$, then $R_Y - \xi(Y)$ annihilates $R_w p_{\xi+\mu}$. Using Proposition 13.2 and the induction hypothesis we now infer that the function $\psi: M_1/H_{\mathbf{M}_1} \to \mathbf{C}$ defined by $\psi(m) = p_\xi(m)$ satisfies the differential equations

$$[\mu'_Q(D) - \gamma(D : \Lambda + \lambda)][R_Y - \xi(Y)]\psi = 0,$$

for $D \in \mathbf{D}(G/H)$, $Y \in \mathfrak{a}_\mathfrak{q}$. From this we deduce that for every $Y \in \mathfrak{a}_\mathfrak{q}$ the function $\varphi_Y = [R(Y) - \xi(Y)]\psi$ is of the form (101). On the other hand, in view of Lemma 13.1, we have that

$$\psi(ma) = a^\xi p_{\lambda,\xi}(Q \mid f_\lambda, m, \log a)$$

for $m \in M_1$, $a \in A_\mathfrak{q}$. Since $\xi \notin W\lambda - \rho_Q$, this must imply that φ_Y is zero for every $Y \in \mathfrak{a}_\mathfrak{q}$. Hence $\psi(ma) = a^\xi \psi(m)$ and we conclude that the polynomial $X \mapsto p_{\lambda,\xi}(Q \mid f_\lambda, e, X)$ is constant. Applying the same equivariance argument as before we finally conclude that the function $p_{\lambda,\xi}(Q \mid f_\lambda)$ is constant in its second variable.

It now remains to prove the statement about the meromorphic continuation. For this we fix s, ν, and $\lambda_0 \in \Omega$. Let Ξ be the set of pairs $(t, \mu) \in W \times \mathbf{N}\Sigma(Q)$ such that $t\lambda_0 - \mu = s\lambda_0 - \nu$. Then by Theorem 13.7 there exists an open neighbourhood Ω_0 of λ_0 in Ω and a constant $r' \in \mathbf{R}$ such that for every $X \in \mathfrak{a}_\mathfrak{q}$ the $C_{r'}^\infty(G)$-valued function

$$\psi(X, \lambda) = \sum_{(s,\mu) \in \Xi} e^{(s\lambda - \mu)(X)} p_{Q,\nu}(f : s : \lambda) \tag{111}$$

extends holomorphically from $\Omega_0 \cap {}'\mathfrak{a}_{\mathfrak{q}c}^*$ to Ω_0 . For $\lambda \in {}'\mathfrak{a}_{\mathfrak{q}c}^*$ the functions $e^{s\lambda - \mu}$, $(s, \mu) \in \Xi$ are linearly independent. We may therefore fix X_l, $l \in \Xi$, such that the determinant

$$\det(e^{(s\lambda - \mu)(X_l)}; (s, \mu) \in \Xi, l \in \Xi)$$

does not vanish identically as a function of λ. By Cramer's rule this implies that the functions $p_{Q,\nu}(f : s : \lambda)$ may be solved meromorphically as $C_{r'}^\infty(G)$-valued functions of $\lambda \in \Omega$ from the system which arises if one substitutes for X the values X_l, $l \in \Xi$, in Eq. (111).

14. Expansions for Eisenstein Integrals

In this section we will apply the material of the previous two sections to study the asymptotic expansions along minimal $\sigma\theta$-stable parabolic subgroups for families of spherical functions like the Eisenstein integral. We define the notion of principal part of such an expansion, and introduce the c-functions.

Let $\Omega \subset \mathfrak{a}_{\mathfrak{q}c}^*$ be a connected open subset. Given $\Lambda \in \mathfrak{b}_{\mathfrak{k}c}^*$ we define $\mathscr{E}_*(G/H, \tau, \Lambda, \Omega)$ to be the space of functions $f: \Omega \times G/H \to \mathbf{V}$ which are τ-spherical in the second variable, and whose components $\eta \circ f (\eta \in \mathbf{V}^*)$ belong to $\mathscr{E}_*(G/H, \Lambda, \Omega)$ (see the definition above Theorem 12.9). Moreover, let $\mathscr{E}_*(G/H, \tau, \Omega)$ denote the space of functions $f: \Omega \times G/H \to \mathbf{V}$ which may be expressed as finite sums $f = \sum_{\Lambda \in \mathfrak{b}_{\mathfrak{k}c}^*} f_\Lambda$, $f_\Lambda \in \mathscr{E}_*(G/H, \tau, \Lambda, \Omega)$ (notice that by Remark 13.8 the range of Λ is restricted). Then we have the following.

Lemma 14.1. *Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\psi \in {}^\circ\mathscr{C}$, $R \in \mathbf{R}$, and let $\pi \in \Pi_\Sigma(\mathfrak{a}_\mathfrak{q})$ be any polynomial such that $\lambda \mapsto \pi(\lambda)\, E(P : \psi : \lambda)$ is regular on $\mathfrak{a}_\mathfrak{q}^*(P, R)$. Then the function $(\lambda, x) \mapsto \pi(\lambda)\, E(P : \psi : \lambda : x)$ belongs to $\mathscr{E}_*(G/H, \tau, \mathfrak{a}_\mathfrak{q}^*(P, R))$.*

Proof. In view of Lemma 4.5 and Proposition 4.7 we may restrict ourselves to the case that ψ is a simultaneous eigenfunction for the $\underline{\mu}_P(D : \lambda)$, $D \in \mathbf{D}(G/H)$, $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. Then there exists a $\Lambda \in \mathfrak{b}_{\mathfrak{k}c}^*$ such that $\underline{\mu}_P(D : \lambda)\psi = \gamma(D : \Lambda + \lambda)\psi$ for all D, λ. In view of Lemma 4.5 this implies that $\pi(\lambda)\eta \circ E(P : \psi : \lambda) \in \mathscr{E}_{\Lambda+\lambda}(G/H)$ for $\eta \in \mathbf{V}^*$, $\lambda \in \mathfrak{a}_\mathfrak{q}^*(P, R)$. Using Proposition 10.3 we infer that for every relatively compact open subset $\Omega \subset \mathfrak{a}_\mathfrak{q}^*(P, R)$ there exists a $r \geq 0$ such that for every $X \in U(\mathfrak{g})$ the function

$$\lambda \mapsto \|L_X[\pi(\lambda)\eta \circ E(P : \psi : \lambda)]\|_r$$

is uniformly bounded on Ω. On the other hand the function $(x, \lambda) \mapsto \pi(\lambda)\eta \circ E(P : \psi : \lambda)(X; x)$ is smooth and in addition holomorphic in λ. By a straightforward application of the Cauchy integral formulas for the coefficients of a power series it finally follows that $\lambda \mapsto \pi(\lambda)\eta \circ E(P : \psi : \lambda)$ is a meromorphic map from Ω into $C_r^\infty(G/H)$. ∎

Theorem 14.2. *Let $f \in \mathscr{E}_*(G/H, \tau, \Omega)$, and assume that $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $w \in \mathscr{W}$. Then there exist unique meromorphic ${}^\circ\mathscr{C}_\omega$-valued functions $P_{Q,w,\mu}(f : s)$ on Ω $(\mu \in \mathbf{N}\Sigma(Q),\ s \in W)$ such that for $\lambda \in {}'\mathfrak{a}_{\mathfrak{q}c}^* \cap \Omega$, $m \in M_\sigma$, $X \in \mathfrak{a}_\mathfrak{q}^*(Q)$ we have*

$$f_\lambda(m \exp tXw) \sim e^{-t\langle\rho_Q, X\rangle} \sum_{s \in W} \sum_{\mu \in \mathbf{N}\Sigma(Q)} e^{t\langle s\lambda - \mu, X\rangle} P_{Q,w,\mu}(f : s : \lambda)(m) \qquad (t \to \infty).$$

Remark 14.3. Since f_λ is spherical, it follows from [2] that the above expansion is actually convergent for t sufficiently large, in view of uniqueness of asymptotics.

Proof of Theorem 14.2. By uniqueness of asymptotics it suffices to prove the existence. Moreover, it suffices to prove the result for $w = 1$ and arbitrary Q. For assume this has been achived, and observe that

$$f_\lambda(m \exp tXw) = \tau(w)\, f_\lambda(w^{-1}mw \exp t\, \mathrm{Ad}(w^{-1})X).$$

Applying the theorem to f, $w^{-1}Qw$, 1 one then obtains the above expansion with

$$P_{Q,w,\mu}(f:s:\lambda)(m) = \tau(w)\, P_{w^{-1}Qw,1,w^{-1}\mu}(f:w^{-1}s:\lambda)(w^{-1}mw). \quad (112)$$

Moreover, one readily checks that the right hand side of (112) belongs to ${}^\circ\mathscr{C}_w$, as a function of m.

From now on we restrict ourselves to the case $w = 1$. Then without loss of generality we may assume that $f \in \mathscr{E}_*(G/H, \tau, \Lambda, \Omega)$ for some $\Lambda \in \mathfrak{b}_{\mathfrak{kc}}^*$. Hence Theorem 13.10 applies to every component $\eta \circ f$ of f. Thus for $\lambda \in {}'\mathfrak{a}_{\mathfrak{qc}}^* \cap \Omega$ we may define smooth functions $P_{Q,1,\mu}(f:s:\lambda): M_1 \to \mathbf{V}$ by

$$\eta \circ P_{Q,1,\mu}(f:s:\lambda) = p_{Q,\mu}(\eta \circ f:s:\lambda) \mid M_1$$

(where we have used the notation of Theorem 13.10). Then for $\lambda \in {}'\mathfrak{a}_{\mathfrak{qc}}^* \cap \Omega$ we have the above asymptotic expansion. By uniqueness of asymptotics it follows that the functions $P_{Q,1,\mu}(f:s:\lambda)$ are left $\tau_{\mathbf{M}}$-spherical and right $M_1 \cap H$-invariant, hence belong to ${}^\circ\mathscr{C}_1$. Finally, the functions $P_{Q,1,\mu}(f:s)$ are extendable to meromorphic ${}^\circ\mathscr{C}_1$-valued functions by Theorem 13.10. ∎

Let f be as in the above theorem. Then for $Q \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$, $w \in \mathscr{W}$ we call the function $f_{Q,w}: \Omega \times M_1 \to \mathbf{V}$ defined by

$$f_{Q,w}(\lambda : ma) = \sum_{s \in W} a^{s\lambda} P_{Q,w,0}(f:s:\lambda)(m) \qquad (m \in M_\sigma,\ a \in A_{\mathfrak{q}})$$

the (Q, w)-principal term of f. If we fix Q, then the associated principal terms $f_{Q,w}$ govern in a sense the asymptotic behaviour of f, in view of the following lemma.

LEMMA 14.4. *Let $Q \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$. Then the sets $K \exp \mathfrak{a}_{\mathfrak{q}}^+(Q)\, wH$, $w \in \mathscr{W}$ are mutually disjoint. Moreover,*

$$G = \bigcup_{w \in \mathscr{W}} K \exp \overline{\mathfrak{a}_{\mathfrak{q}}^+(Q)}\, wH. \quad (113)$$

Proof. If $X \in \mathfrak{a}_q$, then $X \in v^{-1}\overline{\mathfrak{a}_q^+(Q)}$ for a suitable $v \in W$. Let $w \in \mathcal{W}$ be a representative for v's canonical image in $W/W_{K \cap H}$. Then $X \in K \exp \overline{\mathfrak{a}_q^+(Q)}\, wH$. Hence A_q is contained in the union in (113). Now use (3) to see that (113) holds.

To see that the first assertion holds, suppose that $K \exp \mathfrak{a}_q^+(Q)\, w_1 H = K \exp \mathfrak{a}_q^+(Q)\, w_2 H$, for $w_1, w_2 \in \mathcal{W}$. Then $w_1^{-1} \exp \mathfrak{a}_q^+(Q) w_1 \subset K w_2^{-1} \exp \mathfrak{a}_q^+(Q)\, w_2 H$, hence $\mathrm{Ad}(w_1^{-1})\, \mathfrak{a}_q^+(Q) \subset \mathrm{Ad}(v w_2^{-1})\, \mathfrak{a}_q^+(Q)$ for some $v \in N_{K \cap H}(\mathfrak{a}_q)$ (cf. Section 1). Since W acts simply transitively on $\mathscr{P}_\sigma(A_q)$ it follows that w_1 and $w_2 v^{-1}$ have the same image in W. Therefore w_1, w_2 represent the same element in $W/W_{K \cap H}$ hence are equal. ∎

If $\varepsilon > 0$, we define $\mathfrak{a}_q^*(\varepsilon) = \{\lambda \in \mathfrak{a}_{qc}^*;\, |\mathrm{Re}\, \lambda| < \varepsilon\}$.

LEMMA 14.5. *Let $0 < \varepsilon < (1/2)\min_{\alpha \in \Sigma} |\alpha|$, and suppose that $f \in \mathscr{E}_*(G/H, \tau, \mathfrak{a}_q^*(\varepsilon))$. Then for every $Q \in \mathscr{P}_\sigma(A_q)$, $w \in \mathcal{W}$ the principal term $f_{Q,w}(\lambda : m)$ has removable singularities (hence is holomorphic) on $\mathfrak{a}_q^*(\varepsilon)$ as a function of λ. Moreover, for all $\lambda \in \mathfrak{a}_q^*(\varepsilon)$, $m \in M$, $X \in \mathfrak{a}_q^+(Q)$ we have that*

$$\lim_{t \to \infty} |d_Q(m \exp tX)\, f_\lambda(m \exp tXw) - f_{Q,w}(\lambda : m \exp tX)| = 0.$$

Proof. As in the proof of Theorem 14.2 we may restrict ourselves to the case $w = 1$. For $\lambda \in \mathfrak{a}_{qc}^*$, let $\Pi(\lambda)$ be the set of $(s, \mu) \in W \times \mathbf{N}\Sigma(Q)$ such that $s\lambda - \mu \in W\lambda$. Then for $\lambda \in \mathfrak{a}_q^*(\varepsilon)$ we have that $\Pi(\lambda) = \Pi(0) = W \times \{0\}$. In view of Theorem 13.7 it follows from the definition of the $P_{Q,1,0}(f : s : \lambda)$ in the proof of Theorem 14.2 that $f_{Q,w}(\lambda : ma)$ has removable singularities as a function of $\lambda \in \mathfrak{a}_q^*(\varepsilon)$. Moreover, if $\eta \in \mathbf{V}^*$ then it follows by holomorphic continuation that

$$\eta \circ f_{Q,w}(\lambda : m \exp tX) = e^{t\rho_Q(X)} \sum_{\xi \in W\lambda - \rho_Q} e^{t\xi(X)} p_{\lambda,\xi}(Q \mid \eta \circ f_\lambda, m, tX), \tag{114}$$

both sides being holomorphic in λ. Now use (107) applied to $\eta \circ f$ taking into account that every exponent $\xi \in X(Q, \lambda) \backslash (W\lambda - \rho_Q)$ satisfies $\xi(X) < 0$, for $\lambda \in \mathfrak{a}_q^*(\varepsilon)$. ∎

Remark 14.6. In particular we see that for imaginary λ the principal term is an appropriate analogue of Harish-Chandra's notion of the constant term (cf. [16, p. 153]).

If $\varphi : \Omega \to \mathbf{C}$ is a non-zero holomorphic function, and $f : \Omega \times G/H \to \mathbf{V}$ a function such that $F = \varphi f \in \mathscr{E}_*(G/H, \tau, \Omega)$ then we define (Q, w)-principal terms by

$$f_{Q,w}(\lambda : m) := \varphi(\lambda)^{-1} F_{Q,w}(\lambda : m).$$

Let now $P, Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $w \in \mathscr{W}$. Then in view of Lemma 14.1 the Eisenstein integral $E(P : \psi)$ has a (Q, w)-principal term

$$E_{Q,w}(P : \psi : ma) = \sum_{s \in W} a^{s\lambda} C_{Q|P,w}(s : \lambda : \psi)(m) \qquad (m \in M_\sigma, a \in A_\mathfrak{q}). \tag{115}$$

Here the $C_{Q|P,w}(s : \lambda)$ are uniquely determined $\mathrm{Hom}({}^\circ\mathscr{C}, {}^\circ\mathscr{C}_w)$-valued meromorphic functions on $\mathfrak{a}_{\mathfrak{q}c}^*$. We now define meromorphic $\mathrm{End}({}^\circ\mathscr{C})$-valued functions $\lambda \mapsto C_{Q|P}(s : \lambda)$ $(s \in W)$ by

$$C_{Q|P,w}(s : \lambda) := \mathrm{pr}_w \circ C_{Q|P}(s : \lambda) \qquad (w \in \mathscr{W}).$$

The above functions will becalled c-functions. In the next section we will show that their behaviour is analogous to the behaviour of Harish-Chandra's c-functions as defined in [17, p. 42].

15. The c-Functions

In this section we investigate the c-functions which were introduced in the previous section. In Proposition 15.7 we relate them to intertwining operators and in Corollary 15.11 we formulate a unitarity result.

Let $P_1, P_2 \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\xi \in \hat{M}_{\mathrm{ps}}$, and $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. From [4, p. 373] we recall the definition of the meromorphic scalar function η by the identity

$$A(P_1 : P_2 : \xi : \lambda) \circ A(P_2 : P_1 : \xi : \lambda) = \eta(P_2 : P_1 : \xi : \lambda) I. \tag{116}$$

This identity also holds if we replace A by B, cf. [4, Proposition 6.2].

From [25] we recall that $A(P_2 : P_1 : \delta : -\bar{\lambda})^* = A(P_1 : P_2 : \xi : \lambda)$. We will say that the group G fulfills condition (B) if for all $P_1, P_2 \in \mathscr{P}_\sigma(A_\mathfrak{q})$ and every $\xi \in \hat{M}_{\mathrm{ps}}$ we have

$$B(P_2 : P_1 : \xi : -\bar{\lambda})^* = B(P_1 : P_2 : \xi : \lambda). \tag{B}$$

In [4, Theorem 6.3] it is proved that this condition is fulfilled if every Cartan subgroup of G is abelian, and $H = G^\sigma$, the full fixed point group. In [7] it is observed that (B) is fulfilled under a weaker but more technical condition. It would be interesting to have a simple condition on the pair (G, H), necessary and sufficient for (B) to hold.

By equivariance the intertwining operator induces an endomorphism $A(P_2 : P_1 : \xi : \lambda)_F$ of the finite dimensional linear space $\mathscr{H}_{\xi, F}$, meromorphically depending on λ.

LEMMA 15.1. *If G satisfies condition* (B), *then the endomorphism* $u(\lambda) = A(P_2 : P_1 : \xi : \lambda)_F \bar{\otimes} B(P_2 : P_1 : \xi : -\bar{\lambda})$ *of* $\mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$ *satisfies*

$$u(-\bar{\lambda})^* u(\lambda) = \eta(P_2 : P_1 : \xi : \lambda)\, \overline{\eta(P_2 : P_1 : \xi : -\bar{\lambda})} I.$$

Proof. Use formula (B) and the analogous formula for the transposed of $A(\lambda)$ in combination with the identity (116) for $A(\lambda)$ and $B(-\bar{\lambda})$. ∎

Recall that η is not identically zero as a function of λ (cf. [4, Proposition 4.8]), and let $U(P_2 : P_1 : \xi : \lambda) : {}^\circ\mathscr{C}(\xi) \to {}^\circ\mathscr{C}(\xi)$ be defined by

$$U(P_2 : P_1 : \xi : \lambda)\psi_T = \eta(P_2 : P_1 : \xi : -\lambda)^{-1} \psi_{A(P_2 : P_1 : \xi : -\lambda) \bar{\otimes} B(P_2 : P_1 : \xi : \bar{\lambda})T},$$

for $T \in \mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$. Then in view of Lemma 4.1, $U(P_2 : P_1 : \xi : \lambda) \in \operatorname{End}({}^\circ\mathscr{C}(\xi))$ depends meromorphically on λ. Moreover, if (B) holds then this endomorphism is unitary for imaginary λ, by Lemma 15.1. We define the linear map $U(P_2 : P_1 : \lambda) : {}^\circ\mathscr{C} \to {}^\circ\mathscr{C}$ by

$$U(P_2 : P_1 : \lambda) \mid {}^\circ\mathscr{C}(\xi) = U(P_2 : P_1 : \xi : \lambda),$$

for each $\xi \in \hat{M}_{\mathrm{ps}}$.

LEMMA 15.2. *Let $P_1, P_2 \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$. Then*

$$E(P_2 : U(P_2 : P_1 : \lambda)\psi : \lambda) = E(P_1 : \psi : \lambda). \tag{117}$$

Proof. It suffices to prove this for $\psi = \psi_T$, with $T = f \bar{\otimes} \eta \in \mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$. From Lemma 4.2 we then infer, suppressing $P_2 : P_1 : \xi$ in the notations, that $\eta(-\lambda) = \eta(P_2 : P_1 : \xi : -\lambda)$ times the left hand side of (117) equals

$$\begin{aligned} &\langle A(-\lambda) f, \pi_{P_2,\xi,\bar{\lambda}}(kx)\, j(P_2 : \xi : \bar{\lambda})\eta \rangle \\ &= \langle A(-\lambda) f, \pi_{P_2,\xi,\bar{\lambda}}(kx)\, A(\bar{\lambda})\, j(P_1 : \xi : \bar{\lambda})\eta \rangle \\ &= \langle A(\bar{\lambda})^* A(-\lambda) f, \pi_{P_1,\xi,\bar{\lambda}}(kx)\, j(P_1 : \xi : \bar{\lambda})\eta \rangle \\ &= \eta(-\lambda)\, E(P_1 : \psi : \lambda). \end{aligned} \tag{118}$$

This implies (117). ∎

COROLLARY 15.3. *Let $P_1, P_2 \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$. Then for all $Q \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$, $s \in W$ we have*

$$C_{Q \mid P_1}(s : \lambda) = C_{Q \mid P_2}(s : \lambda) \circ U(P_2 : P_1 : \lambda). \tag{119}$$

Moreover, if (B) *holds, then the map $U(P_2 : P_1 : \lambda)$ is unitary for imaginary λ.*

Proof. This follows from Lemma 15.2 by uniqueness of asymptotics. ∎

Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, and fix $w \in N_K(\mathfrak{a}_\mathfrak{q})$. We recall from [4, Lemma 6.10] that the intertwining operator $L(w): C^{-\infty}(P:\xi:\lambda) \to C^{-\infty}(wPw^{-1}: w\xi : w\lambda)$ induces a unitary linear map $L(\xi, w): V(\xi) \to V(w\xi)$. Moreover, $L(w)$ maps $\mathscr{H}_{\xi,F}$ unitarily onto $\mathscr{H}_{w\xi,F}$. We define the unitary map $\mathscr{L}(\xi, w): {}^\circ\mathscr{C}(\xi) \to {}^\circ\mathscr{C}(w\xi)$ by

$$\mathscr{L}(\xi, w)\psi_T = \psi_{(L(w) \otimes L(\xi, w))T}$$

for $T \in \mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$. We define the unitary bijection

$$\mathscr{L}(w): {}^\circ\mathscr{C} \to {}^\circ\mathscr{C}$$

by $\mathscr{L}(w) \mid {}^\circ\mathscr{C}(\xi) = \mathscr{L}(\xi, w)$.

LEMMA 15.4. *Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $w \in N_K(\mathfrak{a}_\mathfrak{q})$. Then*

$$E(P:\psi:\lambda) = E(wPw^{-1}: \mathscr{L}(w)\psi : w\lambda). \tag{120}$$

Proof. The proof is similar to the proof of Lemma 15.2. ∎

By uniqueness of asymptotics we now obtain:

COROLLARY 15.5. *Let $P, Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $w \in N_K(\mathfrak{a}_\mathfrak{q})$. Then*

$$C_{Q|P}(s:\lambda) = C_{Q|wPw^{-1}}(sw^{-1}: w\lambda) \circ \mathscr{L}(w), \tag{121}$$

for $s \in W$, $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$.

For $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, let the bi-invariant Haar measure $d\bar{n}$ of $\bar{N}_Q$ be normalized as in [25, Sect. 4]. Then the positive real number

$$c(A_\mathfrak{q}) = \left(\int_{\bar{N}_Q} e^{2\rho_Q H_Q(\bar{n})}\, d\bar{n}\right)^{-1} \tag{122}$$

is independent of Q; here $H_Q: G \to \mathfrak{a}_Q$ is defined by $x \in N_Q \exp H_Q(x) M_Q K$ $(x \in G)$.

Given $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$ we shall say that $a \in A_\mathfrak{q}$ tends to infinity along P, notation $a \xrightarrow{P} \infty$, if $a^\alpha \to \infty$ for all $\alpha \in \Sigma(P)$.

LEMMA 15.6. *Let $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, and assume that $\langle \operatorname{Re}\lambda - \rho_Q, \alpha\rangle > 0$ for all $\alpha \in \Sigma(Q)$. If $f \in C(Q:\xi:\lambda)$, $g \in C(Q:\xi:-\bar{\lambda})$, then*

$$\lim_{a \xrightarrow{\bar{Q}} \infty} a^{\lambda-\rho_Q}\langle f, R(a)g\rangle = c(A_\mathfrak{q})\langle [A(\bar{Q}:Q:\xi:\lambda)f](e), g(e)\rangle_{\mathscr{H}_\xi}, \tag{123}$$

the integral defining the intertwining operator being absolutely convergent.

Proof. Without loss of generality we may assume that $\Sigma(Q)$ is compatible with the positive system Σ_0^+ (cf. Section 1). Let $\mathfrak{a}_0^{*+}$ denote the positive Weyl chamber in $\mathfrak{a}_0^*$, and let ${}^+\mathfrak{a}_0$ be the closed dual cone in $\mathfrak{a}_0$, i.e., ${}^+\mathfrak{a}_0 = \{X \in \mathfrak{a}_0;\ \nu(X) \geqslant 0 : \forall \nu \in \mathfrak{a}_0^{*+}\}$. Let $H_0 : G \to \mathfrak{a}_0$ be the map defined by $x \in N_0 \exp H_0(x) K$ $(x \in G)$. Then it is a well known result of Harish-Chandra that for $\bar{n} \in \bar{N}_0$ we have

$$-H_0(\bar{n}) \in {}^+\mathfrak{a}_0$$

(see, e.g., [20, Chap. IV, Corollary 6.6]). Now let the maps $\kappa_Q, \mu_Q, H_Q, \nu_Q$ from G into K, $\exp(\mathfrak{m}_{1Q} \cap \mathfrak{p})$, $\mathfrak{a}_Q$, $\bar{N}_Q$ respectively be defined by

$$x = \nu_Q(x) \exp H_Q(x)\, \mu_Q(x)\, \kappa_Q(x) \qquad (x \in G). \tag{124}$$

Then $H_Q(x)$ is the orthogonal projection of $H_0(x)$ onto $\mathfrak{a}_Q \subset \mathfrak{a}_0$. Hence $\rho_Q \circ H_0 = \rho_Q \circ H_Q$ (cf. Section 1).

The assumption on $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$ implies that $\operatorname{Re}\lambda - \rho_Q \in \overline{\mathfrak{a}_0^{*+}}$. Hence for $\bar{n} \in \bar{N}_Q$ we have that

$$\|f(\bar{n})\| = e^{\langle \operatorname{Re}\lambda + \rho_Q, H_Q(\bar{n})\rangle} \|f(\kappa_Q(\bar{n}))\| \leqslant e^{2\rho_Q H_0(\bar{n})} \sup_{k \in K} \|f(k)\|,$$

and it follows that

$$A(\bar{Q} : Q : \xi : \lambda) f(e) = \int_{\bar{N}_Q} f(\bar{n})\, d\bar{n}$$

with absolutely convergent integral.

We now recall that the map $\bar{n} \mapsto (K \cap M_Q)\, \kappa_Q(\bar{n})$ is a diffeomorphism from $\bar{N}_Q$ onto an open dense subset of $(K \cap M_Q)\backslash K$ and has Jacobian $c(A_\mathfrak{q})\, e^{2\rho_Q H(\bar{n})}$ (cf. [17, p. 45]). Hence by transformation of variables and by using the decomposition (124) for $x = \bar{n}$, the transformation rules for f, g and the unitarity of ξ we infer that

$$a^{\lambda - \rho_Q} \langle f, R(a) g\rangle = c(A_\mathfrak{q}) \int_{\bar{N}_Q} \langle f(\bar{n}), g(a^{-1}\bar{n}a)\rangle_{\mathscr{H}_\xi}\, d\bar{n}. \tag{125}$$

Now observe that

$$\|g(a^{-1}\bar{n}a)\| = e^{\langle \operatorname{Re}\lambda - \rho_Q, H_Q(a^{-1}\bar{n}a)\rangle} \|g(\kappa_Q(a^{-1}\bar{n}a))\| \leqslant \sup_{k \in K} \|g(k)\|,$$

using again that $\operatorname{Re}\lambda - \rho_Q \in \overline{\mathfrak{a}_0^{*+}}$. By the dominated convergence theorem we may take the limit under the integral sign in (125) as $a \xrightarrow{Q} \infty$, and (123) follows. ∎

PROPOSITION 15.7. *Let* $T \in \mathscr{H}_{\xi,F} \bar{\otimes} V(\xi)$. *Then*

$$C_{\bar{Q}|Q}(1:\lambda)\,\psi_T = c(A_q)\,\psi_{(A(\bar{Q}:Q:\xi:-\lambda)\otimes I)T}. \tag{126}$$

Proof. We may assume that $T = f \bar{\otimes} \eta$, with $f \in \mathscr{H}_{\xi,F}$, $\eta \in V(\xi)$. Assume that $\operatorname{Re}\lambda + \rho_Q$ is strictly $\bar{Q}$-dominant. Then $g_\lambda = j(Q:\xi:\bar{\lambda})\eta$ belongs to $C(Q:\xi:\bar{\lambda})$, by [4, Proposition 5.6]. Let $f_\lambda \in C(Q:\xi:-\lambda)$ be the function defined by $f_\lambda \mid K = f$. Then from (35) we obtain, for $w \in \mathscr{W}$, $m \in M$, $a \in A_{\mathfrak{q}}$, that

$$E(Q:\psi_T:\lambda)(maw)(k) = \langle R_{km}^{-1} f_\lambda, R(a)[R_w g_\lambda]\rangle.$$

Applying Lemma 15.6 and observing that $R_w g_\lambda(e) = \mathrm{pr}_w \eta$ we obtain that

$$\begin{aligned}
&\lim_{a \xrightarrow{\bar{Q}} \infty} a^{-\lambda-\rho_Q} E(Q:\psi_T:\lambda)(maw)(k)\\
&\quad = \langle A(\bar{Q}:Q:\xi:-\lambda)\, f_\lambda(m^{-1}k^{-1}), \mathrm{pr}_w \eta\rangle\\
&\quad = \gamma_{(A(\bar{Q}:Q:\xi:-\lambda)\otimes \mathrm{pr}_w)T}(m)(k).
\end{aligned} \tag{127}$$

On the other hand, from the asymptotic behaviour of the Eisenstein integral (cf. (115) and Lemma 14.5) we see that the left hand side of (127) equals

$$\mathrm{pr}_w \circ C_{\bar{Q}|Q}(1:\lambda)\psi_T(m)(k).$$

This implies the result for $\operatorname{Re}\lambda$ strictly $\bar{Q}$-dominant. Now apply meromorphic continuation. ∎

Let $P_1, P_2 \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$ and let $\xi \in \hat{M}_{\mathrm{fu}}$, the set of (equivalence classes of) finite dimensional irreducible unitary representations of M. Then according to [25] we have that

$$\eta(P_2:P_1:\xi:\lambda) = \eta(P_1:P_2:\xi:\lambda). \tag{128}$$

Now let $\alpha \in \Sigma$ be a reduced root, and define the closed subgroup $G(\alpha)$ of G as in [4, p. 392]. Then $G(\alpha)$ is σ- and θ-invariant and of Harish-Chandra's class, and $\mathfrak{a}_{\mathfrak{q}}(\alpha) = (\ker\alpha)^\perp$ is maximal abelian in $\mathfrak{g}(\alpha) \cap \mathfrak{p} \cap \mathfrak{q}$. Thus $G(\alpha)$ is of σ-split rank 1. Let $P(\alpha) \subset G(\alpha)$ be the $\sigma\theta$-stable parabolic subgroup associated with the root α as in [4, p. 392]. Given $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, put $\lambda_\alpha = \lambda \mid \mathfrak{a}_{\mathfrak{q}}(\alpha)$. We define the function η_α by

$$\eta_\alpha(\xi:\lambda) = \eta(G(\alpha):\bar{P}(\alpha):P(\alpha):\xi:\lambda_\alpha) \qquad (\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*).$$

Notice that (128) implies

$$\eta_\alpha(\xi:\lambda) = \eta_{-\alpha}(\xi:\lambda). \tag{129}$$

Given a subset $S \subset \Sigma$ we shall write S_r for the set of reduced roots in S.

LEMMA 15.8. $\eta(P_2 : P_1 : \xi : \lambda) = \prod_{\alpha \in \Sigma_r(\bar{P}_2) \cap \Sigma_r(P_1)} \eta_\alpha(\xi : \lambda)$.

Proof. The proof is standard and follows [25], but with respect to the root system of $\mathfrak{a}_\mathfrak{q}$. First we use the product decomposition of intertwining operators to reduce to the case that P_1 and P_2 are σ-adjacent (cf. [4, p. 390]). Let then α be the reduced root in $\Sigma(\bar{P}_2) \cap \Sigma(P_1)$, and let $G(\alpha)$ be as above. Then restriction induces surjective linear maps $i^* : C^\infty(P_j : \xi : \lambda) \to C^\infty(G(\alpha) : P_j(\alpha) : \xi : \lambda_\alpha)$ where $P_j(\alpha) = P_j \cap G(\alpha)$, $j = 1, 2$. Moreover, the associated intertwining operators are related by $A(P_2(\alpha) : P_1(\alpha) : \xi : \lambda_\alpha) \circ i^* = i^* \circ A(P_2 : P_1 : \xi : \lambda)$ and a similar formula with P_1, P_2 interchanged. Since $P_1(\alpha) = P(\alpha)$, $P_2(\alpha) = \bar{P}(\alpha)$, this implies the result. ∎

In view of (129) the function

$$\eta(\xi : \lambda) = \prod_{\alpha \in \Sigma_r^+} \eta_\alpha(\xi : \lambda)$$

is independent of the chosen of positive roots. In particular it follows that

$$\eta(\xi : \lambda) = \eta(\bar{Q} : Q : \xi : \lambda) \tag{130}$$

for every $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$.

LEMMA 15.9. *Let $\xi \in \hat{M}_{\mathrm{fu}}$. Then for every $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$ we have*

$$A(\bar{Q} : Q : \xi : -\bar{\lambda})^* \circ A(\bar{Q} : Q : \xi : \lambda) = \eta(\xi : \lambda) I.$$

Proof. Use [4, Proposition 4.6(ii)] in combination with (116) and (130). ∎

LEMMA 15.10. *Let $\xi \in \hat{M}_{\mathrm{fu}}$. Then for every $w \in W$ we have $\eta(w\xi : w\lambda) = \eta(\xi : \lambda)$ $(\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*)$.*

Proof. Use [4, Lemma 4.10] in combination with the previous result. ∎

COROLLARY 15.11. *For all $P, Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\xi \in \hat{M}_{\mathrm{ps}}$, $s \in W$, we have that $C_{Q|P}(s : \lambda)$ defines a linear map ${}^\circ\mathscr{C}(\xi) \to {}^\circ\mathscr{C}(s\xi)$ depending meromorphically on $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$. Moreover, if* (B) *holds, then on ${}^\circ\mathscr{C}(\xi)$ we have*

$$C_{Q|P}(s : -\bar{\lambda})^* C_{Q|P}(s : \lambda) = c(A_\mathfrak{q})^2 \eta(\xi : \lambda) I. \tag{131}$$

Proof. From Proposition 15.7 and Lemma 4.1 we infer that $C_{Q|\bar{Q}}(1 : \lambda)$ maps ${}^\circ\mathscr{C}(\xi)$ into itself. Thus, combining Proposition 15.7 with Lemma 15.9 we obtain the result with $P = \bar{Q}$, and $s = 1$. Applying Corollary 15.3 we obtain the result for all P, Q and $s=1$. Let $s \in W$, and let $w \in N_K(\mathfrak{a}_\mathfrak{q})$ be a representative for s. Then by (121) we have that

$$C_{Q|P}(s:\lambda) = C_{Q|sPs^{-1}}(1:s\lambda) \circ \mathscr{L}(w).$$

Now $\mathscr{L}(w)$ maps ${}^\circ\mathscr{C}(\xi)$ unitarily onto ${}^\circ\mathscr{C}(s\xi)$. By the first part of the proof this implies that $C_{Q|P}(s:\lambda)$ maps ${}^\circ\mathscr{C}(\xi)$ onto ${}^\circ\mathscr{C}(s\xi)$. Moreover, if (B) holds, then

$$C_{Q|P}(s:-\bar{\lambda})^* C_{Q|P}(s:\lambda) = c(A_\mathfrak{q})^2 \eta(s\xi : s\lambda) I$$

on ${}^\circ\mathscr{C}(\xi)$. Now use Lemma 15.10 to complete the proof. ∎

16. A Normalized Eisenstein Integral

With Proposition 15.7 in mind, we define the normalized Eisenstein integral

$$E^1(P:\psi:\lambda) := E(P:C_{\bar{P}|P}(1:\lambda)^{-1}\psi:\lambda), \tag{132}$$

for $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\psi \in {}^\circ\mathscr{C}$, $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$. Notice that the present normalization is slightly different from the ones introduced by Harish-Chandra (cf. [15, p. 135, 18, p. 152]). Nevertheless the effect of the present normalization still is that the functional equations for the normalized Eisenstein integral are cast in a nice form. Moreover, if G satisfies (B), then the associated normalized c-functions $C^1_{Q|P}(s:\lambda)$ turn out to be unitary for imaginary λ (Theorem 16.3). We also show that the normalization does not affect the nature of the initial estimates for the Eisenstein integral (Proposition 16.1 and Corollary 16.2).

Recall the definition (44) of $\mathfrak{a}^*_\mathfrak{q}(P, R)$.

PROPOSITION 16.1. *Let $R \in \mathbf{R}$. Then $\lambda \mapsto C_{\bar{P}|P}(1:\lambda)^{-1}$ is a meromorphic* End$({}^\circ\mathscr{C})$*-valued function of Σ-polynomial growth on $\mathfrak{a}^*_\mathfrak{q}(P, R)$.*

We postpone the proof to the end of this section.

COROLLARY 16.2. *Lemma* 4.5, *Proposition* 10.3, *and Lemma* 14.1 *hold with the normalized Eisenstein integral E^1 instead of E.*

Proof. From Proposition 15.7 and the displayed formula for μ_P in the proof of Lemma 4.5 we infer that for every $D \in \mathbf{D}(G/H)$ the endomorphisms $\mu_P(D:\lambda)$ and $C_{\bar{P}|P}(1:\lambda)$ of ${}^\circ\mathscr{C}$ commute. Hence Lemma 4.5 holds for E^1. Proposition 10.3 now follows for E^1 if we use

Proposition 16.1, and Lemma 14.1 follows from these results with unaltered proof. ∎

In view of the above result the normalized Eisenstein integrals possess (Q, w)-principal terms ($Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $w \in \mathscr{W}$) as defined in Section 14. They are given by

$$\begin{aligned} &E^1_{Q,w}(P : \psi : \lambda)(ma) \\ &\quad = \sum_{s \in W} a^{s\lambda} [C^1_{Q|P,w}(s : \lambda)\psi](m) \qquad (m \in M_\sigma,\, a \in A_\mathfrak{q}), \end{aligned} \tag{133}$$

where

$$C^1_{Q|P}(s : \lambda) := C_{Q|P}(s : \lambda) \circ C_{\bar{P}|P}(1 : \lambda)^{-1} \tag{134}$$

are called normalized c-functions. The following unitarity result is the analogue of [18, Lemma 6].

THEOREM 16.3. *Let $P, Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$, and suppose that G satisfies condition* (B) *of the previous section. Then*

$$C^1_{Q|P}(s : -\bar{\lambda})^* \circ C^1_{Q|P}(s : \lambda) = I, \tag{135}$$

for $\lambda \in \mathfrak{a}^*_{\mathfrak{q}c}$. In particular $C^1_{Q|P}(s : \lambda)$ is unitary for imaginary λ.

Proof. It suffices to prove the above identity on ${}^\circ\mathscr{C}(\xi)$, for $\xi \in \hat{M}_{\mathrm{ps}}$. But then the identity is a direct consequence of definition (134) and Corollary 15.11. ∎

We now arrive at the functional equation for the normalized Eisenstein integral.

PROPOSITION 16.4. *Let $P_1, P_2 \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\psi \in {}^\circ\mathscr{C}$, $s \in W$. Then*

$$E^1(P_2 : C^1_{P_2|P_1}(s : \lambda)\psi : s\lambda) = E^1(P_1 : \psi : \lambda).$$

Proof. Let $w \in N_K(\mathfrak{a}_\mathfrak{q})$ be a representative for s. Then by application of Lemma 15.4 and Corollary 15.5 we obtain that

$$\begin{aligned} E^1(P_2 : \psi : s\lambda) &= E(P_2 : C_{\bar{P}_2|P_2}(1 : s\lambda)^{-1}\psi : s\lambda) \\ &= E(w^{-1}P_2 w : \mathscr{L}(w)^{-1} C_{\bar{P}_2|P_2}(1 : s\lambda)^{-1}\psi : \lambda) \\ &= E(w^{-1}P_2 w : C_{\bar{P}_2|w^{-1}P_2 w}(s : \lambda)^{-1}\psi : \lambda). \end{aligned} \tag{136}$$

Applying Lemma 15.2 to (136) and using that

$$U(P_1 : w^{-1}P_2 w : \lambda)\, C_{\bar{P}_2|w^{-1}P_2 w}(s : \lambda)^{-1} = C_{\bar{P}_2|P_1}(s : \lambda)^{-1}$$

(see Lemma 15.3) we find that

$$E(P_2 : \psi : s\lambda) = E(P_1 : C_{\bar{P}_2|P_1}(s:\lambda)^{-1}\psi : \lambda)$$
$$= E^1(P_1 : C^1_{\bar{P}_2|P_1}(s:\lambda)^{-1}\psi : \lambda). \quad \blacksquare$$

COROLLARY 16.5. *Let $Q, P_1, P_2 \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $s, t \in W$. Then*

$$C^1_{Q|P_2}(t : s\lambda)\, C^1_{\bar{P}_2|P_1}(s:\lambda) = C^1_{Q|P_1}(ts : \lambda).$$

In the rest of this section we shall estimate the inverted c-function $C_{\bar{P}|P}(1:\lambda)^{-1}$. According to Proposition 15.7 this comes down to estimating intertwining operators and their inverses on the level of K-finite functions.

Suppose that $\xi \in \hat{M}_{\mathrm{fu}}$, let $F \subset \hat{K}$ be a finite subset, and write $A(P_2 : P_1 : \xi : \lambda)_F$ for the restriction of the intertwining operator to $C(K:\xi)_F$. Moreover, if $R \in \mathbf{R}$ put

$$\mathfrak{a}^*_\mathfrak{q}(P_2, P_1, R) = \{\lambda \in \mathfrak{a}^*_{\mathfrak{q}c};\ \operatorname{Re}\langle\lambda, \alpha\rangle < R \text{ for } \alpha \in \Sigma(P_2) \cap \Sigma(\bar{P}_1)\}.$$

LEMMA 16.6. *Let $R \in \mathbf{R}$. Then the* $\operatorname{End}(C(K:\xi)_F)$*-valued functions $\lambda \mapsto A(P_2 : P_1 : \xi : \lambda)_F$ and $\lambda \mapsto A(P_2 : P_1 : \xi : \lambda)_F^{-1}$ are of Σ-polynomial growth on $\mathfrak{a}^*_\mathfrak{q}(P_2, P_1, R)$.*

Proof. We shall prove this by using an embedding of the induced representation into the (non-unitary) principal series. Let notations be as in [4, Proof of Lemma 4.5]. Thus $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_\mathfrak{q}$, and $(P_j)_\mathfrak{p} = P_\mathbf{M} A N_j$ are minimal parabolic subgroups containing $A_0 = \exp \mathfrak{a}_0$ as defined in [4, p. 372]. Let $(N_j)_\mathfrak{p}$ be the unipotent radical of $(P_j)_\mathfrak{p}$, $j = 1, 2$. Then $(N_2)_\mathfrak{p} \cap (\bar{N}_1)_\mathfrak{p} = N_2 \cap \bar{N}_1$. Hence if $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_0)$ is a root occurring in $(\mathfrak{n}_2)_\mathfrak{p} \cap (\bar{\mathfrak{n}}_1)_\mathfrak{p}$, then $\alpha \mid \mathfrak{a}_\mathfrak{q} \in \Sigma(P_2) \cap \Sigma(\bar{P}_1)$. In addition there exists a suitable $R' \in \mathbf{R}$ such that $\langle \operatorname{Re}\lambda - \rho_\mathbf{M}, \alpha\rangle < R'$ for $\lambda \in \mathfrak{a}^*_\mathfrak{q}(P_2, P_1, R)$. Using the embeddings in the principal series described by the diagram in [4, p. 373], we see that we may reduce the proof to the case that $\sigma = \theta$. Then $\mathfrak{a}_0 = \mathfrak{a}_\mathfrak{q}$ and P_1, P_2 are minimal parabolic subgroups.

Without loss of generality we may assume that $F = \{\delta\}$, where $\delta \in \hat{K}$. Let V_δ be a representation space for δ. By the usual product decomposition for intertwining operators we may restrict ourselves to the case that P_1, P_2 are adjacent. Let α be the reduced root in $\Sigma(\bar{P}_2) \cap \Sigma(P_1)$.

By the Peter–Weyl theorem and Frobenius reciprocity we have a natural bijective linear map

$$\varphi : V_\delta \otimes \operatorname{Hom}_\mathbf{M}(V_\delta, \mathscr{H}_\xi) \to C(K:\xi)_\delta$$

intertwining $\delta \otimes I$ with R. It is given by $\varphi(v \otimes f)(k) = f(\delta(k)v)$. By equivariance the endomorphism $\varphi^{-1} \circ A(P_2 : P_1 : \xi : \lambda) \circ \varphi$ is of the form $I \otimes J(\lambda)$, where $J(\lambda) \in \mathrm{End}(\mathrm{Hom}_M(V_\delta, \mathscr{H}_\xi))$ depends meromorphically on $\lambda \in \mathfrak{a}_{0c}^*$. Moreover, an easy calculation shows that $J(\lambda) = c(\lambda)^* \otimes I$, where $c(\lambda) \in \mathrm{End}_M(V_\delta)$. For $\langle \mathrm{Re}\, \lambda, \alpha \rangle > 0$ this endomorphism is given by the absolutely convergent integral

$$c(\lambda) = \int_{N_2 \cap \bar{N}_1} e^{(\lambda + \rho_1) H_1(\bar{n})} \delta(\kappa_1(\bar{n}))\, d\bar{n}.$$

Here $\rho_1 = \rho_{P_1}$ and the maps $H_1 : G \to \mathfrak{a}_0$, $\kappa_1 : G \to K$ are defined by $x \in N_1 \exp H_1(x)\, \kappa_1(x)$, for $x \in G$.

Now let $G_1(\alpha) = Z_G(\ker \alpha)$, $K(\alpha) = K \cap G_1(\alpha)$, $N_\alpha = N_1 \cap G_1(\alpha)$, and $A_0(\alpha) = \exp(\mathfrak{a}_0 \cap \ker \alpha^\perp)$. Then

$$G(\alpha) = N_\alpha A_0(\alpha)\, K(\alpha)$$

is the Iwasawa decomposition of a split rank one subgroup of Harish-Chandra's class. This decomposition is compatible with $G = N_1 A_0 K$, so the associated maps $H_\alpha : G(\alpha) \to \mathfrak{a}_0(\alpha)$ and $\kappa_\alpha : G(\alpha) \to K(\alpha)$ are the restrictions to $G(\alpha)$ of H_1 and κ_1, respectively. Let $\rho_\alpha \in \mathfrak{a}_0(\alpha)^*$ be defined by $\rho_\alpha(X) = (1/2)\, \mathrm{tr}[\mathrm{ad}(X) \mid \mathfrak{n}_\alpha]$. Then with $G(\alpha)$ and $\delta' = \delta \mid K(\alpha)$ we may associate the c-function $C_{\delta'} : \mathfrak{a}_0(\alpha)_c^* \to \mathrm{End}_M(V_\delta)$ defined by

$$C_{\delta'}(v) = \int_{\bar{N}_\alpha} e^{(v + \rho_\alpha) H_\alpha(\bar{n})} \delta'(\kappa_\alpha(\bar{n}))\, d\bar{n}.$$

Now $N_2 \cap \bar{N}_1 = \bar{N}_\alpha$ and $\rho_\alpha = \rho_1 \mid \mathfrak{a}_0(\alpha)$, and we see that

$$c(\lambda) = C_{\delta'}(\lambda \mid \mathfrak{a}_0(\alpha)) \qquad (\lambda \in \mathfrak{a}_{0c}^*).$$

According to [32, 29] the matrix entries of $C_{\delta'}(v)$ are linear combinations of products of functions of the form

$$\frac{\Gamma(r\langle v, \alpha \rangle + s)}{\Gamma(r\langle v, \alpha \rangle + t)}, \tag{137}$$

where $r > 0$, $s, t \in \mathbf{R}$. This implies that $C_{\delta'}(v)$ is of $\{\alpha\}$-polynomial growth on sets of the form $\langle \mathrm{Re}\, v, \alpha \rangle > R$, $R \in \mathbf{R}$ (see also the argument in [1]). Moreover, in [11] it is proved that $\det C_{\delta'}(v)$ is a product of functions of the form (137) and by Cramer's rule it follows that $C_{\delta'}(v)^{-1}$ is of $\{\alpha\}$-polynomial growth on sets $\langle \mathrm{Re}\, v, \alpha \rangle > R$. These estimates give us the desired estimates for the intertwining operator and its inverse. ∎

Proof of Proposition 16.1. It suffices to prove the assertion for the restriction of the inverted c-function to each invariant subspace ${}^\circ\mathscr{C}(\xi)$,

$\xi \in \hat{M}_{ps}$. Now by Proposition 15.7 and the previous lemma it follows that $\lambda \mapsto C_{\bar{P}|P}(1:\lambda)^{-1}$ is of Σ-polynomial growth on $-\mathfrak{a}_{\mathfrak{q}}^*(\bar{P}, P, R) = \mathfrak{a}_{\mathfrak{q}}^*(P, R)$. ∎

17. Schwartz Functions

In this section we characterize the generalization to G/H of Harish-Chandra's space of Schwartz functions in the group case. In particular this provides us with the dual notion of temperedness on G/H.

Throughout this section V will be a complete locally convex (Hausdorff) space, and $\mathcal{N}(V)$ will denote the set of continuous seminorms on V. Given $s \in \mathcal{N}(V)$ we shall sometimes use the notation $|v|_s = s(v)$ $(v \in V)$.

Let $\tau: G \to [0, \infty[$ be defined by

$$\tau(kah) = |\log a| \qquad (k \in K, a \in A_{\mathfrak{q}}, h \in H).$$

For $1 \leqslant p < \infty$ we define the space $\mathscr{C}^p(G/H, V)$ of L^p-Schwartz functions on G/H to be the space of all C^∞ functions $f: G/H \to V$ (where C^∞ means that all partial derivatives exist), such that for all $u \in U(g)$, $r \geqslant 0$, and $s \in \mathcal{N}(V)$ the function $(1+\tau)^r |uf|_s$ has finite L^p-norm; here we recall that $uf = L_u f$. In particular we shall write $\mathscr{C}(G/H, V)$ for the L^2-Schwartz space.

The space $\mathscr{C}^p(G/H, V)$ equipped with the seminorms

$$f \mapsto \|(1+\tau)^r |uf|_s\|_p \qquad (u \in U(\mathfrak{g}), r \geqslant 0) \tag{138}$$

is a complete locally convex space. If V is Fréchet, then the same holds for $\mathscr{C}^p(G/H, V)$. The space $\mathscr{C}^p(G/H) := \mathscr{C}^p(G/H, \mathbf{C})$ was introduced in [2, p. 246].

The purpose of this section is to establish a different characterization of the space $\mathscr{C}^p(G/H, V)$ in terms of sup norms. Let Ξ denote Harish-Chandra's bi-K-invariant elementary spherical function φ_0 on G (cf. [30, p. 329]). Define the real analytic function $\Theta: G/H \to]0, \infty[$ by

$$\Theta(x) = \sqrt{\Xi(x\sigma(x)^{-1})} \qquad (x \in G). \tag{139}$$

We now define $\mathscr{C}_0^p(G/H, V)$ to be the space of smooth functions $f: G/H \to V$ for which all seminorms

$$\mu_{s,u,r}^p(f) := \sup_{G/H} \Theta^{-2/p}(1+\tau)^r |uf|_s$$

$(s \in \mathcal{N}(V)$, $u \in U(\mathfrak{g})$, $r \geqslant 0)$ are finite. Equipped with these seminorms the space $\mathscr{C}_0^p(G/H, V)$ is a complete locally convex space; it is Fréchet if V is Fréchet. The main result of this section is the following generalization of a well known result of Harish-Chandra (cf. [30, Theorem 9, p. 348]).

THEOREM 17.1. *The spaces $\mathscr{C}^p(G/H, V)$ and $\mathscr{C}^p_0(G/H, V)$ are equal, and their topologies are the same.*

The rest of this section will be devoted to the proof of this result. First we need some properties of the function Θ. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_\mathfrak{q}$. Let Σ_0 be the root system of $\mathfrak{a}_0$ in $\mathfrak{g}$ and let d be one half times the number of indivisible roots in Σ_0. Then the following result describes the asymptotic behaviour of Θ.

PROPOSITION 17.2. *Let $Q \in \mathscr{P}_\sigma(A_\mathfrak{q})$. Then there exists a constant $C > 0$ such that for all $a \in \mathrm{cl}\, A_\mathfrak{q}^+(Q)$ we have that*

$$a^{-\rho_Q} \leqslant \Theta(a) \leqslant Ca^{-\rho_Q}(1 + \tau(a))^d.$$

Proof. Fix a system Σ_0^+ of positive roots for Σ_0 which is compatible with $\Sigma(Q)$. Then for the associated positive Weyl chambers we have $\mathfrak{a}_\mathfrak{q}^+(Q) \subset \mathrm{cl}\, \mathfrak{a}_0^+$. Let $\rho_0 \in \mathfrak{a}_0^*$ be half the sum of the roots in Σ_0^+, counted with multiplicities. Then $\rho_Q = \rho_0 \mid \mathfrak{a}_\mathfrak{q}$.

If $a \in \mathrm{cl}\, A_\mathfrak{q}^+(Q)$, then $a\sigma(a)^{-1} = a^2 \in \mathrm{cl}\, A_0^+$, and we have that $\Theta(a)^2 = \Xi(a^2)$. We now obtain the above estimates as a straightforward consequence of the well known estimates for Ξ on $\mathrm{cl}\, A_0^+$, see [30, Theorem 30, p. 339]. ∎

We shall also need the following (more elementary) properties of Θ. They are straightforward consequences of the corresponding properties of Ξ, cf. [30, p. 329].

PROPOSITION 17.3. *The function Θ is real analytic and has the following properties.*

(1) $0 < \Theta(x) = \Theta(\sigma(x)) \leqslant 1$ $(x \in G)$.

(2) *Let E be a compact subset of G. Then there exists* a $c > 0$ *such that for all $x \in G/H$, $y \in E$ we have*

$$c^{-1}\Theta(x) \leqslant \Theta(yx) \leqslant c\Theta(x).$$

(3) *Let $u \in U(\mathfrak{g})$. Then there exists a $C > 0$ such that*

$$|u\Theta(x)| \leqslant C\Theta(x) \qquad (x \in G/H).$$

(4) *$\Theta(x)$ depends on x only through* $\mathrm{Ad}(x\sigma(x)^{-1})$.

Finally we recall some properties of τ from [3, Proposition 2.1]. Let $\tau_G: G \to \mathbf{R}$ be defined by $\tau_G(k_1 a k_2) = \|\log a\|$ for $k_1, k_2 \in K$, $a \in A_0$.

PROPOSITION 17.4. *The function τ is continuous, and left K- and right*

H-invariant. Moreover, $\tau(e)=0$ *and* $\tau(x)>0$ *for* $x\notin KH$. *Finally, if* $x\in G/H$, $y\in G$, *then*

$$\tau(x)=\tau(\sigma(x)),$$
$$\tau(yx)\leqslant\tau_G(y)+\tau(x).$$

Notice that from the last inequality in the above proposition it follows that

$$1+\tau(yx)\leqslant(1+\tau_G(y))(1+\tau(x)). \qquad (140)$$

From Propositions 17.3 and 17.4 it follows that the space $\mathscr{C}_0^p(G/H, V)$ is invariant under the left regular representation L of G.

Let G_+ denote the closed subgroup $(K\cap H)\exp(\mathfrak{p}\cap\mathfrak{q})$ of G. Its Lie algebra is $\mathfrak{g}_+$ (cf. (1)). If S is a subgroup of G we write $S_+=S\cap G_+$. Thus $H_+=K_+=H\cap K$. Put $X=G/H$ and $X_+=G_+/H_+$. We shall view the Riemannian symmetric space X_+ as a subspace of X.

Consider the action of the group K_+ on $K\times X_+$ by $k_+\cdot(k, x_+)=(kk_+^{-1}, k_+x_+)$. Then the map $(k, x_+)\mapsto kx_+$ induces a diffeomorphism

$$K\times_{K\cap H}X_+\xrightarrow{\approx}X;$$

this is a straightforward consequence of the fact that the map (4.3) in [12] is a diffeomorphism. It follows that there exists a unique left K-invariant real analytic function $J_-: X\to\,]0,\infty[$ such that

$$\int_X f(x)\,dx=\int_K\int_{X_+} f(kx_+)\,J_-(x_+)\,dx_+\,dk \qquad (141)$$

for all $f\in C_c(X)$. Here dx_+ denotes normalized left G_+-invariant measure on X_+. Let Σ_+^+ be a choice of positive roots for the root system Σ_+ of $\mathfrak{a}_\mathfrak{q}$ in $\mathfrak{g}_+$. Then on the associated positive Weyl chamber $A_\mathfrak{q}^+$ we have that

$$J=J_-J_+,$$

where $J(J_+)$ denotes the Jacobian of the $G=K\,\mathrm{cl}(A_\mathfrak{q}^+)H$ decomposition (resp. $G_+=K_+\,\mathrm{cl}(A_\mathfrak{q}^+)\,K_+$ decomposition). From the formulas for these Jacobians (cf. [13, Theorem 2.6]) we obtain that (for a suitable choice of normalization for dx_+)

$$J_-(a)=\prod_{\alpha\in\Sigma^+}(a^\alpha+a^{-\alpha})^{m_-(\alpha)} \qquad (a\in A_\mathfrak{q}). \qquad (142)$$

Here Σ^+ is a choice of positive roots for $\Sigma=\Sigma(\mathfrak{g}, \mathfrak{a}_\mathfrak{q})$ which is compatible with Σ_+^+, and $m_-(\alpha)=\dim(\mathfrak{g}_\alpha\cap\mathfrak{g}_-)$.

Now let Ξ_+ denote Harish-Chandra's spherical function for G_+. We extend Ξ_+ to a left K-invariant real analytic function on X.

PROPOSITION 17.5. *There esist constants $m \in \mathbf{N}$, $C > 0$ such that on $X = G/H$ we have*

$$C^{-1}(1+\tau)^{-m}\Theta \leqslant J_-^{-1/2}\Xi_+ \leqslant C(1+\tau)^m\Theta.$$

Proof. This follows easily from (142) combined with the estimate to Θ in Proposition 17.2 and the analogous estimate for Ξ_+. ▮

COROLLARY 17.6. *There exists a $m \in \mathbf{N}$ such that*

$$(1+\tau)^{-m}\Theta^2 \in L^1(G/H).$$

Proof. Use the analogous result for Ξ_+ in combination with the above estimate and formula (141). ▮

COROLLARY 17.7. *The space $\mathscr{C}_0^p(X, V)$ is a subspace of $\mathscr{C}^p(X, C)$, the embedding being continuous.*

Thus we have established (the easy) part of Theorem 17.1. We will prove the converse inclusion by reduction to the space X_+ via (141). In this way we avoid some of the technicalities which would arise from a reduction to $A_{\mathfrak{q}}^+$ via the $K\,\mathrm{cl}(A_{\mathfrak{q}}^+)H$-decomposition (compare with the proof in [30, pp. 346–348]). This is due to the fact that the Jacobian J_- allows a nice estimate from below (Proposition 17.5).

We start with a simple lemma. Let $X_1, ..., X_n$ be an orthonormal basis for $\mathfrak{k}$, and define $\Omega \in U(\mathfrak{k})$ by

$$\Omega = 1 - X_1^2 - \cdots - X_n^2.$$

If $\delta \in \hat{K}$, let $c(\delta)$ denote the constant by which Ω acts on the $\mathfrak{k}$-module associated with δ.

Let $L_\infty^p(X, V)$ denote the space of $f \in C^\infty(X, V)$ such that $|uf|_s \in L^p(X)$ for all $u \in U(\mathfrak{g})$, $s \in \mathscr{N}(V)$. Put $L_\infty^p(X) = L_\infty^p(X, \mathbf{C})$. If f is a complex valued measurable function on X_+ we put

$$\|f\|_{X_+,p} = \left(\int_{X_+} J_-(x_+)\,|f(x_+)|^p\,dx_+\right)^{1/p}.$$

LEMMA 17.8. *There exist constants $m \in \mathbf{N}$, $C > 0$ such that for each $\delta \in \hat{K}$ and every $f \in L_\infty^p(X)_\delta$ we have*

$$\|f\|_{X_+,p} \leqslant Cc(\delta)^m\,\|f\|_p.$$

Proof. Proceed as in the proof of Lemma 6 of [30, p. 346]. ∎

COROLLARY 17.9. *There exist constants $m \in \mathbf{N}$, $C > 0$ such that for all $f \in L^p_\infty(X, V)$ we have*

$$\|s(f)\|_{X_+, p} \leqslant C\, \|s(\Omega^n f)\|_p \qquad (s \in \mathscr{A}^{\sim}(V)).$$

Proof. Let m be as in the previous lemma and fix $n \in \mathbf{N}$ such that

$$\sum_{\delta \in \hat{K}} c(\delta)^{m-n} \dim(\delta)^2 < \infty.$$

We have $f = \sum_{\delta \in \hat{K}} \alpha_\delta * f$, where α_δ denotes $\dim \delta$ times the character of δ's contragredient. Hence

$$\begin{aligned} \|s(f)\|_{X_+, p} &\leqslant \sum_{\delta \in \hat{K}} \|s(\alpha_\delta * f)\|_{X_+, p} \\ &\leqslant C \sum_{\delta \in \hat{K}} c(\delta)^m \, \|s(\alpha_\delta * f)\|_p \\ &\leqslant C \sum_{\delta \in \hat{K}} c(\delta)^{m-n} \, \|s(\alpha_\delta * \Omega^n f)\|_p \\ &\leqslant C \Big(\sum_{\delta \in \hat{K}} c(\delta)^{m-n} \dim(\delta)^2 \Big) \, \|s(\Omega^n f)\|_p. \end{aligned}$$ ∎

In the following we need a function φ having the same growth behaviour as τ, but allowing differentiations. Let $\mathfrak{v}$ be a θ- and σ-stable central subalgebra of $\mathfrak{g}$ such that $G \simeq {}^\circ G \times \exp \mathfrak{v}$ (cf. [2, p. 227]). Given an element $Y \in \mathfrak{v}$ we write $Y = Y_{\mathfrak{h}} + Y_{\mathfrak{q}}$, with $Y_{\mathfrak{h}} \in \mathfrak{v} \cap \mathfrak{h}$, $Y_{\mathfrak{q}} \in \mathfrak{v} \cap \mathfrak{q}$. We define the function $\varphi \colon G \to \mathbf{R}$ by

$$\varphi(x \exp Y) = \sqrt{1 + \|Y_{\mathfrak{q}}\|^2} - \log \Theta(x) \qquad (x \in {}^\circ G,\ Y \in \mathfrak{v}).$$

LEMMA 17.10. *The function φ is real analytic, and left K- and right H-invariant. Moreover, there exists a $c > 0$ such that on G we have*

$$c^{-1}(1 + \tau) \leqslant \varphi \leqslant c(1 + \tau).$$

Finally, if $u \in U(\mathfrak{g})\mathfrak{g}$, then the function $u\varphi$ is uniformly bounded.

Proof. This follows from Propositions 17.2 and 17.3. ∎

LEMMA 17.11. *Let $s \in \mathcal{N}(V)$. Then there exist $v_j \in U(\mathfrak{g})$, $s_j \in \mathcal{N}(V)$ $(1 \leqslant j \leqslant r)$, and $m \in \mathbf{N}$ such that for all $f \in L^p_\infty(X, V)$ we have*

$$\sup_X \Theta^{-2/p} |f|_s \leqslant \max_{1 \leqslant j \leqslant r} \|(1+\tau)^m s_j(v_j f)\|_p.$$

Proof. It suffices to prove a similar estimate for the supremum over X_+; the general estimate then follows from replacing f by $L_k f$ $(k \in K)$.

Write $\Theta_- = \Theta \Xi_+^{-1}$. Then from Proposition 17.5 it follows that there exist $c > 0$, $l \in \mathbf{N}$ such that

$$c^{-1}(1+\tau)^{-l} \leqslant J_-^{1/2} \Theta_- \leqslant c(1+\tau)^l.$$

The analogue of the lemma for X_+ is valid by a result of Harish-Chandra, cf. [30, Theorem 9, p. 348]. Hence there exist $u_1, ..., u_q \in U(\mathfrak{g}_+)$, $v_1, ..., v_q \in \mathcal{N}(V)$, and $n \in \mathbf{N}$ such that for $f \in L^p_\infty(X, V)$ we have

$$\begin{aligned} \sup_{X_+} \Theta^{-2/p} |f|_s &\leqslant C' \max_{1 \leqslant j \leqslant q} \|(1+\tau)^n v_j(u_j[\Theta_-^{-2/p} f])\|_{L^p(X_+)} \\ &\leqslant C'' \max_{1 \leqslant j \leqslant q} \|(1+\tau)^{n'} \Theta_-^{2/p} v_j(u_j[\Theta_-^{-2/p} f])\|_{X_+, p}, \end{aligned} \tag{143}$$

where $n' = n + l$.

We now observe that for every $w \in U(\mathfrak{g}_+)$ there exists a constant $C_w > 0$ such that

$$|L_w \Theta_-^{-2/p}| \leqslant C_w \Theta_-^{-2/p}.$$

(This follows from Proposition 17.3(3) and the analogous estimate for Ξ_+ by repeatedly using the Leibniz rule.) Hence there exist $u'_1, ..., u'_r \in U(\mathfrak{g}_+)$ and $s_1, ..., s_r \in \mathcal{N}(V)$ (not depending on f), such that (143) may be estimated by

$$C_1 \max_{1 \leqslant j \leqslant r} \|s_j(\varphi^{n'} u'_j f)\|_{X_+, p}.$$

Taking into account that φ is left K-invariant and using Corollary 17.9 and Lemma 17.10 we can estimate the latter expression by

$$C_2 \max_{1 \leqslant j \leqslant r} \|(1+\tau)^{n'} s_j(\Omega^n u'_j f)\|_p$$

with C_2 a constant independent of f. This is the required estimate. ∎

Completion of the Proof of Theorem 17.1. Let $n \in \mathbf{N}$, $s \in \mathcal{N}(V)$. Then it suffices to prove that $f \mapsto \sup_X |(1+\tau)^n \Theta^{-2/p} f|_s$ is a continuous seminorm on $\mathscr{C}^p(X, V)$. Now apply the previous lemma to $\varphi^n f$, and use Lemma 17.10. ∎

18. Uniform Temperedness of Eigenfunctions

The purpose of this section is to improve upon initial estimates for families of eigenfunctions like the Eisenstein integrals, using the differential equations satisfied by them. In particular this will imply that Eisenstein integrals are tempered, with uniformity in λ.

Let $\mathfrak{b}$ be as in Section 2, write $W(\mathfrak{b}) = W(\mathfrak{g}_c, \mathfrak{b}_c)$, and let $\gamma: \mathbf{D}(G/H) \to S(\mathfrak{b})^{W(\mathfrak{b})}$ be Harish-Chandra's isomorphism. If $\varepsilon > 0$, we recall that

$$\mathfrak{a}_{\mathfrak{q}}^*(\varepsilon) = \{\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*;\ |\mathrm{Re}(\lambda)| < \varepsilon\}.$$

Fix $\Lambda \in i\mathfrak{b}_{\mathfrak{k}}^*$. Then by $\mathscr{E}(G/H, \Lambda, \varepsilon) = \mathscr{E}(\Lambda, \varepsilon)$ we denote the space of C^∞-functions $f: \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon) \times G/H \to \mathbf{C}$ such that

(1) f is holomorphic in its first variable; and

(2) for every $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$ we have

$$Df_\lambda = \gamma(D : \Lambda + \lambda) f_\lambda \qquad (D \in \mathbf{D}(G/H)). \tag{144}$$

Here $f_\lambda = f(\lambda, \cdot)$. A function $f \in \mathscr{E}(\Lambda, \varepsilon)$ will be called uniformly tempered of scale s if for every $u \in U(\mathfrak{g})$ there exist constants $n \in \mathbf{N}$, $C > 0$ such that

$$|L_u f_\lambda(x)| \leqslant C\, |(\lambda, x)|^n \Theta(x)\, e^{s\,|\mathrm{Re}\,\lambda|\,\tau(x)}$$

for all $x \in G/H$ and $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$. Here we have written $|(\lambda, x)| = (1 + |\lambda|)(1 + \tau(x))$. The space of these functions will be denoted by $\mathscr{T}(\Lambda, \varepsilon, s)$.

Remark 18.1. Let $\mathscr{C}'(G/H)$ be the space of tempered distributions on G/H, i.e., the continuous linear dual of $\mathscr{C}(G/H)$, provided with the strong dual topology. If $f \in \mathscr{T}(\Lambda, \varepsilon, s)$, then it follows from Corollary 17.6 that $\lambda \mapsto f_\lambda$ is a holomorphic map from $\mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$ into $\mathscr{C}'(G/H)$ (via a choice of invariant measure we identify functions with distributions in the usual way).

Let S be a finite subset of $U(\mathfrak{g})$, and let C_n be a sequence of positive constants. Then the family $\nu = (\nu_{\varepsilon,n};\ \varepsilon > 0,\ n \in \mathbf{N})$ of seminorms $\nu_{\varepsilon,n}: C^\infty(\mathfrak{a}_{\mathfrak{q}}^*(\varepsilon) \times G/H) \to [0, \infty]$ defined by

$$\nu_{\varepsilon,n}(f) = C_n \max_{u \in S} \sup_{\substack{x \in G/H \\ \lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)}} |(\lambda, x)|^{-n} \Theta(x)^{-1} e^{-s\,|\mathrm{Re}\,\lambda|\,\tau(x)}\, |L_u f_\lambda(x)|$$

will be called a string of $\mathscr{T}(s)$-seminorms. For later use we need the following lemma.

LEMMA 18.2. *Let $\Lambda \in i\mathfrak{b}_{\mathfrak{k}}^*$, $s > 0$, and $\varepsilon > \varepsilon' > 0$. If $f \in \mathcal{T}(\Lambda, \varepsilon, s)$, then for every $u \in U(\mathfrak{g})$, $b \in S(\mathfrak{a}_{\mathfrak{q}}^*)$ there exist constants $n \in \mathbf{N}$, $C > 0$ such that*

$$|f(\lambda; b, u; x)| \leqslant C\,|(\lambda, x)|^n\,\Theta(x)\,e^{s\,|\mathrm{Re}\,\lambda|\,\tau(x)} \qquad (x \in G/H,\ \lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon')).$$

Proof. When $\deg b = 0$ this is immediate from the definition of $\mathcal{T}(\Lambda, \varepsilon, s)$. For general b the result follows by an application of Cauchy's integral formula involving a polydisc centered at λ and of radius $\min((2\sqrt{m})^{-1}(\varepsilon - \varepsilon'), (1 + \tau(x))^{-1})$, $m = \dim \mathfrak{a}_{\mathfrak{q}}$. ∎

The purpose of this section is to give a useful criterion for functions to be in the class of uniformly tempered functions.

A function $f \in \mathscr{E}(\Lambda, \varepsilon)$ will be called uniformly moderate of exponential rate $r \in \mathbf{R}$, if for every $u \in U(\mathfrak{g})$ there exist constants $n \in \mathbf{N}$, $C > 0$ such that

$$|L_u f_\lambda(x)| \leqslant C(1 + |\lambda|)^n\,e^{r\tau(x)}$$

for all $x \in G/H$ and $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$. The space of such functions will be denoted by $\mathcal{M}(\Lambda, \varepsilon, r)$. If S is a finite subset of $U(\mathfrak{g})$ and C_n a sequence of positive constants, then the family of seminorms $\mu = (\mu_{\varepsilon,n};\ \varepsilon > 0,\ n \in \mathbf{N})$ defined by

$$\mu_{\varepsilon,n}(f) = C_n \max_{u \in S} \sup_{\substack{x \in G/H \\ \lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)}} (1 + |\lambda|)^{-n}\,e^{r\tau(x)}\,|L_u f_\lambda(x)|$$

will be called a string of $\mathcal{M}(r)$-seminorms. The main result of this section will be that every function $f \in \mathscr{E}(\Lambda, \varepsilon)$ which is uniformly moderate is automatically uniformly tempered. More precisely we have the following.

THEOREM 18.3. *Let $r \in \mathbf{R}$. Then there exists a $s > 0$ such that for $\varepsilon > 0$ sufficiently small one has*

$$\mathcal{M}(\Lambda, \varepsilon, r) \subset \mathcal{T}(\Lambda, \varepsilon, s).$$

Moreover, for every string ν of $\mathcal{T}(s)$-seminorms there exists a string μ of $\mathcal{M}(r)$-seminorms and a constant $N \in \mathbf{N}$, such that for sufficiently small $\varepsilon > 0$ one has

$$\nu_{\varepsilon,n+N}(f) \leqslant \mu_{\varepsilon,n}(f),$$

for every $f \in \mathscr{E}(\Lambda, \varepsilon)$ and all $n \in \mathbf{N}$.

It suffices to prove this theorem when $G = {}^\circ G$. For the proof we need yet another type of function spaces. Let $P \in \mathscr{P}_\sigma(A_{\mathfrak{q}})$, and $\eta \in \mathfrak{a}_{\mathfrak{q}}^*$, $s \geqslant 0$. Then we define $\mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$ to be the space of functions $f \in \mathscr{E}(\Lambda, \varepsilon)$ such that for every $u \in U(\mathfrak{g})$ there exist constants $n \in \mathbf{N}$, $C > 0$ such that

$$|L_u f_\lambda(ka)| \leqslant C\,|(\lambda, a)|^n\,a^\eta\,e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}$$

for all $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon)$, $k \in K$, and $a \in \mathrm{cl}\, A_\mathfrak{q}^+(P)$. If S is a finite subset of $U(\mathfrak{g})$, and C_n a sequence of positive constants, then the family $\nu = (\nu_{\varepsilon,n})$ of seminorms defined by

$$\nu_{\varepsilon,n}(f) = \max_{u \in S} \sup_{\substack{a \in \mathrm{cl}\, A^+(P) \\ k \in K, \lambda \in \mathfrak{a}_q^*(\varepsilon)}} |(\lambda, a)|^{-n} a^{-\eta} e^{-s\,|\mathrm{Re}\,\lambda|\,|\log a|} |L_u f_\lambda(ka)|$$

is called a string of $\mathscr{E}_P(\eta, s)$-seminorms.

We first compare the spaces $\mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$ with the spaces $\mathscr{M}(\Lambda, \varepsilon, r)$ and $\mathscr{T}(\Lambda, \varepsilon, s)$. For this it will be necessary to vary the parabolic subgroup P. Select $P_0 \in \mathscr{P}_\sigma(A_\mathfrak{q})$ and set $\mathscr{P}(P_0) = \{w^{-1} P_0 w; w \in \mathscr{W}\}$. Then from (113) we deduce that

$$G = \bigcup_{P \in \mathscr{P}(P_0)} K\overline{A_\mathfrak{q}^+(P)}H.$$

The following lemma is now straightforward to prove (use Proposition 17.2):

Lemma 18.4. *Let $r \in \mathbf{R}$. Then there exists for every $P \in \mathscr{P}(P_0)$ a $\eta_P \in \mathfrak{a}_\mathfrak{q}^*$ such that for every $\varepsilon > 0$ we have*

$$\mathscr{M}(\Lambda, \varepsilon, r) \subset \bigcap_{P \in \mathscr{P}(P_0)} \mathscr{E}_P(\Lambda, \varepsilon, \eta_P, 0).$$

Moreover, fix $\varepsilon' > 0$ and let for every $P \in \mathscr{P}(P_0)$ a string ν_P of $\mathscr{E}_P(\eta_P, 0)$-seminorms be given. Then there exists a string of $\mathscr{M}(r)$-seminorms such that for every $0 < \varepsilon \leqslant \varepsilon'$ we have

$$\max_{P \in \mathscr{P}(P_0)} \nu_{P,\varepsilon,n}(f) \leqslant \mu_{\varepsilon,n}(f),$$

for all $f \in \mathscr{E}(\Lambda, \varepsilon)$, $n \in N$.

The following lemma is also straightforward to check.

Lemma 18.5. *Let $s \geqslant 0$, $\varepsilon > 0$. Then*

$$\bigcap_{P \in \mathscr{P}(P_0)} \mathscr{E}_P(\Lambda, \varepsilon, -\rho_P, s) \subset \mathscr{T}(\Lambda, \varepsilon, s).$$

Moreover, for every string ν of $\mathscr{T}(s)$-seminorms there exist strings ν_P of $\mathscr{E}_P(-\rho_P, s)$-seminorms ($P \in \mathscr{P}(P_0)$) such that

$$\nu_{\varepsilon,n}(f) \leqslant \max_{P \in \mathscr{P}(P_0)} \nu_{P,\varepsilon,n}(f)$$

for all $\varepsilon > 0$, $f \in \mathscr{E}(\Lambda, \varepsilon)$, and $n \in \mathbf{N}$.

In the following proposition it is asserted that estimates can be improved step by step along each maximal $\sigma\theta$-stable parabolic subgroup. Its proof owes much to [34, Theorem 4.3.5].

Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$. Then there is a one to one correspondence between the maximal $\sigma\theta$-stable parabolic subgroups containing P and the set $\Delta(P)$ of simple roots in $\Sigma(P)$. If $Q = M_Q A_Q N_Q$ is such a maximal parabolic subgroup, then the corresponding simple root β_Q is the unique root in $\Delta(P)$ which does not vanish on $\mathfrak{a}_{Qq}$. Conversely let $\Theta = \Delta \backslash \{\beta_Q\}$. Then

$$\mathfrak{a}_{Qq} = \bigcap_{\alpha \in \Theta} \ker \alpha,$$

and

$$\mathfrak{n}_Q = \bigoplus_{\alpha \in \Sigma(P) \backslash \mathbf{N}\Theta} \mathfrak{g}^\alpha.$$

Let $\mathfrak{a}_{Qq}^+ = \{X \in \mathfrak{a}_{Qq};\ \beta_Q(X) > 0\}$. If $\eta \in \mathfrak{a}_\mathfrak{q}^*$, we define $i_Q(\eta) \in \mathfrak{a}_\mathfrak{q}^*$, its improvement along Q, by

$$\begin{aligned} i_Q(\eta) &= \eta \qquad \text{on } \ker \beta_Q; \\ &= \max(-\rho_P, \eta - \tfrac{1}{2}\beta_Q) \qquad \text{on } \mathfrak{a}_{Qq}^+. \end{aligned}$$

PROPOSITION 18.6. *Let Q be a maximal $\sigma\theta$-stable parabolic subgroup containing $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, and let $\eta \in \mathfrak{a}_\mathfrak{q}^*$, $s \geqslant 0$. Then there exists $s' > 0$ such that for ε sufficiently small we have*

$$\mathscr{E}_P(\Lambda, \varepsilon, \eta, s) \subset \mathscr{E}_P(\Lambda, \varepsilon, i_Q(\eta), s').$$

Moreover, if v is a string of $\mathscr{E}_P(i_Q(\eta), s')$-seminorms, then there exists a string v' of $\mathscr{E}_P(\eta, s)$-seminorms and a constant $N \in \mathbf{N}$, such that for sufficiently small $\varepsilon > 0$ we have

$$v_{\varepsilon, n+N}(f) \leqslant v'_{\varepsilon, n}(f)$$

for all $f \in \mathscr{E}(\Lambda, \varepsilon)$ and $n \in \mathbf{N}$.

Before giving the proof of this proposition we will derive Theorem 18.3 from it.

COROLLARY 18.7. *Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$, $\eta \in \mathfrak{a}_\mathfrak{q}^*$, and $s > 0$. Then there exists a constant $s' > 0$ such that for ε sufficiently small we have*

$$\mathscr{E}_P(\Lambda, \varepsilon, \eta, s) \subset \mathscr{E}_P(\Lambda, \varepsilon, -\rho_P, s').$$

Moreover, if v *is a* $\mathscr{E}_P(-\rho_P, s')$*-seminorm string, then there exist a string* v' *of* $\mathscr{E}_P(\eta, s)$*-seminorms and a constant* $N \in \mathbf{N}$ *such that*

$$v_{\varepsilon, n+N}(f) \leqslant v'_{\varepsilon, n}(f),$$

for $\varepsilon > 0$ *sufficiently small,* $f \in \mathscr{E}(\Lambda, \varepsilon)$*, and* $n \in \mathbf{N}$.

Proof. First we observe that by repeatedly applying Proposition 18.6 we see that its assertions remain valid if we redefine $i_Q(\eta)$ by

$$\begin{aligned} i_Q(\eta) &= \eta \qquad \ker \beta_Q; \\ &= -\rho_P \qquad \text{on } \mathfrak{a}_{Qq}^+. \end{aligned}$$

Let β_j $(1 \leqslant j \leqslant l)$ be an enumeration of $\Delta(P)$, and let Q_j be the maximal parabolic in $\mathscr{P}_\sigma$ with $\beta_{Q_j} = \beta_j$. Then we define a sequence η_j $(0 \leqslant j \leqslant l)$ in $\mathfrak{a}_\mathfrak{q}^*$ recursively by $\eta_0 = \eta$ and for $i \geqslant 1$,

$$\begin{aligned} \eta_i &= \eta_{i-1} \qquad \text{on } \ker \beta_i; \\ &= -\rho_P \qquad \text{on } \mathfrak{a}_{Q_i q}. \end{aligned}$$

We claim that $\eta_l = -\rho_P$. The corollary then follows by applying the improved version of Proposition 18.6 repeatedly. Indeed, let $H_1 \cdots H_l$ be the basis for $\mathfrak{a}_\mathfrak{q}$ which is dual to $\beta_1 \cdots \beta_l$ (we assumed $G = {}^\circ G$). Then $\ker \beta_i = \bigoplus_{j \neq i} \mathbf{R} H_j$, and $\mathfrak{a}_{Q_i q} = \mathbf{R} H_i$. Hence by induction it follows that $\eta_i = -\rho_P$ on $\bigoplus_{j \leqslant i} \mathbf{R} H_j$. ∎

Proof of Theorem 18.3. The theorem follows straightforwardly when we combine the above corollary with Lemmas 18.4 and 18.5. ∎

For the proof of Proposition 18.6, we need the following companion to Proposition 12.4.

PROPOSITION 18.8. *Let* $Q \in \mathscr{P}_\sigma$. *Then there exist:*

(1) *a finite dimensinal linear subspace* $V \subset \mathbf{D}(M_{1Q}/H_{1Q})$ *containing* 1;

(2) *an algebra homomorphism* $b(\lambda, \cdot)$ *from* $U(\mathfrak{m}_{1Q})^{\mathfrak{h}_Q}$ *into* End(V), *depending polynomially on* $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$; *and*

(3) *a bilinear map* $y_\lambda \colon U(\mathfrak{m}_{1Q})^{\mathfrak{h}_Q} \times V \to \bar{\mathfrak{n}}_Q U(\bar{\mathfrak{n}}_Q + \mathfrak{m}_{1Q})$ *depending polynomially on* $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$,

such that for all $\lambda \in \mathfrak{a}_{\mathfrak{q}c}^*$, $D \in U(\mathfrak{m}_{1Q})^{\mathfrak{h}_Q}$, *and* $v \in V$ *we have*

$$Dv = b(\lambda, D)v + y_\lambda(D, v) \qquad \text{mod } J_{\Lambda + \lambda},$$

where $J_{\Lambda+\lambda}$ denotes the left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{h}$ and

$$\{D-\gamma(D:\Lambda+\lambda); D\in U(\mathfrak{g})^{\mathfrak{h}}\}.$$

Finally, the set of $\mathfrak{a}_{Q\mathfrak{q}}$-weights of $b(\lambda,\cdot)$ equals $(W(\mathfrak{b})(\Lambda+\lambda)-\rho_Q)\mid\mathfrak{a}_{Q\mathfrak{q}}$.

Proof. Using duality this can be obtained from Corollary 11.15 in the same way as Proposition 12.4 is obtained from Proposition 11.7. The assertion on the weights is then a consequence of (86). ∎

The remaining part of this section will be devoted to the proof of Proposition 18.6. Let $P\in\mathscr{P}_\sigma(A_\mathfrak{q})$ and let Q be a fixed maximal $\sigma\theta$-stable parabolic subgroup containing P. Let $\eta\in\mathfrak{a}_\mathfrak{q}^*$, $s\geqslant 0$. Throughout the proof we assume that $0<\varepsilon\leqslant\varepsilon'$. Here ε' is a positive constant on which conditions will be imposed in the course of the proof. Let V be the subset of $\mathbf{D}(M_{1Q}/H_Q)$ as defined in Proposition 18.8 and fix $H\in\mathfrak{a}_{Qq}^+$, with $|H|=1$. We define the operator φ from $\mathscr{E}(\Lambda,\varepsilon)$ into $C^\infty(\mathfrak{a}_\mathfrak{q}^*(\varepsilon)\times M_{1Q}/H_Q)\otimes V^*$ by

$$\langle\varphi(f)(\lambda,m),v\rangle=f_\lambda(m;v)\qquad(v\in V).$$

Similarly we define the operator ψ from $\mathscr{E}(\Lambda,\varepsilon)$ into $C^\infty(\mathfrak{a}_\mathfrak{q}^*(\varepsilon)\times M_{1Q}/H_Q)\otimes V^*$ by

$$\langle\psi(f)(\lambda,m),v\rangle=f(m;y_\lambda(H,v)).$$

Then both φ and ψ are left $(\mathfrak{m}_{Q1},K_Q)$-equivariant maps. We agree to write $\varphi_\lambda(f,\cdot)$ for $\varphi(f)(\lambda,\cdot)$ and $\psi_\lambda(f,\cdot)$ for $\psi(f)(\lambda,\cdot)$. Moreover, let $\beta=\beta_Q$.

LEMMA 18.9. *There exists a string ν of $\mathscr{E}_P(\eta,s)$-seminorms and a constant $d\in\mathbf{N}$ such that for all $\varepsilon\in\,]0,\varepsilon']$, $f\in\mathscr{E}(\Lambda,\varepsilon)$, and $n\in\mathbf{N}$ we have*

$$|\varphi_\lambda(f,a)|\leqslant\nu_{\varepsilon,n}(f)\,|(\lambda,a)|^n\,a^\eta\,e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}\tag{145}$$

$$|\psi_\lambda(f,a)|\leqslant\nu_{\varepsilon,n}(f)\,|(\lambda,a)|^{n+d}a^{\eta-\beta}\,e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|},\tag{146}$$

for all $a\in\mathrm{cl}\,A_\mathfrak{q}^+(P)$, $\lambda\in\mathfrak{a}_\mathfrak{q}^*(\varepsilon)$.

Proof. We first observe that every element $u\in\bar{\mathfrak{n}}_Q^kU(\bar{\mathfrak{n}}_P+\mathfrak{m}_1)$ can be expressed as a sum of terms u_ξ, $\xi\in\mathbf{N}\Sigma(P)$, where each u_ξ belongs to the $-\xi-k\beta$ weight space for $\mathrm{ad}(\mathfrak{a}_\mathfrak{q})$. Hence for $a\in\mathrm{cl}\,A_\mathfrak{q}^+(P)$ we have

$$\begin{aligned}|f_\lambda(a;u)|&=|a^{-k\beta}\sum_\xi a^{-\xi}f_\lambda(u_\xi^\vee;a)|\\&\leqslant a^{-k\beta}\,|(\lambda,a)|^n\,a^\eta\,e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}\,\nu'_{\varepsilon,n}(f),\end{aligned}\tag{147}$$

for a suitable string ν' of $\mathscr{E}_P(\eta,s)$-seminorms, only depending on u.

In order to prove the first estimate it suffices to estimate

$\langle \varphi_\lambda(f)(a), v\rangle = f_\lambda(a; v)$ for a fixed $v \in V$. Now v has a representative $u \in U(\bar{\mathfrak{n}}_P + \mathfrak{m}_1)$. Hence (145) follows if we apply the above with $k = 0$.

Let d be the polynomial degree of $\lambda \mapsto y_\lambda(H, \cdot)$. Then for a fixed $v \in V$ we may express $y_\lambda(H, v)$ as a sum of terms $p(\lambda)u$, with $u \in \bar{\mathfrak{n}}_Q U(\bar{\mathfrak{n}}_P + \mathfrak{m}_1)$ and $p \in S(\mathfrak{a}_\mathfrak{q})$ of degree at most d. Hence (146) follows if we apply the first part of the proof with $k = 1$. ∎

For $f \in \mathscr{E}(\Lambda, \varepsilon)$ we have the differential equation

$$\frac{d}{dt}\varphi_\lambda(f, m \exp tH) = \Gamma(\lambda)\, \varphi_\lambda(f, m \exp tH) + \psi_\lambda(f, m \exp tH)$$

for all $m \in M_{1Q}$ and $t \in \mathbf{R}$. Here $\Gamma(\lambda) = b(\lambda, H)^*$ has eigenvalues contained in the set

$$[w(\Lambda + \lambda) - \rho](H), \qquad w \in W(\mathfrak{b}),$$

where we have written $\rho = \rho_P$. The above differential equation can be rewritten as an integral equation

$$\varphi_\lambda(f, m \exp tH)$$
$$= e^{t\Gamma(\lambda)}\varphi_\lambda(f, m) + e^{t\Gamma(\lambda)} \int_0^t e^{-\tau\Gamma(\lambda)}\psi_\lambda(f, m \exp \tau H)\, d\tau. \tag{148}$$

We decompose $W(\mathfrak{b})$ as a disjoint union

$$W(\mathfrak{b}) = W_+ \cup W_-,$$

as follows. First of all we observe that $W(\mathfrak{b})$ leaves $\mathfrak{b}_\mathbf{R} := i\mathfrak{b}_k \oplus \mathfrak{a}_\mathfrak{q}$ invariant. Therefore $(w\Lambda - \rho)(H)$ is a real number for every $w \in W(\mathfrak{b})$. We define the subsets $W_\pm$ of $W(\mathfrak{b})$ by

$$w \in W_+ \Leftrightarrow (w\Lambda - \rho)(H) > \eta(H) - \tfrac{3}{4}\beta(H),$$
$$w \in W_- \Leftrightarrow (w\Lambda - \rho)(H) \leqslant \eta(H) - \tfrac{3}{4}\beta(H).$$

Fix a constant $\sigma \in \mathbf{R}$ with

$$\eta(H) - \tfrac{3}{4}\beta(H) < \sigma < \eta(H) - \tfrac{1}{2}\beta(H) \tag{149}$$

and such that $(w\Lambda - \rho)(H) > \sigma$ for $w \in W_+$ and $(w\Lambda - \rho)(H) < \sigma$ for $w \in W_-$. Our first condition on ε' is that

$$\begin{aligned} &(w\Lambda - \rho)(H) > \sigma + 3\varepsilon' \qquad (w \in W_+); \\ &(w\Lambda - \rho)(H) < \sigma - 3\varepsilon' \qquad (w \in W_-). \end{aligned} \tag{150}$$

Let $E_{\pm}(\lambda)$ denote the projection in V^* onto the sum of the generalized eigenspaces of $\Gamma(\lambda)$ corresponding to the eigenvalues

$$\{(w(\Lambda+\lambda)-\rho)(H), w \in W_{\pm}\}.$$

LEMMA 18.10. *The projections $E_{\pm}(\lambda) \in \mathrm{End}(V^*)$ depend holomorphically on $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon')$, and we have that $E_+(\lambda)+E_-(\lambda)=I$. Moreover, there exist constants $C \geqslant 0$ and $L \in \mathbf{N}$ such that*

$$|e^{-t\Gamma(\lambda)}E_+(\lambda)| \leqslant C\, e^{-(\sigma+\varepsilon')t}(1+|\lambda|)^L; \tag{151}$$

and

$$|e^{t\Gamma(\lambda)}E_-(\lambda)| \leqslant C\, e^{(\sigma-\varepsilon')t}(1+|\lambda|)^L, \tag{152}$$

for all $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon')$, $t \geqslant 0$.

Proof. The eigenvalues of $\Gamma(\lambda)$ are $\xi_w(\lambda)=(w(\Lambda+\lambda)-\rho)(H)$, $w \in W(\mathfrak{b})$. There real parts are given by $\mathrm{Re}\,\xi_w(\lambda)=(w\Lambda+\mathrm{Re}\,w\lambda-\rho)(H)$. Hence in view of conditions (150) we have that

$$\mathrm{Re}\,\xi_w(\lambda)+2\varepsilon' < \sigma < \mathrm{Re}\,\xi_v(\lambda)-2\varepsilon' \tag{153}$$

for every $w \in W_-$, $v \in W_+$, and $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon')$ (here we used that $|H|=1$). All assertions now follow by application of the results of Appendix 20. ∎

For $t \in \mathbf{R}$ we write $h_t=\exp(tH)$. Our second condition on ε' is

$$\varepsilon'(2+s) < \tfrac{1}{4}\beta(H). \tag{154}$$

Then the following is valid.

PROPOSITION 18.11. *For every $\varepsilon \in\,]0, \varepsilon']$, $f \in \mathscr{E}(\Lambda, \varepsilon, \eta, s)$, and $a \in \mathrm{cl}\, A_{\mathfrak{q}}^+(P)$, the integral*

$$I_\lambda(f, a)=\int_0^\infty e^{-\tau\Gamma(\lambda)}E_+(\lambda)\,\psi_\lambda(f, ah_\tau)\,d\tau \tag{155}$$

is absolutely convergent. Moreover, the function

$$\varphi_\lambda^\infty(f, a)=E_+(\lambda)\,\varphi_\lambda(f, a)+I_\lambda(f, a) \tag{156}$$

depends holomorphically on $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$, and there exists a string ν' of $\mathscr{E}_P(\eta, s)$-seminorms such that for all $\varepsilon \in\,]0, \varepsilon']$, $f \in \mathscr{E}(\Lambda, \varepsilon)$, $a \in \mathrm{cl}\, A_{\mathfrak{q}}^+(P)$, and $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$ we have

$$|\varphi_\lambda^\infty(f, a)| \leqslant \nu'_{\varepsilon,n}(f)\,|(\lambda, a)|^{n+d+L}\,a^\eta\,e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}. \tag{157}$$

Proof. Using (146) we infer that for $a \in \mathrm{cl}\, A_{\mathfrak{q}}^+(P)$, $\tau \geqslant 0$ we have

$$|\psi_\lambda(f, ah_\tau)| \leqslant \nu_{\varepsilon,n}(f)\, |(\lambda, a)|^{n+d}\, a^\eta\, e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}\, A_n(\tau), \tag{158}$$

where

$$\begin{aligned} A_n(\tau) &= (1+\tau)^{n+d}\, e^{\tau[\eta(H) - \beta(H) + s\,|\mathrm{Re}\,\lambda|]} \\ &\leqslant (1+\tau)^{n+d}\, e^{\tau(\sigma - 2\varepsilon')}. \end{aligned} \tag{159}$$

The latter inequality is a consequence of (149) and (154). By application of (151) we infer that the integrand of (155) can be estimated from above by

$$\nu_{\varepsilon,n}(f)\, |(\lambda, a)|^{n+d+L}\, a^\eta\, e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}\, e^{-3\varepsilon'\tau}.$$

This implies the estimate for $I_\lambda(f, a)$. The estimate for $E_+(\lambda)\, \varphi_\lambda(f, a)$ follows from (145) and (151) with $t = 0$. ∎

For $f \in \mathscr{E}_P(A, \varepsilon, \eta, s)$, $a \in \mathrm{cl}\, A_{\mathfrak{q}}^+(P)$, $t \geqslant 0$, $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$ we define $R_\lambda(f, t, a) = R_\lambda^0(f, t, a) + R_\lambda^+(f, t, a) + R_\lambda^-(f, t, a)$, where

$$R_\lambda^0(f, t, a) = e^{t\Gamma(\lambda)} E_-(\lambda)\, \varphi_\lambda(f, a),$$

$$R_\lambda^+(f, t, a) = -\int_t^\infty e^{(t-\tau)\Gamma(\lambda)} E_+(\lambda)\, \psi_\lambda(f, ah_\tau)\, d\tau,$$

$$R_\lambda^-(f, t, a) = \int_0^t e^{(t-\tau)\Gamma(\lambda)} E_-(\lambda)\, \psi_\lambda(f, ah_\tau)\, d\tau.$$

From the integral equation (148) it follows that

$$\varphi_\lambda(f, ah_t) = e^{t\Gamma(\lambda)} \varphi_\lambda^\infty(f, a) + R_\lambda(f, t, a). \tag{160}$$

LEMMA 18.12. *There exists a string ν' of $\mathscr{E}_P(\eta, s)$-seminorms such that for all $\varepsilon \in\,]0, \varepsilon']$, $f \in \mathscr{E}_P(A, \varepsilon, \eta)$ we have*

$$|R_\lambda(f, t, a)| \leqslant \nu'_{\varepsilon,n}(f)\, a^\eta\, |(\lambda, a)|^{n+d+L}\, e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}\, e^{t(\sigma+\varepsilon')}, \tag{161}$$

for $n \in \mathbf{N}$, $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$, $a \in \mathrm{cl}\, A_{\mathfrak{q}}^+(P)$, and $t \geqslant 0$. Moreover, $R_\lambda(f, t, a)$ depends holomorphically on λ.

Proof. From (145) and (152) it is immediate that R_λ^0 satisfies an estimate like (161).

From (159) and (151) we obtain that for all $\tau \geqslant t$ we have

$$|A_n(\tau)\, e^{(t-\tau)\Gamma(\lambda)} E_+(\lambda)| \leqslant C(1 + |\lambda|)^L\, e^{(\sigma+\varepsilon')t} (1+\tau)^{n+d}\, e^{-3\varepsilon'\tau}.$$

Combining this estimate with (158) we see that the integral for $R_\lambda^+(f, t, a)$

converges absolutely and depends holomorphically on λ. Moreover, we find that

$$|R_\lambda^+(f, t, a)| \leqslant C_n e^{(\sigma+\varepsilon')t} \nu_{\varepsilon,n}(f)\, |(\lambda, a)|^{n+d+L} a^\eta e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|},$$

with suitable constants C_n only depending on n and ε'.

In order to prove similar assertions for $R_\lambda^-(f, t, a)$ we combine the estimates (159) and (152) to see that for $0 \leqslant \tau \leqslant t$ we have

$$|A_n(\tau)\, e^{(t-\tau)\Gamma(\lambda)} E_-(\lambda)| \leqslant C(1+|\lambda|)^L e^{(\sigma-\varepsilon')t}(1+\tau)^{n+d} e^{-\varepsilon'\tau}.$$

The integral over τ of the above expression from 0 to t is majorized by $C_n e^{\sigma t}(1+|\lambda|)^L$, with a suitable constant only depending on n and ε'. Combining these estimates with (158) we find that

$$|R_\lambda^-(f, t, a)| \leqslant C_n e^{\sigma t} \nu_{\varepsilon,n}(f)\, |(\lambda, a)|^{n+d} a^\eta e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}$$

and the proof is complete. ▮

In order to estimate $e^{t\Gamma(\lambda)} \varphi_\lambda^\infty(f, a)$, we proceed as follows. Put $\tilde{\eta} = i_Q(\eta)$. Then

$$\tilde{\eta}(H) = \max(-\rho(H), \eta(H) - \tfrac{1}{2}\beta(H)).$$

We split the set W_+ as a disjoint union $W_+ = W_1 \cup W_2$, where

$$w \in W_1 \Leftrightarrow (w\Lambda - \rho)(H) \leqslant \tilde{\eta}(H),$$
$$w \in W_2 \Leftrightarrow (w\Lambda - \rho)(H) > \tilde{\eta}(H).$$

Let $W_{\mathfrak{a}}$ denote the normalizer of $\mathfrak{a}_\mathfrak{q}$ in $W(\mathfrak{b})$. Then $W = W(\mathfrak{g}, \mathfrak{a}_\mathfrak{q})$ is a quotient of $W_{\mathfrak{a}}$. Notice that for $w \in W_+ \cap W_{\mathfrak{a}}$ we have $w\Lambda(H) = \Lambda(w^{-1}H) = 0$, hence

$$W_+ \cap W_{\mathfrak{a}} \subset W_1.$$

Our third condition on the magnitude of ε' is

$$(w_1\Lambda - \rho)(H) + 2\varepsilon' < (w_2\Lambda - \rho)(H) - 2\varepsilon', \tag{162}$$

for all $w_1 \in W_1$, $w_2 \in W_2$.

LEMMA 18.13. *For $i = 1, 2$, let $E_i(\lambda)$ be the projection in V^* onto the sum of the generalized eigenspaces for $\Gamma(\lambda)$ corresponding to the eigenvalues $w(\Lambda+\lambda)(H) - \rho(H)$, $w \in W_i$. Then $E_1(\lambda)$ and $E_2(\lambda)$ depend holomorphically on $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon')$. Moreover*

$$E_1(\lambda) + E_2(\lambda) = E_+(\lambda), \tag{163}$$

and there exist constants $C_1 > 0$, $L' \in \mathbf{N}$ *such that*

$$|e^{t\Gamma(\lambda)} E_1(\lambda)| \leqslant C_1 (1 + |\lambda|)^{L'} (1+t)^{d'} \, e^{t(|\operatorname{Re}\lambda| + \tilde{\eta}(H))}, \tag{164}$$

for all $t \geqslant 0$, $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon')$; *here* $d' = \dim V$.

Proof. We use the notations of the proof of Lemma 18.10. Let W_0 be the complement of W_2 in $W(\mathfrak{b})$. Then $W_0 = W_- \cup W_1$. Let $E_0(\lambda)$ be the projection in V^* onto the generalized eigenspaces for $\Gamma(\lambda)$ corresponding to the eigenvalues $\xi_w(\lambda)$, $w \in W_0$. Then $E_0(\lambda) + E_2(\lambda) = I$. From (153) and (162) we deduce that

$$\operatorname{Re} \xi_{w_2}(\lambda) - \operatorname{Re} \xi_{w_0}(\lambda) > 2\varepsilon'$$

for every $w_2 \in W_2$, $w_0 \in W_0$, and $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon')$. Moreover, if $w \in W_0$, then

$$\operatorname{Re} \xi_w(\lambda) \leqslant \tilde{\eta}(H) + |\operatorname{Re} \lambda|,$$

for all $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$. Applying the results of Appendix 20 we infer that E_0 and E_2 are holomorphic and that we have an estimate of the form

$$|e^{t\Gamma(\lambda)} E_0(\lambda)| \leqslant C'(1+t)^{d'} (1 + |\lambda|)^{L} \, e^{t(\tilde{\eta}(H) + |\operatorname{Re}\lambda|)}. \tag{165}$$

From (151) with $t = 0$ we infer that

$$|E_+(\lambda)| \leqslant C(1 + |\lambda|)^L. \tag{166}$$

We now observe that $E_1(\lambda) = E_0(\lambda) \circ E_+(\lambda)$. Consequently the desired estimate follows from (165) and (166), with $L' = 2L$. ∎

PROPOSITION 18.14. *Let* $\varepsilon \in \,]0, \varepsilon']$, $f \in \mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$. *Then*

$$E_2(\lambda) \, \varphi_\lambda^\infty(f, a) = 0,$$

for all $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\varepsilon)$, $a \in \operatorname{cl} A_{\mathfrak{q}}^+(P)$.

We will prove this by reduction to K-types. The following lemma will make the reduction possible. Recall the definitions of Ω, α_δ, $c(\delta)$ $(\delta \in \hat{K})$ from the proof of Corollary 17.9. For $j \in \mathbf{N}$ define $\hat{K}_j = \{\delta \in \hat{K};\, c(\delta) > j\}$. Then $\vartheta_j = \hat{K} \backslash \hat{K}_j$ is a finite set. Given $f \in \mathscr{E}(\Lambda, \varepsilon)$ define $\mathbf{P}_j f \in \mathscr{E}(\Lambda, \varepsilon)$ by $\mathbf{P}_j f(\lambda, x) = \sum_{\delta \in \vartheta_j} \alpha_\delta * f_\lambda(x)$.

LEMMA 18.15. *The map* $\mathbf{P}_j$ *maps* $\mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$ *into itself. Moreover, for every string* ν *of* $\mathscr{E}_P(\eta, s)$*-seminorms there exists a string* ν' *of* $\mathscr{E}_P(\eta, s)$*-seminorms such that for all* $j \geqslant 0$ *and all* $f \in \mathscr{E}(\Lambda, \varepsilon)$ *we have*

$$\nu_{\varepsilon, n}(f - \mathbf{P}_j f) \leqslant \frac{1}{j} \, \nu'_{\varepsilon, n}(f).$$

Proof. Choose $m \in \mathbf{N}$ such that $\sum_{\delta \in \hat{K}} c(\delta)^{-m}$ converges. We have that $f - \mathbf{P}_j f = \sum_{\delta \in \hat{K}_j} \alpha_\delta * f$. Hence

$$\begin{aligned} v_{\varepsilon,n}(f - \mathbf{P}_j f) &\leqslant \frac{1}{j} \sum_{\delta \in \hat{K}_j} c(\delta)^{-m} v_{\varepsilon,n}(\alpha_\delta * \Omega^{m+1} f) \\ &\leqslant \frac{1}{j} \left(\sum_{\delta \in \hat{K}_j} c(\delta)^{-m} \right) v''_{\varepsilon,n}(\Omega^{m+1} f) \end{aligned}$$

with v'' a suitable string of seminorms independent of f. From this the result easily follows. ∎

Proof of Proposition 18.14. By holomorphy we may restrict ourselves to the case that ε' is so small that in addition to the conditions previously imposed we have

$$(w\Lambda - \rho)(H) > \tilde{\eta}(H) + \varepsilon' \qquad \text{for all} \quad w \in W_2. \tag{167}$$

Fix $0 < \varepsilon < \varepsilon'$, and let $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon)$. Using the above lemma in combination with the estimate (157) we infer that $\varphi_\lambda^\infty(\mathbf{P}_j f, a) \to \varphi_\lambda^\infty(f, a)$ as $j \to \infty$. Hence we may as well assume that f is K-finite from the left. Fix $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon) \cap \mathfrak{a}_{\mathfrak{qc}}^{*\prime}(\Lambda)$ (with notations as in (99): it suffices to prove the assertion for λ in this dense subset). According to Lemma 12.3 there exists a $r > 0$ such that $f_\lambda \in \mathscr{E}^\infty_{\Lambda+\lambda,r}(G/H)$. Let $u \in U(\mathfrak{g})$. Then from the proof of Theorem 13.7 it follows that the exponents of $L_u f_\lambda$ along P are all contained in the set $W\lambda - \rho - \mathbf{N}\Sigma(P)$. According to [2, Theorem 6.3] this implies the existence of a constant $C > 0$ such that

$$|L_u f_\lambda(a)| \leqslant C a^{-\rho} e^{\varepsilon' |\log a|}$$

for all $a \in \operatorname{cl} A_\mathfrak{q}^+(P)$. By the same argument as in the proof of Lemma 18.9 this leads to an estimate

$$|\varphi_\lambda(f)(a)| \leqslant C a^{-\rho} e^{\varepsilon' |\log a|} \qquad (a \in \operatorname{cl} A_\mathfrak{q}^+(P)).$$

Now fix $a \in \operatorname{cl} A_\mathfrak{q}^+(P)$. Then $ah_t \in \operatorname{cl} A_\mathfrak{q}^+(P)$ for $t \geqslant 0$, so it follows that

$$|\varphi_\lambda(f)(ah_t)| \leqslant C\, e^{(\varepsilon' - \rho(H))t} \qquad (t \geqslant 0)$$

with $C > 0$ a suitable constant. In view of (160) and the estimate (161) we infer the existence of a $C > 0$ such that

$$|e^{t\Gamma(\lambda)} \varphi_\lambda^\infty(f, a)| \leqslant C\, e^{Rt} \qquad (t \geqslant 0),$$

where $R=\max(\varepsilon'-\rho(H), \sigma+\varepsilon') \leqslant \tilde{\eta}(H)+\varepsilon'$. In view of the identity (163) and the estimate (164), we now see that

$$|e^{t\Gamma(\lambda)}E_2(\lambda)\, \varphi_\lambda^\infty(f, a)| \leqslant C\, e^{(\tilde{\eta}(H)+\varepsilon')t}.$$

But $t \mapsto \varphi(t) := e^{t\Gamma(\lambda)}E_2(\lambda)\, \varphi_\lambda^\infty(f, a)$ is a polynomial exponential function with exponents whose real parts are all strictly greater than $\tilde{\eta}(H)+\varepsilon'$, in view of (167). Hence by uniqueness of asymptotics (cf. [14, p. 305, Corollary]) it follows that $\varphi=0$. ∎

COROLLARY 18.16. *For all* $\varepsilon \in \left]0, \varepsilon'\right]$, $f \in \mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$, *and all* $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon)$, $a \in \operatorname{cl} A_\mathfrak{q}^+(P)$, $t \geqslant 0$ *we have*

$$\begin{aligned}&|e^{t\Gamma(\lambda)}E_+(\lambda)\, \varphi_\lambda^\infty(f, a)| \\ &\quad \leqslant C_1 \nu'_{\varepsilon,n}(f)\, |(\lambda, a)|^{n+d+L+L'} a^\eta\, e^{s\,|\mathrm{Re}\,\lambda|\,|\log a|}(1+t)^{d'}\, e^{t(|\mathrm{Re}\,\lambda|+\tilde{\eta}(H))}.\end{aligned}$$

Proof. In view of Proposition 18.14 and (163) we have that the left hand side in the above inequality equals the norm of $e^{t\Gamma(\lambda)}E_1(\lambda)\, \varphi_\lambda^\infty(f, a)$. The result now follows by combining the estimates (157) and (164). ∎

Completion of the Proof of Proposition 18.6. From (149) it follows that $\sigma < \tilde{\eta}(H)$. The final condition on ε' is

$$\sigma+\varepsilon' < \tilde{\eta}(H).$$

From the equality (160), the estimate (161), and the above corollary, we infer that there exists a string μ of $\mathscr{E}_P(\eta, s)$-seminorms such that for $\varepsilon \in \left]0, \varepsilon'\right]$, $f \in \mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$, $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon)$ we have

$$|\varphi_\lambda(f, a_0 h_t)| \leqslant \mu_{\varepsilon,n}(f)\, |(\lambda, a_0)|^{n+d+L+L'}\, a_0^\eta\, e^{s|\mathrm{Re}\,\lambda|\,|\log a_0|}(1+t)^{d'}\, e^{M(\lambda)t},$$

for $t \geqslant 0$, $a_0 \in \operatorname{cl} A_\mathfrak{q}^+(P) \cap \exp(\ker \beta)$. Here

$$\begin{aligned}M(\lambda) &= \max(\sigma+\varepsilon', |\mathrm{Re}\,\lambda| + \tilde{\eta}(H)) \\ &= |\mathrm{Re}\,\lambda| + \tilde{\eta}(H) = |\mathrm{Re}\,\lambda| + i_Q(\eta)(H)\end{aligned}$$

by the final requirement on ε'. Every element $a \in \operatorname{cl} A_\mathfrak{q}^+(P)$ can be written as $a = a_0 h_t$, with a_0 and t subject to the above restrictions. Moreover, since $\langle \log a_0, \log H \rangle \geqslant 0$, we have $|\log a_0| \leqslant |\log a||$ and $t \leqslant |\log a|$. Since f_λ is a component of $\varphi_\lambda(f)$, the above estimate yields (with $N = d + d' + L + L'$)

$$|f_\lambda(a)| \leqslant \mu_{\varepsilon,n}(f)\, |(\lambda, a)|^{n+N} a^{iQ(\eta)}\, e^{(s+1)\,|\mathrm{Re}\,\lambda|\,|\log a|}, \tag{168}$$

for all $\varepsilon \in \left]0, \varepsilon'\right]$, $f \in \mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$, $\lambda \in \mathfrak{a}_\mathfrak{q}^*(\varepsilon)$, and $a \in A_\mathfrak{q}^+(P)$. Fix $u \in U(\mathfrak{g})$, then $I \otimes L(k^{-1}) L_u$ leaves $\mathscr{E}_P(\Lambda, \varepsilon, \eta, s)$ invariant, for every $k \in K$. Hence in the above estimate we may replace f by $[I \otimes L(k^{-1}) L_u] f$. One easily checks that there exists a string of seminorms μ' such that $\mu_{\varepsilon,n}([I \otimes L(k^{-1}) L_u] f) \leqslant \mu'_{\varepsilon,n}(f)$ for all $k \in K$. We therefore obtain the estimate

$$|(\lambda, a)|^{-(n+N)} a^{-i\varrho(\eta)} e^{-(s+1)|\mathrm{Re}\,\lambda|\,|\log a|} |L_u f_\lambda(ka)| \leqslant \mu'_{\varepsilon,n}(f).$$

This completes the proof; notice that we may take $s' = s + 1$. ∎

19. The Fourier Transform

By the results of the previous section the normalized Eisenstein integrals belong to the class of uniformly tempered functions. This allows us to define a Fourier transform which maps a space of spherical Schwartz functions continuously into a Euclidean Schwartz space.

Let $\mathbf{V}$ and τ be as in Section 3. If $f, g \colon G/H \to \mathbf{V}$ are τ-spherical functions such that the function $x \mapsto \langle f(x), g(x) \rangle$ is integrable on G/H, then we write

$$\langle f, g \rangle_2 := \int_{G/H} \langle f(x), g(x) \rangle \, dx.$$

Let $P \in \mathscr{P}_\sigma(A_\mathfrak{q})$ be fixed. If $f \in C_c^\infty(G/H, \tau)$, the space of compactly supported smooth τ-spherical functions $G/H \to \mathbf{V}$, then we define its Fourier transform $\mathscr{F}f = \mathscr{F}_P f$ to be the meromorphic function $\mathfrak{a}_{\mathfrak{q}c}^* \to {}^\circ\mathscr{C}$ given by

$$\langle \mathscr{F}f(\lambda), \psi \rangle = \langle f, E^1(P : \psi : -\bar{\lambda}) \rangle_2 \qquad (\psi \in {}^\circ\mathscr{C}). \tag{169}$$

Notice that by Proposition 10.3 and Corollary 16.2, $\mathscr{F}f$ is of Σ-exponential growth on every set of the form $\mathfrak{a}_\mathfrak{q}^*(P, R)$, $R \in \mathbf{R}$.

Let $\pi \in \Pi_\Sigma(\mathfrak{a}_\mathfrak{q})$ be any polynomial such that $\lambda \mapsto \pi(\lambda) E^1(P : \psi : \lambda)$ is regular on $i\mathfrak{a}_\mathfrak{q}^*$, for every $\psi \in {}^\circ\mathscr{C}$ (for its existence see Proposition 10.3 and Corollary 16.2). Let $\mathscr{C}(G/H, \tau)$ denote the space of τ-spherical L^2-Schwartz functions $G/H \to \mathbf{V}$ and let $\mathscr{S}(i\mathfrak{a}_\mathfrak{q}^*)$ denote the usual space of Schwartz functions on $i\mathfrak{a}_\mathfrak{q}^*$. Then we have the following.

Theorem 19.1. *The map $f \mapsto \pi\mathscr{F}f \mid i\mathfrak{a}_\mathfrak{q}^*$ extends (uniquely) to a continuous linear map from $\mathscr{C}(G/H, \tau)$ into $\mathscr{S}(i\mathfrak{a}_\mathfrak{q}^*) \otimes {}^\circ\mathscr{C}$.*

Remark. The above result actually holds with $\pi = 1$. This will be proved elsewhere.

We prove the theorem in the course of this section. Basic for the proof is the following uniform estimate for the normalized Eisenstein integral. We agree to write $E_\pi(\psi : \lambda) = \pi(\lambda) E^1(P : \psi : \lambda)$, and $\mathscr{F}_\pi f = \pi\mathscr{F}f \mid i\mathfrak{a}_\mathfrak{q}^*$.

THEOREM 19.2. *Let* $u \in S(\mathfrak{a}_{\mathfrak{q}}^*)$, $X \in U(\mathfrak{g})$. *Then there exist constants* $N \in \mathbf{N}$, $C > 0$ *such that*

$$|E_\pi(\psi : \lambda; u : X; x)| \leqslant C \, \|\psi\| \, |(\lambda, x)|^N \Theta(x),$$

for $\psi \in {}^\circ\mathscr{C}$, $x \in G$, *and* $\lambda \in i\mathfrak{a}_{\mathfrak{q}}^*$.

Proof. In view of Lemma 4.5, Proposition 4.7, and Corollary 16.2 it suffices to prove the estimate for a fixed ψ with the property that $E_\pi(\psi : \lambda)$ satisfies a system of differential equations of the form (144). Moreover, E_π being spherical, it suffices to prove the estimate for $f(\lambda, x) = E_\pi(\psi : \lambda)(x)(1)$. Being of Σ-polynomial growth the function $\lambda \mapsto f_\lambda$ has its singularities in $\mathfrak{a}_{\mathfrak{q}}^*(P, 1)$ on a finite union of hyperplanes of the form $\langle \lambda, \alpha \rangle = c$. Hence there exists a $\varepsilon > 0$ such that $f \in \mathscr{E}(A, \varepsilon)$. In view of Proposition 10.3, Corollary 16.2, and Lemma 6.1 we have that $f \in \mathscr{M}(A, \varepsilon, r)$ for a suitable $r > 0$ (shrink ε if necessary). By application of Theorem 18.3 we infer that $f \in \mathscr{T}(A, \varepsilon, s)$ for suitable $\varepsilon, s > 0$. The desired estimate now follows by application of Lemma 18.2. ▮

From the above theorem, Corollary 17.6, and the characterization of the Schwartz space in Theorem 17.1 one straightforwardly deduces that $\mathscr{F}_\pi$ allows a unique extension to a continuous linear map $\mathscr{C}(G/H, \tau) \to C^\infty(i\mathfrak{a}_{\mathfrak{q}}^*) \otimes {}^\circ\mathscr{C}$, defined by the formula (169). The stronger assertion that the Fourier transform maps continuously into the Schwartz space will be proved in the usual manner by using partial integrations.

LEMMA 19.3. *Let* $D \in \mathbf{D}(G/H)$. *Then for every* $f \in \mathscr{C}(G/H, \tau)$ *we have*

$$\mathscr{F}_\pi(Df)(\lambda) = \underline{\mu}_{\mathbf{P}}(D : \lambda) \, \mathscr{F}_\pi f(\lambda) \qquad (\lambda \in i\mathfrak{a}_{\mathfrak{q}}^*).$$

Proof. By continuity of $\mathscr{F}_\pi$ as a map into $C^\infty(i\mathfrak{a}_{\mathfrak{q}}^*) \otimes {}^\circ\mathscr{C}$, it suffices to prove this for a fixed f in the dense (cf. [2, Lemma 7.1]) subspace $C_c^\infty(G/H, \tau)$. From (169) and Lemma 4.5 we infer that then $\mathscr{F}_\pi(Df)(\lambda) = \underline{\mu}_P(D^* : -\bar{\lambda})^* \; \mathscr{F}_\pi f(\lambda)$. Here D^* denotes the formal adjoint of D with respect to $\langle \cdot, \cdot \rangle_2$, and the second star denotes the adjoint with respect to the unitary structure of ${}^\circ\mathscr{C}$. We must therefore show that $\underline{\mu}_P(D^* : -\bar{\lambda}) = \underline{\mu}_P(D : \lambda)^*$.

From the definition of $\underline{\mu}_P$ in Section 4 one readily checks that it suffices to show that $\mu_P(D^*) = \mu_P(D)^*$, where the second star denotes the formal (Hermitian) adjoint in $\mathbf{D}(M_1/H_{M_1})$. Moreover, without loss of generality we may assume that D has real coefficients. The canonical anti-automorphism $X \mapsto X^\vee$ of $U(\mathfrak{g})$ induces automorphisms of $\mathbf{D}(G/H)$ and $\mathbf{D}(M_1/H_{M_1})$, which are both denoted by $D \mapsto D^\vee$. Let $\eta \colon \mathfrak{g}_{\mathbf{c}} \to \mathfrak{g}_{\mathbf{c}}$ be the conjugation associated with the real form $\mathfrak{g}$. Being real, D has a representative $X \in U(\mathfrak{g})^H$ with $\eta(X) = X$. Hence $X^\vee$ is a representative for D^*, and $D^* = D^\vee$. Moreover, the decomposition (20) is η-stable, so that $\mu_P(D)$

is real, and we see that it suffices to show that $\mu_P(D^\vee) = \mu_P(D)^\vee$. In view of (21) this equality follows from the fact that the maps γ and γ_P commute with the canonical anti-automorphism (reduce to the Riemannian case as in the proof of Lemma 2.1, and then use [20, p. 307]). ▮

LEMMA 19.4. *Let Ω be the canonical image of the Casimir in* $\mathbf{D}(G/H)$. *Then there exists a $R>0$ such that for $\lambda \in i\mathfrak{a}_\mathfrak{q}^*$ with $|\lambda| \geqslant R$ we have that $\underline{\mu}_\mathbf{P}(\Omega : \lambda)$ is invertible and*

$$|\lambda|^2 \, \|\underline{\mu}_\mathbf{P}(\Omega : \lambda)^{-1}\| \leqslant 2 \qquad (|\lambda| \geqslant R).$$

Proof. This is a straightforward consequence of the easy fact that $\underline{\mu}_\mathbf{P}(\Omega, \lambda) - (\lambda, \lambda)$ belongs to $\mathrm{End}({}^\circ\mathscr{C}) \otimes S_1(\mathfrak{a}_\mathfrak{q})$: here $(\cdot, \cdot)$ denotes the complex bilinear extension of the dual of the positive definite form $B \mid \mathfrak{a}_\mathfrak{q} \times \mathfrak{a}_\mathfrak{q}$, and the index 1 indicates the space of elements of order at most 1. ▮

Completion of the Proof of Theorem 19.1. Let R be as in Lemma 19.4. Then by continuity of $\mathscr{F}_\pi$ as a map into $C^\infty(i\mathfrak{a}_\mathfrak{q}^*) \otimes {}^\circ\mathscr{C}$, it suffices to prove the following statement. Let $M \in \mathbf{N}$, $u \in S(\mathfrak{a}_\mathfrak{q}^*)$. Then there exists a continuous seminorm s on $\mathscr{C}(G/H, \tau)$ such that

$$|\mathscr{F}_\pi f(\lambda; u)| \leqslant (1 + |\lambda|)^{-M} s(f)$$

for all $f \in \mathscr{C}(G/H, \tau)$ and all $\lambda \in i\mathfrak{a}_\mathfrak{q}^*$ with $|\lambda| \geqslant R$.

We shall prove this by induction on the degree of u. In view of Theorem 19.2 and Corollary 17.6 there exists a seminorm s_0 such that for $f \in \mathscr{C}(G/H, \tau)$ we have

$$|\mathscr{F}_\pi f(\lambda; u)| \leqslant (1 + |\lambda|)^N s_0(f) \qquad (\lambda \in i\mathfrak{a}_\mathfrak{q}^*).$$

Using Lemma 19.4 we now obtain that

$$|\underline{\mu}_\mathbf{P}(\Omega : \lambda)^{-n} \mathscr{F}_\pi(\Omega^n f)(\lambda; u)| \leqslant (1 + |\lambda|)^{N-2n} s_1(f) \qquad (|\lambda| \geqslant R) \quad (170)$$

for a suitable seminorm s_1. In view of Lemma 19.3 this proves the result already when $\deg u = 0$.

To prove the assertion in generality we assume that it has been established for operators of degree at most d. Let u have degree $d+1$. We observe that $\mathscr{F}_\pi f(\lambda; u)$ can be rewritten as $\underline{\mu}_\mathbf{P}(\Omega : \lambda)^{-n} \mathscr{F}_\pi(\Omega^n f))(\lambda; u)$ modulo a finite sum of terms of the form

$$\underline{\mu}_\mathbf{P}(\Omega : \lambda)^{-n} q(\lambda) \, \mathscr{F}_\pi(f)(\lambda; v)$$

with $q \in S(\mathfrak{a}_\mathfrak{q}) \otimes \mathrm{End}({}^\circ\mathscr{C})$ and with $v \in S(\mathfrak{a}_\mathfrak{q}^*)$ of degree at most d. The proof is now completed by using (170) together with the induction hypothesis. ▮

20. APPENDIX: SPECTRAL PROJECTIONS

The purpose of this section is to provide estimates for spectral projections associated with parameter dependent endomorphisms of a finite dimensional complex vector space V of dimension $n \geqslant 2$.

Let X be an open subset of a finite dimensional real vector space, Ω an open subset of a finite dimensional complex vector space, and

$$\Gamma : X \times \Omega \to \operatorname{End}(V)$$

a C^∞-map which is holomorphic in its second variable. We assume that continuous functions $\xi_1, ..., \xi_k : X \times \Omega \to \mathbf{C}$ are given so that $\{\xi_j(x, \lambda); 1 \leqslant j \leqslant k\}$ is the set of eigenvalues for $\Gamma(x, \lambda)$, for every $(x, \lambda) \in X \times \Omega$ (here we do not count them with multiplicities).

Let $1 \leqslant l \leqslant k$ be a fixed integer, and define $P_-(x, \lambda) \in \operatorname{End}(V)$ to be the projection onto the sum of the generalized eigenspaces corresponding to the eigenvalues $\xi_j(x, \lambda)$, $1 \leqslant j \leqslant l$, along the remaining generalized eigenspaces. Let $P_+(x, \lambda)$ be the complementary projection. Then $P_-(x, \lambda) + P_+(x, \lambda) = I$ for all $(x, \lambda) \in X \times \Omega$.

LEMMA 20.1. *Suppose that for every $(x, \lambda) \in X \times \Omega$ we have*

$$\{\xi_j(x, \lambda); 1 \leqslant j \leqslant l\} \cap \{\xi_j(x, \lambda); l < j \leqslant k\} = \varnothing.$$

Then the functions $P_\pm(x, \lambda)$ depend smoothly on (x, λ) and holomorphically on λ.

Proof. Fix $(x_0, \lambda_0) \in X \times \Omega$. Then there exists a bounded open subset D of $\mathbf{C}$ with (compact) smooth boundary ∂D such that for $(x, \lambda) = (x_0, \lambda_0)$ we have

$$\xi_j(x, \lambda) \in D \ (j \leqslant l) \qquad \text{and} \qquad \xi_j(x, \lambda) \notin \operatorname{cl} D \ (l < j). \tag{171}$$

By continuity (171) still holds for (x, λ) in a sufficiently small open neighbourhood $N(x_0, \lambda_0)$ of (x_0, λ_0). Then for $(x, \lambda) \in N(x_0, \lambda_0)$ we have

$$P_-(x, \lambda) = \frac{1}{2\pi i} \int_{\partial D} (zI - \Gamma(x, \lambda))^{-1} \, dz,$$

where ∂D is provided with the induced orientation. All assertions now easily follow. ∎

We now come to a result involving estimates. We assume that there exists a constant $C_0 > 0$ and positive integers p, q such that

$$\|\Gamma(x, \lambda)\| \leqslant C_0(1 + |\lambda|)^p$$
$$|\xi_j(x, \lambda)| \leqslant C_0(1 + |\lambda|)^q \qquad (1 \leqslant j \leqslant k)$$

for all $(x, \lambda) \in X \times \Omega$. Define

$$\xi_-(x, \lambda) = \max_{1 \leqslant j \leqslant l} \operatorname{Re} \xi_j(x, \lambda),$$

and

$$\xi_+(x, \lambda) = \min_{l < j \leqslant k} \operatorname{Re} \xi_j(x, \lambda).$$

PROPOSITION 20.2. *Assume that*

$$\xi_-(x, \lambda) < \xi_+(x, \lambda)$$

for all $(x, \lambda) \in X \times \Omega$, *and put*

$$\delta(x, \lambda) = \min(1, \xi_+(x, \lambda) - \xi_-(x, \lambda)).$$

Then there exist constants $C > 0$, $L \in N$ *such that*

$$\|e^{t\Gamma(x, \lambda)} P_-(x, \lambda)\| \leqslant C \left(\frac{1+t}{\delta(x, \lambda)}\right)^n (1 + |\lambda|)^L e^{t\xi_-(x, \lambda)} \tag{172}$$

$$\|e^{-t\Gamma(x, \lambda)} P_+(x, \lambda)\| \leqslant C \left(\frac{1+t}{\delta(x, \lambda)}\right)^n (1 + |\lambda|)^L e^{-t\xi_+(x, \lambda)}, \tag{173}$$

for all $(x, \lambda) \in X \times \Omega$ *and* $t \geqslant 0$. *In fact one may take* $L = q + (n-1) \max(p, q)$.

Proof. It suffices to prove (172) since (173) will then follow if we replace $\Gamma(x, \lambda)$ by $-\Gamma(x, \lambda)$. Put

$$\mu(x, \lambda) = \tfrac{1}{2}(\xi_-(x, \lambda) + \xi_+(x, \lambda)).$$

There exists a constant $C_1 > 0$ such that

$$|\xi_j(x, \lambda) - \mu(x, \lambda)| \leqslant C_1(1 + |\lambda|)^q$$

for all $(x, \lambda) \in X \times \Omega$ and $1 \leqslant j \leqslant k$. For $(x, \lambda) \in X \times \Omega$ and $t \geqslant 0$ we define $D(t, x, \lambda)$ to be the set of $z \in \mathbf{C}$ with

$$|z - \mu(x, \lambda)| < C_1(1 + |\lambda|)^q + \frac{\delta(x, \lambda)}{2(1+t)}$$

and

$$\operatorname{Re} z < \xi_-(x, \lambda) + \frac{\delta(x, \lambda)}{2(1+t)}.$$

Then clearly $\xi_j(x, \lambda) \in D(t, x, \lambda)$ for $j \leqslant l$ and $\xi_j(x, \lambda) \notin \operatorname{cl} D(t, x, \lambda)$ for $l < j \leqslant k$. Hence

$$e^{t\Gamma(x,\lambda)} P_-(x, \lambda) = \frac{1}{2\pi i} \int_{\partial D(t,x,\lambda)} e^{tz}(zI - \Gamma(x, \lambda))^{-1}\, dz. \tag{174}$$

Now there exists a constant $C_2 > 0$ such that

$$\operatorname{length}(\partial D(t, x, \lambda)) \leqslant C_2(1 + |\lambda|)^q \tag{175}$$

for all $(x, \lambda) \in X \times \Omega$, $t \geqslant 0$. Hence it suffices to estimate the integrand of (174). It is straightforward to see that for $z \in \partial D(t, x, \lambda)$ we have

$$|e^{tz}| \leqslant e^{t\xi_-(x,\lambda) + 1/2}. \tag{176}$$

To estimate the remaining part of the integrand we recall that by Cramer's rule there exists a polynomial map $M: \operatorname{End}(V) \to \operatorname{End}(V)$ such that for every $A \in GL(V)$ one has $A^{-1} = (\det A)^{-1} M(A)$. Since M has degree $\leqslant n-1$ there exists a constant $C_3 > 0$ such that $(r = \max(p, q))$

$$\|M(zI - \Gamma(x, \lambda))\| \leqslant C_3(1 + |\lambda|)^{r(n-1)}$$

for all $(x, \lambda) \in X \times \Omega$, $t \geqslant 0$, and $z \in \partial D(t, x, \lambda)$. On the other hand, if $z \in \partial D(t, x, \lambda)$ then $zI - \Gamma(x, \lambda)$ has the eigenvalues $z - \xi_j(x, \lambda)$ $(1 \leqslant j \leqslant k)$. All of those have absolute value not less then $(1/2)\, \delta(x, \lambda)(1+t)^{-1}$. Hence

$$|\det(zI - \Gamma(x, \lambda))| \geqslant \left(\frac{\delta(x, \lambda)}{2(1+t)}\right)^n$$

and we infer that

$$\|(zI - \Gamma(x, \lambda))^{-1}\| \leqslant 2^n C_3 \left(\frac{1+t}{\delta(x, \lambda)}\right)^n (1 + |\lambda|)^{r(n-1)}, \tag{177}$$

for all $(x, \lambda) \in X \times \Omega$, $t \geqslant 0$, and $z \in \partial D(t, x, \lambda)$. The estimate (172) now follows from (175), (176), and (177). ∎

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REFERENCES

1. J. ARTHUR, A Paley–Wiener theorem for real reductive groups, *Acta Math.* **150** (1983), 1–89.
2. E. P. VAN DEN BAN, Asymptotic behaviour of matrix coefficients related to reductive symmetric spaces, *Proc. Kon. Nederl. Akad. Wetensch.* **90** (1987), 225–249.
3. E. P. VAN DEN BAN, Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula, *Ark. Mat.* **25** (1987), 175–187.
4. E. P. VAN DEN BAN, The principal series for a reductive symmetric space. I. H-fixed distribution vectors, *Ann. Sci. École Norm. Sup.* **21** (1988), 359–412.
5. E. P. VAN DEN BAN AND H. SCHLICHTKRULL, Asymptotic expansions and boundary values of eigenfunctions on a Riemannian symmetric space, *J. Reine Angew. Math.* **380** (1987), 108–165.
6. E. P. VAN DEN BAN AND H. SCHLICHTKRULL, Local boundary data of eigenfunctions on a Riemannian symmetric space, *Invent. Math.* **98** (1989), 639–657.
7. E. P. VAN DEN BAN AND H. SCHLICHTKRULL, Multiplicities in the Plancherel decomposition for a semisimple symmetric space, preprint, Univ. of Utrecht, 1991; *in* "Conf. Proc. Univ. of Maryland, College Park, 1991," in press.
8. E. P. VAN DEN BAN AND H. SCHLICHTKRULL, The most continuous part of the Plancherel decomposition for a reductive symmetric space, in preparation.
9. M. BERGER, Les espaces symétriques non-compacts, *Ann. Sci. École Norm. Sup.* **74** (1957), 85–177.
10. W. CASSELMAN AND D. MILIČIĆ, Asymptotic behaviour of matrix coefficients of admissible representations, *Duke Math. J.* **49** (1982), 869–930.
11. L. COHN, Analytic theory of the Harish-Chandra C-function, *in* "Lecture Notes in Math.," Vol. 429, Springer-Verlag, Berlin/Heidelberg/New York, 1974.
12. M. FLENSTED-JENSEN, Spherical functions on a real semisimple Lie group: A method of reduction to the complex case, *J. Funct. Anal.* **30** (1978), 106–146.
13. M. FLENSTED-JENSEN, Discrete series for semisimple symmetric spaces, *Ann. of Math.* **111** (1980), 253–311.
14. HARISH-CHANDRA, Spherical functions on a semisimple Lie group, *Amer. J. Math.* **80** (1958), 241–310 and 553–613.
15. HARISH-CHANDRA, On the theory of the Eisenstein integral, *in* "Lecture Notes in Math.," Vol. 266, pp. 123–149, Springer-Verlag, Berlin/Heidelberg/New York, 1972.
16. HARISH-CHANDRA, Harmonic analysis on real reductive groups. I. The theory of the constant term, *J. Funct. Anal.* **19** (1975), 103–204.
17. HARISH-CHANDRA, Harmonic analysis on real reductive groups. II. Wave-packets in the Schwartz space, *Invent. Math.* **36** (1976), 1–55.
18. HARISH-CHANDRA, Harmonic analysis on real reductive groups. III. The Maass–Selberg relations and the Plancherel formula, *Ann. of Math.* **104** (1976), 117–201.
19. S. HELGASON, Differential operators on homogeneous spaces, *Acta Math.* **102** (1959), 239–299.
20. S. HELGASON, "Groups and Geometric Analysis," Academic Press, Orlando, FL, 1984.
21. B. HOOGENBOOM, Intertwining functions on compact Lie groups, CWI tract 5, Amsterdam, 1984.
22. M. KASHIWARA, A. KOWATA, K. MINEMURA, K. OKAMOTO, T. OSHIMA, AND M. TANAKA, Eigenfunctions of invariant differential operators on a symmetric space, *Ann. of Math.* **107** (1978), 1–39.
23. B. KOSTANT, On the tensor product of a finite and an infinite dimensional representation, *J. Funct. Anal.* **20** (1975), 257–285.
24. A. W. KNAPP, "Representation Theory of Semisimple Groups," Princeton Mathematical Series, Vol. 36, Princeton Univ. Press, Princeton, NJ, 1986.

25. A. W. Knapp and E. M. Stein, Intertwining operators for semisimple groups, II, *Invent. Math.* **60** (1980), 9–84.
26. G. Olafsson, Fourier and Poisson transformation associated to a semisimple symmetric space, *Invent. Math.* **90** (1987), 605–629.
27. T. Oshima and J. Sekiguchi, Eigenspaces of invariant differential operators on a semisimple symmetric space, *Invent. Math.* **57** (1980), 1–81.
28. H. Schlichtkrull, "Hyperfunctions and Harmonic Analysis on Symmetric Spaces," Birkhäuser, Boston, 1984.
29. G. Schiffmann, Intégrales d'entrelacement et fonctions de Whittaker, *Bull. Soc. Math. France* **99** (1971), 3–72.
30. V. S. Varadarajan, Harmonic analysis on real reductive groups, *in* "Lecture Notes in Math.," Vol. 576, Springer-Verlag, New York/Berlin, 1977.
31. D. A. Vogan and N. R. Wallach, Intertwining operators for real reductive groups, *Adv. in Math.* **82** (1990), 203–243.
32. N. R. Wallach, On Harish-Chandra's generalized C-functions, *Amer. J. Math.* **97** (1975), 386–403.
33. N. R. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, "Lecture Notes in Math.," Vol. 1024, pp. 287–369, Springer-Verlag, New York/Berlin, 1983.
34. N. R. Wallach, "Real Reductive Groups," Academic Press, Orlando, FL, 1988.
35. G. Warner, "Harmonic Analysis on Semi-Simple Lie Groups, I," Springer-Verlag, Berlin/Heidelberg/New York, 1972.
36. C.-B. Zhu, Functional equations satisfied by intertwining operators on reductive groups, preprint, Univ. of Maryland, 1991.
37. G. Zuckerman, Tensor products of finite and infinite dimensional representations of semisimple Lie groups, *Ann. of Math.* **106** (1977), 295–308.