Normalizations of Eisenstein integrals for reductive symmetric spaces

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Abstract

We construct minimal Eisenstein integrals for a reductive symmetric space G/H as matrix coefficients of the minimal principal series of G. The Eisenstein integrals thus obtained include those from the σ-minimal principal series. In addition, we obtain related Eisenstein integrals, but with different normalizations. Specialized to the case of the group, this wider class includes Harish-Chandra's minimal Eisenstein integrals.

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Introduction

Eisenstein integrals play a fundamental role in harmonic analysis on reductive symmetric spaces of the form $X = G/H$; here G is assumed to be a real reductive group of the Harish-Chandra class, and H an (essentially connected) open subgroup of the group G^σ of an involution σ of G. The notion of Eisenstein integral goes back to Harish-Chandra, who used it to describe the contribution of generalized principal series to the Plancherel decomposition of a real reductive group ${}^\backprime G$. In this setting an Eisenstein integral is essentially a matrix coefficient of an induced representation of the form $\mathrm{Ind}_{{}^\backprime P}^{{}^\backprime G}({}^\backprime\omega)$, with ${}^\backprime P$ a proper parabolic subgroup of ${}^\backprime G$ and ${}^\backprime\omega$ a suitable representation of ${}^\backprime P$.

For general symmetric spaces G/H, the notion of Eisenstein integral was introduced in [6] for minimal σ-parabolic subgroups of G, i.e., minimal parabolic subgroups of G with the property that $\sigma(P) = \bar{P}$. The notion was later generalized to arbitrary σ-parabolic subgroups in [14], [15] and found application in the Plancherel theorem for G/H, see [16] and [12]. In this setting of reductive symmetric spaces, the Eisenstein integrals appear essentially as matrix coefficients of K-finite matrix coefficients with H-fixed distribution vectors.

A group ${}^\backprime G$ of the Harish-Chandra class may be viewed as a homogeneous space for the left times right action of $G = {}^\backprime G \times {}^\backprime G$ on ${}^\backprime G$, and is thus realized as the symmetric space G/H with H the diagonal in G. The definition of Eisenstein integral for the symmetric space G/H yields a matrix coefficient on ${}^\backprime G$ which is closely related to Harish-Chandra's Eisenstein integral, but not equal to it. The two obtained types of Eisenstein integrals differ by a normalization which can be described in terms of intertwining operators, see [8] for details. In the present paper we develop a notion of minimal Eisenstein integrals for reductive symmetric spaces, which cover both the existing notion for symmetric spaces and Harish-Chandra's notion for the group.

An even stronger motivation for the present article lies in the application of its results to a theory of cusp forms for symmetric spaces, initiated by M. Flensted-Jensen. In our forthcoming paper [7] we will use our results on Eisenstein integrals to generalize the results of [2] and [1] to reductive symmetric spaces of σ-split rank one (i.e., $\dim \mathfrak{a}_\mathrm{q} = 1$).

We will now explain our results in more detail. Let θ be a Cartan involution of G commuting with σ and let K be the associated maximal compact subgroup of G. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$$

be the eigenspace decompositions into the ± 1-eigenspaces for the infinitesimal involutions θ and σ, respectively. Furthermore, let $\mathfrak{a}_\mathrm{q}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_\mathrm{q}$. We put $A_\mathrm{q} := \exp \mathfrak{a}_\mathrm{q}$ and $A := \exp \mathfrak{a}$.

For the description of the minimal σ-principal series one needs the (finite) set of minimal σ-parabolic subgroups of G containing A_q; this set is denoted by $\mathscr{P}_\sigma(A_\mathrm{q})$. In the case of the group ${}^\backprime G$ one may take $A = {}^\backprime A \times {}^\backprime A$, with ${}^\backprime\mathfrak{a}$ maximal abelian in ${}^\backprime\mathfrak{p}$.

Then $\mathscr{P}_\sigma(A)$ consists of all parabolic subgroups of the form ${}^{\backprime}P \times {}^{\backprime}\bar{P}$, with ${}^{\backprime}P$ a minimal parabolic subgroup from ${}^{\backprime}G$ containing ${}^{\backprime}A$. To obtain Harish-Chandra's Eisenstein integral one would need to also consider minimal parabolic subgroups of the form ${}^{\backprime}P \times {}^{\backprime}P$.

Our goal is then to define Eisenstein integrals by means of suitable H-fixed distribution vectors for all minimal parabolics of G containing A. The (finite) set of these is denoted by $\mathscr{P}(A)$. For the case of the group one has $\mathscr{P}_\sigma(A_{\mathrm{q}}) \subsetneq \mathscr{P}(A)$, but for general symmetric spaces G/H, the parabolic subgroups from $\mathscr{P}_\sigma(A_{\mathrm{q}})$ will in general not be minimal.

A parabolic subgroup $P \in \mathscr{P}(A)$ is called $\mathfrak{q}$-extreme if it is contained in a parabolic subgroup P_0 from $\mathscr{P}_\sigma(A_{\mathrm{q}})$, see Section 1 for details. For such a parabolic, each representation $\mathrm{Ind}_{P_0}^G(\xi \otimes \lambda \otimes 1)$ of the σ-principal series can be embedded in the representation $\mathrm{Ind}_P^G(\xi_M \otimes (\lambda - \rho_{P\mathrm{h}}) \otimes 1)$ of the minimal principal series, through induction by stages. Here ξ is a finite dimensional unitary representation of the Langlands component $M_0 := M_{P_0}$, and ξ_M denotes the restriction of ξ to $M := M_P$. Furthermore, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $\rho_{P\mathrm{h}} := \rho_P - \rho_{P_0}$. This is discussed in Section 4.

In Section 5 the H-fixed generalized vectors of the first of these induced representations are shown to allow a natural realization in the latter. To describe it, one needs to parametrize the open H-orbits on G/P_0. We will avoid this complication in the introduction, and work under the simplifying assumption that HP_0 is the single open orbit. This condition is always fulfilled in the case of the group; in general the open orbits are given by PvH, for v in a finite set $\mathscr{W} \simeq W(\mathfrak{a}_{\mathrm{q}})/W_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$.

Let $C^{-\infty}(P_0 : \xi : \lambda)$ denote the space of generalized vectors for the induced representations $\mathrm{Ind}_{P_0}^G(\xi \otimes \lambda \otimes 1)$. The H-fixed elements in this space needed for the definition of the Eisenstein integral are parametrized by $V(\xi) = \mathscr{H}_\xi^{M_0 \cap H}$. Given $\eta \in V(\xi)$, one has a family

$$j(P_0 : \xi : \lambda : \eta) \in C^{-\infty}(P_0 : \xi : \lambda)^H, \qquad (\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*),$$

defined in [5]. In a suitable sense it depends meromorphically on $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. This family has image $j_H(P : \xi_M : \lambda : \eta)$ in $C^{-\infty}(P : \xi_M : \lambda - \rho_{P\mathrm{h}})^H$. By definition the latter defines a continuous conjugate linear functional on the space $C^\infty(P : \xi_M : -\bar{\lambda} + \rho_{P\mathrm{h}})$. In (5.5) we show that for λ in a suitable region $\Omega_P \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ this functional is given by an absolutely convergent integral

$$\langle j_H(P : \xi_M : \lambda : \eta)\,,\, f\rangle = \int_{H_P\backslash H} \bar{f}_{\eta,\omega}, \tag{I.1}$$

for $f \in C^\infty(P : \xi_M : -\bar{\lambda} + \rho_{P\mathrm{h}})$. Here $H_P := H \cap P$, and $\bar{f}_{\eta,\omega}$ is a natural interpretation of the function $\langle \eta\,,\, f\rangle|_H \in C^\infty(H)$ as a density on the quotient manifold $H_P\backslash H$.

To extend formula (I.1) to the setting of a parabolic subgroup $Q \in \mathscr{P}(A)$ which is not $\mathfrak{q}$-extreme, two problems need to be solved. First of all a suitable domain Ω_Q for the convergence needs to be determined. Next, the resulting family $j_H(Q : \xi_M : \lambda)$ needs to be extended meromorphically in the parameter $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

In the present paper both these problems are solved by using a suitable partial ordering $\succeq$ on $\mathscr{P}(A)$ whose maximal elements are the $\mathfrak{q}$-extremal parabolic subgroups, see Section 2 for details. Let $P \in \mathscr{P}_\sigma(A)$ be such that $P \succeq Q$. Then the definition of the ordering guarantees that $H_P \subset H_Q$ and that the fiber $H_P\backslash H_Q$ of the natural fiber bundle $H_P\backslash H \to H_Q\backslash H$ is diffeomorphic to $N_Q \cap \overline{N}_P$ in a natural way, see Section 6. We use a general Fubini type theorem for densities on fiber bundles, discussed in the appendix of this paper, to decompose the integral (I.1) in terms of a fiber integral over $N_Q \cap \overline{N}_P$ followed by an integral over the base manifold $H_Q\backslash H$, see Thm. 6.7. The first of these integrals turns out to be the integral for the standard intertwining operator

$$A(Q:P:\xi_M:-\overline{\lambda}+\rho_{P\mathrm{h}}): C^\infty(P:\xi_M:-\overline{\lambda}+\rho_{P\mathrm{h}}) \to C^\infty(Q:\xi_M:-\overline{\lambda}+\rho_{P\mathrm{h}}),$$

whereas the second integral turns out to be (I.1) with Q in place of P. According to Theorem 7.1 this results in the formula

$$j_H(P:\xi_M:\lambda:\eta) = j_H(Q:\xi_M:\lambda:\eta)\circ A(Q:P:\xi_M:-\overline{\lambda}+\rho_{P\mathrm{h}}), \tag{I.2}$$

with convergent integrals for $\lambda \in \Omega_P$. Convergence of the integral for $j_H(Q:\xi_M:\lambda:\eta)$ is thus obtained through Fubini's theorem, as a consequence of the known convergence of the other two integrals. Furthermore, since the appearing standard intertwining operator has an inverse which is meromorphic in λ, formula (I.1) also allows us to conclude in Theorem 7.8 that

$$\lambda \mapsto j_H(Q:\xi_M:\lambda:\eta) \tag{I.3}$$

depends meromorphically on $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$.

Once the meromorphic extension of (I.3) has been established for general $Q \in \mathscr{P}(A)$ we apply a recent convexity theorem [3, Thm. 10.1] to determine a large domain on which (I.3) is holomorphic, see Corollary 7.6). The convexity theorem describes the image of H under the projection $\mathfrak{H}_{Q,\mathfrak{q}}: G \to \mathfrak{a}_\mathfrak{q}$ determined by the Iwasawa decomposition $G = K(A\cap H)\exp(\mathfrak{a}_\mathfrak{q})N_Q$ as a convex polyhedral cone described in terms of a subset of the set of roots of $\mathfrak{a}$ in $\mathfrak{n}_Q$. This description allows one to decide whether this cone properly contains the origin or equals it. In the latter case it is shown that (I.3) is holomorphic on all of $\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$, see Remark 7.9.

The definition of the meromorphic family of H-fixed generalized vectors (I.3) allows us to define Eisenstein integrals $E(Q:\lambda)$ essentially as matrix coefficients with K-finite vectors in the induced representation under consideration. In particular, the Eisenstein integral depends meromorphically on λ. Holomorphy of (I.3) implies holomorphy of the corresponding Eisenstein integral, see Corollary 8.5.

The relation (I.2) leads to a relation between the Eisenstein integral $E(Q:\lambda)$ and the Eisenstein integral $E(P_0,\lambda)$, earlier defined in [6] and [10]. This relation amounts to a different normalization of the Eisenstein integral expressed in terms of a C-function, see Corollary 8.14.

Finally, in Section 9, we discuss the case of the group, and express the obtained Eisenstein integrals in terms of Harish-Chandra's Eisenstein integrals, see Corollary

9.8. In this case, the Eisenstein integral $E(Q : \lambda)$ coincides with Harish-Chandra's if and only if Q is a $\succeq$-minimal element of $\mathscr{P}(A)$. The latter means that Q is of the form ${}^\backprime Q \times {}^\backprime Q$, with ${}^\backprime Q \in \mathscr{P}({}^\backprime A)$; see Corollary 9.6. The result on holomorphy established above, is consistent with the holomorphic dependence of Harish-Chandra's Eisenstein integral, see Remark 9.7.

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1 Notation and preliminaries

In this section we collect some of the notation that will be used throughout this article.

We assume that G is a reductive Lie group of the Harish-Chandra class. Let σ be an involutive automorphism of G and let θ be a Cartan involution that commutes with σ, let $K := G^\theta$ be the associated maximal compact subgroup. Let H be an open subgroup of the fixed point subgroup G^σ. We assume H to be essentially connected. (See [4, p.24].) If S is any closed subgroup of G, we agree to write

$$H_S := S \cap H.$$

A Lie subgroup will in general be denoted by a Roman upper case letter; the associated Lie algebra by the corresponding lower case gothic letter. We write σ for the involution $T_e\sigma$ of $\mathfrak{g}$. Likewise, we write θ for the Cartan involution $T_e\theta$ of $\mathfrak{g}$. Now $\mathfrak{g}$ decomposes as a direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{q}$ is the -1-eigenspace of σ and $\mathfrak{h}$ is the $+1$-eigenspace; thus, $\mathfrak{h}$ is the Lie algebra of H. Moreover, $\mathfrak{g}$ admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the -1-eigenspace of θ and $\mathfrak{k}$ the $+1$-eigenspace; thus, $\mathfrak{k}$ is the Lie algebra of K. Since σ and θ commute, $\mathfrak{g}$ also decomposes as a direct sum of vector spaces

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

We fix a non-degenerate G-invariant bilinear form B on $\mathfrak{g}$, which coincides with the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, and for which the above decomposition is orthogonal. Furthermore, we equip $\mathfrak{g}$ with the positive definite inner product given by

$$\langle \cdot\,, \cdot \rangle := -B(\cdot\,, \theta(\cdot)).$$

We fix a maximal abelian subspace $\mathfrak{a}_{\mathrm{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$ and a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing $\mathfrak{a}_{\mathrm{q}}$. Then $\mathfrak{a}$ decomposes as

$$\mathfrak{a} = \mathfrak{a}_{\mathrm{h}} \oplus \mathfrak{a}_{\mathrm{q}},$$

where $\mathfrak{a}_{\mathrm{h}} = \mathfrak{a} \cap \mathfrak{h}$. This decomposition induces natural embeddings of the associated dual spaces spaces $\mathfrak{a}_{\mathrm{h}}^*$ and $\mathfrak{a}_{\mathrm{q}}^*$ into $\mathfrak{a}^*$ so that

$$\mathfrak{a}_{\mathrm{h}}^* \simeq \{\lambda \in \mathfrak{a}^* : \lambda|_{\mathfrak{a}_{\mathrm{q}}} = 0\} \quad \text{and} \quad \mathfrak{a}_{\mathrm{q}}^* \simeq \{\lambda \in \mathfrak{a}^* : \lambda|_{\mathfrak{a}_{\mathrm{h}}} = 0\}.$$

Let $A := \exp(\mathfrak{a})$, $A_{\mathrm{q}} := \exp(\mathfrak{a}_{\mathrm{q}})$ and $A_{\mathrm{h}} := \exp(\mathfrak{a}_{\mathrm{h}})$.

If P is a parabolic subgroup (not necessarily minimal), then we write N_P for the unipotent radical of P. If P contains A and $\mathfrak{b}$ is a subalgebra of $\mathfrak{a}$, then we write $\Sigma(P, \mathfrak{b})$ for the set of weights of $\mathfrak{b}$ in $\mathfrak{n}_P$. Furthermore, still assuming $P \supset A$, we will write $\Sigma(P)$ for $\Sigma(P, \mathfrak{a})$, unless clarity of exposition requires otherwise. If τ is an involution of $\mathfrak{g}$ preserving $\mathfrak{a}$, we agree to write

$$\Sigma(P, \tau) := \Sigma(P) \cap \tau\Sigma(P). \tag{1.1}$$

For a root $\alpha \in \Sigma(\mathfrak{a}) \cap \mathfrak{a}_{\mathrm{q}}^*$, we note that $\sigma\theta\alpha = \alpha$, so that $\sigma\theta$ leaves the root space $\mathfrak{g}_\alpha$ invariant. We define the subset $\Sigma(P)_{\sigma,-} = \Sigma(P)_-$ of $\Sigma(P, \sigma\theta)$ by

$$\Sigma(P)_- := \{\alpha \in \Sigma(P, \sigma\theta) : \alpha \in \mathfrak{a}_{\mathrm{q}}^* \Rightarrow \sigma\theta|_{\mathfrak{g}_\alpha} \neq I\}. \tag{1.2}$$

Let M denote the centralizer of A in K and let $\mathscr{P}(A)$ denote the set of minimal parabolic subgroups $P \subset G$ with $A \subset P$. Then each subgroup $P \in \mathscr{P}(A)$ has a Langlands decomposition of the form $P = MAN_P$.

Definition 1.1. A parabolic subgroup $P \in \mathscr{P}(A)$ is said to be $\mathfrak{q}$-extreme if

$$\Sigma(P, \sigma\theta) = \Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^*.$$

The set of these parabolic subgroups is denoted by $\mathscr{P}_\sigma(A)$.

We will finish this section by comparing $\mathscr{P}_\sigma(A)$ with the set $\mathscr{P}_\sigma(A_{\mathrm{q}})$ of minimal $\sigma\theta$-stable parabolic subgroups of G containing A_{q}. We recall from [5] that the latter set is finite and in bijective correspondence with the set of positive systems for $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. Indeed, if Π is such a positive system then the corresponding parabolic subgroup P_Π from $\mathscr{P}_\sigma(A_{\mathrm{q}})$ equals

$$P_\Pi = Z_G(\mathfrak{a}_{\mathrm{q}}) N_\Pi,$$

where

$$\mathfrak{n}_\Pi := \oplus_{\alpha \in \Pi} \mathfrak{g}_\alpha, \qquad \text{and} \quad N_\Pi := \exp(\mathfrak{n}_\Pi).$$

The Langlands decomposition of P_Π is given by

$$P_\Pi = M_0 A_0 N_\Pi,$$

where the groups M_0 and A_0 are described as follows. First, $A_0 = \exp(\mathfrak{a}_0)$ where

$$\mathfrak{a}_0 = \bigcap_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \cap \mathfrak{a}_{\mathrm{h}}^*} \ker \alpha.$$

Next $M_0 = Z_K(\mathfrak{a}_{\mathrm{q}}) \exp(\mathfrak{m}_0)$, where $\mathfrak{m}_0 := Z_{\mathfrak{g}}(\mathfrak{a}_{\mathrm{q}}) \cap \mathfrak{a}_0^{\perp}$. Thus,

$$Z_G(\mathfrak{a}_{\mathrm{q}}) \simeq M_0 \times A_0.$$

Conversely, if $P_0 \in \mathscr{P}_\sigma(A_{\mathrm{q}})$ then the associated positive system is given by

$$\Sigma(P_0, \mathfrak{a}_{\mathrm{q}}) := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}}) \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_{P_0}\}.$$

Lemma 1.2. *Let $P \in \mathscr{P}(A)$. Then the following conditions are equivalent.*

(a) $P \in \mathscr{P}_\sigma(A)$;

(b) *there exists a $P_0 \in \mathscr{P}_\sigma(A_{\mathrm{q}})$ such that $P \subset P_0$.*

Proof. First assume (a). Then $\Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^* = \Sigma(P, \sigma\theta)$ and we see that the set Π of non-zero restrictions $\alpha|_{\mathfrak{a}_{\mathrm{q}}}$, for $\alpha \in \Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^*$, is a positive system for $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. Now $N_P = (N_P \cap M_0) N_\Pi$ and we see that $P \subset P_\Pi$ and (b) follows.

Next assume (b). We first note that

$$\Sigma(P_0, \mathfrak{a}_{\mathrm{q}}) = \{\alpha|_{\mathfrak{a}_{\mathrm{q}}} : \alpha \in \Sigma(P_0), \alpha|_{\mathfrak{a}_{\mathrm{q}}} \neq 0\}.$$

The minimality of P_0 implies that $\Sigma(P_0, \mathfrak{a}_{\mathrm{q}})$ is a positive system for the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, hence

$$\Sigma(\mathfrak{g}, \mathfrak{a}) \setminus \mathfrak{a}_{\mathrm{h}}^* = \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})\} = \Sigma(P_0) \cup -\Sigma(P_0).$$

By assumption $P \subset P_0$. This implies $\Sigma(P_0) \subset \Sigma(P)$ and $\Sigma(P) \cap -\Sigma(P_0) = \emptyset$. Hence,

$$\Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^* = \Sigma(P_0).$$

Moreover, since P_0 is $\sigma\theta$-stable the above equality implies $\Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^* \subset \Sigma(P, \sigma\theta)$. As the converse inclusion is obvious, the parabolic P is $\mathfrak{q}$-extreme and (a) follows. $\square$

2 Minimal parabolic subgroups

Lemma 2.1. *Let $P \in \mathscr{P}(A)$. The set $\Sigma(P)$ is the disjoint union of $\Sigma(P, \sigma)$ and $\Sigma(P, \sigma\theta)$.*

Proof. Let $\alpha \in \Sigma(P)$. Then either $\sigma\alpha \in \Sigma(P)$ or $\sigma\theta\alpha = -\sigma\alpha \in \Sigma(P)$. The two cases are exclusive, and in the first case we have $\alpha \in \Sigma(P, \sigma)$, while in the second $\alpha \in \Sigma(P, \sigma\theta)$. $\square$

We define the partial ordering $\succeq$ on $\mathscr{P}(A)$ by

$$P \succeq Q \iff \Sigma(Q, \sigma\theta) \subset \Sigma(P, \sigma\theta) \text{ and } \Sigma(P, \sigma) \subset \Sigma(Q, \sigma). \tag{2.1}$$

It is easy to see that this condition on P and Q implies that $H_{N_P} \subset H_{N_Q}$. The latter condition implies that we have a natural surjective H-map $H/H_{N_P} \to H/H_{N_Q}$.

Lemma 2.2. *Let $P, Q \in \mathscr{P}(A)$, and assume that $P \succeq Q$. Then*

(a) $\Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^* = \Sigma(Q) \cap \mathfrak{a}_{\mathrm{q}}^*$;

(b) $\Sigma(P) \cap \mathfrak{a}_{\mathrm{h}}^* = \Sigma(Q) \cap \mathfrak{a}_{\mathrm{h}}^*$.

Proof. Let $\alpha \in \Sigma(Q) \cap \mathfrak{a}_{\mathrm{q}}^*$. Then $\sigma\theta\alpha = \alpha$ so that $\alpha \in \Sigma(Q, \sigma\theta) \subset \Sigma(P, \sigma\theta)$. We infer that $\Sigma(Q) \cap \mathfrak{a}_{\mathrm{q}}^* \subset \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$. Since both sets in this inclusion are positive systems for the root system $\Sigma \cap \mathfrak{a}_{\mathrm{q}}^*$, the converse inclusion follows by a counting argument.

Assertion (b) is proved in a similar fashion, using σ in place of $\sigma\theta$ and referring to the second inclusion of (2.1) instead of the first. $\square$

Lemma 2.3. *Let $P, Q \in \mathscr{P}(A)$. Then the following statements are equivalent.*

(a) $P \succeq Q$;

(b) $\Sigma(P) \cap \Sigma(\bar{Q}) \subset \Sigma(P, \sigma\theta)$ *and* $\Sigma(\bar{P}) \cap \Sigma(Q) \subset \Sigma(Q, \sigma)$;

(c) $\Sigma(P) \cap \Sigma(\bar{Q}) = \Sigma(P, \sigma\theta) \cap \Sigma(\bar{Q}, \sigma)$.

Proof. First assume (a). Let $\alpha \in \Sigma(P) \cap \Sigma(\bar{Q})$. Then $\sigma\alpha \in \Sigma(P)$ would lead to $\alpha \in \Sigma(Q)$, contradiction. Hence, $\alpha \in \Sigma(P, \sigma\theta)$. The second inclusion of (b) follows in a similar fashion.

Next, (b) is equivalent to $\Sigma(P) \cap \Sigma(\bar{Q}) \subset \Sigma(P, \sigma\theta) \cap \Sigma(\bar{Q}, \sigma)$, which is readily seen to be equivalent to (c).

Finally, assume (c) and let $\alpha \in \Sigma(P, \sigma)$. Then $\alpha \in \Sigma(P) \setminus \Sigma(P, \sigma\theta)$, hence $\alpha \notin \Sigma(\bar{Q})$ by the equality of (c) and it follows that $\alpha \in \Sigma(Q)$. Likewise, $\sigma\alpha \in \Sigma(Q)$ and we conclude that $\alpha \in \Sigma(Q, \sigma)$. On the other hand, let $\alpha \in \Sigma(Q, \sigma\theta)$. Then $\alpha \in \Sigma(Q) \setminus \Sigma(Q, \sigma)$. The equality in (c) is equivalent to

$$\Sigma(\bar{P}) \cap \Sigma(Q) = \Sigma(\bar{P}, \sigma\theta) \cap \Sigma(Q, \sigma),$$

which shows that $\alpha \in \Sigma(P)$. Likewise, $\sigma\theta\alpha \in \Sigma(P)$ and we see that $\alpha \in \Sigma(P, \sigma\theta)$. This proves (a). $\square$

Lemma 2.4. *Let $P, Q, R \in \mathscr{P}(A)$ be such that $P \succeq R$. Then the following assertions are equivalent:*

(a) $P \succeq Q \succeq R$;

(b) $\Sigma(P) \cap \Sigma(\bar{Q}) \subset \Sigma(P) \cap \Sigma(\bar{R})$.

Proof. Assume (a). By Lemma 2.3, the first set in (b) equals $\Sigma(P,\sigma\theta)\cap\Sigma(\bar{Q},\sigma)$, which by (2.1) is contained in $\Sigma(P,\sigma\theta)\cap\Sigma(\bar{R},\sigma)$. The latter set equals the second set of (b), again by application of Lemma 2.3. Assertion (b) follows.

For the converse implication, assume (b). Then it is well known and easy to show that

$$\Sigma(P)\cap\Sigma(\bar{R}) = (\Sigma(P)\cap\Sigma(\bar{Q}))\cup(\Sigma(Q)\cap\Sigma(\bar{R})) \qquad \text{(disjoint union)} \tag{2.2}$$

Indeed, it is obvious that the set on the left-hand side of (2.2) is contained in the union on the right-side. For the converse inclusion, we first note that (b) implies $\Sigma(\bar{P})\cap\Sigma(Q)\subset\Sigma(R)$. Now assume that $\alpha\in\Sigma(Q)\cap\Sigma(\bar{R})$. Then $\alpha\notin\Sigma(\bar{P})$ so that $\alpha\in\Sigma(P)\cap\Sigma(\bar{R})$. Hence, the second inclusion of (2.2) follows as well.

Still assuming (b), we claim that $P\succeq Q$. To see this, let $\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})$. Then $\alpha\in\Sigma(P)\cap\Sigma(\bar{R})\subset\Sigma(P,\sigma\theta)\cap\Sigma(\bar{R},\sigma)$ by Lemma 2.3. Assume now in addition $\alpha\notin\Sigma(\bar{Q},\sigma)$. Then $\sigma\theta\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})\subset\Sigma(P)\cap\Sigma(\bar{R})$, hence $\alpha\in\Sigma(\bar{R},\sigma\theta)$, contradicting the earlier conclusion that $\alpha\in\Sigma(\bar{R},\sigma)$. Thus the assumption cannot hold, so that $\alpha\in\Sigma(P,\sigma\theta)\cap\Sigma(\bar{Q},\sigma)$. In view of Lemma 2.3 this establishes the claim.

We will now infer (a) by establishing that $Q\succeq R$. For this, let $\alpha\in\Sigma(Q)\cap\Sigma(\bar{R})$. Then $\alpha\in\Sigma(P)\cap\Sigma(\bar{R})$ by (2.2), which implies that $\alpha\in\Sigma(P,\sigma\theta)\cap\Sigma(\bar{R},\sigma)$ by Lemma 2.3. Assume now that $\alpha\notin\Sigma(Q,\sigma\theta)$. Then $\sigma\alpha\in\Sigma(Q)\cap\Sigma(\bar{R})\subset\Sigma(P)$, so that $\alpha\in\Sigma(P,\sigma)$, contradicting the earlier conclusion that $\alpha\in\Sigma(P,\sigma\theta)$. Thus, the assumption cannot hold, so that $\alpha\in\Sigma(Q,\sigma\theta)\cap\Sigma(\bar{R},\sigma)$. Applying Lemma 2.3 with (Q,R) in place of (P,Q), we finally obtain that $Q\succeq R$. $\square$

Remark 2.5. Recall that two parabolic subgroups $P,Q\in\mathscr{P}(A)$ are said to be adjacent if $\Sigma(P)\cap\Sigma(\bar{Q})$ has a one dimensional span in $\mathfrak{a}^*$.

If $P,Q\in\mathscr{P}(A)$ then there exists a sequence $P=P_0,P_1,\dots,P_n=Q$ of parabolic subgroups in $\mathscr{P}(A)$ such that for all $0\le j<n$ we have $\Sigma(P)\cap\Sigma(\bar{P}_j)\subset\Sigma(P)\cap\Sigma(\bar{P}_{j+1})$ and such that P_j and P_{j+1} are adjacent. If in addition $P\succ Q$, then it follows from repeated application of the lemma above that

$$P=P_0\succ P_1\succ\cdots\succ P_n=Q.$$

Our next objective in this section is to show that every parabolic subgroup from $\mathscr{P}(A)$ is dominated by a $\mathfrak{q}$-extreme one, see Definition 1.1

Given $Q\in\mathscr{P}(A)$, we denote the positive Weyl chamber for $\Sigma(Q)$ in $\mathfrak{a}$ by $\mathfrak{a}^+(Q)$. Furthermore, we put

$$\mathfrak{a}_\mathrm{q}^+(Q)=\{H\in\mathfrak{a}_\mathrm{q}\mid\alpha(H)>0,\ \forall\alpha\in\Sigma(Q,\sigma\theta)\}. \tag{2.3}$$

It is readily verified that this set contains the image of $\mathfrak{a}^+(Q)$ under the projection $\mathrm{pr}_\mathrm{q}:\mathfrak{a}\to\mathfrak{a}_\mathrm{q}$; in particular, it is non-empty.

Let $\mathfrak{a}_\mathrm{q}^\mathrm{reg}$ be the set of regular elements in $\mathfrak{a}_\mathrm{q}$, relative to the root system $\Sigma(\mathfrak{a}_\mathrm{q})$. The connected components of this set are the chambers for the root system $\Sigma(\mathfrak{a}_\mathrm{q})$. The collection of these is denoted by $\Pi_0(\mathfrak{a}_\mathrm{q}^\mathrm{reg})$. It is clear that $\mathfrak{a}_\mathrm{q}^\mathrm{reg} \cap \mathfrak{a}_\mathrm{q}^+(Q)$ is the disjoint union of the chambers contained in $\mathfrak{a}_\mathrm{q}^+(Q)$.

We define

$$\mathscr{P}_\sigma(A,Q) := \{P \in \mathscr{P}_\sigma(A) \mid \ P \succeq Q\ \}$$

Lemma 2.6. *Let $Q \in \mathscr{P}(A)$. Then the assignment $P \mapsto \mathfrak{a}_\mathrm{q}^+(P)$ defines a bijection from the set $\mathscr{P}_\sigma(A,Q)$ onto the set $\{C \in \Pi_0(\mathfrak{a}_\mathrm{q}^\mathrm{reg}) \mid C \subset \mathfrak{a}_\mathrm{q}^+(Q)\}$.*

Proof. We abbreviate $\mathscr{C}(Q) := \{C \in \Pi_0(\mathfrak{a}_\mathrm{q}^\mathrm{reg}) \mid C \subset \mathfrak{a}_\mathrm{q}^+(Q)\}$. Let $P \in \mathscr{P}_\sigma(A,Q)$. Then a root $\alpha \in \Sigma(P)$ restricts to a non-zero root on $\mathfrak{a}_\mathrm{q}$ if and only if $\alpha \in \Sigma(P) \setminus \mathfrak{a}_\mathrm{h}^*$. The latter set equals $\Sigma(P) \setminus \Sigma(P,\sigma) = \Sigma(P,\sigma\theta)$. Therefore, $\mathfrak{a}_\mathrm{q}^+(P)$ is a connected component of $\mathfrak{a}_\mathrm{q}^\mathrm{reg}$. Furthermore, from $P \succeq Q$ it follows that $\Sigma(P,\sigma\theta) \supset \Sigma(Q,\sigma\theta)$, which in turn implies that $\mathfrak{a}_\mathrm{q}^+(P) \subset \mathfrak{a}_\mathrm{q}^+(Q)$. It follows that $\mathfrak{a}_\mathrm{q}^+(P) \in \mathscr{C}(Q)$. It remains to be shown that the map

$$P \mapsto \mathfrak{a}_\mathrm{q}^+(P),\ \mathscr{P}_\sigma(A,Q) \to \mathscr{C}(Q) \tag{2.4}$$

is bijective. For injectivity, assume that $P_1, P_2 \in \mathscr{P}_\sigma(A,Q)$ and that $\mathfrak{a}_\mathrm{q}^+(P_1) = \mathfrak{a}_\mathrm{q}^+(P_2)$. Let $\alpha \in \Sigma(P_1)$. If $\alpha \in \mathfrak{a}_\mathrm{h}^*$, then $\alpha \in \Sigma(Q) \cap \mathfrak{a}_\mathrm{h}^* \subset \Sigma(P_2)$. If $\alpha \notin \mathfrak{a}_\mathrm{h}^*$, then $\alpha \in \mathfrak{a}_\mathrm{q}^+(P_1,\sigma\theta)$ and it follows that $\alpha > 0$ on $\mathfrak{a}_\mathrm{q}^+(P_1) = \mathfrak{a}_\mathrm{q}^+(P_2)$, which implies that $\alpha \in \Sigma(P_2)$. Thus, we see that $\Sigma(P_1) \subset \Sigma(P_2)$ which implies $P_1 = P_2$.

For surjectivity, let C be a chamber in $\mathscr{C}(Q)$. Let Π_C denote the set of roots $\alpha \in \Sigma(\mathfrak{a})$ that are strictly positive on C. The set $\Pi_\mathrm{h} := \Sigma(Q) \cap \mathfrak{a}_\mathrm{h}^*$ is a choice of positive roots for the root system $\Sigma(\mathfrak{a}) \cap \mathfrak{a}_\mathrm{h}^*$. Hence, there exists an element $Y \in \mathfrak{a}_\mathrm{h}^*$ such that

$$\Pi_\mathrm{h} = \{\alpha \in \Sigma(\mathfrak{a}) \cap \mathfrak{a}_\mathrm{h}^* \mid \alpha(Y) > 0\}.$$

Fix $X \in C$ and put $X_t = X + tY$ for $t \in \mathbb{R}$. Then there exists $\varepsilon > 0$ such that for $|t| < \varepsilon$ we have $\alpha(X_t) > 0$ for all $\alpha \in \Pi_C$. Fix $0 < t < \varepsilon$. Then it follows that X_t is regular for $\Sigma(\mathfrak{a})$ and that the associated choice of positive roots $\Pi := \{\alpha \in \Sigma(\mathfrak{a}) \mid \alpha(X_t) > 0\}$ is the disjoint union of Π_C and Π_h. Let P be the parabolic subgroup in $\mathscr{P}(A)$ with $\Sigma(P) = \Pi$. Then $\Sigma(P) \cap \mathfrak{a}_\mathrm{h}^* = \Pi_\mathrm{h} = \Sigma(Q) \cap \mathfrak{a}_\mathrm{h}^*$. Furthermore, if $\alpha \in \Sigma(P) \setminus \mathfrak{a}_\mathrm{h}^*$, then $\alpha \in \Pi_C$. Hence, $\sigma\theta\alpha(X_t) = \alpha(-\sigma(X_t)) = \alpha(X_{-t}) > 0$, and we see that $\alpha \in \Sigma(P,\sigma\theta)$. It readily follows that $P \in \mathscr{P}_\sigma(A,Q)$. $\square$

We finish this section by investigating these structures in the setting where H is replaced by a conjugate vHv^{-1}, with $v \in N_K(\mathfrak{a}) \cap N_{\mathfrak{a}_\mathrm{q}}$. Let such an element v be fixed. Then v normalizes $\mathfrak{a}_\mathrm{h}$ as well. Let $C_v : G \to G$ denote conjugation by v, and put

$$\sigma_v := C_v \circ \sigma \circ C_v^{-1}. \tag{2.5}$$

Then σ_v is an involution of G which commutes with the Cartan involution θ; moreover, since v normalizes $Z_K(\mathfrak{a}_\mathrm{q})$, the conjugate group vHv^{-1} is readily seen to be an essentially connected open subgroup of G^{σ_v}. The infinitesimal involution associated with σ_v is given by $\sigma_v = \mathrm{Ad}(v) \circ \sigma \circ \mathrm{Ad}(v)^{-1}$. Since $\mathrm{Ad}(v)$ normalizes $\mathfrak{a}_\mathrm{q}$ and $\mathfrak{a}_\mathrm{h}$, it follows that

$$\sigma_v|_\mathfrak{a} = \sigma|_\mathfrak{a} \tag{2.6}$$

and that $\mathfrak{a}_{\mathrm{q}}$ is maximal abelian in $\mathfrak{p} \cap \ker(\sigma_v + I)$.

It follows from (1.1) and (2.6) that

$$\Sigma(Q, \sigma_v) = \Sigma(Q, \sigma) \quad \text{and} \quad \Sigma(Q, \sigma_v \theta) = \Sigma(Q, \sigma\theta). \tag{2.7}$$

From this we see that the ordering on $\mathscr{P}(A)$ defined by (2.1) with σ_v in place of σ coincides with the ordering $\succeq$. It is also clear that $P \mapsto v^{-1}Pv$ preserves $\mathscr{P}_\sigma(A)$.

Lemma 2.7. *Let $Q \in \mathscr{P}(A)$ and $v \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_{\mathrm{q}})$. Then*

$$\Sigma(Q)_{\sigma_v,-} = v\Sigma(v^{-1}Qv)_- \tag{2.8}$$

Let $S_v := \{\alpha \in \Sigma(\mathfrak{a}) \cap \mathfrak{a}_{\mathrm{q}}^* \mid \sigma_v\theta|_{\mathfrak{g}_\alpha} \neq I\}$. Then it is readily seen that $S_v = vS_e$. From (1.2) we now deduce that

$$\begin{aligned} \Sigma(Q)_{\sigma_v,-} \cap \mathfrak{a}_{\mathrm{q}}^* &= \Sigma(Q) \cap vS_e = v(\Sigma(v^{-1}Qv) \cap S_e) \\ &= v\Sigma(v^{-1}Qv)_- \cap \mathfrak{a}_{\mathrm{q}}^*. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Sigma(Q)_{\sigma_v,-} \setminus \mathfrak{a}_{\mathrm{q}}^* &= \Sigma(Q, \sigma_v\theta) \setminus \mathfrak{a}_{\mathrm{q}}^* = \Sigma(Q, \sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^* \\ &= v(\Sigma(v^{-1}Qv, \sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^*) = v\Sigma(v^{-1}Qv)_- \setminus \mathfrak{a}_{\mathrm{q}}^* \end{aligned}$$

and we deduce (2.8).

3 Induced representations and densities

Let $P = M_PA_PN_P$ be a parabolic subgroup with the indicated Langlands decomposition and let $(\xi, \mathscr{H}_\xi)$ be a unitary representation in a finite dimensional Hilbert space $\mathscr{H}_\xi$. The assumption of finite dimensionality is natural for the purpose of this paper. Moreover, the following definitions, though valid in general, will merely be needed for the case that P belongs to either $\mathscr{P}(A)$ or $\mathscr{P}_\sigma(A_{\mathrm{q}})$.

For $\mu \in \mathfrak{a}_{P\mathbb{C}}^*$ we denote by $C(P : \xi : \mu)$ the space of continuous functions $f : G \to \mathscr{H}_\xi$ transforming according to the rule

$$f(manx) = a^{\mu+\rho_P}\xi(m)f(x),$$

for all $x \in G$ and $(m, a, n) \in M_P \times A_P \times N_P$. For $s \in \mathbb{N} \cup \{\infty\}$, the subspace of C^s-functions is denoted by $C^s(P : \xi : \mu)$. The right regular representation R of G in this space is the C^s- version of the normalized induced representation $\mathrm{Ind}_P^G(\xi \otimes \mu \otimes 1)$.

We put $K_{M_P} := K \cap M_P$ and denote by $C(K : \xi) = C(K : K_{M_P} : \xi)$ the space of continuous functions $f : K \to \mathscr{H}_\xi$ transforming according to the rule

$$f(mk) = \xi(m)f(k), \qquad (k \in K,\, m \in K_{M_P}).$$

The subspace of C^s-functions is denoted by $C^s(K:\xi)$. All function spaces introduced so far are assumed to be equipped with the usual Fréchet topologies (Banach when $s<\infty$). Then the restriction map $f\mapsto f|_K$ gives topological linear isomorphisms

$$C^s(P:\xi:\mu)\ \xrightarrow{\simeq}\ C^s(K:\xi), \tag{3.1}$$

intertwining the K-actions from the right. Through these, the right regular actions of the group G may be transferred to continuous representations of G on $C^s(K:\xi)$, denoted $\pi_{P,\xi,\mu}$. This realisation $\pi_{P,\xi,\mu}$ is called the compact picture of the C^s-version of the parabolically induced representation $\mathrm{Ind}_P^G(\xi\otimes\mu\otimes 1)$. Let dk denote the normalized Haar measure on K, and let $\langle\,\cdot\,,\,\cdot\,\rangle_\xi$ denote the inner product of $\mathscr{H}_\xi$. Then it is well known that the sesquilinear pairing $C(K:\xi)\times C(K:\xi)\to\mathbb{C}$ given by

$$\langle f\,,g\rangle_\xi := \int_K \langle f(k)\,,g(k)\rangle_\xi\,dk, \tag{3.2}$$

is equivariant for the representations $\pi_{P,\xi,\mu}$ and $\pi_{P,\xi,-\bar{\mu}}$. Accordingly, the above formula gives an equivariant sesquilinar pairing

$$C(P:\xi:\mu)\times C(P:\xi:-\bar{\mu})\to\mathbb{C}. \tag{3.3}$$

We will usually omit the index ξ in the notation of the pairing (3.2).

We denote by $C^{-s}(P:\xi:\mu)$ the continuous conjugate-linear dual of the Fréchet space $C^s(P:\xi:-\bar{\mu})$, equipped with the strong dual topology and with the natural dual representation. Likewise, we denote by $C^{-s}(K:\xi)$ the continuous conjugate-linear dual of $C^s(K:\xi)$.

By using the pairing (3.3) we obtain equivariant continuous linear injections

$$C(P:\xi:\mu)\hookrightarrow C^{-s}(P:\xi:\mu),$$

for $s\in\mathbb{N}\cup\{\infty\}$. Likewise, by using the pairing (3.2) we obtain K-equivariant continuous linear injections $C(K:\xi)\hookrightarrow C^{-s}(K:\xi)$. Through the indicated pairings it is readily seen that the isomorphism (3.1) for $s=0$ extends to a topological linear isomorphism

$$C^{-s}(P:\xi:\mu)\ \xrightarrow{\simeq}\ C^{-s}(K:\xi), \tag{3.4}$$

for all $s\in\mathbb{N}\cup\{\infty\}$. By transfer we obtain a continuous representation $\pi^{-s}_{P,\xi,\mu}$ of G in the second space in (3.4), such that the isomorphism becomes G-equivariant. It is readily verified that this representation is dual to the representation $\pi_{P,\xi,-\bar{\mu}}$ on $C^s(K:\xi)$. We will usually omit the superscript $-s$ in the notation of this dual representation.

For $s,t\in\mathbb{N}$ with $s<t$, the inclusion map $C^t(K:\xi)\to C^s(K:\xi)$ is a compact linear map of Banach spaces which has a dense image and therefore dualizes to a compact linear injection

$$C^{-s}(K:\xi)\to C^{-t}(K:\xi).$$

In view of [23, Thm. 11], the locally convex space $C^{-\infty}(K:\xi)$, equipped with the strong dual topology, coincides with the inductive limit of the Banach spaces $C^{-s}(K:$

$\xi)$. Furthermore, by [23, Lemma 3] each bounded subset of $C^{-\infty}(K:\xi)$ is a bounded subset of $C^{-s}(K:\xi)$ for some s.

Let now Ω be a complex manifold. Then by the above mentioned property of bounded subsets of the inductive limit, a function $\varphi:\Omega\to C^{-\infty}(K:\xi)$ is holomorphic if for each $z_0\in\Omega$ there exists an open neighborhood Ω_0 of z_0 in Ω and a natural number $s\in\mathbb{N}$ such that φ maps Ω_0 holomorphically into the Banach space $C^{-s}(K:\xi)$. A densely defined function f from Ω to $C^{-\infty}(K:\xi)$ is said to be meromorphic if for each $z_0\in\Omega$ there exists an open neighborhood Ω_0 and a holomorphic function $q:\Omega_0\to\mathbb{C}$ such that qf extends holomorphically from $\mathrm{Dom}(f)\cap\Omega_0$ to Ω_0.

For later use, we record some observations involving the contragredient M_P-representation $\xi^\vee$, whose space $\mathscr{H}_{\xi^\vee}$ is the linear dual of $\mathscr{H}_\xi$. The map $v\mapsto\langle v,\,\cdot\rangle$ is a M_P-equivariant conjugate-linear isomorphism from $\mathscr{H}_\xi$ onto its dual, $\mathscr{H}_{\xi^\vee}$. This isomorphism induces a K-equivariant topological conjugate-linear isomorphism from $C^\infty(K:\xi)$ onto $C^\infty(K:\xi^\vee)$. The latter isomorphism is equivariant for the representations $\pi_{P,\xi,-\bar{\mu}}$ and $\pi_{P,\xi^\vee,-\mu}$, respectively, for every $\mu\in\mathfrak{a}^*_{P\mathbb{C}}$. Through this isomorphism, the pairing (3.2) is transferred to the bilinear pairing

$$C^\infty(K:\xi)\times C^\infty(K:\xi^\vee)\to\mathbb{C} \tag{3.5}$$

given by

$$\langle f\,,g\rangle=\int_K\langle f(k)\,,g(k)\rangle\,dk. \tag{3.6}$$

Furthermore, this pairing is equivariant for the representations $\pi_{P,\xi,\mu}$ and $\pi_{P,\xi^\vee,-\mu}$. Through it, we see that $C^{-\infty}(K:\xi)$ is naturally identified with the continuous linear dual of $C^\infty(K:\xi^\vee)$. Moreover, this identification realizes the representation $\pi^{-\infty}_{P,\xi,\nu\mu}$ as the contragredient of $\pi^\infty_{P,\xi^\vee,-\mu}$. Accordingly, we obtain the G-equivariant topological linear isomorphism

$$C^{-\infty}(P:\xi:\mu)\simeq C^\infty(P:\xi^\vee:-\bar{\mu})'.$$

In the rest of this section we assume that $P\in\mathscr{P}(A)$ and that $(\xi,\mathscr{H}_\xi)$ is a (not necessarily irreducible) unitary representation of M in a finite dimensional Hilbert space $\mathscr{H}_\xi$.

One of the goals of this paper is to study H-invariant distribution vectors of principal series representations. A first step in the construction of these is the following. Recall that H_P denotes the intersection $H\cap P$ and consider the corresponding homogeneous space $H_P\backslash H$. We denote the associated canonical projection by $\pi:H\to H_P\backslash H$. Given $x\in H$ we write $[x]=\pi(x)$. Furthermore, for $h\in H$ we use the following notation for the right multiplication map,

$$r_h:H_P\backslash H\to H_P\backslash H,\quad [x]\mapsto[x]h=[xh]$$

We refer to the appendix, the text preceding (A.4), for the notion of a density on $H_P\backslash H$ and the associated notion of the density bundle $\mathscr{D}_{H_P\backslash H}$. The notion of the pull-back bundle $\pi^*\mathscr{D}_{H_P\backslash H}\to H$ is defined in the same appendix, in the text before (A.5).

Let $\mathfrak{h}_P$ denote the Lie algebra of $H_P = H \cap P$, then $d\pi(e)$ induces a linear isomorphism $\mathfrak{h}/\mathfrak{h}_P \simeq T_{[e]}(H_P\backslash H)$. We fix a positive density $\omega = \omega_{H_P\backslash H} \in \mathscr{D}_{\mathfrak{h}/\mathfrak{h}_P}$.

If $S \subset \Sigma(P)$, we define the subspace $\mathfrak{n}_S \subset \mathfrak{g}$ to be the direct sum of the root spaces $\mathfrak{g}_\alpha$, for $\alpha \in S$, and we define $\rho_S \in \mathfrak{a}^*$ by

$$\rho_S(X) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}(X)|_{\mathfrak{n}_S}), \quad (X \in \mathfrak{a}).$$

Furthermore, we agree to abbreviate

$$\rho_{P\mathrm{h}} := \rho_{\Sigma(P)\cap\mathfrak{a}_{\mathrm{h}}^*}.$$

In the following result we will describe certain densities associated with principal series representations.

Lemma 3.1. *Let $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$, $f \in C(P : \xi : -\bar{\lambda} + \rho_{P\mathrm{h}})$ and $\eta \in \mathscr{H}_\xi^{H_M}$. Then*

$$\overline{f}_{\eta,\omega} : h \mapsto \langle \eta\,,\, f(h)\rangle_\xi\, dr_h([e])^{-1*}\omega$$

defines a continuous density on the homogeneous space $H_P\backslash H$.

Proof. For each $h \in H$, put $\varphi(h) = \overline{f}_{\eta,\omega}(h)$. Then $\varphi(h)$ defines a density on the tangent space $T_{H_Ph}(H_P\backslash H)$ and $\varphi : H \to \pi^*(\mathscr{D}_{H_P\backslash H})$ defines continuous section of the pull-back bundle. It suffices to show that $\varphi(h_Ph) = \varphi(h)$ for all $h_P \in H_P$. We note that $H_P = H \cap P = H_M A_{\mathrm{h}} H_{N_P}$. Accordingly, write $h_P = man$, then

$$\begin{aligned}\varphi(h_Ph) &= a^{-\lambda+\rho_{P\mathrm{h}}+\rho_P}\big\langle \xi(m)^{-1}\eta, f(h)\big\rangle\, dr_h([e]h_P)^{-1*}dr_{h_P}([e])^{-1*}\omega \\ &= a^{\rho_{P\mathrm{h}}+\rho_P}\Delta(h_P)\varphi(h),\end{aligned} \tag{3.7}$$

where

$$\Delta(h_p) = |\det \mathrm{Ad}(h_P)|_{\mathfrak{h}/\mathfrak{h}_P}| = |\det \mathrm{Ad}(h_P)|_{\mathfrak{h}_P}|^{-1}.$$

Since H_{N_P} is nilpotent, whereas H_M is compact, it follows that

$$\Delta(h_P) = \Delta(a) = |\det \mathrm{Ad}(a)|_{\mathfrak{h}_P}|^{-1}.$$

Using the decomposition $\mathfrak{h}_P = (\mathfrak{h}\cap\mathfrak{m}) \oplus \mathfrak{a}_{\mathrm{h}} \oplus (\mathfrak{h}\cap\mathfrak{n}_P)$ we finally see that

$$\Delta(h_P) = |\det \mathrm{Ad}(a)|_{\mathfrak{h}\cap\mathfrak{n}_P}|^{-1} = a^{-\delta},$$

where $\delta = \mathrm{tr}\,(\mathrm{ad}(\,\cdot\,)|_{\mathfrak{h}\cap\mathfrak{n}_P}) \in \mathfrak{a}_{\mathrm{h}}^*$. We now use that

$$\mathfrak{h}\cap\mathfrak{n}_P = \bigoplus_{\alpha\in(\Sigma(P)\cap\mathfrak{a}_{\mathrm{h}}^*)} \mathfrak{g}_\alpha \;\oplus\; \Big(\bigoplus_{\alpha\in\Sigma(P,\sigma)\setminus\mathfrak{a}_{\mathrm{h}}^*} \mathfrak{g}_\alpha\Big)^\sigma$$

For each $\alpha \in \Sigma(P,\sigma)\setminus\mathfrak{a}_{\mathrm{h}}^*$, we have $\sigma\alpha \neq \alpha$, and the direct sum $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha}$ is σ-invariant, so that its intersection with $\mathfrak{h}$ is given by

$$(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})^\sigma = \{X + \sigma(X) \mid X \in \mathfrak{g}_\alpha\}.$$

The action of an element $H \in \mathfrak{a}_{\mathrm{h}}$ on this space has trace $\dim(\mathfrak{g}_\alpha)\,\alpha(H)$. We conclude that

$$\delta = \big(2\rho_{P\mathrm{h}} + \rho_{\Sigma(P,\sigma)\setminus\mathfrak{a}_{\mathrm{h}}^*}\big)\big|_{\mathfrak{a}_{\mathrm{h}}}.$$

Using the decomposition

$$\rho_P = \rho_{P\mathrm{h}} + \rho_{\Sigma(P,\sigma)\setminus\mathfrak{a}_{\mathrm{h}}^*} + \rho_{\Sigma(P,\sigma\theta)}$$

we see that

$$\big(\rho_P + \rho_{P\mathrm{h}}\big)\big|_{\mathfrak{a}_{\mathrm{h}}} - \delta = \rho_{\Sigma(P,\sigma\theta)}\big|_{\mathfrak{a}_{\mathrm{h}}} = 0.$$

Combining this with (3.7) we infer that φ is left H_P-invariant. □

Remark 3.2. Recall the definition of $\Sigma(P)_-$ in (1.2). In this paper Section 5 we will show that for all $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, for which there exists $P_0 \in \mathscr{P}_\sigma(A_{\mathrm{q}})$ with $\Sigma(P,\sigma\theta) \subset \Sigma(P_0)$ such that

$$\forall \alpha \in \Sigma(P)_- : \qquad \langle \mathrm{Re}\,\lambda + \rho_{P_0}\,,\,\alpha\rangle \leq 0,$$

the above density $f_{\eta,\omega}$ is integrable over $H_P\backslash H$.

4 Comparison of principal series representations

In this section we will compare the principal series representations with the σ-principal series defined in [5]. The latter involve parabolic subgroups P_0 from $\mathscr{P}_\sigma(A_{\mathrm{q}})$. Each of these has a Langlands decomposition of the form $P_0 = M_0A_0N_{P_0}$, see the end of Section 1 for details.

We will now investigate the structure of the group M_0 in more detail. Our starting point is the following lemma.

Lemma 4.1. *Let $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \cap \mathfrak{a}_{\mathrm{h}}^*$. Then $\mathfrak{g}_\alpha \subset \mathfrak{h}$.*

Proof. Let α be as in the assertion. Then $\sigma\alpha = \alpha$ so that σ leaves the root space $\mathfrak{g}_\alpha$ invariant. Thus, it suffices to show that $\mathfrak{g}_\alpha \cap \mathfrak{q} = 0$. Assume that $X \in \mathfrak{g}_\alpha \cap \mathfrak{q}$. Then $(X - \theta X)$ belongs to $\mathfrak{p} \cap \mathfrak{q}$ and centralizes $\mathfrak{a}_{\mathrm{q}}$. As the latter space is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$, it follows that

$$X - \theta X \in \mathfrak{a}_{\mathrm{q}} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) = 0.$$

□

Let $\mathfrak{m}_{0n}$ be the ideal in $\mathfrak{m}_0$ generated by $\mathfrak{a} \cap \mathfrak{m}_0$. Since $\mathfrak{a} \cap \mathfrak{m}_0$ has trivial intersection with the center of $\mathfrak{m}_0$, the ideal $\mathfrak{m}_{0n}$ equals the sum of the simple ideals of non-compact type in $\mathfrak{m}_0$. It has a unique complementary ideal; this is contained in the centralizer of $\mathfrak{a} \cap \mathfrak{m}_0$ in $\mathfrak{m}_0$, hence in $\mathfrak{m}$.

Lemma 4.2. *The ideal $\mathfrak{m}_{0n}$ is contained in $\mathfrak{m}_0 \cap \mathfrak{h}$.*

Proof. The algebra $\mathfrak{m}_0$ admits the decomposition

$$\mathfrak{m}_0 = \mathfrak{m} \oplus (\mathfrak{a} \cap \mathfrak{m}_0) \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \cap \mathfrak{a}_\mathfrak{h}^*} \mathfrak{g}_\alpha. \tag{4.1}$$

Each appearing root space $\mathfrak{g}_\alpha$ equals $[\mathfrak{a} \cap \mathfrak{m}_0, \mathfrak{g}_\alpha]$. Hence, $\mathfrak{m}_{0n}$ contains the subspace

$$\mathfrak{s} := (\mathfrak{a} \cap \mathfrak{m}_0) \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \cap \mathfrak{a}_\mathfrak{h}^*} \mathfrak{g}_\alpha.$$

It follows that $\mathfrak{m}_{0n}$ contains the subalgebra $\tilde{\mathfrak{s}}$ of $\mathfrak{m}_0$ generated by $\mathfrak{s}$. On the other hand, since $\mathfrak{m}_0 = \mathfrak{m} + \mathfrak{s}$ and $\mathfrak{m}$ normalizes $\mathfrak{s}$, the algebra $\tilde{\mathfrak{s}}$ is an ideal of $\mathfrak{m}_{0n}$. We conclude that $\mathfrak{m}_{0n}$ equals the algebra $\tilde{\mathfrak{s}}$ generated by $\mathfrak{s}$.

Now $\mathfrak{a} \cap \mathfrak{m}_0 \subset \mathfrak{h}$ and each of the root spaces in (4.1) is contained in $\mathfrak{h}$ by Lemma 4.1. Therefore, $\mathfrak{s} \subset \mathfrak{h}$ and we conclude that $\mathfrak{m}_{0n} = \tilde{\mathfrak{s}} \subset \mathfrak{h}$. □

Let M_{0n} be the connected subgroup of M_0 with Lie algebra $\mathfrak{m}_{0n}$.

Lemma 4.3.

(a) *M_{0n} is a closed normal subgroup of M_0.*

(b) $M_0 = MM_{0n} \simeq M \times_{M \cap M_{0n}} M_{0n}$.

(c) *The inclusion map $M \to M_0$ induces a group isomorphism*

$$M/M \cap M_{0n} \simeq M_0/M_{0n}.$$

(d) $H_{M_0} = H_M M_{0n}$

(e) *The inclusion map $M \to M_0$ induces a diffeomorphism*

$$M/H_M \simeq M_0/H_{M_0}.$$

(f) *The group M_{0n} acts trivially on M_0/M_{0n} and on M_0/H_{M_0}.*

Proof. The normality of M_{0n} follows since $\mathfrak{m}_{0n}$ is an ideal of $\mathfrak{m}_0$. Since $\mathfrak{m}_0$ is reductive, there exists an ideal $\mathfrak{m}_{0c}$ complementary to $\mathfrak{m}_{0n}$. The group M_{0n} is equal to the connected component of $Z_{M_0}(\mathfrak{m}_{0c})$ and therefore M_{0n} is closed. This proves assertion (a). From $\mathfrak{m}_0 = \mathfrak{m} + \mathfrak{m}_{0n}$ and (a) it follows that MM_{0n} is an open subgroup of M_0. Since M_0 is of the Harish-Chandra class, and $M = Z_{K \cap M_0}(\mathfrak{a} \cap \mathfrak{m}_0)$, it follows that M intersects every connected component of M_0. Hence, $M_0 = MM_{0n}$ and (b) readily follows. Assertion (c) follows from (b) and (a). We now turn to assertion (d). From Lemma 4.2 it follows that $M_{0n} \subset H$. In view of (b) we now see that

$$H_{M_0} = [MM_{0n}] \cap H = H_M M_{0n},$$

Hence (d). From (c) and (d) we obtain a natural fiber bundle $M/M\cap M_{0n} \to M/H_M$ which corresponds to factorization by the group $F := H_M/(M\cap M_{0n})$. Likewise, we obtain a natural fiber bundle $M_0/M_{0n} \to M_0/H_{M_0}$ which corresponds to factorization by group $F_0 := H_{M_0}/M_{0n}$. The isomorphism of (c) maps F onto F_0, hence (e) follows.

Since M_{0n} is normal in M_0, it acts trivially on the quotient M_0/M_{0n}. The second assertion of (f) follows from this as $M_{0n} \subset H_{M_0}$. $\square$

Given a continuous Fréchet M_0-module V, we denote its space of smooth vectors by V^∞. This comes equipped with the structure of a continuous Fréchet M_0-module in the usual way. The continuous linear dual is denoted by $V^{\infty\prime}$; this is called the space of distribution vectors of V.

Corollary 4.4. *Let (ξ, V) be an irreducible continuous representation of M_0 in a Fréchet space V such that*

$$(\mathrm{V}^{\infty\prime})^{H_{M_0}} \neq 0. \tag{4.2}$$

Then $\xi|_{M_{0n}}$ is trivial and $\xi|_M$ is irreducible. In particular, ξ is finite dimensional and unitarizable.

Proof. Let η be a non-zero element of the space in (4.2). Then there is a unique injective continuous linear M_0-equivariant map $j: V^\infty \to C^\infty(M_0/H_{M_0})$ such that $j^*(\delta_{[e]}) = \eta$, with $\delta_{[e]}$ denoting the Dirac measure of M_0/H_{M_0} at $[e] := eH_{M_0}$. Since M_{0n} acts trivially on M_0/H_{M_0} it follows that M_{0n} acts trivially on V^∞ hence on V. We conclude that $\xi|_{M_{0n}}$ is trivial. By application of Lemma 4.3 it follows that $\xi|_M$ is irreducible. $\square$

The above result provides motivation for considering only finite dimensional unitary representations of M_0. We note that any such representation restricts to the trivial representation on M_{0n}, since the latter group is connected semisimple of the non-compact type. Since M_0/M_{0n} is a compact group, it follows that

$$\widehat{M}_{0\mathrm{fu}} \simeq (M_0/M_{0n})^\wedge, \tag{4.3}$$

where $\widehat{M}_{0\mathrm{fu}}$ denote the set of equivalence classes of finite dimensional irreducible unitary representations of M_0.

Lemma 4.5. *The restriction map $\xi \mapsto \xi_M := \xi|_M$ induces an injection*

$$\widehat{M}_{0\mathrm{fu}} \hookrightarrow \widehat{M}. \tag{4.4}$$

The image of this injection equals $(M/M\cap M_{0n})^\wedge$.

Proof. It follows from Lemma 4.3 (c) that the restriction map induces an isomorphism

$$(M_0/M_{0n})^\wedge \simeq (M/M\cap M_{0n})^\wedge.$$

The latter set may be viewed as the subset of $\widehat{M}$ consisting of equivalence classes of irreducible unitary representations that are trivial on $M\cap M_{0n}$. Now use (4.3). $\square$

From now on we will use the map (4.4) to view $\widehat{M}_{0\mathrm{fu}}$ as a subset of $\widehat{M}$.

Lemma 4.6. *Let $(\xi, \mathscr{H}_\xi)$ be a finite dimensional unitary representation of M_0 (not necessarily irreducible). Then*

$$\mathscr{H}_\xi^{H_{M_0}} = \mathscr{H}_\xi^{H_M}. \tag{4.5}$$

Proof. The space on the left-hand side of the equation is clearly contained in the space on the right-hand side. For the converse inclusion, let $\eta \in \mathscr{H}_\xi$ be an H_M-fixed vector. Then η is fixed under the group $H_M M_{0n}$, which equals H_{M_0} by Lemma 4.3 (d). $\square$

Let $W(\mathfrak{a}_\mathrm{q})$ denote the Weyl group of the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$. Then $W(\mathfrak{a}_\mathrm{q}) \simeq N_K(\mathfrak{a}_\mathrm{q})/Z_K(\mathfrak{a}_\mathrm{q})$, naturally. We denote by $W_{K\cap H}(\mathfrak{a}_\mathrm{q})$ the image of $N_{K\cap H}(\mathfrak{a}_\mathrm{q})$ in $W(\mathfrak{a}_\mathrm{q})$. Let $W(\mathfrak{a}) \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ denote the Weyl group of the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Then restriction to $\mathfrak{a}_\mathrm{q}$ induces an epimorphism from the normalizer of $\mathfrak{a}_\mathrm{q}$ in $W(\mathfrak{a})$ onto $W(\mathfrak{a}_\mathrm{q})$. We may therefore select a finite subset $\mathscr{W} \subset N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_\mathrm{q})$ such that $e \in \mathscr{W}$ and such that the map $v \mapsto \mathrm{Ad}(v)|_{\mathfrak{a}_\mathrm{q}}$ induces a bijection

$$\mathscr{W} \xrightarrow{1-1} W(\mathfrak{a}_\mathrm{q})/W_{K\cap H}(\mathfrak{a}_\mathrm{q}). \tag{4.6}$$

Let ξ be a finite dimensional unitary representation of M_0 (not necessarily irreducible). Then following [5] we define

$$V(\xi, v) := \mathscr{H}_\xi^{M_0 \cap vHv^{-1}} = \mathscr{H}_\xi^{M \cap vHv^{-1}}. \tag{4.7}$$

Here we note that the second equality is valid by Lemma 4.6 applied with vHv^{-1} in place of H. We equip the space in (4.7) with the restriction of the inner product on $\mathscr{H}_\xi$. Finally we define the formal direct sum of Hilbert spaces

$$V(\xi) := \oplus_{v\in\mathscr{W}} V(\xi, v). \tag{4.8}$$

For $v \in \mathscr{W}$, let

$$\mathrm{i}_v : V(\xi, v) \to V(\xi) \quad \text{and} \quad \mathrm{pr}_v : V(\xi) \to V(\xi, v)$$

denote the natural inclusion and projection map, respectively.

Our goal will be to study H-fixed distribution vectors in representations induced from minimal parabolic subgroups $P \in \mathscr{P}(A)$. For this it will be convenient to compare these representations to representations induced from minimal $\sigma\theta$-stable parabolic subgroups, by using the method of induction by stages.

Let $P \in \mathscr{P}_\sigma(A)$, see Definition 1.1, and let $P_0 \in \mathscr{P}_\sigma(A_\mathrm{q})$ be such that $P \subset P_0$. Let $(\xi, \mathscr{H}_\xi)$ be a finite dimensional unitary representation of M_0 and let $C^\infty(P_0 : \xi : \lambda)$ be defined as in the first part of Section 3 for P_0 in place of P. We agree to write $\xi_M := \xi|_M$. Observe that $P \cap M_0$ is a minimal parabolic subgroup of M_0 with split component

$A \cap M_0$. Moreover, since the set of roots of $\mathfrak{a} \cap \mathfrak{m}_0$ in $N_P \cap M_0$ equals $\Sigma(P) \cap \mathfrak{a}_{\mathrm{h}}^*$, it follows that

$$\rho_{P \cap M_0} = \rho_{P\mathrm{h}}.$$

Hence, there is a natural M_0-equivariant embedding

$$i : \xi \hookrightarrow \mathrm{Ind}_{M_0 \cap P}^{M_0}(\xi_M \otimes -\rho_{P\mathrm{h}} \otimes 1),$$

see [5, Lemma 4.4]. Concretely, the map i from $\mathscr{H}_\xi$ into the space $C^\infty(M_0 \cap P : \xi_M : -\rho_{P\mathrm{h}})$ of smooth vectors for the principal series representation on the right-hand side is given by

$$i(v)(m_0) = \xi(m_0)v, \qquad (v \in \mathscr{H}_\xi,\ m_0 \in M_0). \tag{4.9}$$

Induction now gives a G-equivariant embedding

$$\mathrm{Ind}_{P_0}^G(\xi \otimes \lambda \otimes 1) \hookrightarrow \mathrm{Ind}_{P_0}^G\big(\mathrm{Ind}_{M_0 \cap P}^{M_0}(\xi_M \otimes -\rho_{P\mathrm{h}} \otimes 1) \otimes \lambda \otimes 1\big).$$

According to the principle of induction by stages, the latter representation is naturally isomorphic with $\mathrm{Ind}_P^G(\xi_M \otimes (\lambda - \rho_{P\mathrm{h}}) \otimes 1)$. The resulting G-equivariant embedding

$$i_\lambda^\# : C^\infty(P_0 : \xi : \lambda) \to C^\infty(P : \xi_M : \lambda - \rho_{P\mathrm{h}}) \tag{4.10}$$

is given by $(i_\lambda^\# f)(x) = \mathrm{ev}_1 \circ i \circ f(x)$ for $f \in C^\infty(P_0 : \xi : \lambda)$ and $x \in G$. Here,

$$\mathrm{ev}_1 : C^\infty(M_0 \cap P : \xi_M : -\rho_{P\mathrm{h}}) \to \mathscr{H}_\xi$$

is given by evaluation at the identity of M_0. Comparing this with (4.9) we see that $i_\lambda^\#$ is the inclusion map.

By $C^\infty(K : K \cap M_0 : \xi)$ we denote the space of smooth functions $K \to \mathscr{H}_\xi$ transforming according to the rule

$$f(mk) = \xi(m)f(k) \qquad (m \in K \cap M_0,\ k \in K).$$

Likewise we write $C^\infty(K : M : \xi_M)$ for the space of smooth functions $K \to \mathscr{H}_\xi$ transforming according to the rule

$$f(mk) = \xi_M(m)f(k) \qquad (m \in M,\ k \in K).$$

Note that restriction to K induces topological linear isomorphisms $C^\infty(P_0 : \xi : \lambda) \to C^\infty(K : K \cap M_0 : \xi)$ and $C^\infty(P : \xi_M : \lambda - \rho_{P\mathrm{h}}) \to C^\infty(K : M : \xi_M)$.

In these compact pictures of the induced representations, $i_\lambda^\#$ becomes the inclusion map

$$i^\# : C^\infty(K : K \cap M_0 : \xi) \hookrightarrow C^\infty(K : M : \xi_M). \tag{4.11}$$

We now see that we have the following commutative diagram

$$\begin{array}{ccc} C^\infty(P_0 : \xi : \lambda) & \xrightarrow{i_\lambda^\#} & C^\infty(P : \xi_M : \lambda - \rho_{P\mathrm{h}}) \\ \downarrow & & \downarrow \\ C^\infty(K : K \cap M_0 : \xi) & \xrightarrow{i^\#} & C^\infty(K : M : \xi_M). \end{array} \tag{4.12}$$

The vertical arrows in this diagram represent the topological linear isomorphisms induced by restriction to K; see [5] for details.

Lemma 4.7. *The space $C^\infty(K : K\cap M_0 : \xi)$ coincides with the subspace of left $K\cap M_{0n}$-invariants in $C^\infty(K : M : \xi_M)$.*

Proof. Let $f \in C^\infty(K : K \cap M_0 : \xi)$. Since ξ is finite dimensional, it follows that $\xi|_{M_{0n}}$ is trivial. Hence, for $k \in K$ and $m_0 \in M_{0n}$ we have that

$$f(m_0 k) = \xi(m_0) f(k) = f(k).$$

This establishes one inclusion. For the converse, assume that $f \in C^\infty(K : M : \xi_M)$ is left $K \cap M_{0n}$-invariant. Let $m_0 \in K \cap M_0$. Then we may write $m_0 = mm_{\rm n}$ with $m \in M$ and $m_{\rm n} \in K \cap M_{0n}$. Let $k \in K$. Then

$$\begin{aligned} f(m_0 k) = f(mm_{\rm n} k) &= \xi(m) f(m_{\rm n} k) = \xi(m) f(k) \\ &= \xi(m)\xi(m_{\rm n}) f(k) = \xi(m_0) f(k). \end{aligned}$$

For the third equality we used that $\xi|_{M_{0n}}$ is trivial. We thus conclude that f belongs to $C^\infty(K : K \cap M_0 : \xi)$. □

Since M normalizes $K \cap M_{0\rm n}$, we see that for $f \in C^\infty(K : M : \xi_M)$ the function $p(f) : K \to \mathscr{H}_\xi$ defined by

$$p(f)(k) = \int_{K\cap M_{0\rm n}} f(m_0 k)\, dm_0, \quad (k \in K),$$

belongs to $C^\infty(K : M : \xi_M)$ again. The associated operator

$$p : C^\infty(K : M : \xi_M) \to C^\infty(K : M : \xi_M) \tag{4.13}$$

is continuous linear and K-equivariant. Since $K \cap M_0 = M(K \cap M_{0\rm n})$, the image of p is contained in the subspace of left $K \cap M_{0\rm n}$-invariants. Furthermore, p is obviously the identity on this subspace, so that p is a projection operator with image equal to the image $\mathrm{im}(i^\#)$ of $i^\#$, see (4.11). It is readily seen that p is symmetric with respect to the pre-Hilbert structure $\langle\cdot\,,\,\cdot\rangle$ on $C^\infty(K : M : \xi_M)$ obtained by restriction of the inner product from $L^2(K)\otimes\mathscr{H}_\xi$. Accordingly, $C^\infty(K : M : \xi_M)$ is the direct sum of $\mathrm{im}(i^\#)$ and its orthocomplement with respect to the give pre-Hilbert structure, and p is the associated orthogonal projection onto $\mathrm{im}(i^\#)$. Let

$$p^\# : C^\infty(K : M : \xi_M) \to C^\infty(K : K \cap M_0 : \xi) \tag{4.14}$$

be the unique linear map such that $p = i^\# \circ p^\#$. For later use, we note that the maps introduced are related by

$$p = i^\# \circ p^\#, \qquad p^\# \circ i^\# = I. \tag{4.15}$$

The map $p^\#$ is K-equivariant and continuous linear, and $i^\#$ and $p^\#$ are adjoint with respect to the pre-Hilbert structures $\langle\cdot\,,\,\cdot\rangle$.

Lemma 4.8. *For all $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$, the following holds.*

(a) *The map $i^\#$ given in* (4.11) *intertwines the representations $\pi_{P_0,\xi,\lambda}$ and $\pi_{P,\xi_M,\lambda-\rho_{P\mathrm{h}}}$.*

(b) *The map $p^\#$ intertwines the representations $\pi_{P,\xi_M,\lambda+\rho_{P\mathrm{h}}}$ and $\pi_{P_0,\xi,\lambda}$.*

Proof. Since the top horizontal map in (4.12) is intertwining, (a) follows. Using (a) we see that for each $x \in G$,

$$i^\# \circ \pi_{P_0,\xi,\lambda}(x) = \pi_{P,\xi_M,\lambda-\rho_{P\mathrm{h}}}(x) \circ i^\#.$$

Taking adjoints and using equivariance of the pairings $\langle\,\cdot\,,\,\cdot\,\rangle$ involved, we infer that

$$\pi_{P_0,\xi,-\bar{\lambda}}(x^{-1}) \circ p^\# = p^\# \circ \pi_{P,\xi_M,-\bar{\lambda}+\rho_{P\mathrm{h}}}(x^{-1})$$

for all $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$ and $x \in G$. From this, (b) follows. $\square$

For each $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$ we denote the unique lift of the map (4.14) to a map $C^\infty(P : \xi_M : \lambda + \rho_{P\mathrm{h}}) \to C^\infty(P_0 : \xi : \lambda)$ by $p^\#_\lambda$. Then it follows from Lemma 4.8 that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(P : \xi_M : \lambda + \rho_{P\mathrm{h}}) & \xrightarrow{p^\#_\lambda} & C^\infty(P_0 : \xi : \lambda) \\ \downarrow & & \downarrow \\ C^\infty(K : M : \xi_M) & \xrightarrow{p^\#} & C^\infty(K : K \cap M_0 : \xi). \end{array} \tag{4.16}$$

Remark 4.9. In view of (4.15) it follows from Lemma 4.8 that p defined in (4.13) intertwines the representations $\pi_{P:\xi_M:\lambda+\rho_{P\mathrm{h}}}$ and $\pi_{P:\xi_M:\lambda-\rho_{P\mathrm{h}}}$ of G in $C^\infty(K : M : \xi_M)$. Thus, it has a unique lift to an equivariant map $C^\infty(P : \xi_M : \lambda + \rho_{P\mathrm{h}}) \to C^\infty(P : \xi_M : \lambda - \rho_{P\mathrm{h}})$. However, we shall never use this lift.

Proposition 4.10. *Let $P_0 \in \mathscr{P}_\sigma(A_q)$ and $P \in \mathscr{P}(A)$ be such that $P \subset P_0$. Let $\xi \in \widehat{M}_{0\mathrm{fu}}$ and $\lambda \in \mathfrak{a}_q^*$. Then the embedding* (4.10) *has a unique extension to a continuous linear map*

$$i^\#_\lambda : C^{-\infty}(P_0 : \xi : \lambda) \to C^{-\infty}(P : \xi_M : \lambda - \rho_{P\mathrm{h}}). \tag{4.17}$$

This extension is G-equivariant and maps homeomorphically onto a closed subspace. As a map from $C^{-\infty}(K : K \cap M_0 : \xi)$ to $C^{-\infty}(K : M : \xi_M)$ the extended map is the unique continuous extension of (4.11). In particular, it is independent of λ.

Proof. Let the map

$$i^\# : C^{-\infty}(K : K \cap M_0 : \xi) \to C^{-\infty}(K : M : \xi_M)$$

be defined as the transposed of $p^\#$. Then $i^\#$ is a continuous linear extension of the bottom horizontal map of (4.12). This continuous extension is unique by density of $C^\infty(K : K \cap M_0 : \xi)$ in $C^{-\infty}(K : K \cap M_0 : \xi)$. Likewise, the adjoint $i^{\#\mathrm{T}}$ of the bottom horizontal map in (4.12) is the continuous linear extension of the projection map $p^{\#}$ which we denote by

$$p^{\#} : C^{-\infty}(K : M : \xi_M) \to C^{-\infty}(K : M_0 \cap K : \xi)$$

as well. Finally, the transpose p^{T} is the unique continuous linear extension of p to a continuous linear map, denoted

$$p : C^{-\infty}(K : M : \xi_M) \to C^{-\infty}(K : M : \xi_M).$$

By transposition we see that the relations (4.15) remain valid for the extensions of these maps to the spaces of generalized functions involved. In particular, it follows that the extended map $i^{\#}$ is a homeomorphism onto the kernel of the extended map $p - I$. In particular, it has closed image.

By transfer under the vertical isomorphisms in the diagram (4.12) we see that $i^{\#}_{\lambda}$ has a unique continuous linear extension (4.17) with closed image. The extension is G-equivariant because it is so on the dense subspace of smooth functions. □

Remark 4.11. By a similar argument it follows that the map $p^{\#}_{\lambda}$ represented by the top horizontal arrow in (4.16) has a unique continuous linear extension to a surjective equivariant map $p^{\#}_{\lambda} : C^{-\infty}(P : \xi_M : \lambda + \rho_{P\mathrm{h}}) \to C^{-\infty}(P_0 : \xi : \lambda)$. However, we shall not need this in the present paper.

5 H-fixed distribution vectors, the q-extreme case

We retain the notation of the previous section. In particular, we assume that $P \in \mathscr{P}_{\sigma}(A)$ and that $P_0 \in \mathscr{P}_{\sigma}(A_{\mathrm{q}})$ contains P. We will now construct H-fixed distribution vectors in P-induced representations, by comparison with the H-distribution vectors in P_0-representations as defined in [5].

We assume that ξ is a finite dimensional unitary representation of M_0 and put $\xi_M = \xi|_M$. Furthermore, we assume that $\eta \in V(\xi, e)$, see (4.7).

Following [5, (5.4)] we define the function $\varepsilon_1(P_0 : \xi : \lambda : \eta)$ for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ by

$$\begin{cases} \varepsilon_1(P_0 : \xi : \lambda : \eta) = 0 & \text{outside} \quad P_0 H \\ \varepsilon_1(P_0 : \xi : \lambda : \eta)(namh) = a^{\lambda + \rho_{P_0}} \xi(m)\eta, \end{cases}$$

for $m \in M_0, a \in A_0, n \in N_0$ and $h \in H$. Clearly, for every $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ the function $\varepsilon_1(P_0 : \xi : \lambda : \eta)$ is continuous outside the set $\partial(P_0 H)$ which has measure zero in G. By right-P-equivariance, the restriction of this function to K is continuous outside $\partial(K \cap P_0 H)$, which has measure zero in K.

Let $\Sigma(P_0, \mathfrak{a}_{\mathrm{q}})_-$ denote the space of $\mathfrak{a}_{\mathrm{q}}$-roots in $\mathfrak{n}_{P_0}$ such that $\ker(\theta\sigma + I) \cap \mathfrak{g}_{\alpha} \neq 0$. In case ξ is irreducible, it follows from [5, Prop. 5.6] that the function $\varepsilon_1(P_0 : \xi : \lambda : \eta)$ is continuous on G for all $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ with $\langle \mathrm{Re}\,\lambda + \rho_{P_0}, \alpha \rangle < 0$ for all $\alpha \in \Sigma(P_0, \mathfrak{a}_{\mathrm{q}})_-$. By decomposition into irreducibles one readily sees that this result is also valid for an arbitrary finite dimensional unitary representation of M_0.

Lemma 5.1. *Let ξ be a finite dimensional unitary representation of M_0 and assume that $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$ satisfies*

$$\forall \alpha \in \Sigma(P_0, \mathfrak{a}_\mathfrak{q})_- : \qquad \langle \mathrm{Re}\,\lambda + \rho_{P_0}\,,\,\alpha\rangle \leq 0. \tag{5.1}$$

Then the function $\varepsilon_1(P_0 : \xi : \lambda : \eta)$ is measurable and locally bounded on G, and its restriction to K is measurable and bounded on K, uniformly for λ in the indicated subset of $\mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$. Finally, $\varepsilon_1(P_0 : \xi : \lambda : \eta)|_K$ depends continuously on λ as a function with values in $L^1(K)\otimes \mathscr{H}_\xi$.

Proof. We may as well assume that ξ is irreducible. The assertions about measurability have been settled above. For the assertions about boundedness, it suffices to consider the restriction of the function to K. From the argument in the proof of [5, Prop. 5.6], which in turn relies on the convexity theorem of [4], it follows that for all λ in the indicated region we have

$$\sup_K \|\varepsilon_1(P_0 : \xi : \lambda : \eta)\|_\xi \leq \|\eta\|_\xi.$$

We obtain the final assertion by observing that $\varepsilon_1(P_0 : \xi : \lambda : \eta)$ depends pointwise continuously on λ and applying Lebesgue's dominated convergence theorem. $\square$

Proposition 5.2. *Let $P \in \mathscr{P}_\sigma(A)$ and $P_0 \in \mathscr{P}_\sigma(A_\mathfrak{q})$ be such that $P \subset P_0$. Let ξ be a finite dimensional unitary representation of M_0 and $\eta \in V(\xi, e)$. Let $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$ be such that*

$$\forall \alpha \in \Sigma(P)_- : \qquad \langle \mathrm{Re}\,\lambda + \rho_{P_0}\,,\,\alpha\rangle \leq 0.$$

Finally, let $f \in C^\infty(P : \xi_M : -\bar{\lambda} + \rho_{P\mathfrak{h}})$. Then the density $\bar{f}_{\eta,\omega}$, defined in Lemma 3.1, is integrable. Moreover,

$$\int_{H_P\backslash H} \bar{f}_{\eta,\omega} = c_\omega \langle i^\#_\lambda\big(\varepsilon_1(P_0 : \xi : \lambda : \eta)\big)\,,\,f\rangle, \tag{5.2}$$

with $c_\omega > 0$ a constant depending on the normalization of the positive density ω.

Proof. By the assumption on λ, the function $\varepsilon = i^\#_\lambda\big(\varepsilon_1(P_0 : \xi : \lambda : \eta)\big)$ is locally integrable on K. It follows that the expression on the right-hand side of (5.2) equals the integral

$$\int_K \langle \varepsilon(k)\,,\,f(k)\rangle_\xi\, dk,$$

where dk denotes normalized Haar measure on K. The integrand is left M-invariant, so that the integral may also be written as the integral over $k \in M\backslash K$, with dk replaced with the normalized invariant density $d\bar{k}$ on $M\backslash K$. This density may be viewed as the section of the density bundle over $M\backslash K$ given by

$$k \mapsto dr_k([e])^{-1*}\omega_{M\backslash K}$$

with $\omega_{M\backslash K}$ a suitable positive density on $\mathfrak{k}/\mathfrak{m} \simeq T_{[e]}(M\backslash K)$. We now obtain that

$$\langle \varepsilon\,,\, f\rangle = \int_{M\backslash K} \langle \varepsilon(k)\,,\, f(k)\rangle_\xi \; dr_k([e])^{-1*}\omega_{M\backslash K}. \tag{5.3}$$

Let $\phi : M\backslash K \to P\backslash G$ be the diffeomorphism induced by the inclusion $K \to G$. Then we find that the pull-back under ϕ of the density in the integral in the right-hand side of (5.3) equals

$$x \mapsto \langle \varepsilon(x)\,,\, f(x)\rangle \; dr_x([e])^{-1*} d\phi([e])^{-1*}\omega_{M\backslash K}.$$

Since $P_0H = PH$, it follows that that the above density is supported by PH. Writing $\omega_{P\backslash G} = d\phi([e])^{-1*}\omega_{M\backslash K}$, we obtain that the integral in (5.3) equals

$$\int_{P\cdot H} \langle \varepsilon(x)\,,\, f(x)\rangle dr_x([e])^{-1*}\omega_{P\backslash G}. \tag{5.4}$$

Let $\psi : H_P\backslash H \to P\backslash G$ be the natural open embedding induced by the inclusion map $H \to G$. Then $|d\psi([e])^*\omega_{P\backslash G}| = c_\omega^{-1}\omega_{P\backslash G}$ for a positive constant c_ω. We now observe that

$$\begin{aligned}
&\psi^*\Big(Px \mapsto \langle \varepsilon(x)\,,\, f(x)\rangle_\xi \; dr_x([e])^{-1*}\omega_{P\backslash G}\Big)\\
&= \; c_\omega^{-1}\cdot\Big(H_Ph \mapsto \langle \varepsilon(h)\,,\, f(h)\rangle_\xi \; dr_h([e])^{-1*}\omega\Big)\\
&= \; c_\omega^{-1}\,\overline{f}_{\eta,\omega}.
\end{aligned}$$

By invariance of integration of densities under diffeomorphisms, we see that (5.4) equals

$$c_\omega^{-1}\int_{H_P\backslash H} \overline{f}_{\eta,\omega}.$$

$\square$

For $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$ such that the conditions of the above theorem are fulfilled, and for $\eta \in V(\xi, e)$, we define the conjugate-linear functional $j_H(P : \xi_M : \lambda : \eta)$ on $C^\infty(P : \xi_M : -\lambda + \rho_{P\mathrm{h}})$ by

$$\langle j_H(P : \xi_M : \lambda : \eta)\,,\, f\rangle = c_\omega^{-1}\int_{H_P\backslash H} \overline{f}_{\eta,\omega}, \tag{5.5}$$

for $f \in C^\infty(P : \xi_M : -\overline{\lambda} + \rho_{P\mathrm{h}})$.

We now recall the definition of the H-fixed distribution vector $j(P_0, \xi, \lambda)$ from [5, Section 5]. For $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$ such that

$$\forall \alpha \in \Sigma(P)_- : \qquad \langle \mathrm{Re}\,\lambda + \rho_{P_0}\,,\, \alpha\rangle \leq 0$$

and for $v \in \mathscr{W}$ and $\eta \in V(\xi, v)$ we define $\varepsilon_v(P_0 : \xi : \lambda : \eta) : G \to \mathscr{H}_\xi$ by

$$\begin{cases} \varepsilon_v(P_0 : \xi : \lambda : \eta) = 0 \qquad \text{outside} \quad P_0vH \\ \varepsilon_v(P_0 : \xi : \lambda : \eta)(namvh) = a^{\lambda+\rho_{P_0}}\xi(m)\eta. \end{cases}$$

We further define

$$j(P_0 : \xi : \lambda)(\eta) = \sum_{v \in \mathscr{W}} \varepsilon_v(P_0 : \xi : \lambda : \eta_v), \qquad \big(\eta \in V(\xi)\big).$$

Then $j(P_0 : \xi : \lambda)$ is a map $V(\xi) \to C^{-\infty}(P_0 : \xi : \lambda)^H$, hence defines an element in $V(\xi)^* \otimes C^{-\infty}(K : K \cap M_0 : \xi)$. The map $\lambda \mapsto j(P_0 : \xi : \lambda)$ extends to a meromorphic $V(\xi)^* \otimes C^{-\infty}(K : K \cap M_0 : \xi)$-valued function on $\mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$. See [5, Section 5] for details. (Strictly speaking the definition in [5] is given for ξ irreducible, but the definition works equally well in general.)

Proposition 5.2 now has the following corollary.

Corollary 5.3. *Let ξ be a finite dimensional unitary representation of M_0. The map $\lambda \mapsto j_H(P : \xi_M : \lambda)$ extends to a meromorphic $V(\xi, e)^* \otimes C^{-\infty}(K : M : \xi_M)$-valued function. Moreover,*

$$j_H(P : \xi_M : \lambda) = i^\# \circ j(P_0 : \xi : \lambda) \circ \mathrm{i}_e$$

as an identity of meromorphic $V(\xi, e)^* \otimes C^{-\infty}(K : M : \xi_M)$-valued functions. In particular,

$$j_H(P : \xi_M : \lambda) \in V(\xi, e)^* \otimes C^{-\infty}(P : \xi_M : \lambda - \rho_{P\mathrm{h}})^H$$

for generic $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$.

Remark 5.4. In the above formulation we have used the notation $i^\#$ rather than $i^\#_\lambda$, to emphasize that $j(P_0 : \xi : \lambda) \circ \mathrm{i}_e$ is viewed as λ-dependent element of the space $V(\xi, e)^* \otimes C^{-\infty}(K : M : \xi_M)$.

Let $v \in \mathscr{W}$. Motivated by the definition of $j(P_0 : \xi : \lambda)$ and the above identity, we define the meromorphic $\mathrm{Hom}\big(V(\xi), C^{-\infty}(K : M : \xi_M)\big)$-valued map $j(P : \xi_M : \cdot)$ by

$$j(P : \xi_M : \lambda) = \sum_{v \in \mathscr{W}} \pi_{P, \xi_M, \lambda - \rho_{P\mathrm{h}}}(v^{-1}) j_{vHv^{-1}}(P : \xi_M : \lambda) \circ \mathrm{pr}_v.$$

Corollary 5.5. *Let ξ be a finite dimensional unitary representation of M_0. Then*

$$j(P : \xi_M : \lambda) = i^\# \circ j(P_0 : \xi : \lambda) \tag{5.6}$$

as an identity of meromorphic $V(\xi)^* \otimes C^{-\infty}(K : M : \xi_M)$-valued functions of $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$. In particular, for $\eta \in V(\xi)$ and generic $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$,

$$j(P : \xi_M : \lambda)(\eta) \in C^{-\infty}(P : \xi_M : \lambda - \rho_{P\mathrm{h}})^H.$$

We recall that the map p in the result below is exclusively used in the compact picture of the induced representations, see Remark 4.9.

Corollary 5.6. *Let $\xi \in \widehat{M}_{0\mathrm{fu}}$ and $\eta \in V(\xi)$. Then for every $x \in G$,*

$$p \circ \pi_{P, \xi_M, \lambda - \rho_{P\mathrm{h}}}(x) j(P : \xi_M : \lambda)(\eta) = \pi_{P, \xi_M, \lambda - \rho_{P\mathrm{h}}}(x) j(P : \xi_M : \lambda)(\eta)$$

as an identity of meromorphic $C^{-\infty}(K : M : \xi_M)$-valued functions of $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$.

Proof. Use (5.6) and note that $\pi_{P, \xi_M, \lambda - \rho_{P\mathrm{h}}}(x) \circ i^\# = i^\# \circ \pi_{P_0, \xi, \lambda}(x)$ and $p \circ i^\# = i^\#$, see (4.15). □

6 An important fibration

In this section we will apply Fubini's theorem, as formulated in the appendix, Theorem A.8, to an important fibration. The main result will be needed for the definition of distribution vectors for induced representations with $P \in \mathscr{P}(A)$ not necessarily contained in a parabolic subgroup from $\mathscr{P}_\sigma(A_\mathrm{q})$.

We assume that $P, Q \in \mathscr{P}(A)$ and that $P \succeq Q$. There exists $X \in \mathfrak{a}_\mathrm{q}$ such that

(a) $\alpha(X) \neq 0$ for all $\alpha \in \Sigma(P) \setminus \mathfrak{a}_\mathrm{h}^*$;

(b) $\alpha(X) > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$.

Since $\Sigma(Q, \sigma\theta) \subset \Sigma(P, \sigma\theta)$, it follows that (a) and (b) are also valid with Q in place of P. We now put

$$\mathfrak{n}_{Q,X} := \bigoplus_{\substack{\alpha \in \Sigma(Q) \\ \alpha(X) > 0}} \mathfrak{g}_\alpha, \quad \text{and} \quad N_{Q,X} := \exp(\mathfrak{n}_{Q,X}).$$

Lemma 6.1. *The multiplication map $(n_1, n_2) \mapsto n_1 n_2$ is a diffeomorphism*

$$H_{N_Q} \times N_{Q,X} \xrightarrow{\simeq} N_Q.$$

This result is contained and proven in [3, Prop. 2.16].

Lemma 6.2. *Let $Q, P \in \mathscr{P}(A)$ be such that $P \succeq Q$. Let $X \in \mathfrak{a}_\mathrm{q}$ be such that (a) and (b) are valid. Then*

$$N_{Q,X} \subset N_{P,X}.$$

Proof. Let $\alpha \in \Sigma(Q)$ be such that $\alpha(X) > 0$. Then it suffices to show that $\alpha \in \Sigma(P)$. Assume this were not the case. Then either $-\alpha \in \Sigma(P, \sigma)$, or $-\alpha \in \Sigma(P, \sigma\theta)$. In the first case it would follow that $-\alpha \in \Sigma(Q, \sigma)$, which contradicts the assumption that $\alpha \in \Sigma(Q)$. In the second it would follow that $-\alpha(X) > 0$ which contradicts the assumption that $\alpha(X) > 0$. □

Lemma 6.3. *The inclusion map $H_{N_Q} \to N_Q$ induces a diffeomorphism*

$$\varphi : H_{N_P} \backslash H_{N_Q} \xrightarrow{\simeq} (N_Q \cap N_P) \backslash N_Q.$$

Proof. It follows from Lemma 6.1 that the natural map $H_{N_Q} \to N_{Q,X} \backslash N_Q$ is a diffeomorphism onto. By application of Lemma 6.2 it now follows that the natural map

$$p : H_{N_Q} \to (N_Q \cap N_P) \backslash N_Q$$

is a surjective submersion. The map p intertwines the natural H_{N_Q}-actions, and the fiber of $[e]$ equals $H_{N_Q} \cap N_P = H_{N_P}$. Thus, φ is induced by p and is a diffeomorphism onto. □

Lemma 6.4. *The inclusion map $N_Q \cap \bar{N}_P \to N_Q$ induces a diffeomorphism*

$$\psi : N_Q \cap \bar{N}_P \xrightarrow{\simeq} (N_Q \cap N_P)\backslash N_Q.$$

Proof. This is well known. $\square$

Lemma 6.5. *Let ϕ and ψ be as in Lemma 6.4 and Lemma 6.3. The map $\Phi := \varphi^{-1} \circ \psi$ is a diffeomorphism from $N_Q \cap \bar{N}_P$ onto $H_{N_P}\backslash H_{N_Q}$. Moreover, let ω be a positive H_{N_Q}-invariant density on the image manifold. Then $\Phi^*(\omega)$ is a choice of Haar measure on $N_Q \cap \bar{N}_P$.*

Proof. Being the composition of two diffeomorphisms, Φ is a diffeomorphism. We note that $\Phi^*(\omega) = \psi^*\varphi^{-1*}(\omega)$. Let dn be a choice of positive N_Q-invariant density on $(N_Q \cap N_P)\backslash N_Q$. Since φ is H_{N_Q} intertwining, it follows that $\varphi^*(dn)$ is a positive H_{N_Q}-invariant density on $H_{N_P}\backslash H_{N_Q}$. By uniqueness of positive invariant densities up to positive scalars, it follows that $\varphi^*(dn) = c\omega$ for some $c > 0$, so that also $\varphi^{-1*}(\omega) = c^{-1}dn$. By equivariance, it follows that $\psi^*(dn)$ is a choice of Haar measure on $N_Q \cap \bar{N}_P$. Thus, $\Phi^*(\omega) = c^{-1}\psi^*(dn)$ is as required. $\square$

In view of this lemma we may fix invariant measures $d\bar{n}$ on $N_Q \cap \bar{N}_P$ and dh on $H_{N_P}\backslash H_{N_Q}$ such that $\Phi^*(dh) = d\bar{n}$.

Lemma 6.6. *Let $f : G \to \mathbb{C}$ be a left N_P-invariant measurable function. Then the following statements are equivalent.*

(a) *f is absolutely integrable over $H_{N_P}\backslash H_{N_Q}$.*

(b) *f is absolutely integrable over $N_Q \cap \bar{N}_P$.*

If any of these statements hold, then with invariant measures normalized as above,

$$\int_{H_{N_P}\backslash H_{N_Q}} f(h)\, dh = \int_{N_Q \cap \bar{N}_P} f(\bar{n})\, d\bar{n}.$$

Proof. As $\Phi^*(dh) = d\bar{n}$ it suffices to show that $\Phi^*(f|_{H_{N_Q}}) = f|_{N_Q \cap \bar{N}_P}$. Since f is left N_P-invariant, this follows from the obvious fact that for $\bar{n} \in N_Q \cap \bar{N}_P$ the canonical images of $\bar{n}$ and $\Phi(\bar{n})$ in $N_P\backslash G$ coincide. $\square$

Fix $P, Q \in \mathscr{P}(A)$ and assume that $P \succeq Q$. Then $H_Q = H \cap Q$ contains $H_P = H \cap P$. We note that $H_P \simeq H_M A_\mathfrak{h} H_{N_P}$ and that H_Q admits a similar decomposition.

We shall now apply the results in the Appendix with H, H_Q and H_P in place of G, H and L, respectively.

Let $\omega_{H_P\backslash H} \in \mathscr{D}_{\mathfrak{h}/\mathfrak{h}_P}$, $\omega_{H_Q\backslash H} \in \mathscr{D}_{\mathfrak{h}/\mathfrak{h}_Q}$ and $\omega_{H_P\backslash H_Q} \in \mathscr{D}_{\mathfrak{h}_Q/\mathfrak{h}_P}$ be such that $\omega_{H_P\backslash H} = \omega_{H_P\backslash H_Q} \otimes \omega_{H_Q\backslash H}$ in accordance with the identification $\mathscr{D}_{\mathfrak{h}/\mathfrak{h}_P} = \mathscr{D}_{\mathfrak{h}_Q/\mathfrak{h}_P} \otimes \mathscr{D}_{\mathfrak{h}/\mathfrak{h}_Q}$ induced by the natural short exact sequence

$$0 \longrightarrow \mathfrak{h}_Q/\mathfrak{h}_P \longrightarrow \mathfrak{h}/\mathfrak{h}_P \longrightarrow \mathfrak{h}/\mathfrak{h}_Q \longrightarrow 0.$$

See (A.2) and Lemma A.2 for details. We observe that

$$H_P\backslash H_Q \simeq H_{N_P}\backslash H_{N_Q}$$

naturally. Using the associated natural isomorphism of the tangent spaces at the origins, we view $\omega_{H_P\backslash H_Q}$ as a density on the quotient $(\mathfrak{h}\cap\mathfrak{n}_Q)/(\mathfrak{h}\cap\mathfrak{n}_P)$. By unimodularity of the groups H_{N_Q} and H_{N_P}, it follows that

$$dn : n \mapsto dr_n([e])^{-1*}\omega_{H_P\backslash H_Q}$$

defines a choice of right H_{N_Q}-invariant density on $H_{N_P}\backslash H_{N_Q}$. We define the character $\Delta_{H_P\backslash H}$ of H_P as in Appendix, Equation (A.9) with H and H_P in place of G and L, respectively. Likewise, the space $\mathscr{M}(H : H_P : \Delta_{H_P\backslash H})$ is defined as in the text subsequent to (A.9).

Theorem 6.7. *Let $f \in \mathscr{M}(H : H_P : \Delta_{H_P\backslash H})$ and let $f_P := f_{\omega_{H_P\backslash H}}$ be the associated measurable density on $H_P\backslash H$. Then the following assertions (a) and (b) are equivalent.*

(a) *The density f_P is absolutely integrable.*

(b) *There exists a left H_Q-invariant set $\mathscr{Z}$ of measure zero in H such that*

  (1) *for every $x \in H \setminus \mathscr{Z}$, the integral*

$$A_x(f) := \int_{H_{N_P}\backslash H_{N_Q}} f(nx)\, dn$$

  is absolutely convergent;

  (2) *the function $A(f) : x \mapsto A_x(f)$ belongs to $\mathscr{M}(H : H_Q : \Delta_{H_Q\backslash H})$.*

  (3) *the density $A(f)_Q := A(f)_{\omega_{H_Q\backslash H}}$ is absolutely integrable.*

If any of the conditions (a) and (b) is fulfilled, then

$$\int_{H_P\backslash H} f_P = \int_{H_Q\backslash H} A(f)_Q.$$

Proof. We will use the notation introduced in the text before the theorem. The inclusion map $H_{N_Q} \to H_Q$ induces a diffeomorphism

$$\phi : H_{N_P}\backslash H_{N_Q} \to H_P\backslash H_Q.$$

Fix $x \in H$ and let $f_{P,x}$ be the density on $H_P\backslash H_Q$ given by

$$f_{P,x}(H_P h) = \Delta_{H_Q\backslash H}(h)^{-1} f(hx) dr_h([e])^{-1*}\omega_{H_P\backslash H_Q}.$$

By nilpotence, $\Delta_{H_Q\backslash H}(n) = 1$ for $n \in H_{N_Q}$. It follows that

$$\phi^*(f_{P,x})(H_{N_P}\cdot n) = f(n) dr_n([e])^{-1*}\omega_{H_P\backslash H_Q} = f(n)dn.$$

In accordance with the notation of Theorem A.8 we denote the integral of $f_{P,x}$ over $H_P\backslash H_Q$ by $I_x(f)$. Then it follows by invariance of integration that the integral for $I_x(f)$ converges absolutely if and only if the integral $A_x(f)$ converges absolutely, while in case of convergence,

$$I_x(f) = \int_{H_{N_P}\backslash H_{N_Q}} \phi^*(f_{P,e}) = A_x(f).$$

All assertions now follow by application of Theorem A.8. $\square$

7 H-fixed distribution vectors, the general case

Recall the definition of $\Sigma(P)_-$ in (1.2).

Theorem 7.1. *Let $P \in \mathscr{P}_\sigma(A)$, let ξ be a finite dimensional unitary representation of M_0 and $\eta \in V(\xi, e)$. Assume that $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$ satisfies*

$$\langle \mathrm{Re}\,\lambda + \rho_P - \rho_{P\mathrm{h}}\,,\, \alpha\rangle \leq 0, \qquad \text{for all } \alpha \in \Sigma(P)_-. \tag{7.1}$$

Furthermore, let $f \in C^\infty(P : \xi_M : -\bar{\lambda} + \rho_{P\mathrm{h}})$. Then

$$j_H(P : \xi_M : \lambda : \eta) f = \int_{H_P\backslash H} \langle \eta\,,\, f(h)\rangle\, dr_h([e])^{-1*}\omega_{H_P\backslash H},$$

with absolutely convergent integral.

Let $Q \in \mathscr{P}(A)$ be a second parabolic subgroup, with $P \succeq Q$. Then for all $x \in G$,

$$A(Q : P : \xi_M : -\bar{\lambda} + \rho_{P\mathrm{h}}) f(x) = \int_{N_Q \cap \bar{N}_P} f(nx)\, dn$$

with absolutely convergent integral. Finally,

$$\begin{aligned} & j_H(P : \xi_M : \lambda : \eta) f \\ = & \int_{H_Q\backslash H} \langle \eta\,, [A(Q : P : -\bar{\lambda} + \rho_{P\mathrm{h}}) f](h)\rangle\, dr_h([e])^{-1*}\omega_{H_Q\backslash H}, \end{aligned} \tag{7.2}$$

with absolutely convergent integral.

Proof. Observe that the function f restricted to $H_P\backslash H$ belongs to $C^\infty(H : H_P : \Delta_{H_P\backslash H})$. The first assertion now follows from Proposition 5.2 and Equation (5.5).

We will now apply Theorem 6.7. For $x \in H$ the fiber integral takes the form

$$A_x(f|_H) = \int_{H_{N_P}\backslash H_{N_Q}} f(nx)\, dn,$$

which by Lemma 6.6 equals

$$\int_{N_Q \cap \bar{N}_P} f(\bar{n}x)\, d\bar{n}.$$

The latter is just the integral for the standard intertwining operator $A(Q:P:\xi_M:-\bar{\lambda}+\rho_{P\mathrm{h}})$ (up to suitable normalization). This integral is known to converge absolutely in case

$$\mathrm{Re}\,\langle -\lambda+\rho_{P\mathrm{h}}\,,\,\alpha\rangle > 0, \quad \forall \alpha\in\Sigma(P)\cap\Sigma(\bar{Q}). \tag{7.3}$$

If $\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})$, then $\alpha\in\Sigma(P)\setminus\Sigma(Q)$ so that $\alpha\notin\mathfrak{a}_{\mathrm{q}}^*$ and $\alpha\notin\Sigma(P,\sigma)$ from which we conclude that $\lambda\in\Sigma(P,\sigma\theta)\setminus\mathfrak{a}_{\mathrm{q}}^*\subset\Sigma(P)_-$. It then follows from (7.1) that

$$\mathrm{Re}\,\langle -\lambda+\rho_{P\mathrm{h}}\,,\,\alpha\rangle > \mathrm{Re}\,\langle -\lambda+\rho_{P\mathrm{h}}-\rho_P\,,\,\alpha\rangle \geq 0$$

and we see that (7.3) is satisfied. This implies the second assertion. The final assertion now follows by application of Theorem 6.7. □

In the following we will need to use the K-fixed function in the induced representation $\mathrm{Ind}_Q^G(1\otimes\mu\otimes 1)$, for $Q\in\mathscr{P}(A)$ and $\mu\in\mathfrak{a}_{\mathbb{C}}^*$. More precisely, given such Q and μ we define the function $\mathbb{1}_{Q,\mu}:G\to\mathbb{C}$ by

$$\mathbb{1}_{Q,\mu}(nak) := a^{\mu+\rho_Q}, \qquad (k\in K,\, a\in A,\, n\in N_Q).$$

Thus, $\mathbb{1}_{Q,\mu}$ is the unique function in $C^\infty(Q:1:\mu)$ satisfying $\mathbb{1}_{Q,\mu}|_K=1$.

Corollary 7.2. *Let $Q\in\mathscr{P}(A)$, $P\in\mathscr{P}_\sigma(A)$ and assume that $P\succeq Q$. Then*

$$h\mapsto \mathbb{1}_{Q,\rho_P}(h)\,dr_h([e])^{-1*}\omega_{H_Q\backslash H} \tag{7.4}$$

defines a density on $H_Q\backslash H$ which is absolutely integrable.

Proof. We apply Theorem 7.1 with $\xi=1$, $\mathscr{H}_\xi=\mathbb{C}$ and $\eta=1$. Furthermore, we take $\lambda=-\rho_P+\rho_{P\mathrm{h}}\in\mathfrak{a}_{\mathrm{q}}^*$ so that $-\bar{\lambda}+\rho_{P\mathrm{h}}=\rho_P$, and we take $f=\mathbb{1}_{P,\rho_P}$. It follows from the mentioned theorem that the integral for $A(Q:P:1:\rho_P)f$ converges absolutely. By equivariance, it gives a K-fixed element of $C^\infty(Q:1:\rho_P)$, so that

$$A(Q:P:1:\rho_P)f = A(Q:P:1:\rho_P)\mathbb{1}_{P,\rho_P} = c(Q:P:\rho_P)\mathbb{1}_{Q,\rho_P},$$

for some constant $c(Q:P:\rho_P)\in\mathbb{C}$. Evaluating this identity in the unit element we find

$$c(Q:P:\rho_P) = \int_{N_Q\cap\bar{N}_P}\mathbb{1}_{P,\rho_P}(\bar{n})\,d\bar{n},$$

which of course is the integral representation of a partial c-function. As the integrand is everywhere positive, it follows that $c(Q:P:\rho_P)$ is a positive real number. It now follows from the final assertion of Theorem 7.1 that

$$h\mapsto c(Q:P:\rho_P)\cdot\mathbb{1}_{Q,\rho_P}(h)\,dr_h([e])^{-1*}\omega_{H_Q\backslash H}$$

defines a density on $H_Q\backslash H$ which is absolutely integrable. By positivity of $c(Q:P:\rho_P)$ all assertions now follow. □

Let $\Gamma(Q)$ denote the cone in $\mathfrak{a}_{\mathrm{q}}$ spanned by the elements $H_\alpha + \sigma\theta H_\alpha$, for $\alpha \in \Sigma(Q)_-$, where the latter set is defined as in (1.2). The (closed) dual cone in $\mathfrak{a}_{\mathrm{q}}^*$ is readily seen to be given by

$$\Gamma(Q)^\circ := \{\lambda \in \mathfrak{a}_{\mathrm{q}}^* \mid \langle\lambda\,,\alpha\rangle \geq 0,\ \forall\alpha \in \Sigma(Q)_-\}. \tag{7.5}$$

Lemma 7.3. *Let $Q \in \mathscr{P}(A)$. Let $\mu \in \Gamma(Q)^\circ$. Then*

$$0 < \mathbb{1}_{Q,\mu-\rho_Q}(h) \leq 1, \qquad (h \in H). \tag{7.6}$$

Proof. It follows from [3, Thm 10.1] that if $h = nak$ with $n \in N_Q$, $a \in A$ and $k \in K$, then

$$\mathrm{pr}_{\mathrm{q}} \log a \in -\Gamma(Q),$$

where pr_{q} denotes the projection $\mathfrak{a} \to \mathfrak{a}_{\mathrm{q}}$. Therefore

$$\mathbb{1}_{Q,\mu-\rho_Q}(h) = a^\mu = e^{\mu(\mathrm{pr}_{\mathrm{q}} \log a)} \leq 1.$$

This establishes the upper bound. The lower bound is trivial. □

The above result will play a crucial role in the proof of a domination expressed in the following lemma.

Lemma 7.4. *Let $Q \in \mathscr{P}(A)$ and let ξ be a finite dimensional unitary representation of M_0. Let $P \in \mathscr{P}_\sigma(A)$ and assume that $P \succeq Q$; thus, in particular, $\rho_{P\mathrm{h}} = \rho_{Q\mathrm{h}}$. Furthermore, assume that $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ satisfies*

$$\mathrm{Re}\,\langle\lambda + \rho_P - \rho_{P\mathrm{h}}\,,\alpha\rangle \leq 0 \qquad \text{for all } \ \alpha \in \Sigma(Q)_-. \tag{7.7}$$

Then for every $f \in C^\infty(Q : \xi_M : -\bar{\lambda} + \rho_{Q\mathrm{h}})$, we have

$$\|f(h)\|_\xi \leq \sup_{k\in K} \|f(k)\|_\xi \cdot \mathbb{1}_{Q,\rho_P}(h), \qquad (h \in H). \tag{7.8}$$

Proof. Since $P \succeq Q$, we have $\rho_{P\mathrm{h}} = \rho_{Q\mathrm{h}}$. Thus, if $k \in K$ and $u \in Q$ then

$$f(uk) = \mathbb{1}_{Q,-\bar{\lambda}+\rho_{P\mathrm{h}}}(uk)\, f(k).$$

It follows that

$$\|f(x)\|_\xi \leq \sup_{k\in K} \|f(k)\|_\xi \cdot \mathbb{1}_{Q,\mu+\rho_P}(x), \qquad (x \in G), \tag{7.9}$$

where $\mu = -\mathrm{Re}\,\lambda - \rho_P + \rho_{P\mathrm{h}}$. For $x \in G$ we have

$$\mathbb{1}_{Q,\mu+\rho_P}(x) = \mathbb{1}_{Q,\mu-\rho_Q}(x)\mathbb{1}_{Q,\rho_P}(x).$$

As $\mu \in \Gamma(Q)^\circ$ by (7.7), it follows by application of Lemma 7.3 that

$$\mathbb{1}_{Q,\mu+\rho_P}(h) \leq \mathbb{1}_{Q,\rho_P}(h), \qquad (h \in H). \tag{7.10}$$

The required estimate (7.8) follows from combining (7.9) and (7.10). □

For the formulation of the next result, we note that the set of $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$ satisfying condition (7.7) is given by

$$\Omega_{P,Q} := -(\rho_P - \rho_{P\mathrm{h}}) - \Gamma(Q)^\circ + i\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*. \tag{7.11}$$

Corollary 7.5. *Let $Q \in \mathscr{P}(A)$, ξ a finite dimensional unitary representation of M_0 and $\eta \in V(\xi, e)$. Let $P \in \mathscr{P}_\sigma(A)$ and assume that $\underline{P} \succeq Q$; thus, in particular, $\rho_{P\mathrm{h}} = \rho_{Q\mathrm{h}}$. Let $\lambda \in \Omega_{P,Q}$. Then for every $f \in C(Q : \xi_M : -\bar{\lambda} + \rho_{Q\mathrm{h}})$, the integral*

$$j_H(Q : \xi_M : \lambda : \eta) f := \int_{H_Q \backslash H} \langle \eta \,,\, f(h) \rangle \, dr_h([e])^{-1*} \omega_{H_Q \backslash H} \tag{7.12}$$

converges absolutely.

Proof. It follows by application of Lemma 7.4 that

$$|\langle \eta \,,\, f(h) \rangle| \leq \|\eta\|_\xi \sup_{k \in K} \|f(k)\|_\xi \cdot \mathbb{1}_{Q,\rho_P}(h), \qquad (h \in H). \tag{7.13}$$

The result now follows from Corollary 7.2. ☐

Working in the setting of the above corollary, if $f \in C(K : M : \xi_M)$, then for $\mu \in \mathfrak{a}_\mathbb{C}^*$ we define $f_\mu \in C(Q : \xi_M : \mu)$ by $f_\mu|_K = f$. Furthermore, we define

$$j_H(Q : \xi_M : \lambda : \eta)(f) := j_H(Q : \xi_M : \lambda : \eta)(f_{-\bar{\lambda} + \rho_{Q\mathrm{h}}})$$

for $\lambda \in \Omega_{P,Q}$. Accordingly, $j_H(Q : \xi_M : \lambda : \eta)$ is viewed as an element of $C^{-0}(K : M : \xi_M)$, see the beginning of Section 3. Given $f \in C(K : M : \xi_M)$, we agree to write

$$\langle j_H(Q : \xi_M : \lambda : \eta) \,,\, f \rangle = j_H(Q : \xi_M : \lambda : \eta)(f)$$

and $\langle f \,,\, j_H(Q : \xi_M : \lambda : \eta) \rangle$ for its conjugate. Then

$$\langle f \,,\, j_H(Q : \xi_M : \bar{\lambda} : \eta) \rangle = \int_{H_Q \backslash H} \langle f_{-\lambda + \rho_{Q\mathrm{h}}}(h) \,,\, \eta \rangle \, dr_h([e])^{-1*} \omega_{H_Q \backslash H}.$$

Corollary 7.6. *Let notation be as in Corollary 7.5. Then*

$$\lambda \mapsto j_H(Q : \xi_M : \lambda : \eta) \tag{7.14}$$

is a continuous $C^{-0}(K : M : \xi_M)$-valued function on the closed subset $\Omega_{P,Q}$ of $\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$. Its restriction to the interior of $\Omega_{P,Q}$ is holomorphic as a $C^{-0}(K : M : \xi_M)$-valued function.

Proof. It is clear that $\lambda \mapsto f_{-\bar{\lambda} + \rho_{P\mathrm{h}}}|_H$ is a holomorphic $C(H) \otimes \mathscr{H}_\xi$-valued function on $\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$ satisfying the uniform estimate

$$|\langle \eta \,,\, f_{-\bar{\lambda} + \rho_{P\mathrm{h}}} \rangle \leq \|\eta\|_\xi \sup_{k \in K} \|f(k)\|_\xi \cdot \mathbb{1}_{Q,\rho_P}(h), \qquad (h \in H),$$

for all $\lambda \in \Omega_{P,Q}$, by application of (7.13). In view of Corollary 7.2 the result now follows by application of the dominated convergence theorem. ☐

The following lemma will be useful for later use. If $Q \in \mathscr{P}(A)$, we have that $\Sigma(Q,\sigma\theta)|_{\mathfrak{a}_\mathrm{q}} \subset \Sigma(\mathfrak{a}_\mathrm{q})$. In accordance with (2.3) we define

$$\mathfrak{a}_\mathrm{q}^{*+}(Q) := \{\lambda \in \mathfrak{a}_\mathrm{q}^* \mid \langle \lambda\,,\,\alpha \rangle > 0,\ \forall \alpha \in \Sigma(Q,\sigma\theta)\}.$$

This set is a non-empty open subset of $\mathfrak{a}_\mathrm{q}^*$, see the text below (2.3).

Lemma 7.7. *Let $Q \in \mathscr{P}(A)$ and $P \in \mathscr{P}_\sigma(A,Q)$. Then*

$$\Omega_{P,Q} \supset -(\rho_P - \rho_{P\mathrm{h}}) - \mathfrak{a}_\mathrm{q}^{*+}(Q) + i\mathfrak{a}_\mathrm{q}^*.$$

Proof. In view of (7.11) it suffices to show that $\Gamma(Q)^\circ \supset \mathfrak{a}_\mathrm{q}^{*+}(Q)$. This is a straightforward consequence of the fact that $\Sigma(Q)_- \subset \Sigma(Q,\sigma\theta)$, by (1.2) and (2.1). $\square$

Theorem 7.8. *Let $Q \in \mathscr{P}(A)$, $P \in \mathscr{P}_\sigma(A)$ such that $P \succeq Q$. Let ξ be a finite dimensional unitary representation of M_0 and $\eta \in V(\xi,e)$. Then the $C^{-\infty}(K:M:\xi_M)$-valued function*

$$\lambda \mapsto j_H(Q : \xi_M : \lambda : \eta), \tag{7.15}$$

defined by (7.12), extends to a meromorphic function on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with values in $C^{-\infty}(K : M : \xi_M)$. Furthermore, up to a positive factor, depending on the normalization of the Haar measure on $N_P \cap \bar{N}_Q$,

$$j_H(P : \xi_M : \lambda : \eta) = A(P : Q : \xi_M : \lambda - \rho_{Qh})\, j_H(Q : \xi_M : \lambda : \eta) \tag{7.16}$$

as an identity of $C^{-\infty}(K:M:\xi_M)$-valued meromorphic functions in $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Finally, the function (7.15) is continuous on the set $\Omega_{P,Q}$ defined in (7.11) and holomorphic on its interior.

Remark 7.9. In particular, if $\Sigma(Q)_- = \emptyset$, it follows that $\Gamma(Q)^\circ = \mathfrak{a}_\mathrm{q}^*$ so that $j_H(Q : \xi_M : \cdot\,)$ is holomorphic everywhere.

Proof. Without loss of generality, we may assume that ξ is irreducible. Then it follows from (7.2) combined with (7.12) that

$$\langle j_H(P : \xi_M : \lambda : \eta)\,,\, f\rangle = \langle j_H(Q : \xi_M : \lambda : \eta)\,, A(Q : P : \xi_M : -\bar{\lambda} + \rho_{Ph}) f\rangle \tag{7.17}$$

for all $\lambda \in \Omega_{P,P}$ and $f \in C^\infty(P : \xi_M : -\bar{\lambda} + \rho_{P\mathrm{h}})$.

The standard intertwining operator $A(Q:P:\xi_M:\nu)$ from the induced representation $\mathrm{Ind}_P^G(\xi_M \otimes \nu \otimes 1)$ to the representation $\mathrm{Ind}_Q^G(\xi_M \otimes \nu \otimes 1)$ may be viewed as a meromorphic function of $\nu \in \mathfrak{a}_\mathbb{C}^*$, with values in the space $\mathrm{End}(C^\infty(K:M:\xi_M))$ (equipped with the strong topology), see [24, Thm. 1.5] and [13, Thm. 1.5]. Its singular locus is contained in a locally finite union of hyperplanes of the form $\mu + \ker\alpha$, with $\mu \in \mathfrak{a}^*$ and $\alpha \in \Sigma(P) \cap \Sigma(\bar{Q})$, see [13, Rem. 1.6]. Since $\Sigma(P) \cap \Sigma(\bar{Q}) \cap \mathfrak{a}_\mathrm{h}^* = \emptyset$ in view of Lemma 2.2 (b), none of these singular hyperplanes contain $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, so that $A(Q:P:\xi_M:\cdot\,)$ restricts to a meromorphic function on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

The operator $A(P:Q:\xi_M:\nu)$ has a similar meromorphic behavior, and since the induced representation $\mathrm{Ind}_Q^G(\xi_M\otimes\nu\otimes 1)$ is irreducible for generic $\nu\in\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ it follows that

$$A(Q:P:\xi_M:\nu)\circ A(P:Q:\xi_M:\nu)=\eta(P:Q:\xi_M:\nu)\,\mathrm{I} \tag{7.18}$$

as an identity of $\mathrm{End}(C^\infty(K:M:\xi_M))$-valued functions of $\nu\in\mathfrak{a}^*_{\mathbb{C}}$. Here $\eta=\eta(P:Q:\xi_M:\cdot\,)$ is a meromorphic $\mathbb{C}$-valued function on $\mathfrak{a}^*_{\mathbb{C}}$. By the usual product decomposition of intertwining operators it follows that η admits a decomposition of the form

$$\eta(\nu)=\prod_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})}\eta_\alpha(\langle\nu\,,\alpha\rangle),$$

where the η_α are meromorphic functions on $\mathbb{C}$. We now fix $g\in C^\infty(K:M:\xi_M)$. By substituting $f=A(P:Q:\xi_M:-\bar{\lambda}+\rho_{Q\mathrm{h}})g$ in (7.17) we infer that

$$\begin{aligned}\langle j_H(P:\xi_M:\lambda:\eta)\,,A(P:Q:\xi_M:-\bar{\lambda}+\rho_{Qh})g\rangle &=\\ &= \langle j_H(Q:\xi_M:\lambda:\eta)\,,\eta(-\bar{\lambda}+\rho_{Q\mathrm{h}})g\rangle.\end{aligned}$$

By using that $A(Q:P:\xi_M:\lambda-\rho_{Q\mathrm{h}})$ is the Hermitian conjugate of $A(P:Q:\xi_M:-\bar{\lambda}+\rho_{Qh})$, see [22, Prop. 7.1 (iv)], and that $\rho_{Q\mathrm{h}}=\rho_{P\mathrm{h}}$, it follows that

$$j_H(Q:\xi_M:\lambda:\eta)=\overline{\eta(-\lambda+\rho_{P\mathrm{h}})}^{-1}A(Q:P:\xi_M:\lambda-\rho_{Q\mathrm{h}})\,j_H(P:\xi_M:\lambda:\eta), \tag{7.19}$$

for generic $\lambda\in\Omega_{P,P}$.

Let $\Omega\subset\mathfrak{a}^*_{\mathrm{q}}$ be a relatively compact open subset. Then there exists a constant $s\in\mathbb{N}$ such that $\lambda\mapsto j_H(P:\xi_M:\lambda:\eta)$ is meromorphic on $\Omega+i\mathfrak{a}^*_{\mathrm{q}}$, with values in the Banach space $C^{-s}(K:M:\xi_M)$, see Section 3 and [6, Thm. 9.1] for details. Furthermore, there exists a constant $r\in\mathbb{N}$ such that $A(Q:P:\xi_M:\lambda-\rho_{P\mathrm{h}})$ depends meromorphically on $\lambda\in\Omega+i\mathfrak{a}^*_{\mathrm{q}}$, as a function with values in the Banach space of bounded linear maps from $C^{-s}(K:M:\xi_M)$ to $C^{-s-r}(K:M:\xi_M)$. Combining these observations with (7.19) we see that $\lambda\mapsto j_H(Q:\xi_M:\lambda:\eta)$ is a meromorphic function on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ with values in $C^{-\infty}(K:M:\xi_M)$, equipped with the strong dual topology. Its continuity on $\Omega_{P,Q}$ and holomorphy on the interior of this set follows from Corollary 7.6. By meromorphic continuation it now follows that (7.17) is valid as an identity of meromorphic functions. Since $A(P:Q:\xi_M:\lambda-\rho_{Q\mathrm{h}})$ is the Hermitian conjugate of the intertwining operator appearing in that identity, whereas the identity holds for all $f\in C^\infty(K:M:\xi_M)$, it follows that (7.16) is valid as an identity of meromorphic $C^{-\infty}(K:M:\xi_M)$-valued functions of $\lambda\in\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. $\square$

Let $Q\in\mathscr{P}$ be fixed for the moment. Then the function (7.15) is independent of the choice of $P\succeq Q$, whereas the description of the domain of holomorphy depends on it. This motivates the definition of the following closed subset of $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$,

$$\Omega_Q:=\bigcup_{P\in\mathscr{P}_\sigma(A,Q)}\Omega_{P,Q}, \tag{7.20}$$

where the union is taken over the finite non-empty set $\mathscr{P}_\sigma(A,Q)$ of parabolic groups $P \in \mathscr{P}_\sigma(A)$ with $P \succeq Q$, see Lemma 2.6. The function (7.15) is continuous on Ω_Q and holomorphic on the interior of this set. We can actually improve on this result.

In fact, let $\Gamma(Q)^\circ$ be as in (7.5). We denote by $B : \mathfrak{a}_\mathrm{q} \to \mathfrak{a}_\mathrm{q}^*$ the linear isomorphism induced by the inner product on $\mathfrak{a}_\mathrm{q}$. Then for $\alpha \in \Sigma(\mathfrak{a}_\mathrm{q})$ we have $B(H_\alpha) = \alpha^\vee$. Therefore, $B(\Gamma(Q))$ is the cone spanned by the $\mathfrak{a}_\mathrm{q}$-roots from $\mathrm{pr}_\mathrm{q}(\Sigma(Q)_-)$.

Let $\widehat{\Omega}_Q$ denote the hull in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ of the set Ω_Q with respect to the functions $\mathrm{Re}\,\langle\,\cdot\,,\alpha\rangle$ with $\alpha \in \Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(Q))$, i.e.,

$$\widehat{\Omega}_Q := \{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \mid \mathrm{Re}\,\langle\lambda\,,\alpha\rangle \leq \sup \mathrm{Re}\,\langle\Omega_Q\,,\alpha\rangle,\ \forall \alpha \in \Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(Q))\}. \tag{7.21}$$

Since the roots from $\Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(Q))$ satisfy $\langle\alpha\,,\cdot\,\rangle \leq 0$ on $-\Gamma(Q)^\circ$ it follows that we can describe the given hull by means of inequalities as follows:

$$\widehat{\Omega}_Q = \{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \mid \mathrm{Re}\,\langle\lambda\,,\alpha\rangle \leq \max_{P \in \mathscr{P}_\sigma(A,Q)} \langle -\rho_P\,,\alpha\rangle,\ \forall \alpha \in \Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(Q))\}. \tag{7.22}$$

Corollary 7.10. *Let $Q \in \mathscr{P}(A)$, $\xi \in \widehat{M}_{0\mathrm{fu}}$ and $\eta \in V(\xi, e)$. Then the $C^{-\infty}(K : M : \xi_M)$-valued function $\lambda \mapsto j_H(Q : \xi_M : \lambda : \eta)$ is holomorphic on an open neighborhood of $\widehat{\Omega}_Q$.*

Proof. From (7.19) we infer that the singular locus of $\lambda \mapsto j_H(Q : \xi_M : \lambda : \eta)$ is the union of a locally finite collection $\mathscr{H}$ of hyperplanes of the form $\mathrm{H}_{\alpha,\mu} = \mu + (\alpha^\perp)_\mathbb{C}$ with $\alpha \in \Sigma(\mathfrak{a}_\mathrm{q})$ and $\mu \in \mathfrak{a}_\mathrm{q}^*$. Indeed, the singular loci of the meromorphic ingredients on the right-hand side of that formula are all of this form, by [11, Lemma 3.2], [13, Rem. 1.6] and (7.18).

Let μ be a singular point of $j_H(Q : \xi_M : \cdot : \eta)$, i.e., a point in the union of the singular hyperplanes. Then there exists a root $\alpha \in \Sigma(\mathfrak{a}_\mathrm{q})$ such that $\mathrm{H}_{\alpha,\mu}$ is a singular hyperplane. By analytic continuation it follows that $\mathrm{H}_{\alpha,\mu} \cap \Omega_Q = \emptyset$. From the fact that the cone $\Gamma(Q)^\circ$ has non-empty interior it follows that the set $\Omega_Q \cap \mathfrak{a}_\mathrm{q}^*$ is connected, hence is contained in one connected component of $\mathfrak{a}_\mathrm{q}^* \setminus \mathrm{H}_{\alpha,\mu}$. Replacing α by $-\alpha$ if necessary, we may assume that

$$\Omega_Q \cap \mathfrak{a}_\mathrm{q}^* \subset \{\lambda \in \mathfrak{a}_\mathrm{q}^* : \langle\alpha, \lambda\rangle \geq c\}$$

for some $c \in \mathbb{R}$. This in turn implies that

$$\Gamma(Q)^\circ \subset \{\lambda \in \mathfrak{a}_\mathrm{q}^* : \langle\alpha, \lambda\rangle \geq 0\} = \{\lambda \in \mathfrak{a}_\mathrm{q}^* : \lambda(H_\alpha) \geq 0\}.$$

Since $\Gamma(Q)^\circ$ has open interior, α does not vanish on $\Gamma(Q)^\circ$. Using that $\Gamma(Q)^\circ$ is a cone, we find

$$\langle\alpha, \Gamma(Q)^\circ\rangle = \mathbb{R}_{\geq 0}.$$

In particular this implies $H_\alpha \in \Gamma(Q)^{\circ\circ} = \Gamma(Q)$ and thus we conclude that $\alpha \in \Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(Q))$.

For any $P \in \mathscr{P}_\sigma(A)$ with $P \succeq Q$, the singular hyperplane $\mathrm{H}_{\alpha,\mu}$ does not intersect $-\rho_P + \rho_{P\mathrm{h}} - \Gamma(Q)^\circ$, hence

$$\langle \alpha, \mu \rangle \notin -\langle \alpha, \rho_P \rangle - \mathbb{R}_{\geq 0}.$$

This implies that $\langle \alpha \,,\, \mu \rangle > -\langle \alpha \,,\, \rho_P \rangle$. We conclude that

$$\langle \alpha \,,\, \mu \rangle > \max_{P \in \mathscr{P}_\sigma(A), P \succeq Q} \langle \alpha \,,\, -\rho_P \rangle$$

so that $\mu \notin \widehat{\Omega}_Q$. Thus, $\widehat{\Omega}_Q$ is disjoint from the singular locus. $\square$

Let $Q \in \mathscr{P}(A)$ and $\xi \in \widehat{M}_{0\mathrm{fu}}$. We define the meromorphic $V(\xi, e)^* \otimes C^{-\infty}(K : M : \xi_M)$-valued function $j_H(Q : \xi_M : \cdot\,)$ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ by

$$j_H(Q : \xi_M : \lambda)(\eta) = j_H(Q : \xi_M : \lambda : \eta)$$

for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and $\eta \in V(\xi, e)$. Furthermore, we define the meromorphic $V(\xi)^* \otimes C^{-\infty}(K : M : \xi_M)$-valued function $j(Q : \xi_M : \cdot\,)$ on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ by

$$j(Q : \xi_M : \lambda) = \sum_{v \in \mathscr{W}} \pi_{Q, \xi_M, \lambda - \rho_{Q\mathrm{h}}}(v^{-1})\, j_{vHv^{-1}}(Q : \xi_M : \lambda) \circ \mathrm{pr}_v. \tag{7.23}$$

Here $j_{vHv^{-1}}$ is defined for the data σ_v, vHv^{-1} in place of σ, H. This definition is allowed since $\mathscr{W} \subset N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_\mathrm{q})$ (see text preceding (4.6)), so that A, A_q, M_0 and $\mathscr{P}(A)$ are invariant under conjugation by v and $\mathfrak{a}_\mathrm{q}$ is maximal abelian in $\mathfrak{p} \cap \mathrm{Ad}(v)\mathfrak{q}$. See also the discussion at the end of Section 2.

In order to formulate our next result, we define, for $v \in \mathscr{W}$, the set $\Omega_{v,Q}$ as Ω_Q in (7.20), with vHv^{-1} in place of H. Likewise, we define $\widehat{\Omega}_{v,Q}$ to be the set $\widehat{\Omega}_Q$ defined as in (7.21), with vHv^{-1} in place of H.

Lemma 7.11. *Let $Q \in \mathscr{P}(A)$. Then for each $v \in \mathscr{W}$, we have*

$$\Omega_{v,Q} = v\Omega_{v^{-1}Qv} \quad \text{and} \quad \widehat{\Omega}_{v,Q} = v\widehat{\Omega}_{v^{-1}Qv}.$$

Proof. In view of Lemma 2.7 the cone $\Gamma(v, Q)$, defined as $\Gamma(Q)$ with σ_v in place of σ, is given by $\Gamma(v, Q) = v\Gamma(v^{-1}Qv)$. Likewise, its dual, defined as in (7.5) is given by

$$\Gamma(v, Q)^\circ = v\Gamma(v^{-1}Qv)^\circ.$$

From (7.11) and (7.20), with σ_v in place of σ, we now find, with obvious notation, $\Omega_{v,P,Q} = v\Omega_{v^{-1}Pv, v^{-1}Qv}$, for $P \in \mathscr{P}_\sigma(A)$ with $P \succeq Q$. Taking the union over such P, we obtain the first asserted equality.

The second equality follows from the first, by taking the hull of the sets $\Omega_{v,Q}$ and $v\Omega_{v^{-1}Qv}$ with respect to the functions $\mathrm{Re}\langle \,\cdot\,,\, \alpha \rangle$ with $\alpha \in \Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(v, Q))$. The first hull equals $\widehat{\Omega}_{v,Q}$ by definition. Using that

$$\Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(v, Q)) = \Sigma(\mathfrak{a}_\mathrm{q}) \cap B(v\Gamma(v^{-1}Qv)) = v(\Sigma(\mathfrak{a}_\mathrm{q}) \cap B(\Gamma(v^{-1}Qv)),$$

we see that the second hull equals $v\widehat{\Omega}_{v^{-1}Qv}$. $\square$

We define the following closed subsets of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$,

$$\Upsilon_Q = \bigcap_{v \in \mathscr{W}} v\Omega_{v^{-1}Qv}, \qquad \widehat{\Upsilon}_Q = \bigcap_{v \in \mathscr{W}} v\widehat{\Omega}_{v^{-1}Qv}. \tag{7.24}$$

The following lemma guarantees in particular that the set Υ_Q, and hence also the bigger set $\widehat{\Upsilon}_Q$, have non-empty interior.

Lemma 7.12. *Let $Q \in \mathscr{P}(A)$. Then for every $P \in \mathscr{P}_\sigma(A, Q)$, we have*

$$\Upsilon_Q \;\supset\; -(\rho_P - \rho_{P\mathrm{h}}) - \mathfrak{a}_{\mathrm{q}}^{*+}(Q) + i\mathfrak{a}_{\mathrm{q}}^*.$$

Proof. Fix $v \in \mathscr{W}$. Then $v^{-1}Pv$ belongs to $\mathscr{P}_\sigma(A, v^{-1}Qv)$, hence it follows from (7.20) and Lemma 7.7

$$\Omega_{v^{-1}Qv} \supset -(\rho_{v^{-1}Pv} - \rho_{v^{-1}Pv\mathrm{h}}) - \mathfrak{a}_{\mathrm{q}}^{*+}(v^{-1}Qv) + i\mathfrak{a}_{\mathrm{q}}^*.$$

Applying v we obtain $v\Omega_{v^{-1}Qv} \supset -(\rho_P - \rho_{P\mathrm{h}}) - \mathfrak{a}_{\mathrm{q}}^{*+}(Q) + i\mathfrak{a}_{\mathrm{q}}^*$. As this is true for each $v \in \mathscr{W}$, the asserted inclusion follows. $\square$

Lemma 7.13. *Let $Q \in \mathscr{P}(A)$ and $\xi \in \widehat{M}_{0\mathrm{fu}}$. Let $\eta \in V(\xi)$.*

(a) *For each $v \in \mathscr{W}$ the defining integral for the corresponding term in (7.23) is absolutely convergent for every $\lambda \in \Upsilon_Q$.*

(b) *The meromorphic $C^{-\infty}(K : M : \xi_M)$-valued function $\lambda \mapsto j(Q : \xi_M : \lambda : \eta)$ is holomorphic on an open neighborhood of the set $\widehat{\Upsilon}_Q$.*

Proof. It follows from (7.23) and Corollary 7.5 that the integral for $j_{vHv^{-1}}(Q : \xi_M : \lambda : \eta_v)$ is convergent for $\lambda \in \Omega_{v,Q}$. This set contains Υ_Q, by (7.24) and Lemma 7.11, and we see that (a) follows.

It follows from Corollary 7.6 applied with σ_v in place of σ that the mentioned function is holomorphic on an open neighborhood of $\widehat{\Omega}_{v,Q}$. From this we deduce that $j(Q : \xi_M \cdot : \eta)$ is holomorphic on an open neighborhood of the intersection of the sets $\widehat{\Omega}_{v,Q}$, for $v \in \mathscr{W}$. This intersection equals $\widehat{\Upsilon}_Q$, by (7.24) and Lemma 7.11. $\square$

We finish this section relating the constructed functions $j(Q : \xi_M : \cdot)$, for different Q, by intertwining operators.

Theorem 7.14. *Let $Q \in \mathscr{P}(A)$ and $\xi \in \widehat{M}_{0\mathrm{fu}}$. Then the following assertions are valid.*

(a) *For every $\eta \in V(\xi)$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, the element $j(Q : \xi_M : \lambda)(\eta)$ of the space $C^{-\infty}(K : M : \xi_M)$ is $\pi_{Q,\xi_M,\lambda-\rho_{Q\mathrm{h}}}(H)$-invariant.*

(b) *If $Q, Q' \in \mathscr{P}(A)$ and $Q' \succeq Q$, then (up to normalization),*

$$j(Q' : \xi_M : \lambda) = A(Q' : Q : \xi_M : \lambda - \rho_{Qh}) \circ j(Q : \xi_M : \lambda), \tag{7.25}$$

as an identity of meromorphic $V(\xi)^* \otimes C^{-\infty}(K : M : \xi_M)$-valued functions in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

Proof. We start with (b). Let $P \in \mathscr{P}_\sigma(A)$ be such that $P \succeq Q'$. Then by application of Lemma 2.4 it follows that $\Sigma(P) \cap \Sigma(\bar{Q'}) \subset \Sigma(P) \cap \Sigma(\bar{Q})$ so

$$A(P:Q:\xi_M:\lambda) = A(P:Q':\xi_M:\lambda) \circ A(Q':Q:\xi_M:\lambda) \tag{7.26}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. See [22, Cor. 7.7] for details. Using (7.16) both with Q and with Q' in place of Q we find

$$A(P:Q':\xi_M:\lambda) \circ j_H(Q':\xi_M:\lambda) = A(P:Q:\xi_M:\lambda) \circ j_H(Q:\xi_M:\lambda)$$

combining this with (7.26) and using that $A(P:Q':\xi_M:\lambda)$ is injective for generic λ, we obtain that

$$j_H(Q':\xi_M:\lambda) = A(Q':Q:\xi_M:\lambda - \rho_{Q\mathrm{h}})\, j_H(Q:\xi_M:\lambda)$$

for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Since the expressions on both sides of the equation are meromorphic $V(\xi,e)^* \otimes C^{-\infty}(K:M:\xi_M)$-valued functions, the identity holds as an identity of meromorphic functions. The identity also holds with H replaced by vHv^{-1}, as an identity of $V(\xi,v)^* \otimes C^{-\infty}(K:M:\xi_M)$-valued meromorphic functions of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. If we apply this to each of the terms of the sum in (7.23) we obtain (7.25). This establishes (b).

We now turn to (a). Fix $P \in \mathscr{P}_\sigma(A)$ such that $P \succeq Q$. Then assertion (a) holds with P in place of Q, in view of Corollary 5.5. To establish assertion (a) for $j(Q:\xi_M:\lambda)(\eta)$ as well, we use (b) with $Q' = P$. Then assertion (a) follows from the fact that $A(P:Q:\xi_M:\lambda - \rho_{Q\mathrm{h}})$ is intertwining and injective for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. $\square$

8 Eisenstein integrals

In this section we will extend the definition of Eisenstein integrals for minimal $\sigma\theta$-stable parabolic subgroups from $\mathscr{P}_\sigma(A_\mathrm{q})$ to similar integrals for minimal parabolic subgroups from $\mathscr{P}(A)$.

First we need to carefully discuss the parameter space for the Eisenstein integral. In view of Lemma 4.3, it follows that the inclusion map $M \to M_0$ induces a diffeomorphism $M/H_M \simeq M_0/H_{M_0}$. This diffeomorphism induces a topological linear isomorphism $C^\infty(M/H_M) \simeq C^\infty(M_0/H_{M_0})$ via which we will identify the elements of these spaces.

Let (τ, V_τ) be a finite dimensional unitary representation of K. Then we define τ_{M_0} to be the restriction of τ to M_0. Likewise, we define τ_M to be the restriction of τ to M. Then τ_{M_0} and τ_M have the same representation space.

We define $C^\infty(M_0/H_{M_0}:\tau_{M_0})$ to be the space of smooth functions $\psi: M_0/H_{M_0} \to V_\tau$ satisfying the transformation rule

$$\psi(kx) = \tau(k)\psi(x) \qquad \big(k \in K \cap M_0,\ x \in M_0/H_{M_0}\big).$$

Similarly, we define $C^\infty(M/H_M : \tau_M)$ to be the space of smooth functions $\psi : M/H_M \to V_\tau$ satisfying the transformation rule

$$\psi(mx) = \tau(m)\psi(x) \qquad \big(m \in M,\, x \in M/H_M\big).$$

We then have the obvious inclusion

$$C^\infty(M_0/H_{M_0} : \tau_{M_0}) \;\subset\; C^\infty(M/H_M : \tau_M).$$

In general, the first of these spaces will be strictly contained in the second. The first of these spaces enters the definition of the Eisenstein integral for minimal $\sigma\theta$-stable parabolic subgroup from $\mathscr{P}_\sigma(A_{\mathrm{q}})$, whereas the second is convenient in the context of induction from a minimal parabolic subgroup from $\mathscr{P}(A)$. The relation between the spaces can be clarified as follows. Since M normalizes $M_{0n} \cap K$ it follows that the space V_τ^0 of $M_{0n} \cap K$-invariants in V_τ is invariant under $\tau(M)$, so that we may define the following representation τ_M^0 of M by restriction:

$$\tau_M^0 := \tau_M|_{V_\tau^0}, \quad \text{where} \quad V_\tau^0 := (V_\tau)^{M_{0n}\cap K}, \tag{8.1}$$

Observe that for every $v \in \mathscr{W}$ we have

$$V_\tau^{M_0\cap K\cap vHv^{-1}} = (V_\tau^0)^{M\cap vHv^{-1}}.$$

Indeed, this follows from the fact that $M_0 \cap K = M(M_{0n} \cap K)$ and that $\tau(M_{0n} \cap K) = 1$ on V_τ^0.

Lemma 8.1. *Let (τ, V_τ) be a finite dimensional unitary representation of K. Then*

$$C^\infty(M_0/H_{M_0} : \tau_{M_0}) \;=\; C^\infty(M/H_M : \tau_M^0), \tag{8.2}$$

Proof. We observe that M_{0n} acts trivially on M_0/H_{M_0} by Lemma 4.3 (f). Therefore, every function in the space on the left-hand side of (8.2) has values in V_τ^0 and we see that the space on the left is indeed contained in the space on the right. For the converse inclusion, let $f : M/H_M \to V_\tau^0$ be a function in the space on the right. If $k_0 \in M_0 \cap K$ we may write $k_0 = k_M k_n$ with $k_M \in M$ and $k_n \in M_{0n} \cap K$. Let $m_0 \in M_0$, then $m_0 = mh$ for a suitable $m \in M$ and $h \in H_{M_0}$. Since $M_{0n} \subset H$, it follows that

$$\begin{aligned} f(k_0 m_0) &= f(k_M k_n m h) = \tau(k_M) f(m(m^{-1}k_n m)h) \\ &= \tau(k_M) f(m) = \tau(k_M)\tau(k_n) f(m_0) \\ &= \tau(k_0) f(m_0). \end{aligned}$$

It follows that f belongs to the space on the left. $\square$

We are now prepared for the definition of the Eisenstein integral related to a fixed parabolic subgroup $P \in \mathscr{P}(A)$. Given $\psi \in C^\infty(M/H_M : \tau_M^0)$ we define the function $\psi_{P,\lambda} : G \to V_\tau$ by

$$\psi_{P,\lambda}(kman) = a^{\lambda - \rho_P - \rho_{P\mathrm{h}}}\, \tau(k)\psi(m).$$

We denote by $C^\infty(G/H:\tau)$ the space of smooth functions $\phi: G/H \to V_\tau$ satisfying the rule

$$\phi(kx) = \tau(k)\phi(x) \qquad (k \in K,\, x \in G/H).$$

Recall the definition of Ω_P from (7.20) with P in place of Q.

Proposition 8.2. *Let $\omega \in \mathscr{D}_{\mathfrak{h}/\mathfrak{h}_P}$. Let $\psi \in C^\infty(M/H_M:\tau_M^0)$ and let $\lambda \in \Omega_P$. Then the following assertions are valid.*

(a) *For each $x \in G$ the function*

$$h \mapsto \psi_{P,\lambda}(xh)\, dl_h(e)^{-1*}\omega$$

defines a V_τ-valued density on H/H_P.

(b) *For each $x \in G$ the density in (a) is integrable.*

(c) *The function $E_H(P:\psi:\lambda): G \to V_\tau$ defined by*

$$E_H(P:\psi:\lambda)(x) := \int_{H/H_P} \psi_{P,\lambda}(xh)\, dl_h(e)^{-1*}\,\omega, \quad (x \in G),$$

in accordance with (a) and (b), belongs to $C^\infty(G/H:\tau)$.

Proof. Before we start with the actual proof, we note that the condition on λ implies the existence of parabolic subgroup $P' \in \mathscr{P}_\sigma(A,P)$ such that $\lambda \in \Omega_{P',P}$, in view of (7.20).

Let $F_\tau \subset \widehat{M}$ denote the finite set of M-types in $\tau^\vee$ and let $\mathscr{H}_\tau$ denote the subspace of $C^\infty(M/H_M)$ consisting of the left M-finite functions of isotype contained in F_τ. Let L be the left regular representation of M in $C^\infty(M/H_M)$ and let $\xi_\tau := L|_{\mathscr{H}_\tau}$ be its restriction to the subspace $\mathscr{H}_\tau$.

Since $C^\infty(M/H_M:\tau_M)$ consists of the M-fixed functions in $C^\infty(M/H_M)\otimes V_\tau$, it follows that

$$C^\infty(M/H_M:\tau_M) \subset \mathscr{H}_\tau \otimes V_\tau.$$

We define the function $\tilde{\psi}_{P,\lambda}: G \to \mathscr{H}_\tau \otimes V_\tau$ by

$$\tilde{\psi}_{P,\lambda}(x)(m) = \psi_{P,\lambda}(xm).$$

Then it follows readily that

$$\tilde{\psi}_{P,\lambda}(xman) = a^{\lambda-\rho_P-\rho_{P\mathrm{h}}}(\xi_\tau(m)^{-1}\otimes 1)\tilde{\psi}_{P,\lambda}(x).$$

We define $\psi^\vee_{P,\lambda}(x) := \tilde{\psi}_{P,\lambda}(x^{-1})$. Then it follows that

$$\psi^\vee_{P,\lambda} \in C^\infty(P:\xi_\tau:-\lambda+\rho_{P\mathrm{h}})\otimes V_\tau. \tag{8.3}$$

Let $\varphi : H/H_P \to H_P\backslash H$ be the diffeomorphism induced by $h \mapsto h^{-1}$. Then $d\varphi(e)^*\omega = \omega$ and for $x \in G$ we see that

$$\begin{aligned}&\varphi^*[h \mapsto \psi_{P,\lambda}(xh)\, dl_h(e)^{-1*}\omega] \\ &= \ [h \mapsto \psi^\vee_{P,\lambda}(hx^{-1})(e)\, dr_h(e)^{-1*}\omega].\end{aligned} \tag{8.4}$$

Let ε denote the element of $\mathscr{H}_\tau^{H_M}$ such that $\langle g\,,\,\varepsilon\rangle = g(e)$ for all $g \in \mathscr{H}_\tau$. We may now apply Corollary 7.5 to the first tensor component of the space in (8.3) with (P', P) in place of (P, Q), with $(\xi_\tau, \mathscr{H}_\tau)$ in place of $(\xi\,, \mathscr{H}_\xi)$, with $R_{x^{-1}}(\psi^\vee_{P,\lambda})$ in place of f (where R denotes the right regular representation) and with ε in place of η. From applying the corollary in this fashion, it follows that the expression on the right-hand side of (8.4) is a V_τ-valued density on $H_P\backslash H$ which is integrable. This implies (a) and (b).

Using that $x \mapsto R_x(\psi^\vee_{P,\lambda})$ is smooth as a function with values in the Fréchet space $C^\infty(P : \xi_\tau : -\lambda + \rho_{P\mathrm{h}}) \otimes V_\tau$, we find that $E_H(P : \psi : \lambda) \in C^\infty(G, V_\tau)$.

The right H-invariance and the τ-spherical behavior are readily checked. $\square$

Remark 8.3. The above procedure would also work more generally for functions $\psi \in C^\infty(M/H_M : \tau_M)$. However, for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ the map $\psi \mapsto E_H(P : \psi : \lambda)\psi$ would then have a (possibly λ-dependent) kernel complementary to $C^\infty(M/H_M : \tau^0_M)$.

For $v \in \mathscr{W}$ the above procedure applies to the data $K, vHv^{-1}, A, A_\mathrm{q}$ in place of K, H, A, A_q. We thus obtain Eisenstein integrals $E_{vHv^{-1}}(P : \psi : \lambda : x)$ for ψ in the parameter space $C^\infty(M/M \cap vHv^{-1} : \tau^0_M)$. The general Eisenstein integral is defined as follows. For $v \in \mathscr{W}$ we equip $L^2(M/M \cap vHv^{-1})$ with the L^2-inner product for the normalized invariant measure, and $L^2(M/M \cap vHv^{-1}) \otimes V_\tau$ with the tensor product inner product. The latter restricts to an inner product on the finite dimensional subspace $C^\infty(M/M \cap vHv^{-1} : \tau^0_M)$. We define

$$\mathscr{A}_{M,2} := \oplus_{v\in\mathscr{W}}\, C^\infty(M/M \cap vHv^{-1} : \tau^0_M). \tag{8.5}$$

Equipped with the direct sum of the given inner products on the summands, this space becomes a finite dimensional Hilbert space.

For $\psi \in \mathscr{A}_{M,2}$ and $\lambda \in \Omega_P$ we define the function $E(P : \psi : \lambda) : G \to V_\tau$ by

$$E(P : \psi : \lambda)(x) = \sum_{v\in\mathscr{W}} E_{vHv^{-1}}(P : \mathrm{pr}_v\psi : \lambda)(xv^{-1}) \qquad (x \in G). \tag{8.6}$$

It is readily verified that this function belongs to $C^\infty(G/H : \tau)$. We will occasionally write $E(P : \psi : \lambda : x)$ for $E(P : \psi : \lambda)(x)$.

We will now relate the Eisenstein integral thus defined to matrix coefficients with H-fixed distribution vectors. For this we will use a suitable realization of the space $\mathscr{A}_{M,2}$. In analogy with (8.5) we define

$$\mathscr{A}_{M_0,2} := \oplus_{v\in\mathscr{W}}\, C^\infty(M_0/M_0 \cap vHv^{-1} : \tau_{M_0}).$$

In view of Lemma 8.1 applied with vHv^{-1} in place of H, for $v \in \mathscr{W}$, we see that

$$\mathscr{A}_{M_0,2} = \mathscr{A}_{M,2}.$$

For $\xi \in \widehat{M}_{0\mathrm{fu}}$ and $v \in \mathscr{W}$, we denote by $C^\infty_\xi(M_0/M_0 \cap vHv^{-1})$ the space of left M_0-finite functions in $C^\infty(M_0/M_0 \cap vHv^{-1})$ of isotopy type ξ. Furthermore, we denote by

$$C^\infty_\xi(M_0/M_0 \cap vHv^{-1} : \tau_{M_0}) \tag{8.7}$$

the intersection of $C^\infty(M_0/M_0 \cap vHv^{-1} : \tau_{M_0})$ with $C^\infty_\xi(M_0/M_0 \cap vHv^{-1}) \otimes V_\tau$. The direct sum of the spaces (8.7) for $v \in \mathscr{W}$ is denoted by $\mathscr{A}_{M_0,2,\xi}$. Then it follows that

$$\mathscr{A}_{M_0,2} = \oplus_{\xi \in \widehat{M}_{0\mathrm{fu}}} \mathscr{A}_{M_0,2,\xi}, \tag{8.8}$$

as an orthogonal direct sum with finitely many non-zero terms.

Similar definitions, with M in place of M_0, lead to spaces

$$C^\infty_\xi(M/M \cap vHv^{-1} : \tau^0_M), \tag{8.9}$$

equal to (8.7) in view of (8.2), for $v \in \mathscr{W}$. The orthogonal direct sum of (8.9) over $v \in \mathscr{W}$ is denoted by $\mathscr{A}_{M,2,\xi}$. Then obviously

$$\mathscr{A}_{M_0,2,\xi} = \mathscr{A}_{M,2,\xi}. \tag{8.10}$$

For $\xi \in \widehat{M}_{0\mathrm{fu}}$ we define $C(K : \xi : \tau)$ to be the space of functions $f : K \to \mathscr{H}_\xi \otimes V_\tau$ transforming according to the rule:

$$f(mk_0k) = [\xi(m) \otimes \tau(k)^{-1}] f(k_0), \qquad (k, k_0 \in K, m \in M_0 \cap K).$$

We recall from [10, Lemma 3, p. 528] that there exists a natural linear isomorphism

$$T \mapsto \psi_T, \quad C(K : \xi : \tau) \otimes \bar{V}(\xi) \xrightarrow{\simeq} \mathscr{A}_{M_0,2,\xi} = \mathscr{A}_{M,2,\xi}, \tag{8.11}$$

given by

$$(\psi_T)_v(m) = \langle f(e), \xi(m)\mathrm{pr}_v(\eta)\rangle, \quad (v \in \mathscr{W}), \tag{8.12}$$

for $T = f \otimes \eta \in C(K : \xi : \tau) \otimes \bar{V}(\xi)$ and $m \in M$. Moreover, $T \mapsto \sqrt{\dim \xi}\ \psi_T$ is an isometry.

The map $i^\#$ introduced in (4.11) is an isometric embedding $C(K : K \cap M_0 : \xi) \to C(K : M : \xi_M)$. Through tensoring with the identity map on V_τ it induces an isometric embedding $C(K : \xi : \tau) \to C(K : \xi_M : \tau_M)$ which we denote by $i^\#$ again.

Theorem 8.4. *Let $\xi \in \widehat{M}_{0\mathrm{fu}}$ and let $T = f \otimes \eta \in C(K : \xi : \tau) \otimes \bar{V}(\xi)$. Then for $x \in G$ and $\lambda \in \Upsilon_P$,*

$$E(P : \psi_T : \lambda : x) = \langle i^\# f, \pi_{P,\xi_M,\bar{\lambda}-\rho_{Ph}}(x) j(P : \xi_M : \bar{\lambda})(\eta)\rangle,$$

where $j(P : \xi_M : \bar{\lambda})(\eta)$ should be viewed as an element of $C^{-\infty}(K : M : \xi_M)$. Moreover, the indicated sesquilinear pairing is taken on the first tensor components of $i^\# f$.

Proof. The set Υ_P is the intersection of the sets $\Omega_{v,P}$, for $v \in \mathscr{W}$, by (7.24) and Lemma 7.11. In view of (8.6) and (7.23) it therefore suffices to restrict to the case that $\eta \in V(\xi, e)$ and prove the result under the (weaker) assumption that $\lambda \in \Omega_P$.

Write $\psi = \mathrm{pr}_e \psi_T$. Then it follows from the proof of Proposition 8.2 that, for x and λ as specified,

$$\begin{aligned} E_H(P : \psi_T : \lambda : x) &= \int_{H_P \backslash H} \varphi^*[h \mapsto \psi_{P,\lambda}(xh)\, dl_h(e)^{-1*}\omega] \\ &= \int_{H_P \backslash H} \psi^\vee_{P,\lambda}(hx^{-1})(e)\, dr_h(e)^{-1*}\omega. \end{aligned}$$

We now calculate the function $\psi^\vee_{P,\lambda}$ in this particular case. As it belongs to $C^\infty(P : \xi_M : -\lambda + \rho_{P\mathfrak{h}}) \otimes V_\tau$ it is sufficient to calculate its restriction to K. Since $\psi = \psi_T$, it follows from (8.12) that

$$\psi(m) = \psi_T(m) = \langle f(e), \xi(m)\eta \rangle.$$

This implies that

$$\psi_{P,\lambda}(k) = \tau(k)\langle f(e), \xi(e)\eta \rangle = \langle f(k^{-1}), \eta \rangle.$$

In turn, this implies that

$$\psi^\vee_{P,\lambda}(k)(e) = \langle f(k), \eta \rangle.$$

We write $[i^\# f]_{P,-\lambda+\rho_{P\mathfrak{h}}}$ for the extension of the function $i^\# f \in C(K : \xi_M : \tau_M)$ to a function in $C^\infty(P : \xi_M : -\lambda + \rho_{P\mathfrak{h}}) \otimes V_\tau$. Then

$$\psi^\vee_{P,\lambda}(x)(e) = \langle [i^\# f]_{P,-\lambda+\rho_{P\mathfrak{h}}}(x), \eta \rangle.$$

Thus, in view of (7.12) we find that

$$\begin{aligned} &E_H(P : \psi_T : \lambda : x) = \\ &\quad = \int_{H_P \backslash H} \big\langle [R(x^{-1})[i^\# f]_{P,-\lambda+\rho_{P\mathfrak{h}}}(h), \eta \big\rangle\, dr_h(e)^{-1*}\,\omega \\ &\quad = \big\langle [R(x^{-1})[i^\# f]_{P,-\lambda+\rho_{P\mathfrak{h}}}, j_H(P : \xi_M : \bar{\lambda})(\eta) \big\rangle \\ &\quad = \big\langle \pi_{P,\xi_M,-\lambda+\rho_{P\mathfrak{h}}}(x^{-1}) i^\# f, j_H(P : \xi_M : \bar{\lambda})(\eta) \big\rangle \\ &\quad = \big\langle i^\# f, \pi_{P,\xi_M,\bar{\lambda}-\rho_{P\mathfrak{h}}}(x) j_H(P : \xi_M : \bar{\lambda})(\eta) \big\rangle. \end{aligned}$$

The proof is complete. □

Corollary 8.5. *Let $P \in \mathscr{P}(A)$ and let $\psi \in \mathscr{A}_{M,2}$. Then the Eisenstein integral $E(P : \psi : \lambda)$ depends meromorphically on $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$ as a function with values in $C^\infty(G/H : \tau)$. As such, it is holomorphic on an open neighborhood of the set $\widehat{\Upsilon}_P$.*

Proof. The assertion about meromorphy follows from the previous result in view of (8.8) and the linear dependence of the Eisenstein integral on ψ. The statement about holomorphy now follows from Lemma 7.13 (b). $\square$

In the following it will sometimes be convenient to write $E(P:\lambda:x)\psi = E(P:\psi:\lambda:x)$ and to adopt the viewpoint that $E(P:\lambda)$ is a meromorphic $\mathrm{Hom}(\mathscr{A}_{M,2}, C^\infty(G:\tau))$-valued function of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

We proceed by relating the Eisenstein integrals defined above to the Eisenstein integrals introduced earlier in [6] and [10] for minimal $\sigma\theta$-stable parabolic subgroups.

Corollary 8.6. *Let $P \in \mathscr{P}_\sigma(A)$ and let P_0 be the unique parabolic subgroup from $\mathscr{P}_\sigma(A_{\mathrm{q}})$ containing P. Then*

$$E(P:\lambda) = E(P_0:\lambda) \tag{8.13}$$

as $\mathrm{Hom}\big(\mathscr{A}_{M_0,2}, C^\infty(G/H:\tau)\big)$*-valued meromorphic functions of* $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

Proof. Let $\xi \in \widehat{M}_{0\mathrm{fu}}$. Then it follows from Corollary 5.5 that

$$j(P:\xi_M:\lambda) = i^\# \circ j(P_0:\xi:\lambda).$$

Let $T = f\otimes\eta \in C(K:\xi:\tau)\otimes\overline{V}(\xi)$. Then it follows by [6, Lemma 4.2] and (4.11) that

$$\begin{aligned}
E(P_0:\lambda:x)\psi_T &= \langle f\,,\pi_{P_0,\xi,\bar\lambda}(x)\,j(P_0:\xi:\bar\lambda)(\eta)\rangle\\
&= \langle i^\# f\,, i^\#\pi_{P_0,\xi,\bar\lambda}(x)\,j(P_0:\xi:\bar\lambda)(\eta)\rangle\\
&= \langle i^\# f\,,\pi_{P,\xi_M,\bar\lambda-\rho_{Ph}}(x)\,j(P:\xi_M:\bar\lambda)(\eta)\rangle\\
&= E(P:\lambda:x)\psi_T.
\end{aligned}$$

$\square$

The Eisenstein integrals for parabolic subgroups from $\mathscr{P}(A)$ can be related to each other as follows.

Proposition 8.7. *Let $Q \in \mathscr{P}(A)$, $P \in \mathscr{P}_\sigma(A)$ and $P \succeq Q$. Then for all $\xi \in \widehat{M}_{0\mathrm{fu}}$, all $T \in C(K:\xi:\tau)\otimes\overline{V}(\xi)$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, we have*

$$E(Q:\lambda)\psi_T = E(P:\lambda)\psi_{[p^\#\circ A(Q:P:\xi_M:-\lambda+\rho_{Ph})^{-1}\circ i^\#\otimes I]T}. \tag{8.14}$$

Here, $p^\#$ is shorthand for the restriction of $p^\#\otimes I_{V_\tau}$ to the subspace $C(K:\xi_M:\tau_M)$ of $C^\infty(K:M:\xi_M)\otimes V_\tau$, see also (4.14). Likewise, the intertwining operator acts on the first tensor component in $C(K:M:\xi_M)\otimes V_\tau$.

Proof. By linearity it suffices to prove this for $T = f \otimes \eta$, with $f \in C(K:\xi:\tau)$ and $\eta \in \overline{V}(\xi)$. It follows from Theorem 8.4 and (7.25) that for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ we have

$$j(Q:\xi_M:\bar\lambda)(\eta) = A(P:Q:\xi_M:\bar\lambda - \rho_{P\mathrm{h}})^{-1} j(P:\xi_M:\bar\lambda)(\eta).$$

For $x \in G$ we now obtain (with the pairing taken on first tensor components)

$$\begin{aligned}
&E(Q:\lambda:x)\psi_T = \\
&= \langle i^\# f\,,\,\pi_{Q,\xi_M,\bar\lambda-\rho_{Q\mathrm{h}}}(x)\, j(Q:\xi_M:\bar\lambda)(\eta)\rangle \\
&= \langle i^\# f\,,\,A(P:Q:\xi_M:\bar\lambda-\rho_{P\mathrm{h}})^{-1}\pi_{P,\xi_M,\bar\lambda-\rho_{P\mathrm{h}}}(x)\, j(P:\xi_M:\bar\lambda)(\eta)\rangle \\
&= \langle A(Q:P:\xi_M:-\lambda+\rho_{P\mathrm{h}})^{-1} i^\# f\,,\,\pi_{P,\xi_M,\bar\lambda-\rho_{P\mathrm{h}}}(x)\, j(P:\xi_M:\bar\lambda)(\eta)\rangle \\
&= \langle p\circ A(Q:P:\xi_M:-\lambda+\rho_{P\mathrm{h}})^{-1} i^\# f\,,\,\pi_{P,\xi_M,\bar\lambda-\rho_{P\mathrm{h}}}(x)\, j(P:\xi_M:\bar\lambda)(\eta)\rangle
\end{aligned} \tag{8.15}$$

The last identity follows by application of Corollary 5.6. Finally, since $p = i^\# \circ p^\#$, the expression in (8.15) equals the expression on the right-hand side of (8.14), in view of Theorem 8.4. $\square$

Let $Q, P \in \mathscr{P}(A)$ be as in Proposition 8.7. Then motivated by the proposition, we define the C-function $C(Q:P:\lambda)$ to be the unique $\mathrm{End}(\mathscr{A}_{M,2})$-valued meromorphic function of $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ such that

$$C(Q:P:\lambda)\psi_T = \psi_{[p^\# \circ A(Q:P:\xi_M:-\lambda+\rho_{P\mathrm{h}})^{-1}\circ i^\# \otimes I]T}$$

for all $\xi \in \widehat{M}_{0\mathrm{fu}}$, all $T \in C(K:\xi:\tau)\otimes \overline{V}(\xi)$ and generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Then (8.14) may be abbreviated as

$$E(Q:\lambda) = E(P:\lambda)C(P:Q:\lambda). \tag{8.16}$$

The following result is a variation of a result of Harish-Chandra, see [19, Lemma 3, p. 47]. The proof given below follows a different strategy, which also works in the setting of [19].

Proposition 8.8. *Let $Q \in \mathscr{P}(A)$ and $P \in \mathscr{P}_\sigma(A)$ such that $P \succeq Q$. Then the meromorphic function $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \ni \lambda \mapsto \det C(Q:P:\lambda)$ is not identically zero.*

Before proceeding with the proof of this proposition, we first list a corollary.

Corollary 8.9. *Let $Q \in \mathscr{P}(A)$ and $P \in \mathscr{P}_\sigma(A)$ such that $P \succeq Q$. Then the endomorphism $C(Q:P:\lambda) \in \mathrm{End}(\mathscr{A}_{M,2})$ is invertible for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and $\lambda \mapsto C(Q:P:\lambda)^{-1}$ is a meromorphic $\mathrm{End}(\mathscr{A}_{M,2})$-valued meromorphic function on $\mathfrak{a}_\mathrm{q}$.*

Proof. This follows from Proposition 8.8 by application of Cramer's rule for the inversion of a matrix. $\square$

The following lemma will play an important role in the proof of Proposition 8.8.

Lemma 8.10. *Let $P \in \mathscr{P}_\sigma(A)$ and let P_0 be the unique minimal $\sigma\theta$-stable parabolic subgroup in $\mathscr{P}_\sigma(A_\mathrm{q})$ containing P. Then the following diagram commutes:*

$$\begin{array}{ccc} C(K : K\cap M_0 : \xi) & \stackrel{A(\bar{P}_0:P_0:\xi:\lambda)}{\longrightarrow} & C(K : K\cap M_0 : \xi) \\ p^{\#}\uparrow & & \uparrow p^{\#} \\ C(K : M : \xi_M) & \stackrel{A(\sigma(P):P:\xi_M:\lambda+\rho_{Ph})}{\longrightarrow} & C(K : M : \xi_M) \end{array}$$

for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

Proof. Let $P_{M_0} = P\cap M_0$. Then it follows that $P = P_{M_0}N_P$ and $\sigma P = P_{M_0}N_{\sigma P}$. Furthermore, by the assertion at the end of the proof of Theorem 4.2 in [5, p. 373], with $P_1 = \bar{P}_0$, $P_2 = P_0$, $(P_1)_\mathfrak{p} = \sigma(P)$ and $(P_2)_\mathfrak{p} = P$, it follows that the following diagram commutes for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$,

$$\begin{array}{ccc} C(K : K\cap M_0 : \xi) & \stackrel{A(P_0:\bar{P}_0:\xi:-\bar{\lambda})}{\longleftarrow} & C(K : K\cap M_0 : \xi) \\ i^{\#}\downarrow & & \downarrow i^{\#} \\ C(K : M : \xi_M) & \stackrel{A(P:\sigma(P):\xi_M:-\bar{\lambda}-\rho_{Ph})}{\longleftarrow} & C(K : M : \xi_M). \end{array}$$

The desired result now follows by taking adjoints with respect to the given equivariant sesquilinear pairings on the spaces involved. □

Proof of Proposition 8.8. Let $\xi \in \widehat{M}_{0\mathrm{fu}}$ and let $\vartheta \subset \widehat{K}$ be a finite set of K-types. Then it suffices to show that the restricted operator

$$p^{\#}\circ A(Q : P : \xi_M : -\lambda+\rho_{P\mathrm{h}})^{-1}\circ i^{\#}|_{C(K:K\cap M_0:\xi)_\vartheta} \tag{8.17}$$

has determinant not identically zero. For this it suffices to show that the composition of $A(\bar{P}_0 : P_0 : \xi : -\lambda)$ with (8.17) has determinant not-identically zero. By Lemma 8.10 this composition may be rewritten as

$$p^{\#}\circ A(\sigma(P) : P : \xi : -\lambda+\rho_{P\mathrm{h}})\circ A(Q : P : \xi_M : -\lambda+\rho_{P\mathrm{h}})^{-1}\circ i^{\#}|_{C(K:K\cap M_0:\xi)_\vartheta}. \tag{8.18}$$

Since $\Sigma(\sigma(P)\cap\Sigma(P) = \Sigma(P)\cap\mathfrak{a}_\mathrm{h}^* \subset \Sigma(Q)\cap\Sigma(P)$, it follows by the usual product decomposition of the standard intertwining operators, see [22, Cor. 7.7], that (8.18) equals

$$p^{\#}\circ A(\sigma P : Q : \xi_M : -\lambda+\rho_{P\mathrm{h}})\circ i^{\#}|_{C(K:K\cap M_0:\xi)_\vartheta}. \tag{8.19}$$

Thus it suffices to show that the determinant of the linear endomorphism of $C(K : K\cap M_0 : \xi)_\vartheta$ given in (8.19) is not identically zero as a meromorphic function of λ. Now this is an immediate consequence of the following result. □

Lemma 8.11. *Let $Q, R \in \mathscr{P}(A)$ be such that $\Sigma(\bar{R})\cap\Sigma(Q) \subset \Sigma(Q,\sigma\theta)$. Then there exists an element $\eta \in \mathfrak{a}_\mathrm{q}^*$ such that $\langle\eta\,,\,\alpha\rangle > 0$ for all $\alpha \in \Sigma(\bar{R})\cap\Sigma(Q)$. Let η be such an element and $d := \dim(N_R\cap\bar{N}_Q)$. Then there exists a constant $c > 0$ such that for every $\mu \in \mathfrak{a}_\mathbb{C}^*$ and all $f \in C(K : M : \xi_M)$*

$$\lim_{t\to\infty} t^{d/2}A(R : Q : \xi_M : \mu+t\eta)f = cf.$$

in $C(K : M : \xi_M)$.

Proof. Since $\Sigma(Q)$ is a positive system for $\Sigma(\mathfrak{g},\mathfrak{a})$, there exists $\xi \in \mathfrak{a}^*$ such that $\langle \xi\,,\,\alpha\rangle > 0$ for $\alpha \in \Sigma(Q)$. Let $\eta = \xi + \sigma\theta\xi$, then $\langle \eta\,,\,\alpha\rangle > 0$ for $\alpha \in \Sigma(Q,\alpha\theta)$. Thus, η satisfies the requirements. Let η be any such element.

Replacing μ by $\mu + t_0\eta$ for a suitable $t_0 > 0$ we see that we may as well assume that $\mathrm{Re}\,\langle \mu\,,\,\alpha\rangle > 0$ for all $\alpha \in \Sigma(\bar{R}) \cap \Sigma(Q)$. In this case, we see that for all $t \geq 0$ the intertwining operator is given by the absolutely convergent integral

$$\begin{aligned}
[A(R:Q:\xi_M:\mu+t\eta)f](k) &= \int_{N_R\cap\bar{N}_Q} f_t(\bar{n}^{-1}k)d\bar{n} \\
&= \int_{N_R\cap\bar{N}_Q} e^{-t\eta\mathscr{H}_Q(\bar{n})}e^{(-\mu-\rho_Q)\mathscr{H}_Q(\bar{n})}f(\kappa_Q(\bar{n})^{-1}k)\,d\bar{n}.
\end{aligned}$$

Here f_t denotes the extension of f to an element of $C^\infty(Q:\xi_M:\mu+t\eta)$. Moreover, the analytic maps $\mathscr{H}_Q : G \to \mathfrak{a}$ and $\kappa_Q : G \to K$ are defined by

$$x \in \kappa_Q(x)\exp\mathscr{H}_Q(x)N_Q, \qquad (x \in G),$$

in accordance with the Iwasawa decomposition $G = KAN_Q$.

By using the properties of the function $h = \eta\mathscr{H}_Q|_{N_R\cap\bar{N}_Q}$ stated in Lemma 8.12 below we will be able to determine the asymptotic behavior for $t \to \infty$ by using the real version of the method of stationary phase.

It follows from Lemma 8.12 (a) that $h \geq 0$. Hence, the intertwining operator is a continuous linear endomorphism of $C(K:M:\xi_M)$, with operator norm bounded by

$$\|A(R:Q:\xi_M:\mu+t\eta)\| \leq \int_{N_R\cap\bar{N}_Q} e^{(-\mathrm{Re}\,\mu-\rho_Q)\mathscr{H}_Q(\bar{n})}\,d\bar{n}.$$

It follows from Lemma 8.12 (b) that there exists an open neighborhood V of 0 in $\mathbb{R}^d$ and an open embedding $\varphi : \mathbb{R}^d \to N_R \cap \bar{N}_Q$, sending 0 to e such that

$$h(\varphi(x)) = \langle Sx\,,\,x\rangle$$

with S a positive definite matrix. Let U be an open neighborhood of e in $N_R \cap \bar{N}_Q$ with closure contained in $\varphi(V)$, and let $r > 0$ be as in condition (c) of the mentioned lemma. Fix $\chi \in C_c^\infty(\varphi(V))$ such that $\chi = 1$ on a neighborhood of the closure of U. Then

$$A(R:Q:\xi_M:\mu+t\eta)f = I_t(f) + R_t(f)$$

with

$$I_t(f) = \int_{N_R\cap\bar{N}_Q} e^{-th(\bar{n})}e^{(-\mu-\rho_Q)\mathscr{H}_Q(\bar{n})}f(\kappa(\bar{n})^{-1}k)\chi(\bar{n})\,d\bar{n}.$$

The remainder term $R_t(f)$ is given by the same integral but with $\chi(\bar{n})$ replaced by $1-\chi(\bar{n})$. As the latter function is zero on U, it follows from the estimate in (b) that

$$\|R_t(f)\| \leq C_1 e^{-tr}\|f\|,$$

with C_1 a positive constant independent of f and t. Accordingly, we may ignore this term and concentrate on $I_t(f)$. By substituting $\varphi(x)$ for $\bar{n}$ we obtain

$$I_t(f) = \int_{\mathbb{R}^n} e^{-t\langle Sx\,,x\rangle} e^{(-\mu-\rho_Q)\mathscr{H}_Q(\varphi(x))} f(\kappa(\varphi(x))^{-1}k)\chi(\varphi(x))J(x)\,dx,$$

where $J(x)$ is a Jacobian. Substituting $t^{-1/2}x$ for x and taking the limit for $t \to \infty$, we see that

$$t^{d/2} I_t(f) \to \int_{\mathbb{R}^d} e^{-\langle Sx\,,x\rangle} f(k)\,J(0)\,dx,$$

uniformly in k. This establishes the result with

$$c = J(0)\int_{\mathbb{R}^d} e^{-\langle Sx\,,x\rangle}\,dx.$$

□

Lemma 8.12. *Let $Q,R \in \mathscr{P}(A)$ and let $\eta \in \mathfrak{a}^*$ be such that $\langle \eta\,,\alpha\rangle > 0$ for all $\alpha \in \Sigma(\bar{R})\cap\Sigma(Q)$. Let $\mathscr{H}_Q : G \to \mathfrak{a}$ be the Iwasawa map determined by $x \in K\exp\mathscr{H}_Q(x)N_Q$, for $x \in G$. Then the function $h = \eta\mathscr{H}_Q|_{N_R\cap\bar{N}_Q}$ has the following properties,*

(a) $h \geq 0$;

(b) *h has an isolated critical point at e with positive definite Hessian;*

(c) *for each open neighborhood U of e in $N_R\cap\bar{N}_Q$ there exists a constant $r > 0$ such that*

$$\bar{n} \in (N_R\cap\bar{N}_Q)\setminus U \implies h(\bar{n}) > r.$$

Proof. For each indivisible root $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ we write $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, $N_\alpha = \exp(\mathfrak{n}_\alpha)$, $\bar{\mathfrak{n}}_\alpha = \theta\mathfrak{n}_\alpha$ and $\bar{N}_\alpha := \exp(\bar{\mathfrak{n}}_\alpha)$. Furthermore, we write $\mathfrak{g}(\alpha)$ for the split rank one subalgebra generated by $\mathfrak{n}_\alpha + \bar{\mathfrak{n}}_\alpha$ and $G(\alpha)$ for the associated analytic subgroup of G. Let $\mathscr{H}_\alpha : G(\alpha) \to \mathfrak{a}\cap\mathfrak{g}(\alpha) = (\ker\alpha)^\perp$ be the Iwasawa projection associated with the Iwasawa decomposition $G(\alpha) = K\cap G(\alpha))(A\cap G(\alpha))N_\alpha$. Then $\mathscr{H}_\alpha = \mathscr{H}_Q|_{G(\alpha)}$.

Let $\alpha_1,\ldots,\alpha_k$ be the indivisible positive roots in $\Sigma(\bar{R})\cap\Sigma(Q)$. Then by the method of S.G. Gindikin and F.I. Karpelevic [17] (see [22, Thm. 7.6] for the version for intertwining operators), there exists a diffeomorphism $\psi : \bar{N}_{\alpha_1}\times\cdots\bar{N}_{\alpha_k} \to N_R\cap\bar{N}_Q$ such that

$$h(\psi(\bar{n}_1,\ldots,\bar{n}_k)) = \sum_{j=1}^{k}\eta\mathscr{H}_{\alpha_j}(\bar{n}_j).$$

To show that h has properties (a), (b) and (c), it suffices to show that each of the functions $h_j = \eta\circ\mathscr{H}_{\alpha_j}|_{\bar{N}_{\alpha_j}}$ has these properties, with $\bar{N}_{\alpha_j}$ in place of $N_R\cap\bar{N}_Q$.

Let $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ be any indivisible root such that $\langle\eta\,,\alpha\rangle > 0$. Then it suffices to show that $h_\alpha : \bar{N}_\alpha \to \mathbb{R}, \bar{n} \mapsto \eta\mathscr{H}_\alpha(\bar{n})$ has properties (a) and (b) with $\bar{N}_\alpha$ in place of $N_R\cap\bar{N}_Q$. The function h_α can be explicitly computed through $\mathrm{SU}(2,1)$-reduction, see [21, Thm. IX.3.8]. From the explicit expression given in [21], properties (a), (b) and (c) are readily verified. □

We will now describe the asymptotic behavior of the various Eisenstein integrals, using the established relations (8.13) and (8.16) between them.

For a parabolic subgroup $R \in \mathscr{P}_\sigma(A_\mathrm{q})$ and for $v \in \mathscr{W}$ we define the functions

$$\Phi_{R,v}(\lambda : \cdot) : A_\mathrm{q}^+(R) \to \mathrm{End}(V_\tau^{M_0 \cap K \cap vHv^{-1}})$$

as in [9, Lemma 10.3]. These functions are smooth on the chamber $A_\mathrm{q}^+(R)$ and as such depend meromorphically on the parameter $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Moreover, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ they have an absolutely converging series expansion of the form

$$\Phi_{R,v}(\lambda : a) = a^{-\rho_R} \sum_{\mu \in \mathbb{N}\Sigma^+(R)} a^{-\mu} \Gamma_{R,\mu}(\lambda),$$

where the $\Gamma_{R,\mu}$ are meromorphic $\mathrm{End}(V_\tau^{M_0 \cap K \cap vHv^{-1}})$-valued functions and $\Gamma_{R,0} = I$.

Let $P_0 \in \mathscr{P}_\sigma(A_\mathrm{q})$ then by [9, Thm. 11.1] and (8.10), there exist unique $\mathrm{End}(\mathscr{A}_{M,2})$-valued meromorphic functions $C_{R|P_0}(s : \cdot)$ on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ such that for all $\psi \in \mathscr{A}_{M,2}$ and each $v \in \mathscr{W}$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ we have

$$E(P_0 : \lambda : av)\psi = \sum_{s \in W(\mathfrak{a}_\mathrm{q})} \Phi_{R,v}(s\lambda : a)[C_{R|P_0}(s : \lambda)\psi]_v(e), \qquad (a \in A_\mathrm{q}^+(R)).$$

Here $W(\mathfrak{a}_\mathrm{q})$ denotes the Weyl group of the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$.

Theorem 8.13. *Let $Q \in \mathscr{P}(A)$ and $R \in \mathscr{P}_\sigma(A_\mathrm{q})$. Then there exist unique meromorphic* $\mathrm{End}(\mathscr{A}_{M,2})$*-valued meromorphic functions $C_{R|Q}(s : \cdot)$ on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, for $s \in W(\mathfrak{a}_\mathrm{q})$, such that for all $\psi \in \mathscr{A}_{M,2}$, each $v \in \mathscr{W}$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ we have*

$$E(Q : \lambda : av)\psi = \sum_{s \in W(\mathfrak{a}_\mathrm{q})} \Phi_{R,v}(s\lambda : a)[C_{R|Q}(s : \lambda)\psi]_v(e), \qquad (a \in A_\mathrm{q}^+(R)).$$

These meromorphic C-functions are generically pointwise invertible, with meromorphic inverses.

Proof. Uniqueness follows by uniqueness of asymptotics, see, e.g., [18, p. 305, Cor.] for details.

For the remaining statements on existence and invertibility, we first consider the case that Q is $\mathfrak{q}$-extreme, i.e., $Q \in \mathscr{P}_\sigma(A)$. Then there exists a unique $Q_0 \in \mathscr{P}_\sigma(A_\mathrm{q})$ containing Q. By applying Corollary 8.6 and the preceding discussion we find that

$$C_{R|Q}(s : \lambda) = C_{R|Q_0}(s : \lambda)$$

satisfies the asymptotic requirements. Invertibility follows from [6, Cor. 15.11].

We now assume that $Q \in \mathscr{P}(A)$ is general. Then there exists a $P \in \mathscr{P}_\sigma(A)$ such that $P \succeq Q$.

By Proposition 8.7 and (8.14) we have

$$E(P : \lambda : x) = E(Q : \lambda : x) \circ C(Q : P : \lambda).$$

In view of Proposition 8.8 we see that

$$C_{R|Q}(s:\lambda) = C_{R|P}(s:\lambda) \circ C(Q:P:\lambda)^{-1}$$

satisfies the asymptotic requirements. The invertibility requirements now follow from the invertibility of $C_{R|P}(s:\lambda)$, established earlier in this proof. $\square$

Corollary 8.14. *Let $P, Q \in \mathscr{P}(A)$. Then there exists a unique meromorphic* $\mathrm{End}(\mathscr{A}_{M,2})$*-valued function $C(P:Q:\cdot)$ on $\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$ such that*

$$E(P:\lambda:x) = E(Q:\lambda:x) \circ C(Q:P:\lambda) \tag{8.20}$$

for all $x \in G/H$ and generic $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$. Furthermore, the following identities are valid as identities of meromorphic $\mathrm{End}(\mathscr{A}_{M,2})$*-valued functions in $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$.*

(a) $C(Q:P:\lambda) = C_{R|Q}(s:\lambda)^{-1} C_{R|P}(s:\lambda), \qquad (s \in W(\mathfrak{a}_\mathfrak{q}),\ R \in \mathscr{P}_\sigma(A_\mathfrak{q}))$;

(b) $C(P_1:P_2:\lambda)C(P_2:P_3:\lambda) = C(P_1:P_3:\lambda), \qquad (P_1, P_2, P_3 \in \mathscr{P}(A))$;

(c) $C(P:Q:\lambda)C(Q:P:\lambda) = C(Q:P:\lambda)C(P:Q:\lambda) = I$.

Proof. Uniqueness follows from Theorem 8.13 combined with uniqueness of asymptotics. We will first establish the existence for $P, Q \in \mathscr{P}_\sigma(A)$. Let P_0, Q_0 be the unique minimal $\sigma\theta$-stable parabolic subgroups in $\mathscr{P}_\sigma(A_\mathfrak{q})$ with $P_0 \supset P$ and $Q_0 \supset Q$. Then by [10, (42) & (70)] there exists a meromorphic function $a : \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^* \to \mathrm{End}(\mathscr{A}_{M,2})$ such that $E(P_0:\lambda) = E(Q_0:\lambda)a(\lambda)$. In view of Corollary 8.6 it follows that (8.20) is valid with $C(Q:P:\lambda) = a(\lambda)$.

By using Proposition 8.7, (8.14) and Corollary 8.9 the existence of $C(Q:P:\lambda)$ can now be inferred for arbitrary $P, Q \in \mathscr{P}(A)$.

Now that the existence has been established, (a) follows from Theorem 8.13 combined with uniqueness of asymptotics. Finally, (b) and (c) follow from the established uniqueness of the C-functions involved. $\square$

9 The case of the group

In this section we will consider the case of the group, viewed as a symmetric space, and compare our definition of the Eisenstein integral for a minimal parabolic subgroup with the one given by Harish-Chandra [20].

Let ${}^\backprime G$ be a group of the Harish-Chandra class, and let $G = {}^\backprime G \times {}^\backprime G$ and H the diagonal in G. Then H equals the fix point group of the involution $\sigma : G \to G$ given by $\sigma(x,y) = (y,x)$. The map $m : (x,y) \mapsto xy^{-1}$ induces a diffeomorphism $G/H \to {}^\backprime G$ which is equivariant for the action of G on G/H by left translation and the action on ${}^\backprime G$ by left times right translation. Accordingly, pull-back by m induces a G-equivariant topological linear isomorphism $m^* : C^\infty({}^\backprime G) \to C^\infty(G/H)$.

We fix a Cartan involution ${}^\backprime\theta$ for ${}^\backprime G$. Let ${}^\backprime\mathfrak{g} = {}^\backprime\mathfrak{k} \oplus {}^\backprime\mathfrak{p}$ be the associated infinitesimal Cartan decomposition and let ${}^\backprime\mathfrak{a}$ be a fixed choice of a maximal abelian subspace of ${}^\backprime\mathfrak{p}$. Then $\theta = {}^\backprime\theta \times {}^\backprime\theta$ is a Cartan involution for G which commutes with σ. The associated Cartan decomposition is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = {}^\backprime\mathfrak{k} \times {}^\backprime\mathfrak{k}$ and $\mathfrak{p} = {}^\backprime\mathfrak{p} \times {}^\backprime\mathfrak{p}$. Furthermore, $\mathfrak{a} = {}^\backprime\mathfrak{a} \times {}^\backprime\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.

The infinitesimal involution σ on $\mathfrak{g} = {}^\backprime\mathfrak{g} \times {}^\backprime\mathfrak{g}$ is given by $(X,Y) \mapsto (Y,X)$, so that its $+1$ eigenspace $\mathfrak{h}$ equals the diagonal of $\mathfrak{g}$, whereas the -1-eigenspace $\mathfrak{q}$ consists of the elements $(X,-X)$, $X \in {}^\backprime\mathfrak{g}$. It follows that $\mathfrak{p} \cap \mathfrak{q} = \{(X,-X) \mid X \in {}^\backprime\mathfrak{p}\}$, and that the subspace

$$\mathfrak{a}_{\mathrm{q}} := \{(X,-X) \mid X \in {}^\backprime\mathfrak{a}\}$$

is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. Furthermore, $\mathfrak{a} = \mathfrak{a}_{\mathrm{h}} \oplus \mathfrak{a}_{\mathrm{q}}$, where $\mathfrak{a}_{\mathrm{h}} = \{(X,X) \mid X \in {}^\backprime\mathfrak{a}\} = \mathfrak{a} \cap \mathfrak{h}$. At the level of groups we accordingly have $A = A_{\mathrm{h}}A_{\mathrm{q}}$, where $A_{\mathrm{h}} = A \cap H = \{(a,a) \mid a \in {}^\backprime A\}$ and $A_{\mathrm{q}} = \{(a,a^{-1}) \mid a \in {}^\backprime A\}$. The root system Σ of $\mathfrak{a}$ in $\mathfrak{g}$ equals $({}^\backprime\Sigma \times \{0\}) \cup (\{0\} \times {}^\backprime\Sigma)$, where ${}^\backprime\Sigma$ denotes the root system of ${}^\backprime\mathfrak{a}$ in ${}^\backprime\mathfrak{g}$. The associated root spaces are given by

$$\mathfrak{g}_{(\alpha,0)} = \mathfrak{g}_\alpha \times \{0\}, \quad \text{and} \quad \mathfrak{g}_{(0,\beta)} = \{0\} \times \mathfrak{g}_\beta, \qquad (\alpha,\beta \in {}^\backprime\Sigma).$$

The positive systems for Σ are the sets of the form $(\Pi_1 \times \{0\}) \cup (\{0\} \times \Pi_2$ where Π_1, Π_2 are positive systems for ${}^\backprime\Sigma$. Accordingly,

$$\mathscr{P}(A) = \{{}^\backprime P \times {}^\backprime Q \mid {}^\backprime P, {}^\backprime Q \in \mathscr{P}({}^\backprime A)\}.$$

Let ${}^\backprime M$ denote the centralizer of ${}^\backprime A$ in ${}^\backprime K$. Then the centralizer of A in K is given by $M = {}^\backprime M \times {}^\backprime M$ and we see that the θ-stable Levi component of any parabolic in $\mathscr{P}(A)$ is equal to MA.

Our first objective is to give a suitable description of the H-fixed distribution vector $j(R:\xi:\lambda)(\eta)$, for $R = {}^\backprime P \times {}^\backprime Q$ a minimal parabolic subgroup from $\mathscr{P}(A)$, for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, and for $\xi \in \widehat{M}$ such that the space $V(\xi)$, defined as in (4.8), is non-trivial.

We observe that $N_K(\mathfrak{a}_{\mathrm{q}})$ and $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ have the same image in $\mathrm{GL}(\mathfrak{a}_{\mathrm{q}})$, so that $\mathscr{W}$, defined as in (4.6), consists of the identity element $e = ({}^\backprime e, {}^\backprime e)$. It follows that $V(\xi) = V(\xi, e)$ as in (4.8), so that

$$V(\xi) = \mathscr{H}_\xi^{H_M}. \tag{9.1}$$

Thus, $V(\xi) \neq 0$ if and only if ξ has a non-trivial H_M-fixed vector. The set of such (classes of) irreducible representations of M is denoted by $\widehat{M}_{H_M}$.

If $\xi \in \widehat{M}_{H_M}$, then

$$\xi \simeq {}^\backprime\xi \,\widehat{\otimes}\, {}^\backprime\xi^\vee, \tag{9.2}$$

for an irreducible unitary representation ${}^\backprime\xi$ of ${}^\backprime M$ in a finite dimensional Hilbert space $\mathscr{H}_{{}^\backprime\xi}$. Using the canonical identification

$$\mathscr{H}_{{}^\backprime\xi} \otimes \mathscr{H}_{{}^\backprime\xi}^* \simeq \mathrm{End}(\mathscr{H}_{{}^\backprime\xi}) \tag{9.3}$$

we shall model ξ as the representation in $\mathscr{H}_\xi := \mathrm{End}(\mathscr{H}_{{}^\backprime\xi})$ given by

$$\xi(m_1, m_2)T = {}^\backprime\xi(m_1) \circ T \circ {}^\backprime\xi(m_2)^{-1},$$

for $T \in \mathrm{End}(\mathscr{H}_{{}^\backprime\xi})$ and $m_1, m_2 \in {}^\backprime M$. In particular, we see that with this convention,

$$V(\xi) = \mathbb{C}I_{{}^\backprime\xi}.$$

The space $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ is identified with the subspace of $\mathfrak{a}_{\mathbb{C}}^*$ consisting of linear functionals on $\mathfrak{a}_{\mathbb{C}}^*$ of the form $({}^\backprime\lambda, -{}^\backprime\lambda) : (X,Y) \mapsto {}^\backprime\lambda(X) - {}^\backprime\lambda(Y)$. We agree to write

$$\lambda = ({}^\backprime\lambda, -{}^\backprime\lambda), \qquad ({}^\backprime\lambda \in {}^\backprime\mathfrak{a}_{\mathbb{C}}^*).$$

As in Section 3, we write $C^{\pm\infty}(K : \xi)$ for $C^{\pm\infty}(K : M : \xi)$ and $C^{\pm\infty}({}^\backprime K : {}^\backprime\xi)$ for $C^{\pm\infty}({}^\backprime K : {}^\backprime M : {}^\backprime\xi)$. Then as indicated in Section 3, we have topological linear isomorphisms

$$C^{-\infty}(K : \xi) \simeq C^\infty(K : \xi)' \quad \text{and} \quad C^{-\infty}({}^\backprime K : {}^\backprime\xi) \simeq C^\infty({}^\backprime K : {}^\backprime\xi)',$$

which restricted to the subspaces of smooth functions are induced by the pairings (3.6) for (K, ξ) and $({}^\backprime K, {}^\backprime\xi)$.

We now consider the topological linear isomorphism

$$\Phi : C^{-\infty}(K : \xi) \ \xrightarrow{\simeq}\ \mathrm{Hom}(C^\infty({}^\backprime K : {}^\backprime\xi), C^{-\infty}({}^\backprime K : {}^\backprime\xi))$$

determined by the Schwartz kernel theorem. It is given by

$$\langle \Phi(h)(f)\,,g\rangle = \langle h\,, g \otimes f\rangle,$$

for $h \in C^{-\infty}(K : \xi)$, $f \in C^\infty({}^\backprime K : {}^\backprime\xi)$ and $g \in C^\infty({}^\backprime K : {}^\backprime\xi^\vee)$, with $g \otimes f$ viewed as an element of $C^\infty(K : \xi^\vee)$.

According to the compact picture explained in Section 3, we may identify Φ with a uniquely determined topological linear isomorphism

$$\Phi_\lambda : C^{-\infty}(R : \xi : \lambda) \xrightarrow{\simeq} \mathrm{Hom}(C^\infty({}^\backprime Q, {}^\backprime\xi, {}^\backprime\lambda), C^{-\infty}({}^\backprime P, {}^\backprime\xi, {}^\backprime\lambda)).$$

The isomorphism Φ_λ is readily seen to be G-equivariant, by G-equivariance of the pairings involved in the definition of Φ, for the appropriate principal series representations. It maps the H-invariants in the space on the left to the subspace of ${}^\backprime G$-intertwining operators on the right.

We write $\langle\,\cdot\,,\,\cdot\,\rangle_\xi$ for the K-equivariant pre-Hilbert structure on $C^\infty(K : \xi)$ given by (3.2) and $\langle\,\cdot\,,\,\cdot\,\rangle_{{}^\backprime\xi}$ for the similar ${}^\backprime K$-equivariant pre-Hilbert structure on $C^\infty({}^\backprime K : {}^\backprime\xi)$. The latter structure extends to continuous sesquilinear pairings $C^{\pm\infty}({}^\backprime K : {}^\backprime\xi) \times C^{\mp\infty}({}^\backprime K : {}^\backprime\xi) \to \mathbb{C}$, also denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_{{}^\backprime\xi}$. As $C^\infty({}^\backprime K : {}^\backprime\xi)$ is a Montel space, it is reflexive, and we may take adjoints with respect to these pairings. Accordingly, given a continuous linear operator $T : C^\infty({}^\backprime K : {}^\backprime\xi) \to C^{-\infty}({}^\backprime K : {}^\backprime\xi)$ we define the continuous linear operator $T^* : C^\infty({}^\backprime K : {}^\backprime\xi) \to C^{-\infty}({}^\backprime K : {}^\backprime\xi)$ by

$$\langle T^* f\,, g\rangle_{{}^\backprime\xi} = \langle f\,, Tg\rangle_{{}^\backprime\xi}, \qquad (f, g \in C^\infty({}^\backprime K : {}^\backprime\xi)).$$

Lemma 9.1. *Let $F \in C^\infty(K:\xi)$ and let $T: C^\infty({}^\backprime K : {}^\backprime\xi) \to C^\infty({}^\backprime K : {}^\backprime\xi)$ be a continuous linear operator. Then*

$$\langle F \,,\, \Phi^{-1}(T^*)\rangle_\xi = \int_{{}^\backprime K} \mathrm{tr}_{{}^\backprime\xi}\, [(T\otimes I)F({}^\backprime k, {}^\backprime k)]\, d^\backprime k. \tag{9.4}$$

Proof. We first consider the isomorphism $\varphi: C^{-\infty}(K) \to \mathrm{Hom}(C^\infty({}^\backprime K), C^{-\infty}({}^\backprime K))$ given by the Schwartz kernel isomorphism. Let f_j denote an $L^2({}^\backprime K)$-orthonormal basis subordinate to the decomposition into the finite dimensional ${}^\backprime K$-isotypical components with respect to the left regular representation. Then for each smooth function $f \in C^\infty({}^\backprime K)$ we have $f = \sum_j \langle f \,,\, f_j\rangle_2 f_j = \sum_j \langle f \,,\, \bar{f_j}\rangle f_j$ with convergence in $L^2({}^\backprime K)$. Here index 2 indicates that the pairing corresponds to the sesquilinear L^2-inner product. It follows that for each K-finite function $F \in C^\infty(K)$ we have

$$\langle F \,,\, \varphi^{-1}(I)\rangle_2 = \langle F \,,\, \sum_j f_j \otimes \bar{f_j}\rangle_2.$$

For $F = f_k \otimes f_l$ this gives $\langle F \,,\, \varphi^{-1}(I)\rangle_2 = \langle f_l \,,\, f_k\rangle = \int_K f_k(k) f_l(k)\, dk$. By continuous linearity and density this implies that

$$\langle F \,,\, \varphi^{-1}(I)\rangle_2 = \int_K F(k,k)\, dk, \qquad (F \in C^\infty({}^\backprime K \times {}^\backprime K)).$$

We next consider the natural isomorphism ψ from $\mathscr{H}_\xi = \mathscr{H}_{{}^\backprime\xi} \otimes \mathscr{H}_{{}^\backprime\xi}^\vee$ onto $\mathrm{End}(\mathscr{H}_{{}^\backprime\xi})$. Then it is readily verified that

$$\langle U \,,\, \psi^{-1}(I_{{}^\backprime\xi})\rangle_\xi = \mathrm{tr}_{\,\xi}(\psi(U)) \qquad (U \in \mathscr{H}_\xi).$$

Here the index ξ indicates that the natural sesquilinear inner product induced by the inner product on $\mathscr{H}_\xi$ is taken. We now consider the Schwartz kernel isomorphism $\widetilde{\Phi}$ from $C^{-\infty}(K, \mathscr{H}_\xi)$ onto $\mathrm{Hom}(C^\infty({}^\backprime K, \mathscr{H}_{{}^\backprime\xi}), C^{-\infty}({}^\backprime K, \mathscr{H}_{{}^\backprime\xi}))$. Then $\widetilde{\Phi}$ is identified with $\varphi \otimes \psi$ in a natural way. Thus, for $F \in C^\infty(K, \mathscr{H}_\xi))$ we have

$$\langle F \,,\, \widetilde{\Phi}^{-1}(I)\rangle_\xi = \int_{{}^\backprime K} \mathrm{tr}_{\,\xi}(\psi(F({}^\backprime k, {}^\backprime k))\, d^\backprime k. \tag{9.5}$$

Identifying $\mathscr{H}_\xi$ with $\mathrm{End}(\mathscr{H}_{{}^\backprime\xi})$ via ψ we agree to rewrite the above expression without the ψ. We view $C^\infty(K:\xi)$ as the space of $M = {}^\backprime M \times {}^\backprime M$-invariants in $C^\infty(K, \mathscr{H}_\xi)$. Likewise we view $C^{\pm\infty}({}^\backprime K : {}^\backprime\xi)$ as the space of ${}^\backprime M$-invariants in $C^{\pm\infty}({}^\backprime K, \mathscr{H}_{{}^\backprime\xi})$ (for the right action of ${}^\backprime M$ on $C^{\pm\infty}({}^\backprime K)$). The ${}^\backprime M$-equivariant inclusion maps and projection maps will be denoted by i and P respectively. Then $\Phi = \widetilde{\Phi} \circ \mathrm{i} = P \circ \widetilde{\Phi} \circ \mathrm{i}$, and we find that for $F \in C^\infty(K:\xi)$

$$\langle F \,,\, \Phi^{-1}(I)\rangle_2 = \langle F \,,\, \widetilde{\Phi}^{-1}(I)\rangle. \tag{9.6}$$

This implies (9.4) with $T = I$. To obtain the general formula, we note that for a continuous linear operator $T \in \mathrm{End}(C^\infty({}^\backprime K : {}^\backprime\xi))$ the Hermitian adjoint T^* is a continuous linear operator in $\mathrm{End}(C^{-\infty}({}^\backprime K : {}^\backprime\xi))$ and

$$\Phi\big((T^* \otimes I)u\big) = T^* \circ \Phi(u) \qquad \big(u \in C^{-\infty}(K:\xi)\big).$$

For $u = \Phi^{-1}(I)$ this yields

$$(T^* \otimes I)\Phi^{-1}(I) = \Phi(T^*).$$

It follows that

$$\langle F\,,\Phi^{-1}(T^*)\rangle = \langle F\,,(T^* \otimes I)\Phi^{-1}(I)\rangle = \langle (T \otimes I)(F)\,,\Phi^{-1}(I)\rangle.$$

Hence, (9.4) follows by application of (9.5) and (9.6). $\square$

Lemma 9.2. *Let ${}^\backprime P, {}^\backprime Q \in \mathscr{P}({}^\backprime A)$. Then for generic ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}_{\mathbb{C}}^*$,*

$$j({}^\backprime P \times {}^\backprime Q : \xi : \lambda)(I_{{}^\backprime\xi}) = \Phi_\lambda^{-1}(A({}^\backprime P : {}^\backprime Q : {}^\backprime\xi : {}^\backprime\lambda)). \tag{9.7}$$

Proof. Put $R = {}^\backprime P \times {}^\backprime Q$ as before. Then in the present case of the group, $\rho_{Rh} = 0$, so that the distribution vector on the left-hand side of (9.7) belongs to $C^{-\infty}(R : \xi : \lambda)$.

It follows from (7.25) applied with $Q' = {}^\backprime P \times {}^\backprime\bar{P}$ and $Q = {}^\backprime P \times {}^\backprime Q$ that

$$j({}^\backprime P \times {}^\backprime\bar{P} : \xi : \lambda) = [I \otimes A({}^\backprime\bar{P} : {}^\backprime Q : {}^\backprime\xi^\vee : -{}^\backprime\lambda)] \circ j({}^\backprime P \times {}^\backprime Q : \xi : \lambda).$$

Since $A({}^\backprime\bar{P} : {}^\backprime Q : {}^\backprime\xi^\vee : -{}^\backprime\lambda)$ has transpose $A({}^\backprime Q : {}^\backprime\bar{P} : {}^\backprime\xi : {}^\backprime\lambda)$ relative to the bilinear pairing $C^\infty({}^\backprime K : {}^\backprime\xi) \otimes C^\infty({}^\backprime K : {}^\backprime\xi^\vee) \to \mathbb{C}$, it follows that

$$\Phi_\lambda\big(j({}^\backprime P \times {}^\backprime\bar{P} : \xi : \lambda)(I_{{}^\backprime\xi})\big) = \Phi_\lambda\big(j({}^\backprime P \times {}^\backprime Q : \xi : \lambda)(I_{{}^\backprime\xi})\big) \circ A({}^\backprime Q : {}^\backprime\bar{P} : {}^\backprime\xi : {}^\backprime\lambda) \tag{9.8}$$

For ${}^\backprime Q = {}^\backprime\bar{P}$ the equality (9.7) has been established in [8, Lemma 1]. Combining this with (9.8) we find that

$$A({}^\backprime P : {}^\backprime\bar{P} : {}^\backprime\xi : {}^\backprime\lambda) = \Phi_\lambda(j({}^\backprime P \times {}^\backprime Q : \xi : \lambda)(I_{{}^\backprime\xi})) \circ A({}^\backprime Q : {}^\backprime\bar{P} : {}^\backprime\xi : {}^\backprime\lambda). \tag{9.9}$$

The intertwining operator on the left-hand side of (9.9) decomposes as the composition

$$A({}^\backprime P : {}^\backprime Q : {}^\backprime\xi : {}^\backprime\lambda) \circ A({}^\backprime Q : {}^\backprime\bar{P} : {}^\backprime\xi : {}^\backprime\lambda),$$

as an $\mathrm{End}(C^\infty(K : \xi))$-valued meromorphic function of ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}_{\mathbb{C}}^*$. Using the invertibility of the second intertwining operator for generic $\lambda \in {}^\backprime\mathfrak{a}_{\mathbb{C}}^*$ we obtain (9.7). $\square$

Corollary 9.3. *Let $f \in C^\infty(K : \xi)$. Then for generic ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}_{\mathbb{C}}^*$,*

$$\langle f\,, j({}^\backprime P \times {}^\backprime Q : \xi : -\bar{\lambda})(I_{{}^\backprime\xi})\rangle = \int_{{}^\backprime K} \mathrm{tr}_{{}^\backprime\xi}\big([A({}^\backprime Q : {}^\backprime P : {}^\backprime\xi : {}^\backprime\lambda) \otimes I)f]({}^\backprime k, {}^\backprime k)\big)\, d\,{}^\backprime k. \tag{9.10}$$

Proof. For generic ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}_{\mathbb{C}}^*$, the continuous linear endomorphism $\underline{T} := A({}^\backprime Q : {}^\backprime P : {}^\backprime\xi : {}^\backprime\lambda)$ of $C^\infty({}^\backprime K : {}^\backprime\xi)$ has Hermitian adjoint $\underline{T}^* = A({}^\backprime P : {}^\backprime Q : {}^\backprime\xi : -{}^\backprime\bar{\lambda})$. The result now follows by combining Lemma 9.2, with $-{}^\backprime\bar{\lambda}$ in place of ${}^\backprime\lambda$, and Lemma 9.1. $\square$

The expression on the left-hand side of (9.10) is very closely related to an Eisenstein integral for the parabolic subgroup $R = {}^\backprime P \times {}^\backprime Q$, defined as in Definition 8.4. This will allow us to express the Eisenstein integral in terms of the group structure of ${}^\backprime G$.

To be more precise, let ξ be as in (9.2) and let (τ, V_τ) be a finite dimensional unitary representation of K. We recall the definition of the space $C(K:\xi:\tau)$ and the definition of the linear isomorphism $T \mapsto \psi_T$ from $C(K:\xi:\tau)\otimes V(\xi)$ onto $\mathscr{A}_{2,M,\xi}$ from (8.11) and the surrounding text (note that $M_0 = M$).

Since $\mathscr{W} = \{e\}$, we have

$$\mathscr{A}_{2,M,\xi} = C^\infty_\xi(M/H_M : \tau_M).$$

Since $V(\xi) = \mathbb{C}I_{{}^\backprime\xi}$, it follows that the following map is a linear isomorphism;

$$f \mapsto \psi_{f\otimes I_{{}^\backprime\xi}}, \quad C(K:\xi:\tau) \xrightarrow{\simeq} C^\infty_\xi(M/H_M:\tau_M). \tag{9.11}$$

It follows from (8.12) that

$$\psi_{f\otimes I_{{}^\backprime\xi}}(m^{-1}) = \langle f(m)\,,\, I_{{}^\backprime\xi}\rangle_{\mathrm{HS}} = \mathrm{tr}_{{}^\backprime\xi}(f(m)) \qquad (m\in M),$$

where the subscript HS means that the Hilbert-Schmidt inner product is taken.

Corollary 9.4. *With notation as in Corollary 9.3, let $f \in C^\infty(K:\xi:\tau)$. Then*

$$\begin{aligned} &E({}^\backprime P\times{}^\backprime Q : \psi_{f\otimes I_{{}^\backprime\xi}} : \lambda)({}^\backprime x, e) = \\ &\quad = \int_{{}^\backprime K} \mathrm{tr}_{{}^\backprime\xi}\Big(\Big[\Big(A({}^\backprime Q : {}^\backprime P : {}^\backprime\xi : -{}^\backprime\lambda)\otimes \pi_{Q,{}^\backprime\xi^\vee,{}^\backprime\lambda}({}^\backprime x)\Big) f\Big]({}^\backprime k, {}^\backprime k)\Big)\, d{}^\backprime k, \end{aligned} \tag{9.12}$$

for ${}^\backprime x \in {}^\backprime G$ and generic $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$.

Proof. We note that $M_0 = M$, so that $K\cap M_0 = M$, $\xi_M = \xi$ and the map $i^\#$ introduced in (4.11) is just the identity map in the present setting. By application of Theorem 8.4 with $R = {}^\backprime P\times{}^\backprime Q$ in place of P, we now find, taking into account that $\rho_{Rh} = 0$, that the Eisenstein integral on the left-hand side of (9.12) equals

$$\langle f\,,\, \pi_{R,\xi,\bar\lambda}({}^\backprime x, e) j(R:\xi:\bar\lambda)(I_\xi)\rangle = \langle f\,,\, \pi_{R,\xi,\bar\lambda}(e, {}^\backprime x^{-1}) j(R:\xi:\bar\lambda)(I_\xi)\rangle, \tag{9.13}$$

by H-invariance of j. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the sesquilinear map $C^\infty(K:\xi:\tau)\times C^{-\infty}(K:\xi)\to V_\tau$ induced by the sesquilinear pairing $C^\infty(K:\xi)\times C^{-\infty}(K:\xi)\to\mathbb{C}$. By equivariance of the pairing, (9.13) equals

$$\langle \pi_{R,\xi,-\lambda}(e,{}^\backprime x) f\,,\, j(R:\xi:\bar\lambda)(I_{{}^\backprime\xi})\rangle = \langle [I\otimes \pi_{Q,{}^\backprime\xi^\vee,{}^\backprime\lambda}({}^\backprime x)] f\,,\, j(R:\xi:\bar\lambda)(I_{{}^\backprime\xi})\rangle$$

By application of (9.10) we infer that the last displayed expression equals the integral on the right-hand side of (9.12). ∎

We shall now relate the Eisenstein integral in (9.12) to Harish-Chandra's Eisenstein integral for the group. We agree to write

$$\tau_1(k)v = \tau(k,e)v, \quad \text{and} \quad v\tau_2(k) := \tau(e,k^{-1})v, \qquad (v \in V_\tau, k \in {}^\backprime K).$$

Then $(\tau_1,\, \tau_2)$ is a unitary bi-representation of ${}^\backprime K$ in V_τ in the sense that τ_1 is a unitary left representation and τ_2 a unitary right representation of ${}^\backprime K$ in V_τ and these two representations commute. Clearly any such bi-representation (τ_1, τ_2) of ${}^\backprime K$ comes from a unique unitary representation τ as above, and $\tau(k_1,k_2)v = \tau(k_1)v\tau(k_2^{-1})$, for $v \in V_\tau$ and $(k_1,k_2) \in K$. Given τ as above, we agree to write τ_M for the restriction of τ to M. Furthermore, we agree to write $\tau_{j{}^\backprime M}$ for the restriction of τ_j to ${}^\backprime M$, for $j = 1,2$. Then τ_M corresponds to the bi-representation $(\tau_{1{}^\backprime M}, \tau_{2{}^\backprime M})$ of ${}^\backprime M$.

Let $C^\infty({}^\backprime M : \tau_M)$ denote the space of smooth functions $\varphi := {}^\backprime M \to V_\tau$ transforming according to the rule

$$\varphi(m_1 m m_2) = \tau_1(m_1)\varphi(m)\tau_2(m_2), \qquad (m, m_1, m_2 \in {}^\backprime M).$$

Then it is readily verified that pull-back under the map $m : (x,y) \mapsto xy^{-1}$ induces a linear isomorphism

$$m^* :\, C^\infty({}^\backprime M : \tau_M) \xrightarrow{\;\simeq\;} C^\infty(M/H_M : \tau_M). \tag{9.14}$$

The inverse of this isomorphism will be denoted by

$$\psi \mapsto {}^\backprime\psi, \quad C^\infty(M/H_M : \tau_M) \xrightarrow{\;\simeq\;} C^\infty({}^\backprime M : \tau_M). \tag{9.15}$$

By ${}^\backprime M \times {}^\backprime M$-equivariance, it follows that the isomorphism (9.15) restricts to an isomorphism

$$C^\infty_\xi(M/H_M : \tau_M) \simeq C^\infty_\xi({}^\backprime M : \tau_M). \tag{9.16}$$

Here the space on the right-hand side is defined as the intersection of $C^\infty({}^\backprime M : \tau_M)$ with the space $C_\xi({}^\backprime M) \otimes V_\tau$, where $C_\xi({}^\backprime M)$ denotes the isotypical component of type ξ for the representation $L \times R$ of M in $C({}^\backprime M)$. Furthermore, the space on the left-hand side of (9.16) is defined similarly.

Since (9.11) is an isomorphism, it now follows that the following map is a linear isomorphism as well,

$$f \mapsto {}^\backprime\psi_{f \otimes I_{{}^\backprime\xi}}, \quad C(K : \xi : \tau) \xrightarrow{\;\simeq\;} C^\infty_\xi({}^\backprime M : \tau_M).$$

We now recall the definition of Harish-Chandra's Eisenstein integral associated with a parabolic subgroup ${}^\backprime Q \in \mathscr{P}({}^\backprime A)$. Given ${}^\backprime\psi \in C^\infty({}^\backprime M : \tau_{{}^\backprime M})$ and ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}^*_{\mathbb{C}}$, we define the function ${}^\backprime\psi_{{}^\backprime\lambda} : {}^\backprime M \to V_\tau$ by

$${}^\backprime\psi_{{}^\backprime\lambda}({}^\backprime n\,{}^\backprime a\,{}^\backprime m\,{}^\backprime k) = {}^\backprime a^{{}^\backprime\lambda + \rho_{{}^\backprime Q}}\,{}^\backprime\psi({}^\backprime m)\tau_2({}^\backprime k), \tag{9.17}$$

for ${}^\backprime k \in {}^\backprime K$, ${}^\backprime m \in {}^\backprime M$, ${}^\backprime a \in {}^\backprime A$ and ${}^\backprime n \in N_{{}^\backprime Q}$. The Harish-Chandra Eisenstein integral for the group ${}^\backprime G$ is now defined by

$$E_{\mathrm{HC}}({}^\backprime Q : {}^\backprime\psi : {}^\backprime\lambda)({}^\backprime x) = \int_{{}^\backprime K} \tau_1({}^\backprime k)^{-1}\,{}^\backprime\psi_{{}^\backprime\lambda}({}^\backprime k\,{}^\backprime x)\, d\,{}^\backprime k, \tag{9.18}$$

for ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}^*_{\mathbb{C}}$ and ${}^\backprime x \in {}^\backprime G$. We will derive a formula for the present type of Eisenstein integral, which will allow comparison with (9.12). In the formulation of the following lemma, we will use the natural identifications (9.3) and

$$C(K : \xi : \tau) = (C^\infty(K : \xi) \otimes V_\tau)^K \simeq \left(C^\infty({}^\backprime K : {}^\backprime\xi) \otimes C^\infty({}^\backprime K : {}^\backprime\xi^\vee) \otimes V_\tau\right)^K.$$

Furthermore, we will write $\mathrm{tr}_{{}^\backprime\xi}$ as shorthand for the map

$$\mathrm{tr}_{{}^\backprime\xi} \otimes I_{V_\tau} : \mathrm{End}(\mathscr{H}_{{}^\backprime\xi}) \otimes V_\tau \to V_\tau.$$

Lemma 9.5. *Let ${}^\backprime\xi \in {}^\backprime\widehat{M}$ and put $\xi = {}^\backprime\xi \otimes {}^\backprime\xi^\vee$. Furthermore, let $f \in C^\infty(K : \xi : \tau)$ and put ${}^\backprime\psi = {}^\backprime\psi_{f \otimes I_{{}^\backprime\xi}}$. Then for all ${}^\backprime\lambda \in {}^\backprime\mathfrak{a}^*_{\mathbb{C}}$,*

$$E_{\mathrm{HC}}({}^\backprime Q : {}^\backprime\psi_{f\otimes I_{{}^\backprime\xi}} : {}^\backprime\lambda)({}^\backprime x) = \int_{{}^\backprime K} \mathrm{tr}_{{}^\backprime\xi}\left([(I \otimes \pi_{{}^\backprime Q, {}^\backprime\xi^\vee, {}^\backprime\lambda}({}^\backprime x))f]({}^\backprime k, {}^\backprime k)\right)\, d\,{}^\backprime k. \tag{9.19}$$

Proof. We agree to write f_λ for the unique function in $C^\infty(G : {}^\backprime Q \times {}^\backprime Q : \xi : \lambda) \otimes V_\tau$ whose restriction to K equals f.

The function ${}^\backprime\psi := {}^\backprime\psi_{f\otimes I_{{}^\backprime\xi}} \in C^\infty_\xi({}^\backprime M : \tau^0_M)$ is completely determined by

$${}^\backprime\psi(e) = \langle f(e,e)\,,I_{{}^\backprime\xi}\rangle_{\mathrm{HS}} = \mathrm{tr}_{{}^\backprime\xi}[f(e,e)].$$

In the second expression, we have used the bilinear map $(\mathscr{H}_\xi \otimes V_\tau) \times \overline{\mathscr{H}_\xi} \to V_\tau$ induced by the Hilbert-Schmidt inner product on $\mathscr{H}_\xi = \mathrm{End}(\mathscr{H}_{{}^\backprime\xi})$.

We now observe that the function ${}^\backprime\psi_\lambda$ defined by (9.17) can be expressed in terms of $f_{-\lambda}$ in the following fashion;

$${}^\backprime\psi_\lambda({}^\backprime x) = \mathrm{tr}_{{}^\backprime\xi}[f_{-\lambda}(e, {}^\backprime x)], \qquad ({}^\backprime x \in {}^\backprime G). \tag{9.20}$$

It follows from the sphericality of f that

$$\mathrm{tr}_{{}^\backprime\xi}[f_{-\lambda}({}^\backprime x\,{}^\backprime k_1, {}^\backprime y\,{}^\backprime k_2)] = \tau_1({}^\backprime k_1)^{-1}\mathrm{tr}_{{}^\backprime\xi}[f_{-\lambda}({}^\backprime x, {}^\backprime y)]\tau_2({}^\backprime k_2),$$

for ${}^\backprime x, {}^\backprime y \in {}^\backprime G$ and ${}^\backprime k_1, {}^\backprime k_2 \in {}^\backprime K$. We thus obtain from (9.20) that

$$\tau_1({}^\backprime k)^{-1}\,{}^\backprime\psi_{{}^\backprime\lambda}({}^\backprime k\,{}^\backprime x) = \mathrm{tr}_{{}^\backprime\xi}[f_{-\lambda}({}^\backprime k, {}^\backprime k\,{}^\backprime x)] = \mathrm{tr}_{{}^\backprime\xi}\left(\left[(I \otimes \pi_{{}^\backprime Q, {}^\backprime\xi^\vee, {}^\backprime\lambda}({}^\backprime x))f\right]({}^\backprime k, {}^\backprime k)\right).$$

Equation (9.19) now follows from (9.18). □

The $\mathfrak{h}$-extreme parabolic subgroups in $\mathscr{P}(A)$ are the parabolic subgroups of the form ${}^\backprime P \times {}^\backprime P$ with ${}^\backprime P \in \mathscr{P}({}^\backprime A)$. For these parabolic subgroups our Eisenstein integrals essentially coincide with the unnormalized Eisenstein integrals of Harish-Chandra. More precisely, the following result is valid.

Corollary 9.6. *Let* ${}^{\backprime}P \in \mathscr{P}({}^{\backprime}A)$ *and* $\psi \in C^\infty(M/H_M : \tau_M^0)$. *Then for all* ${}^{\backprime}x, {}^{\backprime}y \in {}^{\backprime}G$ *we have*

$$E({}^{\backprime}P \times {}^{\backprime}P : \psi : \lambda)({}^{\backprime}x, {}^{\backprime}y) = E_{\mathrm{HC}}({}^{\backprime}P : {}^{\backprime}\psi : {}^{\backprime}\lambda)({}^{\backprime}x\,{}^{\backprime}y^{-1}), \tag{9.21}$$

with $\lambda = ({}^{\backprime}\lambda, -{}^{\backprime}\lambda)$, *as an identity of meromorphic* V_τ*-valued functions of* ${}^{\backprime}\lambda \in {}^{\backprime}\mathfrak{a}_{\mathbb{C}}^*$.

Proof. The space $C^\infty(M/H_M : \tau_M^0)$ is spanned by the functions of the form $\psi_{f\otimes I_{{}^{\backprime}\xi}}$, where ${}^{\backprime}\xi \in {}^{\backprime}M^\wedge$, $\xi = {}^{\backprime}\xi \otimes {}^{\backprime}\xi^\vee$ and $f \in C(K : \xi : \tau)$. By linearity it therefore suffices to establish (9.21) for $\psi = \psi_{f\otimes I_{{}^{\backprime}\xi}}$, with ${}^{\backprime}\xi$ and f as mentioned. Moreover, by right H-invariance of the Eisenstein integral on the left-hand side, it suffices to prove the result for ${}^{\backprime}y = e$. The claim now follows by comparison of (9.12) and (9.19). □

Remark 9.7. In particular, we see that the Eisenstein integral on the left is holomorphic as a function of $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$. As $\Sigma({}^{\backprime}P \times {}^{\backprime}P)_- = \emptyset$, this can also be derived by combining Theorem 8.4 with Remark 7.9.

Corollary 9.8. *With notation as in Corollary 9.3, let* $f \in C^\infty(K : \xi : \tau)$. *Let* $\psi_{f\otimes I_{{}^{\backprime}\xi}} \in C_\xi^\infty(M/H_M;\tau_M)$ *be defined as in (9.11). Then*

$$\begin{aligned} E({}^{\backprime}P \times {}^{\backprime}Q : \psi_{f\otimes I_{{}^{\backprime}\xi}} : \lambda)({}^{\backprime}x, {}^{\backprime}y) = \\ = \;\; E_{\mathrm{HC}}({}^{\backprime}Q : {}^{\backprime}\psi_{[(A({}^{\backprime}Q:{}^{\backprime}P:{}^{\backprime}\xi:-{}^{\backprime}\lambda)\otimes I)f]\otimes I_{{}^{\backprime}\xi}} : {}^{\backprime}\lambda)({}^{\backprime}x\,{}^{\backprime}y^{-1}), \end{aligned} \tag{9.22}$$

for generic ${}^{\backprime}\lambda \in {}^{\backprime}\mathfrak{a}_{\mathbb{C}}^*$, $\lambda = ({}^{\backprime}\lambda, -{}^{\backprime}\lambda)$ *and all* ${}^{\backprime}x, {}^{\backprime}y \in {}^{\backprime}G$.

Proof. By right H-invariance of the Eisenstein integral on the left-hand side, it suffices to prove the result for ${}^{\backprime}y = e$. It follows from (9.12) that

$$E({}^{\backprime}P \times {}^{\backprime}Q : \psi_{f\otimes I_{{}^{\backprime}\xi}} : \lambda)({}^{\backprime}x, e) = E({}^{\backprime}Q \times {}^{\backprime}Q : \psi_{[(A({}^{\backprime}Q:{}^{\backprime}P:{}^{\backprime}\xi:-{}^{\backprime}\lambda)\otimes I)f]\otimes I_{{}^{\backprime}\xi}} : \lambda)({}^{\backprime}x, e).$$

The identity now follows from (9.21). □

Appendix: Fubini's theorem for densities

In this appendix our purpose is to establish a Fubini type theorem for repeated integration in the setting of a Lie group G with two closed subgroups H and L such that $H \subset L$. The Fubini theorem concerns repeated integration for densities on the total space of the natural fiber bundle

$$\pi : L\backslash G \to H\backslash G, \tag{A.1}$$

with fibers diffeomorphic to $H\backslash L$. It expresses the integral over the total space as an iterated integration, first over the fibers and then over the base space. In case of unimodular groups there is a well known version of such a Fubini theorem involving invariant densities on the quotient spaces. In the case of non-unimodular groups such densities do not exist. Nevertheless, in this setting an appropriate formulation of iterated integration can be given as well.

To describe it, we will first formulate and establish a Fubini theorem for general fiber bundles, and then specialize to the above situation.

If V is real linear space of finite dimension n, then by $\mathscr{D}_V$ we denote the space of complex-valued densities on V, i.e., the (complex linear) space of functions $\lambda : \wedge^n(V) \to \mathbb{C}$ transforming according to the rule $\lambda(t\xi) = |t|\lambda(\xi)$, for all $t \in \mathbb{R}$ and $\xi \in \wedge^n V$. A density λ is said to be positive if $\lambda(\xi) > 0$ for all non-zero $\xi \in \wedge^n V$. By pull-back under the natural map $V^n \to \wedge^n V$ we see that a density may also be viewed as a map $V^n \to \mathbb{C}$ transforming according to the rule $\lambda \circ T^n = |\det T|\lambda$, for all $T \in \mathrm{End}(V)$. This will be our viewpoint from now on. Note that $\mathscr{D}_V$ has dimension 1 over $\mathbb{C}$. If W is a second real linear space of the same dimension n and $A : V \to W$ a linear map, then pull-back under A is the map $A^* : \mathscr{D}_W \to \mathscr{D}_V$ defined by

$$A^*\mu = \mu \circ A^n, \qquad (\mu \in \mathscr{D}_W).$$

Lemma A.1. *Let E, F be finite dimensional real linear spaces. Then $\mathscr{D}_{E\oplus F} \simeq \mathscr{D}_E \otimes \mathscr{D}_F$ naturally.*

Proof. Let p and q be the dimensions of E and F respectively and put $n = p + q$. We consider the natural isomorphism $\mu : \wedge^p E \otimes \wedge^q F \to \wedge^n(E \oplus F)$. Given $\alpha \in \mathscr{D}_E$ and $\beta \in \mathscr{D}_F$, we define $\alpha \boxtimes \beta : \wedge^p E \otimes \wedge^q F \to \mathbb{C}$ by $\alpha \boxtimes \beta(\xi \times \eta) = \lambda(\xi)\mu(\eta)$. We note that this definition is unambiguous, and that $(\alpha \boxtimes \beta)\circ(t\,\cdot) = |t|(\alpha \boxtimes \beta)$, so that $(\alpha, \beta) \mapsto \alpha \boxtimes \beta \circ \mu^{-1}$ defines a bilinear map from $\mathscr{D}_E \times \mathscr{D}_F$ to $\mathscr{D}_{E\oplus F}$. The induced map $\mathscr{D}_E \otimes \mathscr{D}_F \to \mathscr{D}_{E\oplus F}$ is a non-trivial linear map between one dimensional complex linear spaces, hence a linear isomorphism. $\square$

From now on we shall identify $\mathscr{D}_{E\oplus F}$ with $\mathscr{D}_E \otimes \mathscr{D}_F$ via the isomorphism given in the proof of the above lemma.

The lemma can be generalized to the setting of short exact sequences as follows. Let

$$0 \to E' \xrightarrow{\ i\ } E \xrightarrow{\ p\ } E'' \to 0 \tag{A.2}$$

be a short exact sequence of finite dimensional real linear spaces of dimensions k, n and $n-k$. We recall that a linear map $f : E'' \to E$ is said to be splitting if $p \circ f = \mathrm{id}_{E''}$. Associated with f is an isomorphism $i \oplus f : E \oplus E'' \to E$, which by pull-back induces a natural isomorphism

$$(i \oplus f)^* : \mathscr{D}_{E'} \otimes \mathscr{D}_{E''} = \mathscr{D}_{E'\oplus E''} \xrightarrow{\ \simeq\ } \mathscr{D}_E. \tag{A.3}$$

Lemma A.2. *The isomorphism (A.3) is independent of the splitting map f.*

Proof. Let g be a second splitting map. Then $(i \oplus f)^* - (i \oplus g)^* = (i \oplus (f-g))^*$. Now $f - g$ maps E'' into $\ker p = i(E')$ so that $i \oplus (f-g)$ maps $E' \oplus E''$ into the subspace $i(E')$ of E. It follows that $(i \oplus (f-g))^* = 0$ so that $(i \oplus f)^* = (i \oplus g)^*$. $\square$

From now on, given a short exact sequence of the form (A.2) we shall identify elements of the spaces $\mathscr{D}_{E'} \otimes \mathscr{D}_{E''}$ and $\mathscr{D}_E$ via the isomorphism $(i \oplus f)^*$, which is independent of the choice of f.

We now turn to manifolds. Let M, N be smooth manifolds and $\varphi : M \to N$ a smooth map. Then by $T\varphi : TM \to TN$ we denote the induced map between the tangent bundles. For a given $x \in M$, this map restricts to the tangent map $T_x\varphi : T_xM \to T_{\varphi(x)}N$, which will also be denoted by $d\varphi(x)$.

By $\mathscr{D}_M$ we denote the complex line bundle of densities on M. The fiber of this bundle at a point $x \in M$ is equal to $\mathscr{D}_{T_xM}$. The space of continuous densities is denoted by $\Gamma(\mathscr{D}_M)$. If $\dim M = \dim N$ then the smooth map $\varphi : M \to N$ induces a pull-back map $\varphi^* : \Gamma(\mathscr{D}_N) \to \Gamma(\mathscr{D}_M)$, given by

$$\varphi^*(\mu)_x = d\varphi(x)^* \mu_{\varphi(x)}, \qquad (\mu \in \Gamma(\mathscr{D}_N),\, x \in M).$$

There is notion of integration of compactly supported continuous densities on manifolds for which the substitution of variables theorem is valid. More precisely, if $\varphi : M \to N$ is a diffeomorphism of smooth manifolds, then

$$\int_N \mu = \int_M \varphi^*(\mu), \qquad (\mu \in \Gamma(N)). \tag{A.4}$$

Let $\pi : F \to B$ be a smooth fiber bundle. Let $\mathscr{D}_F$ denote the density bundle on F. We may introduce a bundle of fiber densities on F as follows. The map π induces the homomorphism $T\pi : TF \to TB$ of vector bundles. The kernel $K = \ker T\pi$ of this bundle is a subbundle of TF. Obviously, the fiber of K at $p \in F$ may be viewed as the tangent space of the fiber $F_{\pi(p)}$ at the point p. The associated bundle $p \mapsto \mathscr{D}_{K_p}$ is a smooth complex line bundle on F, which we shall call the bundle of fiber densities on F. We shall denote this bundle by $\mathscr{D}_F^B$.

On the other hand, the fiber product or pull-back bundle $\pi^*(\mathscr{D}_B) := F \times_\pi \mathscr{D}_B$ of $\mathscr{D}_B$ under π is a complex line bundle on F. We shall denote the associated canonical line bundle homomorphism $\pi^*(\mathscr{D}_B) \to \mathscr{D}_B$ by $\tilde{\pi}$.

The short exact sequence $0 \to K \to TF \to \pi^*(TB) \to 0$ of vector bundles on F naturally induces a line bundle isomorphism

$$\mathscr{D}_F^B \otimes \pi^*(\mathscr{D}_B) \simeq \mathscr{D}_F, \tag{A.5}$$

via which we shall identify elements of these spaces. Here naturality means that for a fiber bundle morphism φ from π to a bundle $\pi' : F' \to B'$ with $\dim F' = \dim F$ and $\dim B' = \dim B$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{D}_F^B \otimes \pi^*(\mathscr{D}_B) & \xrightarrow{\simeq} & \mathscr{D}_F \\ {\scriptstyle (T\varphi)^* \otimes (T\varphi)^*} \downarrow & & \downarrow {\scriptstyle (T\varphi)^*} \\ \mathscr{D}_{F'}^{B'} \otimes (\pi')^*(\mathscr{D}_{B'}) & \xrightarrow{\simeq} & \mathscr{D}_{F'} \end{array} \tag{A.6}$$

Let now $b \in B$ and let F_b the fiber $\pi^{-1}(b)$ of F above b. Obviously, the restriction of $\mathscr{D}_F^B$ to this fiber is naturally isomorphic to $\mathscr{D}_{F_b}$, the density bundle of the fiber. On the other hand, via $\tilde{\pi}$ the restriction of the bundle $\pi^*(\mathscr{D}_B)$ to F_b may be identified with the trivial bundle $F_b \times \mathscr{D}_{T_bB}$. Accordingly, we obtain natural isomorphisms $\mathscr{D}_{F_b} \otimes \mathscr{D}_{T_bB} \simeq \mathscr{D}_F|_{F_b}$, and

$$\Gamma(\mathscr{D}_F|_{F_b}) \simeq \Gamma(\mathscr{D}_{F_b}) \otimes_{\mathbb{C}} \mathscr{D}_{T_bB}.$$

Integration over the fiber gives a natural linear map

$$I_b : \Gamma_c(\mathscr{D}_{F_b}) \to \mathbb{C}, \qquad \mu \mapsto \int_{F_b} \mu.$$

By transfer we obtain a natural map $I_b \otimes \mathrm{id} : \Gamma_c(\mathscr{D}_F|_{F_b}) \to \mathscr{D}_{T_bB}$. We now define the push-forward map $\pi_* : \Gamma_c(\mathscr{D}_F) \to \mathrm{sect}(\mathscr{D}_B)$, by

$$\pi_*(\mu)(b) := (I_b \otimes \mathrm{id})(\mu|_{F_b}). \tag{A.7}$$

Here $\mathrm{sect}(\mathscr{D}_B)$ denotes the space of all (not-necessarily continuous) sections of $\mathscr{D}_B$.

By the naturality of the constructions and the invariance of integration as formulated in (A.4), one readily checks that the notion of push-ward of compactly supported densities is invariant under isomorphisms of bundles.

Lemma A.3. *Let φ be an isomorphism from the fiber bundle $\pi : F \to B$ to a second fiber bundle $\pi' : F' \to B'$ and let $\varphi_\circ$ denote the induced diffeomorphism $B \to B'$. Then the following diagram commutes:*

$$\begin{array}{ccc} \Gamma_c(\mathscr{D}_{F'}) & \xrightarrow{\varphi^*} & \Gamma_c(\mathscr{D}_F) \\ \pi'_* \downarrow & & \downarrow \pi_* \\ \Gamma_c(\mathscr{D}_{B'}) & \xrightarrow{\varphi_\circ^*} & \Gamma_c(\mathscr{D}_B) \end{array}$$

We can now establish the following Fubini type theorem for the integration of densities over fiber bundles.

Lemma A.4. *The map π_* maps $\Gamma_c(\mathscr{D}_F)$ (respectively $\Gamma_c^\infty(\mathscr{D}_F)$) continuous linearly to $\Gamma_c(\mathscr{D}_B)$ (respectively $\Gamma_c^\infty(\mathscr{D}_B)$). Moreover, for all $\mu \in \Gamma_c(\mathscr{D}_F)$,*

$$\int_F \mu = \int_B \pi_*(\mu). \tag{A.8}$$

Proof. By using partitions of unity, and invoking invariance of integration, cf. (A.4), and Lemma A.3, we may reduce the proof to the case that B is open in $\mathbb{R}^n$ and that $F = B \times V$, with V an open subset of Euclidean space $\mathbb{R}^k$. In that case the result comes to down to continuous and smooth parameter dependence and Fubini's theorem for Riemann integrals of continuous functions. $\square$

Corollary A.5. *Let μ be a measurable section of $\mathscr{D}_F$. Then the following statements are equivalent.*

(a) *The density μ is absolutely integrable.*

(b) *For almost every $b \in B$ the integral for $\pi_*(\mu)_b$ is absolutely convergent and the resulting density $\pi_*(\mu)$ is absolutely integrable over B.*

If any of these conditions is fulfilled, then (A.8) is valid.

Proof. This follows by reduction to Fubini's theorem through the use of partitions of unity, as in the proof of Lemma A.4 $\square$

We will now apply the above result to the particular setting of a Lie group G with closed subgroups H and L such that $H \subset L$. As said at the start of this appendix, this setting gives rise to the natural fiber bundle (A.1) with fiber diffeomorphic to $L\backslash H$.

Let $\Delta_{L\backslash G} : L \to \mathbb{R}_+$ be the positive character given by

$$\Delta_{L\backslash G}(l) = |\det \mathrm{Ad}_G(l)_{\mathfrak{g}/\mathfrak{l}}|^{-1}, \qquad (l \in L), \tag{A.9}$$

where $\mathrm{Ad}_G(l)_{\mathfrak{g}/\mathfrak{l}} \in \mathrm{GL}(\mathfrak{g}/\mathfrak{l})$ denotes the map induced by the adjoint map $\mathrm{Ad}_G(l) \in \mathrm{GL}(\mathfrak{g})$. Given a character ξ of L we denote by $C(G : L : \xi)$ the space of continuous functions $f : G \to \mathbb{C}$ transforming according to the rule

$$f(lx) = \xi(l) f(x),$$

for $x \in G$ and $l \in L$. We denote by $\mathscr{M}(G)$ the space of measurable functions $G \to \mathbb{C}$ and by $\mathscr{M}(G : L : \xi)$ the space of $f \in \mathscr{M}(G)$ transforming according to the same rule.

Given $f \in \mathscr{M}(G)$ and $\omega \in \mathscr{D}_{\mathfrak{g}/\mathfrak{l}}$, we denote by f_ω the function $G \to \mathscr{D}_{L\backslash G}$ defined by

$$f_\omega(x) = f(x)\, dr_x(e)^{-1*}\omega.$$

Lemma A.6. *Let $\omega \in \mathscr{D}_{\mathfrak{g}/\mathfrak{l}} \setminus \{0\}$. Then the map $f \mapsto f_\omega$ defines a continuous linear isomorphism from $C(G : L : \Delta_{L\backslash G})$ onto $\Gamma(\mathscr{D}_{L\backslash G})$.*

Proof. Write $\Delta = \Delta_{L\backslash G}$. In the proof we will use the notation $[e]$ for the image of e in $L\backslash G$. Moreover, we will use the canonical identification $T_{[e]}(L\backslash G) \simeq \mathfrak{g}/\mathfrak{l}$. Let ω be as stated, and let $f \in C(G : L : \Delta)$. Then for $x \in G$ we have $f_\omega(x) \in \mathscr{D}_{T_{[x]}(L\backslash G)}$. Let $l \in L$, then

$$\begin{aligned}
f_\omega(lx) &= \Delta(l) f(x)\, dr_{lx}([e])^{-1*}\omega \\
&= \Delta(l) f(x) dr_x([e])^{-1*}\, dr_l([e])^{-1*}\omega \\
&= \Delta(l) f(x)\, dr_x([e])^{-1*}\mathrm{Ad}(l)^*\omega \\
&= f_\omega(x)
\end{aligned}$$

It follows that f_ω factors through a smooth map $L\backslash G \to \mathscr{D}_{L\backslash G}$, with $f_\omega(x)$ a density on $T_{[x]}(L\backslash G)$. Accordingly, f_ω defines a section of $\mathscr{D}_{L\backslash G}$, which clearly is continuous. The bijectivity of the map $f \mapsto f_\omega$ from $C(G : L : \Delta)$ onto $\Gamma(\mathscr{D}_{L\backslash G})$ is obvious. $\square$

Our next goal is to calculate the push-forward $\pi_*(f_\omega)$, for $\omega \in \mathscr{D}_{\mathfrak{g}/\mathfrak{l}}$ and $f \in C(G : L : \Delta_{L\backslash G})$, and $\pi : L\backslash G \to H\backslash G$ the canonical projection.

We note that π is a fiber bundle with total space $F = L\backslash G$, base space $B = H\backslash G$ and fiber diffeomorphic to $L\backslash H$. Thus, we have the natural isomorphism (A.5).

If $x \in G$, then the diffeomorphism $r_x^F : F \to F, z \mapsto zx$ defines an isomorphism of fiber bundles over the diffeomorphism r_x^B defined by right multiplication on B, i.e., the following diagram commutes,

$$\begin{array}{ccc} F & \xrightarrow{r_x^F} & F \\ \downarrow & & \downarrow \\ B & \xrightarrow{r_x^B} & B. \end{array}$$

In the sequel we shall use the commutativity of the diagram (A.6) with $F = F'$, $B = B'$ and $\varphi = r_x^F$.

In particular, it follows that $(dr_x^F)^* \otimes (dr_x^F)^* \in \mathrm{End}(\mathscr{D}_F^B \otimes \pi^*\mathscr{D}_B)$ corresponds to the naturally induced automorphism $(dr_x^F)^*$ of $\mathscr{D}_F$.

We fix non-zero elements $\omega_{L\backslash G} \in \mathscr{D}_{\mathfrak{g}/\mathfrak{l}}$, $\omega_{L\backslash H} \in \mathscr{D}_{\mathfrak{h}/\mathfrak{l}}$ and $\omega_{H\backslash G} \in \mathscr{D}_{\mathfrak{g}/\mathfrak{h}}$ such that

$$\omega_{L\backslash H} \otimes \omega_{H\backslash G} = \omega_{L\backslash G} \tag{A.10}$$

with respect to the identification determined by the short exact sequence $0 \to \mathfrak{h}/\mathfrak{l} \to \mathfrak{g}/\mathfrak{l} \to \mathfrak{g}/\mathfrak{h} \to 0$. This short exact sequence may be identified with the short exact sequence of tangent spaces

$$0 \to T_{[e]}(L\backslash H) \xrightarrow{di([e])} T_{[e]}(L\backslash G) \xrightarrow{d\pi([e])} T_{[e]}(H\backslash G) \to 0,$$

where $i : L\backslash H \hookrightarrow L\backslash G$ denotes the natural embedding of $L\backslash H$ onto the fiber $\pi^{-1}([e])$. Accordingly, formula (A.10) may be viewed as an identity of elements associated with the decomposition

$$(\mathscr{D}_F^B)_{[e]} \otimes (\mathscr{D}_B)_{[e]} = (\mathscr{D}_F)_{[e]}.$$

Lemma A.7. *Let $\omega_{L\backslash H}, \omega_{H\backslash G}$ and $\omega_{L\backslash H}$ satisfy (A.10). Then for all $h \in H$ and $x \in G$,*

$$dr_{hx}([e])^{-1*}\omega_{L\backslash H} = \Delta_{H\backslash G}(h)^{-1}\left(dr_{hx}([e])^{-1*}\omega_{L\backslash H} \otimes dr_x([e])^{-1*}\omega_{H\backslash G}\right), \tag{A.11}$$

in accordance with the decomposition $(\mathscr{D}_F^B)_{[hx]} \otimes (\mathscr{D}_B)_{[x]} = (\mathscr{D}_F)_{[hx]}$, corresponding to (A.5).

Proof. Let $h \in H$, then $dr_h(e)^{-1*}(\omega_{H\backslash G}) = \mathrm{Ad}(h)^*\omega_{H\backslash G} = \Delta_{H\backslash G}(h)^{-1}\omega_{H\backslash G}$ and we see that

$$dr_h([e])^{-1*}\omega_{L\backslash G} = \Delta_{H\backslash G}(h)^{-1}\left(dr_h([e])^{-1*}\omega_{L\backslash H} \otimes \omega_{H\backslash G}\right). \tag{A.12}$$

Let now $x \in G$, then in view of the G-equivariance of the fiber bundle $F \to B$ formula (A.11) follows by application of $dr_x([h])^{-1*}$ to both sides of the identity (A.12). $\square$

Theorem A.8. *Let $\omega_{L\backslash G}, \omega_{H\backslash G}$ and $\omega_{L\backslash H}$ satisfy (A.10). Let $\varphi \in \mathscr{M}(G : L : \Delta_{L\backslash G})$ and let $\varphi_{\omega_{L\backslash G}}$ be the associated measurable density on $L\backslash G$. Then the following assertions (a) and (b) are equivalent*

(a) *The density $\varphi_{\omega_{L\backslash G}}$ is absolutely integrable.*

(b) *There exists a left H-invariant subset $\mathscr{Z}$ of measure zero in G such that*

   (1) *for every $x \in G \setminus \mathscr{Z}$, the integral*

$$I_x(\varphi) = \int_{L\backslash H \ni [h]} \Delta_{H\backslash G}(h)^{-1} \varphi(hx)\, dr_h([e])^{-1*} \omega_{L\backslash H}, \tag{A.13}$$

   is absolutely convergent;

   (2) *the function $I(\varphi) : x \mapsto I_x(\varphi)$ belongs to $\mathscr{M}(G : H : \Delta_{H\backslash G})$;*

   (3) *the associated density $I(\varphi)_{\omega_{H\backslash G}}$ is absolutely integrable.*

Furthermore, if any of the conditions (a) and (b) are fulfilled, then

$$\int_{L\backslash G} \varphi_{\omega_{L\backslash G}} = \int_{H\backslash G} I(\varphi)_{\omega_{H\backslash G}}. \tag{A.14}$$

Proof. We retain the notation introduced before the statement of the theorem. Then for $x \in G$ and $h \in H$ the associated density at Lhx is given by

$$\varphi_{\omega_{L\backslash G}}(hx) = \Delta_{L\backslash H}(h)^{-1} \varphi(hx) \left(dr_{hx}([e])^{-1*} \omega_{L\backslash H} \otimes dr_x([e])^{-1*} \omega_{H\backslash G} \right), \tag{A.15}$$

in accordance with the decomposition corresponding to (A.5).

We will deduce the result from applying Corollary A.5 to the fibre bundle given by the canonical projection $\pi : F := L\backslash G \to B := H\backslash G$ and to the measurable density $\mu := \varphi_{\omega_{L\backslash G}}$ on F.

The crucial step is to prove the claim that for $x \in G$, the integral for the push-forward $\pi_*(\varphi_{\omega_{L\backslash G}})(Hx)$ converges absolutely if and only if the integral for $I_x(\varphi)$ converges absolutely. We will first establish this claim.

It follows from (A.7) that the push-forward of $\varphi_{\omega_{L\backslash G}}$ under π is the density on $H\backslash G$ given by the following fiber integral

$$\pi_*(\varphi_{\omega_{L\backslash G}})(Hx) = \left(\int_{\pi^{-1}(Hx)} \nu_x \right) \otimes dr_x([e])^{-1*} \omega_{H\backslash G}, \tag{A.16}$$

where $\pi^{-1}(Hx) = r_x(L\backslash H)$, and where ν_x is the density on $r_x(L\backslash H)$ given by

$$\nu_x(Lhx) = \Delta_{L\backslash H}(h)^{-1} \varphi(hx)\, dr_{hx}([e])^{-1*} \omega_{L\backslash H}.$$

The convergence and value of this integral depends on x through its class Hx. We now observe that r_x defines a diffeomorphism from the fiber $\pi^{-1}(He)$ onto the fiber $\pi^{-1}(Hx)$. Moreover,

$$[r_x^* \nu_x](Lh) = dr_x([h])^{-1*} \nu_x(Lhx) = \Delta_{L\backslash H}(h)^{-1} \varphi(hx)\, dr_h([e])^{-1*} \omega_{L\backslash H}.$$

Thus, $I_x(\varphi)$ equals the integral of $r_x^* \nu_x$ over $L\backslash H$, and by invariance of integration we see that it converges absolutely if and only if the integral for $\pi_*(\varphi_{\omega_{L\backslash G}})(Hx)$ converges absolutely. Moreover, in case of convergence we have

$$I_x(\varphi) = \int_{L\backslash H} r_x^*(\nu_x) = \int_{\pi^{-1}(Hx)} \nu_x,$$

so that

$$\pi_*(\varphi_{\omega_{L\backslash G}})(Hx) = I_x(\varphi)\, dr_x([e])^{-1*}\omega_{H\backslash G} = I(\varphi)_{\omega_{H\backslash G}}(Hx). \tag{A.17}$$

This establishes the claim.

The equivalence of (a) and (b) now readily follows from the similar equivalence in Corollary A.5. Finally, if any of these conditions is fulifilled, both are, and in view of (A.17), the identity (A.14) follows from the final assertion of Corollary A.5. □

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