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Vector Valued Poisson Transforms on Riemannian Symmetric Spaces of Rank One

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Let $G \times_K W$ be the associated vector bundle to the K-representation (σ, W) over the Riemannian symmetric space G/K of rank one. In this paper a complete description of the eigensections of the Casimir operator is given using a generalization of the Poisson transform. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let G/K be a Riemannian symmetric space of the noncompact type with Furstenberg boundary K/M. Let (σ, W) be a finite dimensional representation of K and $G \times_K W$ the associated vector bundle. When σ is the trivial representation, which we refer to as the scalar case, there is a well-developed theory describing the eigenfunctions of all invariant differential operators on G/K using Poisson transforms. In this paper the first steps are taken to generalize this theory to sections of the bundle $G \times_K W$. For Riemannian symmetric spaces of rank one we give a complete description of the eigensections of the Casimir operator using a generalization of the Poisson transform.

The statement that all eigenfunctions on G/K are Poisson integrals of hyperfunctions on the boundary K/M was conjectured by Helgason, who proved it for G of real rank one in [7]. In its full generality the conjecture was proven by Kashiwara e.a. ([14]). In [15] Oshima and Sekiguchi described the image under the Poisson transform of distributions on K/M as consisting of functions satisfying certain growth conditions towards the boundary. In [2] Van den Ban and Schlichtkrull gave a new proof of this description using asymptotic methods, introduced by Wallach ([20]), rather than the advanced micro-local analysis of [14, 15].

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The generalization of these results to vector bundles has only just started. In [17] Shimeno extended the results to line bundles over Hermitian symmetric spaces using similar methods as in the scalar case. He was able to describe the algebra of invariant differential operators and to compute the **c**-functions involved, both of which, in general, we do not know enough. For the example of differential forms on real and complex hyperbolic spaces, Gaillard in [5, 6] gave a description of harmonic forms. This paper extends his results to arbitrary bundles and gives a more precise answer.

As we do not know the algebra of invariant differential operators on $G \times_K W$, we restrict ourselves to the well known Casimir operator of G. Although this is but a meager substitute for the whole algebra, the reader shall see that the problem of describing eigensections of the Casimir has an interesting answer which is a highly nontrivial generalization of the scalar case. We are able to give a bijective transform from sections on a certain simple bundle over the boundary K/M onto the eigensections of the Casimir, no matter how the **c**-functions (mis)behave. However, we have to impose the restrictive hypothesis on the eigenvalue that there are no tempered eigensections for this eigenvalue which are not square integrable. There remain, however, enough "difficult" eigenvalues.

For a generic eigenvalue it is fairly easy to realize the eigenspace as the image of an injective Poisson transform. By a detailed analysis of the behaviour of these Poisson transforms near a degenerate eigenvalue we are able to use them to construct a bijective transform to the eigenspace for this degenerate eigenvalue.

We make this more precise in the following outline of the results and contents of the paper.

We can identify the space of smooth sections of $G \times_K W$ with

$$C^\infty \operatorname{Ind}_K^G(\sigma) = \{f : G \to W \mid f \text{ smooth}, f(xk) = \sigma(k)^{-1} f(x)\}.$$

For $\mu \in \mathbb{C}$, let $C_\mu^\infty \operatorname{Ind}_K^G(\sigma)$ denote the subspace of eigensections of the Casimir with eigenvalue μ. For a fixed μ_0 we want to describe $C_{\mu_0}^\infty \operatorname{Ind}_K^G(\sigma)$. For simplicity we assume that there are no tempered eigensections for this eigenvalue (for a definition of tempered eigensections see Remark 5.5).

In Section 3 we define a Poisson transform as a continuous, linear, and G-equivariant map from $C^\infty \operatorname{Ind}_P^G(\tau)$ to $C^\infty \operatorname{Ind}_K^G(\sigma)$. Here τ is a representation of a minimal parabolic subgroup P and $C^\infty \operatorname{Ind}_P^G(\tau)$ are the smooth vectors in $\operatorname{Ind}_P^G(\tau)$.

In Section 4 we prove the existence of asymptotic expansions of eigensections satisfying certain growth conditions. The main ideas of the proof come from [20, 2, 1]. By mapping an eigenfunction in $C_\mu^\infty \operatorname{Ind}_K^G(\sigma)$ to the sum of some of the coefficients in its expansion, where the choice of the coefficients depends on μ_0, we define a K-equivariant boundary value operator

$$\beta_{\mu,\mu_0}\colon C^\infty_\mu \operatorname{Ind}_K^G(\sigma) \to C^\infty \operatorname{Ind}_M^K(\sigma_{|M}).$$

In a certain explicit sense the β_{μ,μ_0} depend holomorphically on μ in a neighbourhood of μ_0.

For a certain direct sum $C^\infty \operatorname{Ind}_P^G(\sigma_\mu)$ of principal series representations, for which $(\sigma_\mu)_{|M} = \sigma_{|M}$, the Poisson transform $\mathscr{P}_\mu$ on $C^\infty \operatorname{Ind}_P^G(\sigma_\mu)$ defined by

$$\mathscr{P}_\mu f(x) = \int_K \sigma(k) f(xk)\, dk$$

maps into $C^\infty_\mu \operatorname{Ind}_K^G(\sigma)$. Since $(\sigma_\mu)_{|M} = \sigma_{|M}$ we can lift elements in $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to elements in $C^\infty \operatorname{Ind}_P^G(\sigma_\mu)$, and composing this lift with $\mathscr{P}_\mu$ we obtain

$$\mathscr{P}_\mu\colon C^\infty \operatorname{Ind}_M^K(\sigma_{|M}) \to C^\infty_\mu \operatorname{Ind}_K^G(\sigma).$$

For generic μ_0 the transform $\mathscr{P}_{\mu_0}$ is an isomorphism and in this case it is essentially inverted by β_{μ_0,μ_0}.

For a degenerate μ_0 one only has that the composite map

$$\beta_{\mu,\mu_0} \circ \mathscr{P}_\mu\colon C^\infty \operatorname{Ind}_M^K(\sigma_{|M}) \to C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$$

is invertible for $\mu \neq \mu_0$ in a neighbourhood of μ_0.

In Section 5 we prove that the possible poles of $(\beta_{\mu,\mu_0} \circ \mathscr{P}_\mu)^{-1}$ at $\mu = \mu_0$ are precisely cancelled by the zeroes of $\mathscr{P}_\mu$ at $\mu = \mu_0$. Moreover, the transform $[\mathscr{P}_\mu \circ (\beta_{\mu,\mu_0} \circ \mathscr{P}_\mu)^{-1}]_{\mu=\mu_0}$ is an isomorphism onto $C^\infty_{\mu_0} \operatorname{Ind}_K^G(\sigma)$. This essentially gives the Poisson transform. The construction of $\mathscr{P}_\mu \circ (\beta_{\mu,\mu_0} \circ \mathscr{P}_\mu)^{-1}$ is not just a normalization by the c-function; the map $\beta_{\mu,\mu_0} \circ \mathscr{P}_\mu$ is a differential operator. We see that for certain degenerate values of the eigenvalue we have to construct a transform which really differs from the generic transform. This is a procedure quite different from the scalar case where the generic Poisson transform works for all eigenvalues, but the problem is to *prove* that it works for the degenerate values.

In Section 6 we give two applications of the Main Theorem. In the first we compute the composition factors of $C^\infty_\mu \operatorname{Ind}_K^G(\sigma)$. The second gives an embedding of the G-representation on $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ defined by the isomorphism $\mathscr{P}_\mu \circ (\beta_{\mu,\mu_0} \circ \mathscr{P}_\mu)^{-1}$ into an induced representation on G/P, and so we get a G-equivariant embedding from $C^\infty_{\mu_0} \operatorname{Ind}_K^G(\sigma)$ into an induced representation space over G/P. The representation of P involved has non-trivial restriction to N.

2. Notations and Preliminaries

Let G be a connected real semisimple Lie group with finite center, and of real rank one. Let K be a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space of rank one. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K (in general, italic capitals denote Lie groups and german lower case letters Lie algebras). Let $G = KAN$ be an Iwasawa decomposition, and for $x \in G$ define $\kappa(x) \in K$, $H(x) \in \mathfrak{a}$ and $n(x) \in N$, by $x = \kappa(x) \exp(H(x))\, n(x)$. Let Σ be the restricted root system of $\mathfrak{a}$ in $\mathfrak{g}$ and Σ^+ be the system of positive roots associated to N. Let α be the indivisible root in Σ^+ and ρ half the sum of the positive roots counted with multiplicities. On the complexified dual $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}$ we define a partial order $\prec$ by $\mu \prec \lambda$ if and only if $\lambda - \mu = l\alpha$ for $l \in \mathbb{Z}_{>0}$. Define $M = Z_K(A)$ and $P = MAN$. Then P is a minimal parabolic subgroup. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in A$ we define $a^{\lambda} = e^{\lambda(\log a)}$.

Let (σ, W) be a (not necessarily irreducible) continuous representation of K in a finite dimensional complex vector space. The smooth sections of the associated vector bundle $G \times_K W$ can be identified with elements of the function space

$$C^{\infty} \operatorname{Ind}_K^G(\sigma) = \{f\colon G \to W \mid f(xk) = \sigma(k)^{-1} f(x),\ x \in G,\ k \in K; f \text{ smooth}\}.$$

Throughout this paper (σ, W) is fixed.

Let $\mathscr{U}(\mathfrak{g})$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$. Let $\mathscr{Z}(\mathfrak{g})$ be the center of $\mathscr{U}(\mathfrak{g})$. Through the left regular representation an element $Z \in \mathscr{Z}(\mathfrak{g})$ acts as an G-invariant differential operator L_Z on $C^{\infty} \operatorname{Ind}_K^G(\sigma)$.

Choose $\mathfrak{t}$ maximal abelian in $\mathfrak{m}$. Then $\mathfrak{h} := (\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}$ is a Cartan subalgebra for $\mathfrak{g}_{\mathbb{C}}$. Choose an order on $\mathfrak{h}$ such that $\bar{\mathfrak{n}}$ is contained in $\mathfrak{g}^-$, the sum of the negative root spaces. Let ρ_M be half the sum of positive roots for $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let $'\gamma_{\mathfrak{h}}\colon \mathscr{Z}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{h})$ be defined by $Z - {'\gamma_{\mathfrak{h}}}(Z) \in \mathfrak{g}_{\mathbb{C}}^- \mathscr{U}(\mathfrak{g})$ for $Z \in \mathscr{Z}(\mathfrak{g})$. Then we recall that for $\Lambda \in \mathfrak{h}^*$, $Z \in \mathscr{Z}(\mathfrak{g})$, we have that $\gamma_{\mathfrak{h}}(Z)(\Lambda) := {'\gamma_{\mathfrak{h}}}(Z)(\Lambda - \rho_M - \rho)$ defines the usual Harish-Chandra isomorphism onto the Weyl group invariant polynomials on $\mathfrak{h}^*$.

Denote by $M^{\wedge}$ the set of equivalence classes of irreducible M-representations. Let $W_{|M} = \Sigma_{\delta \in M^{\wedge}} W(\delta)$ be the decomposition of σ into M-isotypical parts. We write

$$\delta \in \sigma$$

if $W(\delta) \neq 0$. Let $\pi_\delta\colon W \to W(\delta)$ denote the projection according to this direct sum decomposition. For each $\delta \in \sigma$ let

$$(\sigma_\delta, W(\delta))$$

be the M-representation $\sigma_{|M}$ restricted to $W(\delta)$. Let $\Lambda_\delta \in \mathfrak{t}_{\mathbb{C}}^* \subset \mathfrak{h}_{\mathbb{C}}^*$ denote the infinitesimal character of δ.

Let τ be a finite dimensional representation of P on the vector space V_τ. Define

$$C^\infty \operatorname{Ind}_P^G(\tau) \tag{1}$$

to be the space of smooth functions $f\colon G \to V_\tau$ transforming according to the rule

$$f(xman) = a^{-\rho}\tau(man)^{-1} f(x) \tag{2}$$

for all $x \in G$ and $man \in MAN$ (note the ρ-shift). The group G acts on (1) by the left regular representation L.

If (δ, V_δ) is a finite dimensional representation of M let

$$C^\infty \operatorname{Ind}_M^K(\delta) = \{f\colon K \to V_\delta \mid f(km) = \delta(m)^{-1} f(k), k \in K, m \in M; f \text{ smooth}\}$$

be equipped with the left regular representation of K.

3. Poisson Transforms

In this section we generalize the notion of Poisson transform to the vector bundle situation and derive some simple properties.

Let (τ, V_τ) be a finite dimensional P-representation. We do not assume that τ is irreducible or that $\tau_{|N}$ is trivial.

Definition 3.1. A Poisson transform on $C^\infty \operatorname{Ind}_P^G(\tau)$ is a continuous, linear, G-equivariant map from $C^\infty \operatorname{Ind}_P^G(\tau)$ to $C^\infty \operatorname{Ind}_K^G(\sigma)$.

Given $\varphi \in \operatorname{Hom}_M(V_\tau, W)$ define

$$\mathscr{P}_\tau^\varphi f(x) = \int_K [\sigma(k) \circ \varphi] f(xk)\, dk \tag{3}$$

for $f \in C^\infty \operatorname{Ind}_P^G(\tau)$. It is readily verified that the operator $\mathscr{P}_\tau^\varphi$ is a Poisson transform on $C^\infty \operatorname{Ind}_P^G(\tau)$.

On the other hand, let $\mathscr{P}$ be a Poisson transform on $C^\infty \operatorname{Ind}_P^G(\tau)$. Define the Poisson kernel $\mathrm{P} \in [C^\infty \operatorname{Ind}_P^G(\tau)]' \otimes W$, the strong topological dual of $C^\infty \operatorname{Ind}_P^G(\tau)$ tensored by W, by

$$\langle \mathrm{P}, f \rangle = \mathscr{P} f(e)$$

for $f \in C^{\infty} \operatorname{Ind}_P^G(\tau)$. By the G-equivariance of $\mathscr{P}$ the kernel P completely determines $\mathscr{P}$ by

$$\mathscr{P}f(x) = \langle \mathrm{P}, L_x^{-1} f \rangle.$$

The map

$$(f, g) \mapsto \langle f, g \rangle := \int_K \langle f(k), g(k) \rangle \, dk$$

defines a non degenerate, G-equivariant pairing

$$C^{\infty} \operatorname{Ind}_P^G(\tau) \times C^{\infty} \operatorname{Ind}_P^G(\tau^{\vee}) \to \mathbb{C}$$

(this is why we incorporated a ρ-shift in the definition of $C^{\infty} \operatorname{Ind}_P^G(\tau)$). This pairing induces a G-isomorphism

$$[C^{\infty} \operatorname{Ind}_P^G(\tau)]' \simeq C^{-\infty} \operatorname{Ind}_P^G(\tau^{\vee}),$$

where $C^{-\infty} \operatorname{Ind}_P^G(\tau)$ is the space of generalized functions $f\colon G \to V_\tau$ transforming according to rule (2).

Now $\langle \mathrm{P}, L_k f \rangle = \mathscr{P}(L_k f)(e) = \mathscr{P}f(k^{-1}) = \sigma(k) \langle \mathrm{P}, f \rangle$, hence

$$\mathrm{P} \in [C^{-\infty} \operatorname{Ind}_P^G(\tau^{\vee}) \otimes W]^K. \tag{4}$$

The fact that $G = K\mathrm{P}$ forces P to be smooth. Its transformation properties imply that P is completely determined by $\mathrm{P}(e) \in [V_{\tau^{\vee}} \otimes W]^M \simeq \operatorname{Hom}_M(V_\tau, W)$.

PROPOSITION 3.2. *The map $\varphi \mapsto \mathscr{P}_\tau^\varphi$ is a 1–1 correspondence from $\operatorname{Hom}_M(V_\tau, W)$ to the space of Poisson transforms on $C^{\infty} \operatorname{Ind}_P^G(\tau)$. Its inverse is given by $\mathscr{P} \mapsto \mathrm{P}(e)$.*

Proof. From the definition of $\mathscr{P}_\tau^\varphi$ in (3) it is immediate that the kernel of $\mathscr{P}_\tau^\varphi$ evaluated at the identity is equal to φ. So the transform $\mathscr{P}_\tau^\varphi$ has a nonzero kernel, and hence is nonzero, if $\varphi \neq 0$. This proves the injectiveness of $\varphi \mapsto \mathscr{P}_\tau^\varphi$.

On the other hand, let $\mathscr{P}$ be a Poisson transform on $C^{\infty} \operatorname{Ind}_P^G(\tau)$ and let P be its kernel.

By (4) we know $\mathrm{P} \in [C^{\infty} \operatorname{Ind}_P^G(\tau^{\vee}) \otimes W]^K$. Hence

$$\mathscr{P}f(x) = \langle \mathrm{P}, L_x^{-1} f \rangle = \int_K \langle \mathrm{P}(k), f(xk) \rangle \, dk = \int_K [\sigma(k)\, \mathrm{P}(e)]\, f(xk)\, dk.$$

This proves that $\mathscr{P} = \mathscr{P}_\tau^{\mathrm{P}(e)}$. So $\varphi \mapsto \mathscr{P}_\tau^\varphi$ is surjective. ∎

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For an M-representation (χ, V_χ) and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ define the P-representation $\chi \otimes \lambda \otimes 1$ on V_χ by

$$\chi \otimes \lambda \otimes 1(man) = a^\lambda \chi(m).$$

For $\delta \in \sigma$ let i_δ: $W(\delta) \to W$ denote the inclusion. Define

$$\mathscr{P}_\lambda^\delta := \mathscr{P}^{i_\delta}_{\sigma_\delta \otimes -\lambda \otimes 1}$$

(note the minus sign).

By the G-equivariance of the Poisson transform we have that $L_Z \circ \mathscr{P} = \mathscr{P} \circ L_Z$ for all $Z \in \mathscr{Z}(\mathfrak{g})$. Now $\mathscr{Z}(\mathfrak{g})$ acts on $C^\infty \operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda \otimes 1)$ by the multiplicative character $\gamma_{\mathfrak{h}}(\cdot)(\Lambda_\delta - \lambda)$. Hence

$$L_Z \mathscr{P}_\lambda^\delta = \gamma_{\mathfrak{h}}(Z)(\Lambda_\delta - \lambda)\, \mathscr{P}_\lambda^\delta. \tag{5}$$

For $f \in C^\infty \operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda \otimes 1)$, it is easy to describe the behaviour of $\mathscr{P}_\lambda^\delta f(xa)$ as $a \to \infty$. Here, by $a \to \infty$ we mean that a^α tends to infinity. The map $(\bar{n}, m) \mapsto \kappa(\bar{n})\, m$ is a diffeomorphism on $\bar{N} \times M$ onto an open dense subset of K. Since the integrand $k \mapsto (\sigma(k) \circ \varphi) f(xk)$ of (3) is right M-invariant one finds

$$\mathscr{P}_\lambda^\delta f(xa) = a^{\lambda - \rho} \int_{\bar{N}} \sigma(\kappa(\bar{n}))\, e^{(-\lambda - \rho) H(\bar{n})} i_\delta f(xa\bar{n}a^{-1})\, d\bar{n}.$$

Now $\lim_{a \to \infty} a\bar{n}a^{-1} = e$ and as in [16, Chap. 5], one checks that the limit and integral can be interchanged provided $\operatorname{Re} \lambda > 0$. Here $\operatorname{Re} \lambda > 0$ means that $(\operatorname{Re} \lambda, \alpha) > 0$, where $(\cdot, \cdot)$ is the usual inner product on $\mathfrak{a}^*$. Define for $\operatorname{Re} \lambda > 0$ the $\mathbf{c}$-function $\mathbf{c}(\lambda) \in \operatorname{End}_M(W)$ by

$$\mathbf{c}(\lambda) = \int_{\bar{N}} \sigma(\kappa(\bar{n}))\, e^{(-\lambda - \rho) H(\bar{n})}\, d\bar{n}.$$

Then

$$\lim_{a \to \infty} a^{-\lambda + \rho} \mathscr{P}_\lambda^\delta f(xa) = \mathbf{c}(\lambda)[i_\delta f](x) \tag{6}$$

Define $\mathbf{c}_\delta(\lambda) \in \operatorname{End}_M(W(\delta))$ by

$$\mathbf{c}_\delta(\lambda) = \pi_\delta \mathbf{c}(\lambda)\ i_\delta.$$

COROLLARY 3.3. *If* $\operatorname{Re} \lambda > 0$ *and* $\mathbf{c}_\delta(\lambda)$ *is invertible, then the Poisson transform* $\mathscr{P}_\lambda^\delta$ *is injective.*

4. Asymptotics

In this section we prove the existence of asymptotic expansions for certain generalized eigenfunctions of the Casimir in $C^\infty \operatorname{Ind}_K^G(\sigma)$. The dependence on a holomorphic parameter is investigated. This section is quite technical and the reader who is only interested in its applications to Poisson transforms can skip the proofs, which are postponed to the end of the section. The ideas and proofs are along the lines of [2] and [1, Sect. 11 and 12].

For any function $f\colon G \to W$ and for $r \in \mathbb{R}$ define the r-norm of f by

$$\|f\|_r = \sup_{x \in G} \|x\|^{-r} \|f(x)\|_W,$$

where $\|\cdot\|_W$ is a K-invariant norm on W and $\|x\|$ is the operator norm of $\mathrm{Ad}(x)$ on $\mathfrak{g}$. Denote the Banach space of continuous functions with finite r-norm by $C_r(G, W)$. The space $C_r(G, W)$ is invariant under the left regular representation L. Denote the Banach space of C^q-vectors for L in $C_r(G, W)$ by $C_r^q(G, W)$. The space $C_r^\infty(G, W)$ of smooth vectors in $C_r(G, W)$ is a Fréchet space. For more details on these function spaces the reader is referred to [2, Sect. 2].

Let $\mathscr{C}$ be the Casimir operator of G. For $\mu \in \mathbb{C}$ define

$$C_{\mu,r}^\infty \operatorname{Ind}_K^G(\sigma) = \{f \in C^\infty \operatorname{Ind}_K^G(\sigma) \cap C_r^\infty(G, W) \mid (\mathscr{C} - \mu) f = 0\},$$

$$C_{\mu,*}^\infty \operatorname{Ind}_K^G(\sigma) = \bigcup_{r \in \mathbb{R}} C_{\mu,r}^\infty \operatorname{Ind}_K^G(\sigma)$$

For technical reasons, which become clear in Section 5, we have to consider a possibly larger space of generalized eigensections. For μ, $\mu_0 \in \mathbb{C}$ define

$$L_{\mu_0}^2 \operatorname{Ind}_K^G(\sigma) = L^2(G, W) \cap C_{\mu_0,*}^\infty \operatorname{Ind}_K^G(\sigma),$$

$$\mathscr{E}_{\mu,r}^{\mu_0} = \{f \in C^\infty \operatorname{Ind}_K^G(\sigma) \cap C_r^\infty(G, W) \mid (\mathscr{C} - \mu) f \in L_{\mu_0}^2 \operatorname{Ind}_K^G(\sigma)\},$$

$$\mathscr{E}_\mu^{\mu_0} = \bigcup_{r \in \mathbb{R}} \mathscr{E}_{\mu,r}^{\mu_0}.$$

Note that $C_{\mu,r}^\infty \operatorname{Ind}_K^G(\sigma) \subset \mathscr{E}_{\mu,r}^{\mu_0}$ and $C_{\mu,*}^\infty \operatorname{Ind}_K^G(\sigma) \subset \mathscr{E}_\mu^{\mu_0}$. If there are no square integrable eigensections with eigenvalue μ_0 then $\mathscr{E}_\mu^{\mu_0}$ coincides with $C_{\mu,*}^\infty \operatorname{Ind}_K^G(\sigma)$.

For $\delta \in \sigma$ let p_δ be the second-degree polynomial on $\mathfrak{a}_\mathbb{C}^*$ defined by

$$p_\delta(\lambda) = \gamma_\mathfrak{b}(\mathscr{C})(\Lambda_\delta - \lambda). \tag{7}$$

The polynomial p_δ is even (cf. (14) below).

Define for $\mu \in \mathbb{C}$

$$X(\mu) = \bigcup_{\delta \in \sigma} \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid p_\delta(\lambda) = \mu\},$$

$$X^\pm(\mu) = \{\lambda \in X(\mu) \mid \pm \operatorname{Re} \lambda \geqslant 0\},$$

$$X_R(\mu) = \{\operatorname{Re} \lambda \mid \lambda \in X(\mu)\}.$$

THEOREM 4.1. *Fix μ, $\mu_0 \in \mathbb{C}$ and $r \in \mathbb{R}$.*

(i) *Let $f \in \mathscr{E}_{\mu, r}^{\mu_0}$, $x \in G$. Then there exist unique W-valued polynomials $p_\mu^\xi(f)(x)$ on $\mathfrak{a}$ such that*

$$f(xa) \sim \sum_\xi p_\mu^\xi(f)(x)(\log a)\, a^{\xi - \rho} \qquad (a \to \infty). \tag{8}$$

Here the summation is over $\xi \in \{\lambda - l\alpha \mid \lambda \in X(\mu) \cup X^-(\mu_0), l \in \mathbb{N}_0\}$.

(ii) *Let $\xi \in \{\lambda - l\alpha \mid \lambda \in X(\mu) \cup X^-(\mu_0), l \in \mathbb{N}_0\}$. Then there exists a constant r' such that $f \mapsto p_\mu^\xi(f)$ is a G-equivariant, continuous, linear map from $\mathscr{E}_{\mu, r}^{\mu_0}$ to $C_{r'}^\infty(G, W) \otimes \operatorname{Pol}(\mathfrak{a})$.*

Remark 4.2. One can prove almost the same theorem for eigenfunctions in $C_{\mu, *}^\infty \operatorname{Ind}_K^G(\sigma)$, only then the summation in (8) is over $\xi \in \{\lambda - l\alpha \mid \lambda \in X(\mu), l \in \mathbb{N}_0\}$. Intuitively speaking the fact that the summation in (8) is over $\xi \in \{\lambda - l\alpha \mid \lambda \in X(\mu) \cup X^-(\mu_0), l \in \mathbb{N}_0\}$ says that the "positive part" of the asymptotic expansions of functions in $\mathscr{E}_\mu^{\mu_0}$ is the same as for functions in $C_{\mu, *}^\infty \operatorname{Ind}_K^G(\sigma)$.

DEFINITION 4.3. Let $f \in \mathscr{E}_\mu^{\mu_0}$.
An element $\xi \in \mathfrak{a}_{\mathbb{C}}^*$ is called an exponent of f if $p_\mu^\xi(f) \neq 0$. An element $\xi \in \mathfrak{a}_{\mathbb{C}}^*$ is called a leading exponent if ξ is maximal in the set of exponents of f (then $\xi \in X(\mu) \cup X^-(\mu_0)$, cf. Lemma 4.23 below). The corresponding coefficient $p_\mu^\xi(f)$ is called a leading coefficient.

A map φ from an open subset Ω of $\mathbb{C}$ to $C_r^\infty(G, W)$ is called holomorphic if for each $q \in \mathbb{N}_0$ it maps Ω holomorphically into the Banach space $C_r^q(G, W)$. With this notion we can define holomorphic families of generalized eigenfunctions. Let Ω be an open subset of $\mathbb{C}$, μ_0 and $r \in \mathbb{R}$. We denote by

$$\mathscr{E}_{\Omega, r}^{\mu_0}$$

the set of all smooth maps $f\colon \Omega \times G \to W$ such that

(i) for each $\mu \in \Omega$ the function f_μ, defined by $f_\mu(x) = f(\mu, x)$, belongs to $\mathscr{E}_{\mu, r}^{\mu_0}$, and

(ii) the map $\mu \mapsto f_\mu$ maps Ω holomorphically to $C_r^\infty(G, W)$.

EXAMPLE 4.4. Fix $\delta\in\sigma$ and $f\in C^\infty\,\mathrm{Ind}_M^K(\sigma_\delta)$. For each $\lambda\in\mathfrak{a}_\mathbb{C}^*$ let $\mathcal{L}_\lambda f\in C^\infty\,\mathrm{Ind}_P^G(\sigma_\delta\otimes-\lambda\otimes 1)$ be defined by $\mathcal{L}_\lambda f(kan)=a^{\lambda-\rho}f(k)$. By (5) the transform $\mathcal{P}_\lambda^\delta$ maps into the eigenfunctions with eigenvalue $p_\delta(\lambda)$. Using the same argument as in [2, Examples 2.2(i) and 2.3], it is easy to prove that there is an $r(\lambda)\in\mathbb{R}$ such that $\mathcal{P}_\lambda^\delta$ maps into $C_{r(\lambda)}^\infty(G, W)$. Let $\Omega_0\subset\mathfrak{a}_\mathbb{C}^*$ be a bounded open subset on which p_δ has a well-defined and holomorphic inverse. Let $\Omega=p_\delta(\Omega_0)$ and $r=\sup_{\lambda\in\Omega_0}r(\lambda)$, which is finite since $\lambda\mapsto r(\lambda)$ is a locally bounded function. Then

$$(\mu, x)\mapsto\mathcal{P}^\delta_{p_\delta^{-1}(\mu)}[\mathcal{L}_{p_\delta^{-1}(\mu)}(f)](x)$$

is an element of $\mathcal{E}^{\mu_0}_{\Omega, r}$.

For each $\mu\in\mathbb{C}$ with $0\notin X_R(\mu)$, and for each $\delta\in\sigma$ define the unique element $\lambda_\delta(\mu)$ in $\mathfrak{a}_\mathbb{C}^*$ by

$$p_\delta(\lambda_\delta(\mu))=\mu \qquad\text{and}\qquad \operatorname{Re}\lambda_\delta(\mu)>0. \tag{9}$$

Note that $\{\mu\,|\,0\notin X_R(\mu)\}$ is dense in $\mathbb{C}$.

LEMMA 4.5. *Let $\mu_0\in\mathbb{C}$. If $0\notin X_R(\mu_0)$ then there is a neighbourhood Ω_0 of μ_0 such that $\mu\mapsto\lambda_\delta(\mu)$ is holomorphic on Ω_0 for each $\delta\in\sigma$.*

Since the p_δ are even polynomials we know that $X(\mu)=\bigcup\{\pm\lambda_\delta(\mu)\}$. Once and for all fix a $\mu_0\in\mathbb{C}$ with the property that $0\notin X_R(\mu_0)$. Define

$$\Xi_\delta(\mu):=\{\lambda_{\delta'}(\mu)-l\alpha\,|\,\delta'\in\sigma, l\in\mathbb{N}_0:\lambda_{\delta'}(\mu_0)-l\alpha=\lambda_\delta(\mu_0)\}, \tag{10}$$

the set of exponents in $X(\mu)-\mathbb{N}_0\alpha$ which for $\mu=\mu_0$ coincide with $\lambda_\delta(\mu_0)$. Note that $\Xi_\delta(\mu_0)=\{\lambda_\delta(\mu_0)\}$.

THEOREM 4.6. *Let Ω_0 be an open neighbourhood of μ_0, such that the functions $\mu\mapsto\lambda_\delta(\mu)$ $(\delta\in\sigma)$, are holomorphic on Ω_0. Let $f:\Omega_0\times G\to W$ be an element of $\mathcal{E}^{\mu_0}_{\Omega_0, r}$ for a fixed $r\in\mathbb{R}$. Then there exist a subset $\Omega\subset\Omega_0$ containing μ_0 and a constant $r'\in\mathbb{R}$ such that for all $\delta\in\sigma$*

$$(\mu, H)\mapsto\sum_{\xi\in\Xi_\delta(\mu)}p_\mu^\xi(f_\mu)(\cdot)(H)\,e^{\xi(H)} \tag{11}$$

is continuous from $\Omega\times\mathfrak{a}$ to $C_{r'}^\infty(G, W)$ and holomorphic in μ.

For each $\delta\in\sigma$, $\mu\in\mathbb{C}$, and $f\in\mathcal{E}^{\mu_0}_\mu$ define $\beta^\delta_{\mu,\mu_0}(f)\in C^\infty(K, W(\delta))$ by

$$\beta^\delta_{\mu,\mu_0}(f)(k)=\pi_\delta\sum_{\xi\in\Xi_\delta(\mu)}p_\mu^\xi(f)(k)(0)$$

for $k \in K$. The function $\beta^{\delta}_{\mu,\mu_0}(f)$ is called the *δ-boundary value of f*. Note that $\beta^{\delta}_{\mu_0,\mu_0}(f)(k) = \pi_\delta\, p^{\lambda_\delta(\mu_0)}_{\mu_0}(f)(k)(0)$.

THEOREM 4.7. (i) *The δ-boundary value map $\beta^{\delta}_{\mu,\mu_0}$ maps $\mathscr{E}^{\mu_0}_{\mu}$ linearly, continuously, and K-equivariantly into $C^\infty \operatorname{Ind}^K_M(\sigma_\delta)$.*

(ii) *There is an open neighbourhood Ω of μ_0 such that for all $r \in \mathbb{R}$ and $f \in \mathscr{E}^{\mu_0}_{\Omega,r}$ the map $\mu \mapsto \beta^{\delta}_{\mu,\mu_0}(f_\mu)$ is a holomorphic map from Ω to the Banach space $C^q \operatorname{Ind}^K_M(\sigma_\delta)$ for each q.*

Proof. (i) All the statements except for the transformation property under M of $\beta^{\delta}_{\mu,\mu_0}(f)$ follow directly from Theorem 4.1. The transformation property is contained in Lemma 4.19 below.

(ii) This follows directly from Theorem 4.6: the map $\beta^{\delta}_{\mu,\mu_0}$ is the map in (11) restricted to $\Omega \times \{0\}$ and composed with restriction to K and projection onto $W(\delta)$. ∎

Define

$$\beta_{\mu,\mu_0} = \sum_{\delta \in \sigma} \beta^{\delta}_{\mu,\mu_0} : \mathscr{E}^{\mu_0}_{\mu} \to C^\infty \operatorname{Ind}^K_M(\sigma_{|M}).$$

Given $\delta \in \sigma$ define $\Delta[\delta] = \{\delta' \in \sigma \mid p_{\delta'} = p_\delta\}$. Note that $p_{\delta'} = p_\delta$ is equivalent to $\gamma_{\mathfrak{b}}(\mathscr{C})(\Lambda_{\delta'}) = \gamma_{\mathfrak{b}}(\mathscr{C})(\Lambda_\delta)$, which is equivalent to $\|\Lambda_{\delta'}\| = \|\Lambda_\delta\|$ (also see Lemma 4.24). Define the M-representation

$$(\sigma_{[\delta]}, W_{[\delta]}) = \bigoplus_{\delta' \in \Delta[\delta]} (\sigma_{\delta'}, W(\delta')).$$

We need the following properties of the coefficients in the asymptotics.

PROPOSITION 4.8. *Let $f \in \mathscr{E}^{\mu_0}_{\mu}$ and suppose λ is a leading exponent of f with $\operatorname{Re}\lambda > 0$. Then $p^{\lambda}_{\mu}(f) \in C^\infty \operatorname{Ind}^G_P(\sigma_{[\delta]} \otimes -\lambda \otimes 1)$ for any $\delta \in \sigma$ with $p_\delta(\lambda) = \mu$ (such a δ exists). In particular, the polynomial part of $p^{\lambda}_{\mu}(f)$ is constant.*

PROPOSITION 4.9. *Let $\mu_0 \in \mathbb{C}$ be such that $0 \notin X_R(\mu_0)$. Let $f \in \mathscr{E}^{\mu_0}_{\mu_0}$. Then $\beta_{\mu_0,\mu_0}(f) = 0$ if and only if f is square integrable.*

PROPOSITION 4.10. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\delta \in \sigma$ be such that* $\operatorname{Re}\lambda > 0$ *and $0 \notin X_R(p_\delta(\lambda))$. Let $g \in C^\infty \operatorname{Ind}^G_P(\sigma_\delta \otimes -\lambda \otimes 1)$.*

If ξ, *with* $\operatorname{Re}\xi > 0$, *is an exponent of* $\mathscr{P}^{\delta}_{\lambda} g$ *then* $\xi = \lambda - l\alpha$ *for some* $l \in \mathbb{N}_0$. *Moreover,* $p^{\lambda}_{p\delta(\lambda)}(\mathscr{P}^{\delta}_{\lambda} g)$ *has a constant polynomial part and takes values in* $W(\delta)$.

Before giving the proofs of the above statements we derive the generic bijectiveness of the Poisson transform as a consequence.

THEOREM 4.11. *Define*

$$\mathscr{P}_\mu = \sum_{\delta \in \sigma} \mathscr{P}^{\delta}_{\lambda_\delta(\mu)} \circ \pi_\delta : C^\infty \operatorname{Ind}_P^G\Big(\bigoplus_{\delta \in \sigma} [\sigma_\delta \otimes -\lambda_\delta(\mu) \otimes 1]\Big) \to C^\infty_{\mu,*} \operatorname{Ind}_K^G(\sigma).$$

Then for generic μ *the Poisson transform* $\mathscr{P}_\mu$ *is bijective.*

Here generic means outside a finite set of horizontal half lines and outside a locally finite set of points.

Proof. The fact that $\mathscr{P}_\mu$ maps into the eigenfunctions with eigenvalue μ follows from (5) and the definition of the $\lambda_\delta(\mu)$ in (9). Using the same argument as in [2, Examples 2.2(i) and 2.3], it is easy to prove that $\mathscr{P}_\mu$ maps into $C^\infty_r(G, W)$ for some $r \in \mathbb{R}$.

Define $\mathbf{c}(\mu) = \sum \mathbf{c}(\lambda_\delta(\mu))\, \pi_\delta \in \operatorname{End}(W)$. The reader should be aware that $\mathbf{c}(\mu)$ is not a meromorphic function of μ.

We need the following four conditions which are fulfilled for generic μ.

(A) The endomorphism $\mathbf{c}(\mu)$ is invertible.

(B) There are no square integrable functions in $C^\infty_\mu \operatorname{Ind}_K^G(\sigma)$.

(C) The elements in $X(\mu)$ are incomparable with respect to $\prec$.

(D) $0 \notin X_R(\mu)$.

Condition (C) implies that each element of $X(\mu)$ is maximal in the set $\{\lambda - l\alpha \mid \lambda \in X(\mu), l \in \mathbb{N}_0\}$. Hence for each $\lambda \in X(\mu)$ the coefficient $p^{\lambda}_{\mu}(f)$ is a leading coefficient whenever it is nontrivial. By Proposition 4.8 and (D) $\beta^{\delta}_{\mu,\mu}(f) = \pi_\delta\, p^{\lambda_\delta(\mu)}_{\mu}(f)(\cdot)_{|K}(0) = \pi_\delta\, p^{\lambda_\delta(\mu)}_{\mu}(f)(\cdot)_{|K}$. By (C) we have that for all $\delta' \notin \Delta[\delta]$ the exponent $\lambda_{\delta'}(\mu)$ is not comparable to $\lambda_\delta(\mu)$. Hence by Proposition 4.10 $\lambda_{\delta'}(\mu)$ is not an exponent of $\mathscr{P}^{\delta}_{\lambda_\delta(\mu)} g$. If $\delta' \in \Delta[\delta]$ and $\delta' \neq \delta$ by Proposition 4.10 we have $\pi_{\delta'}\, p^{\lambda_{\delta'}(\mu)}_{\mu}(\mathscr{P}^{\delta}_{\lambda_\delta(\mu)} g) = 0$. We see that for $\delta' \neq \delta$ we have $\pi_{\delta'}\, p^{\lambda_{\delta'}(\mu)}_{\mu}(\mathscr{P}^{\delta}_{\lambda_\delta(\mu)} g) = 0$, hence $\beta^{\delta'}_{\mu,\mu}(\mathscr{P}^{\delta}_{\lambda_\delta(\mu)} g) = 0$. Let $\mathscr{R}: C^\infty \operatorname{Ind}_P^G(\tau) \to C^\infty \operatorname{Ind}_M^K(\tau_{|M})$ denote the restriction to K (here τ is arbitrary). Using (6) we see that $p^{\lambda_\delta(\mu)}_{\mu}(\mathscr{P}^{\delta}_{\lambda_\delta(\mu)} g) = \mathbf{c}(\lambda_\delta(\mu))\, i_\delta \mathscr{R} g$ for all $g \in C^\infty \operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda_\delta(\mu) \otimes 1)$. Hence

$$\beta_{\mu,\mu} \circ \mathscr{P}_\mu = \mathbf{c}(\mu)\, \mathscr{R}, \tag{12}$$

and by (A) the injectiveness of $\mathscr{P}_\mu$ follows. Let $\mathscr{L}_\delta: C^\infty \operatorname{Ind}_M^K(\sigma_\delta) \to C^\infty \operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda_\delta(\mu) \otimes 1)$ be defined by $\mathscr{L}_\delta\, g(kan) = a^{\lambda_\delta(\mu) - \rho} g(k)$. Using (12) we see that the function

$$f - (\mathscr{P}_\mu \circ \mathbf{c}(\mu)^{-1}) \left[\sum_{\delta \in \sigma} i_\delta \mathscr{L}_\delta \beta^\delta_{\mu,\mu}(f) \right]$$

lies in the kernel of $\beta_{\mu,\mu}$. By Proposition 4.9, (B), and (D), this implies that $f = (\mathscr{P}_\mu \circ \mathbf{c}(\mu)^{-1})[\sum \mathscr{L}_\delta \beta^\delta_{\mu,\mu}(f)]$, proving that $\mathscr{P}_\mu$ is surjective. ∎

The proof of the above theorem breaks down if one of the four conditions above is not fulfilled. The Main Theorem in Section 5 gives an answer in all of these degenerate cases except when $0 \in X_R(\mu)$.

In the remaining part of this section we give the proofs of the above theorems and propositions.

We first introduce a generalization of the Harish–Chandra homomorphism. We identify elements of $\mathscr{U}(\mathfrak{g}) \otimes E$, $E = \operatorname{End}(W)$, with the left invariant differential operators on $C^\infty(G, W)$. Let I_σ be the left ideal in $\mathscr{U}(\mathfrak{g}) \otimes E$ generated by

$$X \otimes 1 + 1 \otimes \sigma(X), \qquad X \in \mathfrak{k}.$$

Then I_σ annihilates $C^\infty \operatorname{Ind}_K^G(\sigma)$. Define $'\Gamma: \mathscr{U}(\mathfrak{g}) \otimes E \to \mathscr{U}(\mathfrak{a}) \otimes E$ by

$$u \otimes \varphi - {}'\Gamma(u \otimes \varphi) \in \bar{\mathfrak{n}} \mathscr{U}(\mathfrak{g}) \otimes E + I_\sigma.$$

Then $\Gamma(\cdot)(\lambda) := {}'\Gamma(\cdot)(\lambda - \rho)$ defines the generalized Harish–Chandra homomorphism. For $Z \in \mathscr{Z}(\mathfrak{g})$ we denote $\Gamma(Z) = \Gamma(Z \otimes I)$.

We investigate the Casimir and the relation between Γ and $\gamma_{\mathfrak{h}}$. Let μ be the map from $\mathscr{Z}(\mathfrak{g})$ to $\mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m})$ defined by $Z - \mu(Z) \in \bar{\mathfrak{n}} \mathscr{U}(\mathfrak{g})$, $Z \in \mathscr{Z}(\mathfrak{g})$. Let I_m be the left ideal in $\mathscr{U}(\mathfrak{a} \oplus \mathfrak{m}) \otimes E$ generated by $X \otimes 1 + 1 \otimes \sigma(X)$, $X \in \mathfrak{m}$. Let $'\Gamma_{\mathfrak{m}}$ be the map from $\mathscr{U}(\mathfrak{a} \oplus \mathfrak{m}) \otimes E$ to $\mathscr{U}(\mathfrak{a}) \otimes E$ defined by $X \otimes \varphi - {}'\Gamma_{\mathfrak{m}}(X \otimes \varphi) \in I_{\mathfrak{m}}$, $X \otimes \varphi \in \mathscr{U}(\mathfrak{a} \oplus \mathfrak{m}) \otimes E$. Let $'\gamma_{\mathfrak{a} \oplus \mathfrak{m}}: \mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m}) \to \mathscr{U}(\mathfrak{h})$ be defined by $Z - {}'\gamma_{\mathfrak{a} \oplus \mathfrak{m}}(Z) \in \mathfrak{m}_{\mathbb{C}}^- \mathscr{U}(\mathfrak{a} \oplus \mathfrak{m})$, where $\mathfrak{m}_{\mathbb{C}}^-$ is the sum of negative rootspaces in $\mathfrak{m}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$, such that $\mathfrak{m}_{\mathbb{C}}^- \subset \mathfrak{g}_{\mathbb{C}}^-$. Finally, let $\gamma_{\mathfrak{t}}$ be the Harish–Chandra homomorphism for $\mathfrak{m}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{m}_{\mathbb{C}}^-$. Then

$$'\Gamma = {}'\Gamma_{\mathfrak{m}} \circ (\mu \otimes I) \qquad \text{on} \quad \mathscr{Z}(\mathfrak{g}) \otimes E,$$

$$'\gamma_{\mathfrak{h}} = {}'\gamma_{\mathfrak{a} \oplus \mathfrak{m}} \circ \mu \qquad \text{on} \quad \mathscr{Z}(\mathfrak{g}).$$

We identify $\mathscr{U}(\mathfrak{a})$ with $S(\mathfrak{a})$, the algebra of polynomials on $\mathfrak{a}_{\mathbb{C}}^*$.

LEMMA 4.12. *For $\delta \in \sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have $\Gamma(\mathscr{C})(\lambda)_{|W(\delta)} = \gamma_{\mathfrak{h}}(\mathscr{C})$ $(\Lambda_\delta + \lambda)\, I_{W(\delta)}$.*

Proof. Since $\mathfrak{a} \perp \mathfrak{m}$, $\mu(\mathscr{C})$ is an element of $\mathscr{U}(\mathfrak{a}) \oplus \mathscr{Z}(\mathfrak{m})$. Let $\pi_\mathfrak{a}$, resp. $\pi_\mathfrak{m}$ be the projections in this direct sum. Using the above relations we find:

$$\begin{aligned}\gamma_\mathfrak{h}(\mathscr{C})(\Lambda_\delta + \lambda)\, I_{W(\delta)} &= {}'\gamma_{\mathfrak{a}\oplus\mathfrak{m}}(\mu[\mathscr{C}])(\Lambda_\delta - \rho_M + \lambda - \rho)\, I_{W(\delta)} \\ &= {}'\gamma_\mathfrak{t}(\pi_\mathfrak{m}\mu[\mathscr{C}])(\Lambda_\delta - \rho_M)\, I_{W(\delta)} + \pi_\mathfrak{a}(\mu[\mathscr{C}])(\lambda - \rho)\, I_{W(\delta)} \\ &= \sigma_\delta(\pi_\mathfrak{m}\mu[\mathscr{C}]) + \pi_\mathfrak{a}(\mu[\mathscr{C}])(\lambda - \rho)\, I_{W(\delta)} \\ &= {}'\Gamma_\mathfrak{m}(\mu[\mathscr{C}])(\lambda - \rho)_{|W(\delta)} \\ &= {}'\Gamma(\mathscr{C})(\lambda - \rho)_{|W(\delta)}. \quad \blacksquare\end{aligned}$$

By the above lemma we see that

$$\Gamma(\mathscr{C}) = \sum p_\delta \otimes \pi_\delta, \tag{13}$$

where the p_δ are the polynomials defined in (7). They can be explicitly computed to be

$$p_\delta = (\alpha, \alpha)(H_0 - \rho(H_0))(H_0 + \rho(H_0)) - \gamma_\mathfrak{t}(\pi_\mathfrak{m}\mu[\mathscr{C}])(\Lambda_\delta), \tag{14}$$

where $H_0 \in \mathfrak{a}$ is determined by $\alpha(H_0) = 1$. Define ${}'p_\delta(\lambda) = p_\delta(\lambda + \rho)$. Then

$${}'\Gamma(\mathscr{C}) = \sum {}'p_\delta \otimes \pi_\delta. \tag{15}$$

Define

$$C^\infty_{\mu,\mu_0,*} \operatorname{Ind}_K^G(\sigma) = \{f \in C^\infty \operatorname{Ind}_K^G(\sigma) \cap C^\infty_*(G, W) \mid (\mathscr{C} - \mu)(\mathscr{C} - \mu_0) f = 0\}.$$

We first prove the existence of asymptotic expansions of functions in $C^\infty_{\mu,\mu_0,*} \operatorname{Ind}_K^G(\sigma)$ and then restrict to functions in $\mathscr{E}_\mu^{\mu_0} \subset C^\infty_{\mu,\mu_0,*} \operatorname{Ind}_K^G(\sigma)$.

We need a proposition similar to Proposition 12.4 in [1]. In the following only the lemmas and proofs are given which really differ from the corresponding ones in [1].

Define

$$X_\mu = (\mathscr{C} - \mu)(\mathscr{C} - \mu_0).$$

Let m: $[S(\mathfrak{a}) \otimes E] \otimes [S(\mathfrak{a}) \otimes E] \to S(\mathfrak{a}) \otimes E$ be the multiplication map. Let $\mathbb{C}_1[x]$, resp. $S_1(\mathfrak{a})$, denote the space of polynomials in x, resp. on $\mathfrak{a}^*_\mathbb{C}$, with degree less than or equal to 1. Define

$$Z \subset S(\mathfrak{a}) \otimes E$$

to be the finite dimensional image of $[S_1(\mathfrak{a}) \otimes E] \otimes \mathbb{C}_1[\mathscr{C} - \mu_0]$ under $m \circ (1 \otimes {}'\Gamma)$.

LEMMA 4.13. *The map* $m \circ (1 \otimes {}'\Gamma)$: $Z \otimes \mathbb{C}[X_\mu] \to S(\mathfrak{a}) \otimes E$ *is an isomorphism.*

Proof. We first prove

$$m\circ(1\otimes{}'\Gamma_{\bar{p}})\colon [S_1(\mathfrak{a})\otimes E]\otimes\mathbb{C}[\mathscr{C}-\mu]\simeq S(\mathfrak{a})\otimes E. \tag{16}$$

For any fixed $p_0\in\mathbb{C}[x]$ of degree k and any $p\in\mathbb{C}[x]$ there are $q_i\in\mathbb{C}_{k-1}[x]$ and $q_i'\in\mathbb{C}[x]$ such that

$$p=\sum_i q_i\cdot q_i'(p_0).$$

This follows from an easy induction on the degree of p.

Hence for all $\delta\in\sigma$ and $p\in S(\mathfrak{a})$ there are $q_i\in S_1(\mathfrak{a})$ and $q_i'\in S(\mathfrak{a})$ such that

$$p=\sum_i q_i\cdot q_i'('p_\delta-\mu),$$

since $'p_\delta-\mu$ has degree two. Hence for arbitrary $\varphi\in E$ we have

$$\begin{aligned} p\otimes\varphi\pi_\delta &= \sum_i q_i\cdot q_i'('p_\delta-\mu)\otimes\varphi\pi_\delta\\ &=\sum_i m(q_i\otimes\varphi\pi_\delta,\, q_i'('p_\delta-\mu)\otimes\pi_\delta)\\ &=\sum_i m(q_i\otimes\varphi\pi_\delta,\, {}'\Gamma_{\bar{p}}[q_i'(\mathscr{C}-\mu)]),\end{aligned}$$

since $'\Gamma_{\bar{p}}[q_i'(\mathscr{C}-\mu)]=\sum_{\delta'\in\sigma}q_i'('p_{\delta'}-\mu)\otimes\pi_{\delta'}$. Hence $p\otimes\varphi\pi_\delta$ is in the image of the map $m\circ(1\otimes{}'\Gamma_{\bar{p}})$ restricted to $[S_1(\mathfrak{a})\otimes E]\otimes\mathbb{C}[\mathscr{C}-\mu]$. For an arbitrary $\omega=\sum_i p_i\otimes\varphi_i$ in $S(\mathfrak{a})\otimes E$ we write $\omega=\sum_i\sum_\delta p_i\otimes\varphi_i\pi_\delta$, and hence ω is in the image of the map $m\circ(1\otimes{}'\Gamma_{\bar{p}})$ restricted to $[S_1(\mathfrak{a})\otimes E]\otimes\mathbb{C}[\mathscr{C}-\mu]$. This proves (16).

We now claim that

$$\mathbb{C}_1[\mathscr{C}-\mu_0]\otimes\mathbb{C}[X_\mu]\to\mathbb{C}[\mathscr{C}-\mu] \tag{17}$$

defined by multiplication of polynomials is an isomorphism. This is equivalent to the trivial fact that $\mathbb{C}_1[x-\mu_0]\otimes\mathbb{C}[(x-\mu_0)(x-\mu)]\to\mathbb{C}[x-\mu]$ is an isomorphism.

Combining (16) and (17), and using the fact that $'\Gamma_{\bar{p}}$ is a homomorphism, we obtain

$$\begin{aligned} S(\mathfrak{a})\otimes E &\simeq [S_1(\mathfrak{a})\otimes E]\otimes\mathbb{C}[\mathscr{C}-\mu]\\ &\simeq [S_1(\mathfrak{a})\otimes E]\otimes(\mathbb{C}_1[\mathscr{C}-\mu_0]\otimes\mathbb{C}[X_\mu])\\ &\simeq ([S_1(\mathfrak{a})\otimes E]\otimes\mathbb{C}_1[\mathscr{C}-\mu_0])\otimes\mathbb{C}[X_\mu]\\ &\simeq Z\otimes\mathbb{C}[X_\mu]. \quad \blacksquare\end{aligned}$$

Let J_μ be the left ideal in $\mathscr{U}(\mathfrak{g}) \otimes E$ generated by I_σ and X_μ. Define

$$\mathrm{Y}_\mu = (\mathscr{U}(\mathfrak{g}) \otimes E)/J_\mu$$

and

$$\mathscr{Y} = \mathscr{U}(\bar{\mathfrak{n}}) \otimes Z.$$

LEMMA 4.14. *The mapping $\Gamma_\mu : \mathscr{Y} \to \mathrm{Y}_\mu$ induced by $u \otimes z \mapsto uz$ is an isomorphism of left $(\mathscr{U}(\bar{\mathfrak{n}}), E)$-modules.*

This result is proved in the same fashion as the proofs of Lemma 11.3 and Corollary 11.4 in [1], using Lemma 4.13.

Define the following $(\mathfrak{g}, K)$ action on $\mathscr{U}(\mathfrak{g}) \otimes E$:

$$\begin{cases} X \cdot u \otimes \varphi = Xu \otimes \varphi, & X \in \mathfrak{g}, \\ k \cdot u \otimes \varphi = \mathrm{Ad}(k)\, u \otimes \sigma(k) \circ \varphi \circ \sigma(k)^{-1}, & k \in K, \end{cases} \tag{18}$$

for all $u \otimes \phi \in \mathscr{U}(\mathfrak{g}) \otimes E$. The ideal J_μ is invariant under this action and we therefore have a $(\mathfrak{g}, K)$-action on Y_μ. The isomorphism $\mathrm{Y}_\mu \simeq \mathscr{Y}$ induces a representation π_μ on $\mathscr{Y}$. The family $(\pi_\mu)_{\mu \in \mathbb{C}}$ is holomorphic.

The finite dimensional $\bar{\mathscr{V}}_k := \mathscr{Y}/\bar{\mathfrak{n}}^k \mathscr{Y}$ inherits a $(\mathfrak{a} \oplus \mathfrak{m}, M)$ representation π_μ^k from $(\mathscr{Y}, \pi_\mu)$. Let $\tilde{\mathscr{V}}_k$ be a finite dimensional subspace of $\mathscr{Y}$ mapped bijectively into $\bar{\mathscr{V}}_k$ by the canonical projection. Let $\xi : \tilde{\mathscr{V}}_k \to \bar{\mathscr{V}}_k$ be the restriction of the canonical map. Define

$$m' : \mathscr{Y} \to \mathscr{U}(\mathfrak{g}) \otimes E : u \otimes (p \otimes \varphi) \mapsto up \otimes \varphi.$$

Let $\mathscr{V}_k$ denote the image under m' of $\tilde{\mathscr{V}}_k$. Let $\eta : \mathscr{V}_k \to \tilde{\mathscr{V}}_k$ be the inverse of $m'|_{\tilde{\mathscr{V}}_k}$.

PROPOSITION 4.15. *Let $k \geqslant 1$. There exist*

1. *an algebra homomorphism $b_k(\mu, \cdot)$ from $\mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m})$ to $\mathrm{End}(\mathscr{V}_k)$*
2. *a bilinear map $y_\mu : \mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m}) \times \mathscr{V}_k \to \bar{\mathfrak{n}}^k \mathscr{U}(\bar{\mathfrak{n}} \oplus \mathfrak{a}) \otimes E$,*

both depending polynomially on μ, such that for all $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, $D \in \mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m})$, and $v \in \mathscr{V}_k$:

$$Dv - b_k(\mu, D)\, v - y_\mu(D, v) \in J_\mu.$$

Proof. Let $p_\mu : \mathscr{U}(\mathfrak{g}) \otimes E \to \mathscr{Y}$ be the map defined by

$$p_\mu(u \otimes \varphi) = \pi_\mu(u)(1 \otimes (1 \otimes \varphi))$$

for $u \otimes \varphi \in \mathscr{U}(\mathfrak{g}) \otimes E$; here we use that $1 \otimes E \subset Z$.

Define for $D \in \mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m})$, $\tilde{v} \in \tilde{\mathscr{V}}_k$ the maps

$$\begin{aligned} \tilde{b}_k(\mu, D) &= \xi^{-1} \circ \pi_\mu^k(D) \circ \xi \in \operatorname{End}(\tilde{\mathscr{V}}_k), \\ \tilde{y}_\mu(D, \tilde{v}) &= p_\mu(D \cdot m'(\tilde{v}) - m' \cdot [\tilde{b}_k(\mu, D)(\tilde{v})]) \in \mathscr{Y}. \end{aligned} \tag{19}$$

Then $b_k(\mu, \cdot)$ and y_μ are defined by

$$b_k(\mu, D) = m' \circ \tilde{b}_k(\mu, D) \circ \eta$$

$$y_\mu(D, v) = m'[\tilde{y}_\mu(D, \eta(v))]$$

for $D \in \mathscr{Z}(\mathfrak{a} \oplus \mathfrak{m})$, $v \in \mathscr{V}_k$.

The reader should carefully study the following diagram:

$$\begin{array}{ccccc} \mathscr{Y}/\bar{\mathfrak{n}}^k\mathscr{Y} = \bar{\mathscr{V}}_k & \simeq & \tilde{\mathscr{V}}_k & \simeq & \mathscr{V}_k \\ & & \cap & & \cap \\ & & \mathscr{Y} & \rightarrow & \mathscr{U}(\mathfrak{g}) \otimes E \\ & \Gamma_\mu & \wr\!\wr & & \Big\downarrow \mathrm{pr} \\ & & Y_\mu & = & \mathscr{U}(\mathfrak{g}) \otimes E/J_\mu. \end{array}$$

All the representations involved arise from the $(\mathfrak{g}, K)$-representation on $\mathscr{U}(\mathfrak{g}) \otimes E$.

The proof of the proposition is essentially the same as the proof of Proposition 12.4 in [1]. ∎

We now compute the eigenvalues of $b_1(\mu, \cdot)_{|\mathfrak{a}}$. As in [3, Lemma 1.2], we then control the eigenvalues of all $b_k(\mu, \cdot)_{|\mathfrak{a}}$. The eigenvalues occur as the exponents in the asymptotic expansions of the eigenfunctions.

PROPOSITION 4.16. *The set of eigenvalues of* $b_1(\mu, \cdot)_{|\mathfrak{a}}$ *is equal to* $\{\lambda - \rho \mid \lambda \in X(\mu) \cup X(\mu_0)\}$.

Proof. The weights of $b_1(\mu, H)$, $H \in \mathfrak{a}$ are equal to the weights of $(\pi_\mu^1)_{|\mathfrak{a}}$ (cf. (19)). Now $\mathscr{Y}/\bar{\mathfrak{n}}\mathscr{Y} \simeq Z$ and using $Z \simeq [S(\mathfrak{a}) \otimes E]/\langle '\Gamma(X_\mu)\rangle$ (Lemma 4.13) we compute for $p \in Z$:

$$\pi_\mu^1(H)\, p = \Gamma_\mu^{-1} H \cdot \Gamma_\mu(1 \otimes p) = \Gamma_\mu^{-1}(Hp) = Hp + \langle '\Gamma(X_\mu)\rangle.$$

It is now straightforward to prove that the weights of this representation are precisely the roots of the polynomials $('p_\delta - \mu)('p_\delta - \mu_0)$, $(\delta \in \sigma)$. ∎

PROPOSITION 4.17. *Fix* μ, $\mu_0 \in \mathbb{C}$ *and* $r \in \mathbb{R}$.

(i) *Let* $f \in C^{\infty}_{\mu, \mu_0, r} \operatorname{Ind}_K^G(\sigma)$, $x \in G$. *Then there exist unique W-valued polynomials* $p^{\xi}_{\mu}(f)(x)$ *on* $\mathfrak{a}$ *such that*

$$f(xa) \sim \sum_{\xi} p^{\xi}_{\mu}(f)(x)(\log a)\, a^{\xi - \rho} \qquad (a \to \infty).$$

Here the summation is over $\xi \in \{\lambda - l\alpha \mid \lambda \in X(\mu) \cup X(\mu_0), l \in \mathbb{N}_0\}$.

(ii) *Let* $\xi \in \{\lambda - l\alpha \mid \lambda \in X(\mu) \cup X(\mu_0), l \in \mathbb{N}_0\}$. *Then there exists a constant* r' *such that* $f \mapsto p^{\xi}_{\mu}(f)$ *is a G-equivariant, continuous, linear map from* $C^{\infty}_{\mu, \mu_0, r} \operatorname{Ind}_K^G(\sigma)$ *to* $C^{\infty}_{r'}(G, W) \otimes \operatorname{Pol}(\mathfrak{a})$.

Proof. This follows using the techniques of [2, Sect. 6], and [1, the proof of Proposition 12.6]. Proposition 4.15 is used to prove the existence of the expansion and Proposition 4.16 gives the leading exponents. ∎

In the scalar case [2] Ban and Schlichtkrull took as eigenvalue $\gamma(\mathscr{C})(\lambda)$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, where γ is the usual Harish–Chandra isomorphism of the pair $(\mathfrak{g}, \mathfrak{a})$. In this case the eigenvalues of the matrix $b_1(\lambda, \cdot)_{|\mathfrak{a}}$ were $w\lambda - \rho$, $w \in \mathscr{W}$, the Weyl group of Σ. In this parametrization of the eigenvalue the weights depend holomorphically on λ. In our case a good parametrization is not as obvious: taking the roots of a second order polynomial depending on $\mu \in \mathbb{C}$ is certainly not polynomial and not always holomorphic in μ. But with the assumption $0 \notin X_R(\mu_0)$ we can take the eigenvalue as a parameter.

Proof of Lemma 4.5. Since $0 \notin X_R(\mu_0)$ there is a neighbourhood of μ_0 on which the functions $\mu \mapsto \lambda_\delta(\mu)$ are well-defined. For $\delta \in \sigma$ let c_δ be the constant $\gamma_1(\pi_m \mu[\mathscr{C}])(\Lambda_\delta) + (\alpha, \alpha)\,\rho(H_0)^2$. Using the explicit formula (14) for the polynomials p_δ we see that $p_\delta(\lambda_\delta(\mu)) = \mu$ is equivalent to $(\alpha, \alpha)[\lambda_\delta(\mu)(H_0)^2] - c_\delta = \mu$. So

$$\lambda_\delta(\mu)(H_0) = +(\alpha, \alpha)^{-1/2}\,[\mu - \mu_0 + \mu_0 + c_\delta]^{1/2}$$

which depends holomorphically on μ in a neighbourhood of μ_0 if and only if $\mu_0 + c_\delta \neq 0$. This is provided by the assumption that $0 \notin X(\mu_0)$. Indeed, $0 \neq p_\delta(0) - \mu_0 = -c_\delta - \mu_0$. ∎

PROPOSITION 4.18. *Let* Ω_0 *be an open neighbourhood of* μ_0, *such that the functions* $\mu \mapsto \lambda_\delta(\mu)$, $(\delta \in \sigma)$ *are holomorphic on* Ω_0. *Let* $f\colon \Omega_0 \times G \to W$ *be an element of* $C^{\infty}_{\Omega, \mu_0, r} \operatorname{Ind}_K^G(\sigma)$ *(defined in the obvious way) for a fixed* $r \in \mathbb{R}$. *Then there exist a subset* $\Omega \subset \Omega_0$ *containing* μ_0, *and a constant* $r' \in \mathbb{R}$ *such that for all* $\delta \in \sigma$,

$$(\mu, H) \mapsto \sum_{\xi \in \Xi_\delta(\mu) \cup \Xi_\delta(\mu_0)} p^{\xi}_{\mu}(f_\mu)(\cdot)(H)\, e^{\xi(H)}$$

is continuous from $\Omega \times \mathfrak{a}$ *to* $C^{\infty}_{r'}(G, W)$ *and holomorphic in* μ.

Proof. The proposition is the analogue of Theorem 12.9 in [2] and is proved along the same lines as in [2, Sect. 5, 6, and 7]. Of importance is the fact that the exponents depend holomorphically on μ, which is provided by Lemma 4.5, and that one takes all coinciding exponents into account. ∎

LEMMA 4.19. *For all $ma \in MA$*

$$p_\mu^\xi(f)(xma)(H) = \sigma(m)^{-1}\, p_\mu^\xi(f)(x)(H + \log a)\, a^{\xi-\rho}.$$

Let $\omega \otimes \varphi \in \bar{\mathfrak{n}}\mathscr{U}(\mathfrak{a} \oplus \bar{\mathfrak{n}}) \otimes E$ be such that

$$\mathscr{C} - '\Gamma(\mathscr{C}) \in \omega \otimes \varphi + I_\sigma.$$

Write $\omega \otimes \varphi = \sum \omega_i \otimes \varphi_i$, where ω_i has weight $-\nu_i$, $\nu_i \in \mathbb{Z}_{>0}\alpha$.

LEMMA 4.20. *Let $f \in C^\infty_{\mu, \mu_0, *}\operatorname{Ind}_K^G(\sigma)$ and ξ an exponent of f. Then for all $H \in \mathfrak{a}$:*

$$\begin{aligned} p_\mu^\xi(\mathscr{C}f)(\cdot)(H) = {}& '\Gamma(\mathscr{C}) \cdot p_\mu^\xi(f)(\cdot)(H) \\ &+ \sum_i \omega_i \otimes \varphi_i \cdot p_\mu^{\xi+\nu_i}(f)(\cdot)(H). \end{aligned}$$

Here the elements of $\mathscr{U}(\mathfrak{g}) \otimes E$ act as left invariant differential operators on the first variable of $p_\mu^\lambda(f)$. We use the convection that if λ is not an exponent then $p_\mu^\lambda(f) = 0$.

The proofs of the above lemmas are elementary and follow from the uniqueness of the asymptotic expansions. Note that if $f \in C^\infty_{\mu, \mu_0, *}\operatorname{Ind}_K^G(\sigma)$ then $\mathscr{C}f \in C^\infty_{\mu, \mu_0, *}\operatorname{Ind}_K^G(\sigma)$.

LEMMA 4.21. *Let ξ be a leading exponent of $C^\infty_{\mu, \mu_0, *}\operatorname{Ind}_K^G(\sigma)$. Then $\xi \in X(\mu) \cup X(\mu_0)$.*

Proof. Since ξ is a leading exponent we have that $\xi + \nu$ is not an exponent for all $\nu \in \mathbb{Z}_{>0}\alpha$. Repeated use of Lemma 4.20 results in

$$\begin{aligned} 0 &= p_\mu^\xi((\mathscr{C} - \mu)(\mathscr{C} - \mu_0)f)(\cdot)(H) \\ &= ['\Gamma(\mathscr{C}) - \mu]['\Gamma(\mathscr{C}) - \mu_0] \cdot p_\mu^\xi(f)(\cdot)(H) \\ &= \sum_{\delta \in \sigma} [p_\delta(\xi) - \mu][p_\delta(\xi) - \mu_0]\, \pi_\delta\, p_\mu^\xi(f)(\cdot)(H) \\ &\qquad + \text{derivatives of the polynomial part of } p_\mu^\xi(f)(\cdot), \end{aligned}$$

where we use (13) and the right A-behaviour of $p_\mu^\xi(f)$ contained in Lemma 4.19. Since the derivatives lower the degree of the polynomial of $p_\mu^\xi(f)(\cdot)$ we must have

$$\sum_{\delta \in \sigma} [p_\delta(\xi)-\mu][p_\delta(\xi)-\mu_0]\, \pi_\delta\, p_\mu^\xi(f)(\cdot)(H)=0. \tag{20}$$

Since ξ is an exponent of f we have $p_\mu^\xi(f)\neq 0$, so there is a $\delta \in \sigma$ such that $\pi_\delta\, p_\mu^\xi(f)\neq 0$. Hence (20) implies that $[p_\delta(\xi)-\mu][p_\delta(\xi)-\mu_0]=0$; in other words, $\xi \in X(\mu)\cup X(\mu_0)$. ∎

LEMMA 4.22. *Let $f\in C^\infty_{\mu_0,\mu_0,*}\operatorname{Ind}_K^G(\sigma)$. Then f is square integrable if and only if all exponents of f have real part less than zero.*

Proof. First assume that all exponents of f have real part less than zero. Then there is an $\varepsilon>0$ such that

$$\|f(ka)\|\leqslant Ca^{-(1+\varepsilon)\rho},$$

for all $a\in A^+$ and $k\in K$. Using the polar decomposition of G this estimate readily implies that f is square integrable (see, for instance, [21, Lemma 5.A.3.2]).

Conversely, denote $H=L^2_{\mu_0}\operatorname{Ind}_K^G(\sigma)$. The left K-finite functions in H are also right K-finite and square integrable. Hence by 5.A.3.4 in [21] all exponents of left K-finite functions in H have real part less than zero.

We can write any smooth function φ on G as a sum $\sum_{\delta\in K^\wedge}\varphi^\delta$ of its K-isotypical parts for the left regular representation. Since by Proposition 4.17(ii) taking the coefficient of an eigenfunction is equivariant, continuous, and linear, we have for $f\in H$ and $\xi\in\mathfrak{a}_{\mathbb{C}}^*$

$$p_{\mu_0}^\xi(f^\delta)=[p_{\mu_0}^\xi(f)]^\delta.$$

Hence any exponent of f occurs as an exponent of f^δ for some $\delta\in K^\wedge$. By the above the exponents of f^δ have real part less than zero, so any exponent of f has real part less than zero. ∎

PROPOSITION 4.23. *Let ξ be a leading of $f\in\mathscr{E}_\mu^{\mu_0}$. Then $\xi\in X(\mu)\cup X^-(\mu_0)$.*

Proof. By Lemma 4.21 we know that $\xi\in X(\mu)\cup X(\mu_0)$. Assume $\xi\in X^+(\mu_0)$. By definition we have that $(\mathscr{C}-\mu)f$ is a square integrable function. By Lemma 4.22 every exponent of $(\mathscr{C}-\mu)f$ has real less than zero. Hence

$$p_\mu^\xi((\mathscr{C}-\mu)f)=0.$$

Since ξ is a leading exponent, Lemma 4.20 implies

$$\begin{aligned}0 = p_\mu^\xi((\mathscr{C}-\mu)f) &= ('\Gamma(\mathscr{C})-\mu)\,p_\mu^\xi(f)\\ &= \sum_{\delta\in\sigma}\,[p_\delta(\xi)-\mu]\,\pi_\delta\,p_\mu^\xi(f)\\ &\qquad + \text{derivatives of the polynomial part of } p_\mu^\xi(f).\end{aligned}$$

Since these derivatives lower the degree of the polynomial part of $p_\mu^\xi(f)$ we must have

$$0 = \sum_{\delta\in\sigma}\,[p_\delta(\xi)-\mu]\,\pi_\delta\,p_\mu^\xi(f). \tag{21}$$

Note that for $\mu=\mu_0$ the proposition follows immediately from Lemma 4.21. So we may assume $\mu\neq\mu_0$. There are two possibilities: either $p_\delta(\xi)\neq\mu$ for all $\delta\in\sigma$ or there is a $\delta\in\sigma$ such that $p_\delta(\xi)=\mu$. In the latter case $\xi\in X(\mu)$ and the proposition is proved. In the former case (21) implies that $p_\mu^\xi(f)=0$ which contradicts the fact that ξ is an exponent, and hence $\xi\notin X^+(\mu_0)$. ∎

Proof of Theorem 4.1 *and Theorem* 4.6. Both theorems follow from Proposition 4.17 and Proposition 4.18 if we can prove that in the asymptotics of $f\in\mathscr{E}_\mu^{\mu_0}$ (which exist since $\mathscr{E}_\mu^{\mu_0}\subset C^\infty_{\mu,\mu_0,*}\operatorname{Ind}_K^G(\sigma)$) a summation over $\xi\in\{\lambda-l\alpha \mid \lambda\in X(\mu)\cup X^-(\mu_0), l\in\mathbb{N}_0\}$ suffices. By Proposition 4.23 we know that any maximal element in the asymptotics of $f\in\mathscr{E}_\mu^{\mu_0}$ is an element of $X(\mu)\cup X^-(\mu_0)$. By definition any exponent is smaller than a maximal element in the set of exponents and hence any exponent of f is an element of $\{\lambda-l\alpha \mid \lambda\in X(\mu)\cup X^-(\mu_0), l\in\mathbb{N}_0\}$. ∎

LEMMA 4.24. *For each $\mu_0\in\mathbb{C}$ with $0\notin X_R(\mu_0)$ there is an open neighbourhood Ω such that the following statements are equivalent for all δ, $\delta'\in\sigma$*

(i) $\lambda_\delta(\mu_0)=\lambda_{\delta'}(\mu_0)$,

(ii) $\lambda_\delta\equiv\lambda_{\delta'}$ *on* Ω,

(iii) $p_\delta=p_{\delta'}$,

(iv) *there is a* $\mu\in\Omega\backslash\{\mu_0\}$ *such that* $\lambda_\delta(\mu)\preccurlyeq\lambda_{\delta'}(\mu)$.

Proof. The elements $\pm\lambda_\delta(\mu_0)$ are the two roots of the second degree polynomial $p_\delta-\mu_0=(\alpha,\alpha)\,H_0^2+c_\delta-\mu_0$ (in the notation of the proof of Lemma 4.5). So if $\lambda_\delta(\mu_0)=\lambda_{\delta'}(\mu_0)$ then the polynomials $p_\delta-\mu_0$ and $p_{\delta'}-\mu_0$ have the same roots and the same leading term, hence they are equal. This proves (i) ⇒ (iii). The implications (iii) ⇒ (ii) ⇒ (i) and (ii) ⇒ (iv) are trivial.

If we prove (iv) $\Rightarrow$ (iii) we are finished. It is straightforward to check that the subset $\{\mu \in \mathbb{C} \mid \delta, \delta' \in \sigma, k \in \mathbb{Z}_{\geqslant 1} : \lambda_\delta(\mu) = \lambda_{\delta'}(\mu) - k\alpha\}$ is locally finite in $\mathbb{C}$. Hence for $\mu_0 \in \mathbb{C}$ there is a neighbourhood Ω of μ_0 such that the intersection of $\Omega \backslash \{\mu_0\}$ with this subset is empty. So $\lambda_\delta(\mu) \preccurlyeq \lambda_{\delta'}(\mu)$ implies $\lambda_\delta(\mu) = \lambda_{\delta'}(\mu)$. As in the proof of (i) $\Rightarrow$ (iii) it now follows that $p_\delta = p_{\delta'}$. ∎

Proof of Proposition 4.8. By Proposition 4.23 and the assumption that $\operatorname{Re} \lambda > 0$ we know that $\lambda \in X^+(\mu)$. Hence there is a $\delta \in \sigma$ such that $\lambda = \lambda_\delta(\mu)$. Since $(\mathscr{C} - \mu) f$ is square integrable we have

$$p_\mu^\lambda((\mathscr{C} - \mu) f) = 0.$$

Using Lemma 4.20, Lemma 4.19, and the explicit forms of $'\Gamma(\mathscr{C})$ in (15) and of p_δ in (14), one computes for all $H \in \mathfrak{a}$:

$$0 = [{}'\Gamma(\mathscr{C}) - \mu]\, p_\mu^\lambda(f)(\cdot)(H) \tag{22}$$

$$= \sum_{\delta' \in \sigma} [p_{\delta'}(\lambda) - \mu]\, \pi_{\delta'}\, p_\mu^\lambda(f)(\cdot)(H)$$

$$+ 2(\alpha, \alpha)\, \lambda(H_0)\, D_{H_0}[p_\mu^\lambda(f)(\cdot)](H) + (\alpha, \alpha)\, D_{H_0}^2[p_\mu^\lambda(f)(\cdot)](H), \tag{23}$$

where D_{H_0} is differentiation in the direction of H_0. The terms in (22) and (23) are linearly independent polynomials, hence each of them must be zero. If $\delta' \notin \Delta[\delta]$ then Lemma 4.24 implies $p_{\delta'}(\lambda) \neq \mu$, hence $\pi_{\delta'}\, p_\mu^\lambda(f)(\cdot)(H)$ must be zero. This proves that $p_\mu^\lambda(f)(x)(H) \in W_{[\delta]}$. Moreover, $D_{H_0}^2\, p_\mu^\lambda(f)(\cdot) = 0$, and since $\lambda(H_0) \neq 0$ we have $D_{H_0}\, p_\mu^\lambda(f)(\cdot) = 0$. This proves that $p_\mu^\lambda(f)$ has constant polynomial part.

The right MA-behaviour of $p_\mu^\lambda(f)$ is contained in Lemma 4.19 (use that $p_\mu^\lambda(f)$ is polynomially constant). The fact that $x \mapsto p_\mu^\lambda(f)(x)(H)$ is right N-invariant follows as in [2, Lemma 8.6]. ∎

Proof of Proposition 4.9. Let $\lambda \in X(\mu_0)$ be a leading exponent of f. Assume $\operatorname{Re} \lambda > 0$.

Let $\delta \in \sigma$ be a corresponding M-type: $\lambda = \lambda_\delta(\mu_0)$. By Proposition 4.8 we have that $p_{\mu_0}^\lambda(f)$ has constant polynomial part, assumes values in $W_{[\delta]}$, and is completely determined by its restriction to K. So

$$\sum_{\delta' \in \Delta[\delta]} \beta_{\mu_0, \mu_0}^{\delta'} f = p_{\mu_0}^\lambda(f)_{|K}.$$

The fact that $\beta_{\mu_0, \mu_0} f = 0$ implies that $\beta_{\mu_0, \mu_0}^{\delta'} f = 0$. So $p_{\mu_0}^\lambda(f) = 0$ which contradicts the fact that λ is a leading exponent. By assumption we have $\operatorname{Re} \lambda \neq 0$, so $\operatorname{Re} \lambda < 0$.

Hence each leading exponent of f has real part less than zero. By Lemma 4.22 this is equivalent with the fact that f is square integrable. This proves the "only if" part.

If f is square integrable then all exponents of f have real part less than zero. Since $\Xi_\delta(\mu_0)=\{\lambda_\delta(\mu_0)\}$ and $\operatorname{Re}\lambda_\delta(\mu_0)>0$ it follows that $\beta^\delta_{\mu_0,\mu_0}f=0$ for all $\delta\in\sigma$. ∎

Proof of Proposition 4.10. The Poisson transform $\mathscr{P}^\delta_\lambda$ maps into the eigenfunctions with eigenvalue $\mu:=p_\delta(\lambda)$. Let λ' be a leading exponent of $\mathscr{P}^\delta_\lambda g$ with real part greater than zero. Let $\delta'\in\sigma$ be such that $p_{\delta'}(\lambda')=\mu$. By Proposition 4.8 we have $p^{\lambda'}_\mu(\mathscr{P}^\delta_\lambda g)\in C^\infty\operatorname{Ind}^G_P(\sigma_{[\delta']}\otimes-\lambda'\otimes 1)$. Because of Theorem 4.1(ii) the map

$$g\mapsto p^{\lambda'}_\mu(\mathscr{P}^\delta_\lambda g)$$

is G-equivariant. Hence it is an intertwining operator between $C^\infty\operatorname{Ind}^G_P(\sigma_\delta\otimes-\lambda\otimes 1)$ and $C^\infty\operatorname{Ind}^G_P(\sigma_{[\delta']}\otimes-\lambda'\otimes 1)$. From Lemma 21 (cf. Lemma 17, *loc. cit.*) in [12] it follows that such intertwining operators only exist if $\lambda'=-\lambda$ or $\lambda'=\lambda-l\alpha$ for some $l\in\mathbb{N}_0$, in the latter case the intertwining operator is a differential operator. Since $\operatorname{Re}\lambda'>0$ and $\operatorname{Re}\lambda>0$ the case $\lambda'=-\lambda$ is excluded and hence $\lambda'=\lambda-l\alpha$ for some $l\in\mathbb{N}_0$.

Hence if λ is an exponent of $\mathscr{P}^\delta_\lambda g$ it is a leading exponent. By Proposition 4.8 we then have that $p^\lambda_\mu(\mathscr{P}^\delta_\lambda g)$ is polynomially constant and takes values in $W_{[\delta]}$. If $\delta\neq\delta'\in\Delta[\delta]$ then $g\mapsto\pi_{\delta'}p^\lambda_\mu(\mathscr{P}^\delta_\lambda g)$ is a G-equivariant map from $C^\infty\operatorname{Ind}^G_P(\sigma_\delta\otimes-\lambda\otimes 1)$ to $C^\infty\operatorname{Ind}^G_P(\sigma_{\delta'}\otimes-\lambda\otimes 1)$. Since $\lambda\neq 0$ by Lemma 21(i) in [12] any such intertwiner must be zero since δ is not equivalent with δ'. So $p^\lambda_\mu(\mathscr{P}^\delta_\lambda g)$ takes values in $W(\delta)$. ∎

5. Main Theorem

In this section we prove our main theorem.

Recall the definition of $\mathscr{P}_\mu$ in Theorem 4.11 and abbreviate $\mathbf{c}_\delta(\mu)=\mathbf{c}_\delta(\lambda_\delta(\mu))$. By (5) and the definition of the $\lambda_\delta(\mu)$ for $\delta\in\sigma$ in (9) the Poisson transform $\mathscr{P}_\mu$ maps into $C^\infty_{\mu,*}\operatorname{Ind}^G_K(\sigma)$. We define the lift $\mathscr{L}_\mu\colon C^\infty\operatorname{Ind}^K_M(\sigma_{|M})\to\bigoplus_{\delta\in\sigma}C^\infty\operatorname{Ind}^G_P(\sigma_\delta\otimes-\lambda_\delta(\mu)\otimes 1)$ by $\mathscr{L}_\mu f(kan)=\sum_{\delta\in\sigma}a^{\lambda_\delta(\mu)-\rho}\pi_\delta f(k)$ for all $kan\in KAN$. Define

$$\mathfrak{P}_\mu=\mathscr{P}_\mu\circ\mathscr{L}_\mu\colon C^\infty\operatorname{Ind}^K_M(\sigma_{|M})\to C^\infty_{\mu,*}\operatorname{Ind}^G_K(\sigma).$$

$$\mathfrak{P}^\delta_\mu=\mathfrak{P}_\mu\circ i_\delta\colon C^\infty\operatorname{Ind}^K_M(\sigma_\delta)\to C^\infty_{\mu,*}\operatorname{Ind}^G_K(\sigma).$$

Definition 5.1. Let $\Omega\subset\mathbb{C}$ be an open subset, $\mu_0\in\Omega$ and $r\in\mathbb{R}$. By $\mathscr{H}(\Omega,r,\mu_0)$ we denote the set of continuous maps Φ from

$\Omega \times C^{\infty}\operatorname{Ind}_M^K(\sigma_{|M})$ to $C_r^{\infty}(G, W)$ satisfying the following conditions for some $q \in \mathbb{N}_0$:

(1) for all $\mu \in \Omega$ the map Φ_μ, defined by $\Phi_\mu f = \Phi(\mu, f)$, is a continuous, linear, and K-equivariant transform from $C^{\infty}\operatorname{Ind}_M^K(\sigma_{|M})$ to $\mathscr{E}_{\mu, r}^{\mu_0}$,

(2) for all $\mu \in \Omega$ and $s \in \mathbb{N}_0$ the transform Φ_μ extends to a continuous, linear map from $C^{q+s}\operatorname{Ind}_M^K(\sigma_{|M})$ to $C_r^s(G, W)$,

(3) the map $\mu \mapsto \Phi_\mu$ is holomorphic from Ω to the Banach space of continuous linear maps from $C^{q+s}\operatorname{Ind}_M^K(\sigma_{|M})$ to $C_r^s(G, W)$ for each $s \in \mathbb{N}_0$.

It is clear that if $f \in C^{\infty}\operatorname{Ind}_M^K(\sigma_{|M})$ is fixed and $\Phi \in \mathscr{H}(\Omega, r, \mu_0)$, the family $(\Phi_\mu f)_{\mu \in \Omega}$ is an element of $\mathscr{E}_{\Omega, r}^{\mu_0}$.

EXAMPLE 5.2. Fix $\delta \in \sigma$. Let $\Omega \in \mathbb{C}$ be bounded and open such that $\mu \to \lambda_\delta(\mu)$ is holomorphic on Ω. Let $r \in \mathbb{R}$ be such that $\mathscr{P}_\mu^\delta$ maps into $C_r^{\infty}(G, W)$ for all $\mu \in \Omega$. Since the Poisson kernel $\mathrm{P}_{\lambda_\delta(\mu)}^\delta$ has order zero as a distribution, the Poisson transform $\mathscr{P}_\mu^\delta$ maps $C\operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda_\delta(\mu) \otimes 1)$ to $C_r(G, W)$. The G-equivariance of the Poisson transform implies that it maps $C^s\operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda_\delta(\mu) \otimes 1)$, equipped with the usual Banach topology, continuously into $C_r^s(G, W)$. Since the restriction of $\mathscr{L}_\mu$ to $C^s\operatorname{Ind}_M^K(\sigma_\delta)$ maps into $C^s\operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda_\delta(\mu) \otimes 1)$ and is a continuous, linear, and K-equivariant operator, it follows that $\mathfrak{P}_\mu^\delta$ is a continuous, linear, and K-equivariant operator from $C^s\operatorname{Ind}_M^K(\sigma_\delta)$ to $C_r^s(G, W)$. Since the kernel of $\mathfrak{P}_\mu^\delta$ depends holomorphically on $\mu \in \Omega$ it is clear that $\mu \mapsto \mathfrak{P}_\mu^\delta$ is holomorphic from Ω to the Banach space of continuous linear maps from $C^s\operatorname{Ind}_M^K(\sigma_\delta)$ to $C_r^s(G, W)$. Hence $(\mathfrak{P}_\mu^\delta)_{\mu \in \Omega} \in \mathscr{H}(\Omega, r, \mu_0)$.

Define

$$S_{\mu, \mu_0} = \beta_{\mu, \mu_0} \circ \mathfrak{P}_\mu : C^{\infty}\operatorname{Ind}_M^K(\sigma_{|M}) \to C^{\infty}\operatorname{Ind}_M^K(\sigma_{|M}).$$

THEOREM 5.3 (Main Theorem). *Fix $\mu_0 \in \mathbb{C}$. Assume $0 \notin X_R(\mu_0)$. Then there is an open neighbourhood Ω of μ_0 and an $r \in \mathbb{R}$ such that S_{μ, μ_0} is invertible for $\mu \in \Omega \backslash \{\mu_0\}$ and the family $((\mu - \mu_0)\mathfrak{P}_\mu \circ S_{\mu, \mu_0}^{-1})_{\mu \in \Omega}$ is an element of $\mathscr{H}(\Omega, r, \mu_0)$. Define*

$$\Phi_{\mu_0} = [(\mu - \mu_0)\mathfrak{P}_\mu \circ S_{\mu, \mu_0}^{-1}]_{\mu = \mu_0}.$$

Then $\Phi_{\mu_0} : C^{\infty}\operatorname{Ind}_M^K(\sigma_{|M}) \to L_{\mu_0}^2\operatorname{Ind}_K^G(\sigma)$ and the family

$$\Psi := (\mathfrak{P}_\mu \circ S_{\mu, \mu_0}^{-1} - (\mu - \mu_0)^{-1}\Phi_{\mu_0})_{\mu \in \Omega}$$

is an element of $\mathscr{H}(\Omega, r, \mu_0)$ *satisfying*

$$\beta_{\mu, \mu_0} \circ \Psi_\mu = \mathrm{Id}. \tag{24}$$

Remark 5.4. For generic μ_0 the map S_{μ_0, μ_0} is equal to $\mathbf{c}(\mu_0)$; this follows from (12). Since $\mathbf{c}(\mu_0)$ is invertible for generic μ_0, the transform Φ_{μ_0} is zero and the Main Theorem implies Theorem 4.11. Later we see that for the degenerate values of μ_0 the map S_{μ, μ_0} does not necessarily fix M-types.

Remark 5.5. The assumption $0 \notin X_R(\mu_0)$ excludes those eigenvalues for which there are tempered eigensections which are not square integrable. Here we introduce the terminology that $f \in \mathscr{E}^{\mu_0}_{\mu_0}$ is called tempered if all exponents have real part less than or equal to zero (cf. [21, 5.1.1]). If all exponents have real part less than zero f is square integrable (Lemma 4.22).

The importance of the Main Theorem becomes apparent in the following proposition.

PROPOSITION 5.6. *Assume* $0 \notin X_R(\mu_0)$. *Composing* Ψ_{μ_0} *with the canonical projection* $\mathscr{E}^{\mu_0}_{\mu_0} \to \mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma)$ *we obtain a linear K-equivariant isomorphism*

$$C^\infty \operatorname{Ind}^K_M(\sigma_{|M}) \simeq \mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma). \tag{25}$$

Moreover, we have a G-equivariant exact sequence:

$$0 \to C^\infty_{\mu_0, *} \operatorname{Ind}^G_K(\sigma)/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma) \to \mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma) \to \operatorname{Im} \Phi_{\mu_0} \to 0.$$

Proof. If $f \in \mathscr{E}^{\mu_0}_{\mu_0}$ then $\beta_{\mu_0, \mu_0}(f - \Psi_{\mu_0}\beta_{\mu_0, \mu_0} f) = 0$. By Proposition 4.9 this implies that $f - \Psi_{\mu_0}\beta_{\mu_0, \mu_0} f$ is a square integrable function in the kernel of $[\mathscr{C} - \mu_0]^2$. Hence, using Lemma 5.13 below, we see that $f - \Psi_{\mu_0}\beta_{\mu_0, \mu_0} f \in L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma)$. It now easily follows that the composition of Ψ_{μ_0} with the canonical map $\mathscr{E}^{\mu_0}_{\mu_0} \to \mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma)$ is an isomorphism.

For the remaining part of the proposition consider the exact sequence

$$0 \to C^\infty_{\mu_0, *} \operatorname{Ind}^G_K(\sigma)/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma) \to \mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma) \to L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma),$$

where the second arrow is the inclusion and the third arrow is given by $\mathscr{C} - \mu_0$. It suffices to prove that the image of the last map is equal to $\operatorname{Im} \Phi_{\mu_0}$. This fact follows immediately from the following lemma and (25).

LEMMA 5.7. $[\mathscr{C} - \mu_0] \circ \Psi_{\mu_0} = \Phi_{\mu_0}$.

Proof of the lemma. For $\mu \neq \mu_0$ we have $[\mathscr{C} - \mu] \circ \Psi_\mu = -[\mu_0 - \mu] (\mu - \mu_0)^{-1} \Phi_{\mu_0} = \Phi_{\mu_0}$. If we take the limit for $\mu \to \mu_0$ the lemma follows. ∎

The lemma also implies that $\operatorname{Im} \Phi_{\mu_0}$ is a G-invariant subspace of $L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma)$. Indeed, the image of Ψ_{μ_0} is G-invariant and $\mathscr{C} - \mu_0$ is a

G-equivariant operator. Hence, the sequence in the proposition is an exact sequence of G-modules. ∎

COROLLARY 5.8. *Fix $\mu_0 \in \mathbb{C}$. Assume $0 \notin X_R(\mu_0)$. Assume moreover that there are no square integrable functions in $C^\infty_{\mu_0} \operatorname{Ind}_K^G(\sigma)$. Then*

$$[\mathfrak{P}_\mu \circ S^{-1}_{\mu, \mu_0}]_{\mu = \mu_0}$$

is a bijective transform from $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ onto $C^\infty_{\mu_0, *} \operatorname{Ind}_K^G(\sigma)$.

Proof. Since there are no square integrable eigenfunctions Φ_{μ_0} must be zero. Therefore $\Psi_\mu = \mathfrak{P}_\mu \circ S^{-1}_{\mu, \mu_0}$ is a holomorphic family of transforms mapping into $\mathscr{E}^{\mu_0}_\mu = C^\infty_{\mu, *} \operatorname{Ind}_K^G(\sigma)$. The transform $\Psi_{\mu_0} = [\mathfrak{P}_\mu \circ S^{-1}_{\mu, \mu_0}]_{\mu = \mu_0}$ is surjective because of (25) and injective because of (24). ∎

The proof of the Main Theorem requires some preparation.

DEFINITION 5.9. Let $\Omega \subset \mathbb{C}$ be an open subset, $\mu_0 \in \Omega$ and $r \in \mathbb{R}$. By $\mathscr{M}(\Omega, r, \mu_0)$ we denote the set of continuous maps Φ from $\Omega \backslash \{\mu_0\} \times C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to $C^\infty_r(G, W)$ for which there exists an $m \in \mathbb{N}_0$ such that the map

$$(\mu, f) \mapsto (\mu - \mu_0)^m \, \Phi(\mu, f)$$

belongs to $\mathscr{H}(\Omega, r, \mu_0)$. The smallest such m is called the order of the pole of Φ and is denoted by m_Φ.

An element $\xi \in \mathfrak{a}^*_{\mathbb{C}}$ is called a singular exponent of Φ at μ if there is a function $f \in C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ such that ξ is an exponent of $[(\mu - \mu_0)^{m_\Phi} \Phi_\mu](f)$.

For $\lambda \in X^+(\mu_0)$ and $\mu \in \Omega$ define the sets

$$X^\lambda(\mu) = \{\lambda_\delta(\mu) \mid \delta \in \sigma \text{ and } \lambda_\delta(\mu_0) \preccurlyeq \lambda\},$$

$$\Delta^\lambda = \{\delta \in \sigma \mid \lambda_\delta(\mu_0) \preccurlyeq \lambda\},$$

$$\Delta(\lambda) = \{\delta \in \sigma \mid \lambda_\delta(\mu_0) = \lambda\}.$$

DEFINITION 5.10. Let Ω, r, μ_0, and λ be as above.

By $\mathscr{T}(\Omega, r, \mu_0, \lambda)$ we denote the set of $\Phi \in \mathscr{M}(\Omega, r, \mu_0)$ satisfying the following conditions:

(1) if ξ is a singular exponent of Φ at $\mu \in \Omega$ and $\operatorname{Re} \xi > 0$ then $\xi \in \{\nu - l\alpha \mid \nu \in X^\lambda(\mu), l \in \mathbb{N}_0\}$,

(2) there is a $\delta \in \Delta^\lambda$ such that for all $\mu \neq \mu_0$ and for each $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ the coefficient at $\lambda_\delta(\mu)$ of $\Phi_\mu(f)$ is equal to f itself.

Let $\Phi \in \mathscr{T}(\Omega, r, \mu_0, \lambda)$. Let E_Φ^+ denote the set of maximal elements in the set of singular exponents of Φ at μ_0 with real part greater than zero. By Definition 5.10(1) we have $E_\Phi^+ \subset \{v - l\alpha \mid v \in X^\lambda(\mu_0), l \in \mathbb{N}_0\}$. This latter set is totally ordered and hence E_Φ^+ consists of at most one element. If E_Φ^+ is empty we call Φ of negative type. If E_Φ^+ consists of one element we call Φ of positive type. Let $\mathscr{T}^+(\Omega, r, \mu_0, \lambda)$ denote the set of $\Phi \in \mathscr{T}(\Omega, r, \mu_0, \lambda)$ of positive type and $\mathscr{T}^-(\Omega, r, \mu_0, \lambda)$ the set of $\Phi \in \mathscr{T}(\Omega, r, \mu_0, \lambda)$ of negative type.

For $\Phi \in \mathscr{T}^+(\Omega, r, \mu_0, \lambda)$ let ξ_Φ be the unique element in E_Φ^+. By Definition 5.10(1) and Proposition 4.23 we have $\xi_\Phi \in \{v - l\alpha \mid v \in X^\lambda(\mu_0), l \in \mathbb{N}_0\} \cap X^+(\mu_0) = X^\lambda(\mu_0)$, so

$$\xi_\Phi \in X^\lambda(\mu_0).$$

For $f \in C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ let $T_\Phi(f)$ be the coefficient of $[(\mu - \mu_0)^{m_\Phi} \Phi_\mu]_{\mu = \mu_0}(f)$ at the exponent ξ_Φ restricted to K. By Theorem 4.1(ii) and Definition 5.1(1) the map $f \mapsto T_\Phi(f)$ is continuous, linear, and K-equivariant from $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to $C^\infty \operatorname{Ind}_M^K(\sigma_{|M}) \otimes \operatorname{Pol}(\mathfrak{a})$. Let $\delta \in \sigma$ be such that $\xi_\Phi = \lambda_\delta(\mu_0)$. By Proposition 4.8 we have

$$T_\Phi(f) \in C^\infty \operatorname{Ind}_M^K(\sigma_{[\delta]}) = \bigoplus_{\delta' \in \Delta(\xi_\Phi)} C^\infty \operatorname{Ind}_M^K(\sigma_{\delta'}) \tag{26}$$

(note that $\Delta[\delta] = \Delta(\xi_\Phi)$ by Lemma 4.24).

LEMMA 5.11. *Let $\Phi \in \mathscr{T}(\Omega, r, \mu_0, \lambda) \cap \mathscr{H}(\Omega, r, \mu_0)$ and suppose that $\delta \in \Delta(\lambda)$. Then there is an open neighbourhood $\Omega_0 \subset \Omega$ of μ_0 such that for all $f \in C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ and $\mu \in \Omega_0$ we have*

$$\sum_{\delta' \in \Delta(\lambda)} \beta_{\mu, \mu_0}^{\delta'}(\Phi_\mu f)(k) = p_\mu^{\lambda_\delta(\mu)}(\Phi_\mu f)(k), \qquad k \in K.$$

In particular, $p_\mu^{\lambda_\delta(\mu)}(\Phi_\mu f)$ depends holomorphically on $\mu \in \Omega_0$.

Proof. Let $\Omega' \subset \Omega$ be such that for all $\mu \in \Omega'$, $\delta \in \Delta(\lambda)$ and $\xi \in \Xi_\delta(\mu)$ we have $\operatorname{Re} \xi > 0$. Let $\delta' \in \Delta(\lambda)$. Then for $\mu \in \Omega'$,

$$\beta_{\mu, \mu_0}^{\delta'}(\Phi_\mu f)(k) = \sum_{\xi \in \Xi_{\delta'}(\mu) \cap Y(\mu)} \pi_{\delta'} p_\mu^\xi(\Phi_\mu f)(k)(0),$$

where $Y(\mu) = \{v - l\alpha \mid v \in X^\lambda(\mu), l \in \mathbb{N}_0\}$ is the set of possible exponents of $\Phi_\mu f$ with real part greater than zero (cf. Definition 5.10 (1)). By Theorem 4.7 the left-hand side and hence the right-hand side depends holomorphically on μ.

We first investigate $\Xi_{\delta'}(\mu) \cap Y(\mu)$. This is easy for $\mu = \mu_0$ since $\Xi_{\delta'}(\mu_0) = \{\lambda_{\delta'}(\mu_0)\} = \{\lambda\}$. Let $\mu \neq \mu_0$ and suppose $\xi \in Y(\mu)$. Then there are

$\delta_1 \in \Delta^{\lambda}$ and $l_1 \in \mathbb{N}_0$ such that $\xi = \lambda_{\delta_1}(\mu) - l_1 \alpha$. If in addition $\xi \in \Xi_{\delta'}(\mu)$ there are $\delta_2 \in \sigma$ and $l_2 \in \mathbb{N}_0$ such that $\xi = \lambda_{\delta_2}(\mu) - l_2\alpha$ and $\lambda_{\delta_2}(\mu_0) - l_2\alpha = \lambda_{\delta'}(\mu_0)$. Hence $\lambda_{\delta_1}(\mu) - l_1\alpha = \lambda_{\delta_2}(\mu) - l_2\alpha$. By Lemma 4.24 there is a neighbourhood Ω_0 of μ_0 such that for $\mu \in \Omega_0$ we have that $\lambda_{\delta_1}(\mu) - l_1\alpha = \lambda_{\delta_2}(\mu) - l_2\alpha$ implies that $\lambda_{\delta_1} \equiv \lambda_{\delta_2}$ on Ω_0. So assume $\mu \in \Omega_0 \cap \Omega'$, then $\lambda_{\delta_1}(\mu_0) = \lambda_{\delta_2}(\mu_0)$. Since $\xi \in \Xi_{\delta'}(\mu)$ we had $\lambda_{\delta_2}(\mu_0) \geqslant \lambda$ and since $\xi \in Y(\mu)$ we had $\lambda_{\delta_1}(\mu_0) \leqslant \lambda$, so $\lambda_{\delta_1}(\mu_0) = \lambda_{\delta_2}(\mu_0) = \lambda$ and hence $l_1 = l_2 = 0$. This proves $\xi = \lambda_{\delta''}(\mu)$ for a $\delta'' \in \Delta(\lambda)$. So we have proven

$$\Xi_{\delta'}(\mu) \cap Y(\mu) = \{\lambda_{\delta''}(\mu) \mid \delta'' \in \Delta(\lambda)\}.$$

By Lemma 4.24 this set equals the singleton $\{\lambda_\delta(\mu)\}$. Hence

$$\beta^{\delta'}_{\mu,\mu_0}(\Phi_\mu f)(k) = \pi_{\delta'}\, p_\mu^{\lambda_\delta(\mu)}(\Phi_\mu f)(k)(0).$$

We now claim that $\lambda_\delta(\mu)$ is a leading exponent of $\Phi_\mu f$ for all $\mu \in \Omega_0 \cap \Omega'$ whenever it is an exponent. For $\mu \neq \mu_0$ this follows from Lemma 4.24. For $\mu = \mu_0$ we have that $\lambda = \lambda_\delta(\mu_0)$ is maximal in $Y(\mu_0)$, so if $\lambda_\delta(\mu_0)$ is an exponent it is leading.

By Proposition 4.8 the fact that $\lambda_\delta(\mu)$ is leading implies that $p_\mu^{\lambda_\delta(\mu)}(\Phi_\mu f)$ has constant polynomial part and takes values in $W_{[\delta]}$. The lemma now follows from the fact $\Delta[\delta]$ is equal to $\Delta(\lambda)$. ∎

LEMMA 5.12. *For each $\Phi \in \mathscr{T}^-(\Omega, r, \mu_0, \lambda)$ there is a $\Phi' \in \mathscr{M}(\Omega, r, \mu_0)$ such that*

1. *for every $\mu \in \Omega$ the transform $(\mu - \mu_0)^{m_\Phi}\, \Phi'_\mu$ maps into $L^2_{\mu_0} \operatorname{Ind}_K^G(\sigma)$.*
2. *$\Phi + \Phi' \in \mathscr{T}^+(\Omega, r, \mu_0, \lambda)$.*

Proof. If Φ is of negative type then $\Phi^{(1)}_{\mu_0} := [(\mu - \mu_0)^{m_\Phi}\, \Phi_\mu]_{\mu = \mu_0}$ has leading exponents in $X^-(\mu_0)$. By Proposition 5.1.3 of [21] this implies that $\Phi^{(1)}_{\mu_0}$ maps into the square integrable functions. Since Φ_μ maps to $\ker[(\mathscr{C} - \mu)(\mathscr{C} - \mu_0)]$ we have that $\Phi^{(1)}_{\mu_0}$ maps into $\ker[(\mathscr{C} - \mu_0)^2]$. By Lemma 5.13 below this implies that $\Phi^{(1)}_{\mu_0}$ maps into $\ker(\mathscr{C} - \mu_0)$. Therefore for $\mu \neq \mu_0$ the transform $\Phi^{(1)}_{\mu} := \Phi_\mu - (\mu - \mu_0)^{-m_\Phi}\, \Phi^{(1)}_{\mu_0}$ maps into $\mathscr{E}^{\mu_0}_{\mu, r}$. The family $\Phi^{(1)}$ is an element of $\mathscr{T}(\Omega, r, \mu_0, \lambda)$ and has a pole of order strictly smaller than m_Φ. Note that since Φ is of negative type, $\Phi^{(1)}$ satisfies (2) of Definition 5.10 with the same δ. If $\Phi^{(1)}$ is of positive type we are finished, if it is of negative type we can continue in this fashion. The process stops because the order of the poles decreases. If we end up with a holomorphic family it is a nonzero element of $\mathscr{H}(\Omega, r, \mu_0)$ because by definition the elements in $\mathscr{T}(\Omega, r, \mu_0, \lambda)$ have at least one nonzero holomorphic coefficient in their expansions at an exponent in $X^+(\mu_0)$. The assignment $\Phi \mapsto \Phi + \Phi'$ does not change the coefficients at exponents in $X^+(\mu_0)$. ∎

LEMMA 5.13. *If* $f \in L^2 \operatorname{Ind}_K^G(\sigma)$ *and* $(\mathscr{C} - \mu_0)^2 f = 0$ *in the distribution sense then* $(\mathscr{C} - \mu_0) f = 0$.

Proof. Via the Haar measure dg we identify functions in $C_c^\infty(G)$ with distributions. Let U_n, $n \geqslant 0$, be a decreasing sequence of open neighbourhoods of the identity element e such that for all open $U \subset G$ containing e there is an n such that $U_n \subset U$. Let φ_n be a sequence of positive smooth functions with compact support in U_n such that $\int \varphi_n = 1$. If T is a distribution then $\varphi_n * T$ (convolution defined as usual) is a sequence of smooth functions which converges to T in distribution sense. Since $\mathscr{C}$ is a central element we have $\mathscr{C}(\varphi_n * T) = \varphi_n * \mathscr{C}T$ as smooth functions for every distribution T. If $f \in L^2(G, W)$ then $\varphi_n * f$ is an element of $C^\infty L^2(G, W)$, the space of C^∞-vectors of the left regular representation of G on $L^2(G, W)$.

We use this descent to $C^\infty L^2(G, W)$ to prove the lemma. It is obvious that for $f, g \in C^\infty L^2(G, W) \setminus \{0\}$ we have that $\langle \mathscr{C}f, g \rangle_{L^2} = \langle f, \mathscr{C}g \rangle_{L^2}$ and that $(\mathscr{C} - \mu_0) f = 0$ implies that μ_0 is real.

We can assume that $L^2(G, W) \cap \ker(\mathscr{C} - \mu_0)$ is nontrivial, so there is a $g \in L^2(G, W) \setminus \{0\}$ such that $(\mathscr{C} - \mu_0) g = 0$. For n big enough $\varphi_n * g$ is a nonzero element in $C^\infty L^2(G, W) \cap \ker(\mathscr{C} - \mu_0)$ and hence μ_0 is real.

Let $f \in L^2(G, W)$ be such that $(\mathscr{C} - \mu_0)^2 f = 0$. Define $f_n = \varphi_n * f$. Then $(\mathscr{C} - \mu_0)^2 f_n = 0$, $f_n \in C^\infty L^2(G, W)$ and the following computation is valid: $\|(\mathscr{C} - \mu_0) f_n\|^2 = \langle (\mathscr{C} - \overline{\mu_0})(\mathscr{C} - \mu_0) f_n, f_n \rangle = \langle (\mathscr{C} - \mu_0)^2 f_n, f_n \rangle = 0$. Hence $(\mathscr{C} - \mu_0) f_n = 0$. Now $(\mathscr{C} - \mu_0) f_n$ converges to $(\mathscr{C} - \mu_0) f$ in the sense of distributions, hence $(\mathscr{C} - \mu_0) f = 0$. ∎

We now start with the proof of the Main Theorem.

Let

$$\mathscr{S} \colon \mathscr{T}(\Omega, r, \mu_0, \lambda) \to \mathscr{T}^+(\Omega, r, \mu_0, \lambda)$$

be defined as the identity on $\mathscr{T}^+(\Omega, r, \mu_0, \lambda)$ and by $\mathscr{S}[\Phi] = \Phi + \Phi'$ for $\Phi \in \mathscr{T}^-(\Omega, r, \mu_0, \lambda)$, where Φ' is the map of Lemma 5.12. Since $(\mu - \mu_0)^{m_\Phi} \Phi'_\mu$ maps into $L^2_{\mu_0} \operatorname{Ind}_K^G(\sigma)$ for all $\mu \in \Omega$ the singular exponents of Φ' at μ are elements of $X^-(\mu_0)$. Hence for $\mu \neq \mu_0$ the coefficients of $(\mathscr{S}[\Phi])_\mu$ at exponents in $X^+(\mu)$ are the same as the coefficients of Φ_μ.

THEOREM 5.14. *There are a neighbourhood* Ω *of* μ_0 *and a constant* $r \in \mathbb{R}$ *such that for all* $\lambda \in X^+(\mu_0)$ *there is a* $\Psi^\lambda \in \mathscr{H}(\Omega, r, \mu_0) \cap \mathscr{T}(\Omega, r, \mu_0, \lambda)$ *with the following properties. For all* $\mu \in \Omega$, $\delta \in \Delta^\lambda$, *and* $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ *we have*

1. $\lambda_\delta(\mu)$ *is a leading exponent of* $\Psi^\lambda_\mu f$,
2. *the coefficient at* $\lambda_\delta(\mu)$ *of* $\Psi^\lambda_\mu f$ *is equal to* f.

Proof. Let Ω_1 be an open neighbourhood of μ_0 on which the functions $\mu \mapsto \lambda_\delta(\mu)$ are holomorphic (which exists by Lemma 4.5). Let Ω_2 be a neighbourhood of μ_0 such that all $\mu \mapsto \mathbf{c}_\delta(\mu)^{-1}$, $\delta \in \Delta$, are meromorphic and regular on $\Omega_2 \backslash \{\mu_0\}$. Let Ω_3 be a neighbourhood of μ_0 such that for all $\mu \in \Omega_3 \backslash \{\mu_0\}$ and all δ, δ' with $\delta' \notin \Delta[\delta]$ the exponents $\lambda_\delta(\mu)$ and $\lambda_{\delta'}(\mu)$ are incomparable (which exists by Lemma 4.24). Let Ω_4 be an open neighbourhood of μ_0 such that for all $\mu \in \Omega_4$, $\delta \in \sigma$, and $\xi \in \Xi_\delta(\mu)$ we have $\operatorname{Re} \xi > 0$. Let $\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$. We may take Ω to be bounded. Let $r(\lambda, \delta) \in \mathbb{R}$ be the growth parameter, defined as in [2, Example 2.2], such that $\mathscr{P}_\lambda^\delta$ maps into $C^\infty_{r(\lambda,\delta)}(G, W)$. Define $r = \sup_{\mu \in \Omega} \max_{\delta \in \sigma} r(\lambda_\delta(\mu), \delta)$, which is finite because $\lambda \mapsto r(\lambda, \delta)$ is a locally bounded function. For $\lambda \in X^+(\mu_0)$ we abbreviate $\mathscr{T}_\lambda = \mathscr{T}(\Omega, r, \mu_0, \lambda)$.

We shall construct the Ψ^λ by induction on the cardinality of $X^\lambda(\mu_0)$.

Induction start. Let $\lambda_0 \in X^+(\mu_0)$ be such that $|X^{\lambda_0}(\mu_0)| = 1$. Fix $\delta \in \Delta^{\lambda_0}$, then $\lambda_\delta(\mu_0) = \lambda_0$ is the only element in $X^{\lambda_0}(\mu_0)$. We know by (6) that the coefficient of $\mathfrak{P}_\mu^\delta f$, for $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$, at $\lambda_\delta(\mu)$ is equal to $\mathbf{c}_\delta(\mu) f$. So the family $(\mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1})_{\mu \in \Omega}$ in $\mathscr{M}(\Omega, r, \mu_0)$ satisfies Condition 2 in Definition 5.10. By Proposition 4.10 Condition 1 is satisfied for $\lambda = \lambda_0$. Hence $(\mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1})_{\mu \in \Omega}$ is an element of $\mathscr{T}_{\lambda_0}$.

Consider $\mathscr{S}[\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1}]$. Since this family is of positive type its leading singular exponent at μ_0 is an element of $X^{\lambda_0}(\mu_0) = \{\lambda_0 = \lambda_\delta(\mu_0)\}$, hence is equal to λ_0. Let m be the order of the pole of $\mathscr{S}[\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1}]$. The coefficient of $(\mu - \mu_0)^m \mathscr{S}[\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1}](f)$ at $\lambda_\delta(\mu)$ is equal to $(\mu - \mu_0)^m f$, since the map $\mathscr{S}$ does not affect the coefficients at exponents with real part greater than zero. Since $\lambda_\delta(\mu_0)$ is a leading singular exponent of $\mathscr{S}[\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1}]$ at μ_0 its coefficient must be nonzero at $\mu = \mu_0$, hence $m = 0$ and $\mathscr{S}[\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1}] \in \mathscr{H}(\Omega, r, \mu_0)$. Hence

$$\Psi^{\lambda_0} := \sum_{\delta \in \Delta^{\lambda_0}} \mathscr{S}[\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1}] \circ \pi_\delta$$

satisfies all the requirements of the theorem.

Induction step. Let $n > 1$ and assume we have constructed Ψ^λ for all $\lambda \in X^+(\mu_0)$ for which $|X^\lambda(\mu_0)| < n$. Let λ_0 be such that $|X^{\lambda_0}(\mu_0)| = n$. For any $\delta \in \Delta(\lambda_0)$ we construct Ψ^{λ_0} restricted to $\Omega \times C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$. The idea is to consider the family

$$\mu \mapsto \mathfrak{P}_\mu^\delta \circ \mathbf{c}_\delta(\mu)^{-1} \circ \pi_\delta, \tag{27}$$

which is an element of $\mathscr{T}_{\lambda_0}$. By a double recursion on the order of the pole and the leading singular exponent at μ_0 of (27) we construct a holomorphic family from it.

We make this more precise. To each $\Phi \in \mathscr{T}^+_{\lambda_0}$ we associate the pair $(m_\Phi, \xi_\Phi) \in \mathbb{N}_0 \times X^{\lambda_0}(\mu_0)$. On $\mathbb{N}_0 \times X^{\lambda_0}(\mu_0)$ we have the lexicographical ordening induced by the orderings on $\mathbb{N}_0$ and $X^{\lambda_0}(\mu_0)$. For each $\delta \in \Delta(\lambda_0)$ we construct a finite sequence $\Psi^\delta(m_k, \lambda_k)$ in $\mathscr{T}^+_{\lambda_0}$, $1 \leqslant k \leqslant s$. Here $\{(m_k, \lambda_k) \mid 1 \leqslant k \leqslant s\}$ is a strictly decreasing sequence in $\mathbb{N}_0 \times X^{\lambda_0}(\mu_0)$ with $(m_s, \lambda_s) = (0, \lambda_0)$, and (m_1, λ_1) is the pair associated to $\mathscr{S}[\mu \mapsto \mathfrak{P}^\delta_\mu \circ \mathbf{c}_\delta(\mu)^{-1}] \circ \pi_\delta$. We start with $\Psi^\delta(m_1, \lambda_1) = \mathscr{S}[\mu \mapsto \mathfrak{P}^\delta_\mu \circ \mathbf{c}_\delta(\mu)^{-1}] \circ \pi_\delta$. For $\mu \in \Omega$ and k, $1 \leqslant k \leqslant s$, let $\Psi^\delta_\mu(m_k, \lambda_k)$ denote the map $\Psi^\delta(m_k, \lambda_k)(\mu, \cdot)$. For every k, $1 \leqslant k \leqslant s$, the family $\Psi^\delta(m_k, \lambda_k)$ satisfies the following conditions:

(A) the pair (m_k, λ_k) is associated to $\Psi^\delta(m_k, \lambda_k)$,

(B) for all $\mu \neq \mu_0$ and $f \in C^\infty \operatorname{Ind}^K_M(\sigma_\delta)$ the coefficient of $\Psi^\delta_\mu(m_k, \lambda_k)(f)$ at $\lambda_\delta(\mu)$ is equal to f,

(C) the restriction of $\Psi^\delta(m_k, \lambda_k)$ to $C^\infty \operatorname{Ind}^K_M(\sigma_{\delta'})$ is zero for every $\delta' \neq \delta$.

Clearly, $\Psi^\delta(m_k, \lambda_k)$ satisfies these conditions for $k = 1$.

In this paragraph we see that the families $\Psi^\delta(0, \lambda_0)$, $\delta \in \Delta(\lambda_0)$, enable us to make the desired induction step. Since it has a pole of zero order the family $\Psi^\delta(0, \lambda_0)$ is an element of $\mathscr{H}(\Omega, r, \mu_0)$. Condition (B) above says that for all $\mu \neq \mu_0$, $\delta \in \Delta(\lambda_0)$, and $f \in C^\infty \operatorname{Ind}^K_M(\sigma_\delta)$ we have

$$p^{\lambda_\delta(\mu)}_\mu(\Psi^\delta_\mu(0, \lambda_0)(f)) = f.$$

Since the $\Psi^\delta(0, \lambda_0)$ are holomorphic Lemma 5.11 applies and the left-hand side of the above equation is holomorphic in μ. Hence the equation is also valid for $\mu = \mu_0$. We see that the coefficient of $\Psi^\delta_{\mu_0}(0, \lambda_0)(f)$ at $\lambda_\delta(\mu_0)$ is equal to f for all $f \in C^\infty \operatorname{Ind}^K_M(\sigma_\delta)$. In particular, $\lambda_\delta(\mu_0)$ is an exponent of $\Psi^\delta_{\mu_0}(0, \lambda_0) f$. Since $\Psi^\delta(0, \lambda_0) \in \mathscr{T}_{\lambda_0}$ this implies that $\lambda_\delta(\mu_0) = \lambda_0$ is a leading exponent. Hence

$$\sum_{\delta \in \Delta(\lambda_0)} \Psi^\delta(0, \lambda_0)$$

satisfies all the requirements of the theorem with the M-representations restricted to the set $\Delta(\lambda_0)$. Now let λ' be the maximal element in $X^{\lambda_0}(\mu_0) \setminus \{\lambda_0\}$. Since $|X^{\lambda'}(\mu_0)| < |X^{\lambda_0}(\mu_0)|$ we have by the induction hypothesis a map $\Psi^{\lambda'} \in \mathscr{T}_{\lambda'} \cap \mathscr{H}$ satisfying all the requirements of the theorem. Since $\lambda_\delta(\mu) \notin X^{\lambda'}(\mu) - \mathbb{N}_0 \alpha$, Condition 1 of Definition 5.10 implies that for all $\mu \in \Omega$ the exponent $\lambda_\delta(\mu)$, $\delta \in \Delta(\lambda_0)$, is not a singular exponent of $\Psi^{\lambda'}$ at μ. Hence

$$\Psi^{\lambda_0} := \sum_{\delta \in \Delta(\lambda_0)} \Psi^\delta(0, \lambda_0) + \Psi^{\lambda'}$$

satisfies all the requirements of the theorem.

We now describe the recursion process. Fix $\delta \in \Delta(\lambda_0)$. Then $\lambda_\delta(\mu_0) = \lambda_0$. Let (m_k, λ_k) be a pair in the sequence, associated to the already constructed $\Psi^\delta(m_k, \lambda_k)$. We know $\lambda_k \preccurlyeq \lambda_0$. We distinguish the cases $\lambda_k = \lambda_0$ and $\lambda_k \prec \lambda_0$, respectively.

If $\lambda_k = \lambda_0$ then Condition A says that $\lambda_0 = \lambda_\delta(\mu_0)$ is a leading exponent of $[(\mu-\mu_0)^{m_k} \Psi^\delta_\mu(m_k, \lambda_k)]_{\mu=\mu_0}(f)$ for an $f \in C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$. Condition C implies that this can only happen for an $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$. Condition B implies that for all $\mu \neq \mu_0$ and $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ we have

$$p_\mu^{\lambda_\delta(\mu)}[(\mu-\mu_0)^{m_k} \Psi^\delta_\mu(m_k, \lambda_k)(f)] = (\mu-\mu_0)^{m_k} f.$$

Since $[(\mu-\mu_0)^{m_k} \Psi^\delta_\mu(m_k, \lambda_k)]_{\mu\in\Omega}$ is an element of $\mathscr{H}(\Omega, r, \mu_0)$, by Lemma 5.11 we know that the left-hand side of this equation depends holomorphically on μ. So $m_k > 0$ would imply that $p_{\mu_0}^{\lambda_\delta(\mu_0)}([(\mu-\mu_0)^{m_k} \Psi^\delta_\mu(m_k, \lambda_k)]_{\mu=\mu_0}(f)) = 0$ for all $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ contradicting the fact that $\lambda_\delta(\mu_0)$ is a leading exponent. Hence $m_k = 0$, $\Psi^\delta(m_k, \lambda_k) = \Psi^\delta(0, \lambda_0)$, and the recursion stops.

Now assume that $\lambda_k \prec \lambda_0$. Then $|X^{\lambda_k}(\mu_0)| < |X^{\lambda_0}(\mu_0)| = n$ so by the induction hypothesis we have a Ψ^{λ_k} satisfying all the assertions of the theorem. Let $T_{(m_k, \lambda_k)}(f)$ be the coefficient of $[(\mu-\mu_0)^{m_k} \Psi^\delta_\mu(m_k, \lambda_k)]_{\mu=\mu_0}(f)$ at λ_k. Then $T_{(m_k, \lambda_k)}$ is the map T_Φ introduced above (26), with $\Phi = \Psi^\delta(m_k, \lambda_k)$. This family Φ belongs to $\mathscr{T}^+_{\lambda_0}$ and $\xi_\Phi = \lambda_k$. By the remarks above (26) we therefore have that $T_{(m_k, \lambda_k)}$ is a K-equivariant, continuous, and linear operator from $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to $\bigoplus_{\delta' \in \Delta(\lambda_k)} C^\infty \operatorname{Ind}_M^K(\sigma_{\delta'})$.

We now consider the family $\Psi' \in \mathscr{M}(\Omega, r, \mu_0)$ defined by

$$\Psi'_\mu f = \Psi^\delta_\mu(m_k, \lambda_k) f - (\mu-\mu_0)^{-m_k} \Psi^{\lambda_k}_\mu[T_{(m_k, \lambda_k)}(f)]. \tag{28}$$

In view of the induction hypothesis the singular exponents with positive real part of the meromorphic family $((\mu-\mu_0)^{-m_k} \Psi^{\lambda_k}_\mu[T_{(m_k, \lambda_k)}(f)])_{\mu\in\Omega}$ are all of the form $\lambda_{\delta'}(\mu) - l\alpha$ with $\delta' \in \Delta^{\lambda_k}$. Hence they are different from $\lambda_\delta(\mu)$, and we see that the coefficient of $\Psi'_\mu f$ at $\lambda_\delta(\mu)$ equals that of $\Psi^\delta_\mu(m_k, \lambda_k) f$, hence equals f for $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$. It now easily follows that $\Psi' \in \mathscr{T}_{\lambda_0}$.

Let $\delta' \in \Delta(\lambda_k)$. Then by the induction hypothesis, the coefficient of $\Psi^{\lambda_k}_{\mu_0}[\pi_{\delta'} \circ T_{(m_k, \lambda_k)} f]$ at $\lambda_{\delta'}(\mu_0) = \lambda_k$ equals $\pi_{\delta'} \circ T_{(m_k, \lambda_k)} f$. Above we saw that $\sum_{\delta' \in \Delta(\lambda_k)} \pi_{\delta'} T_{(m_k, \lambda_k)} f = T_{(m_k, \lambda_k)} f$, hence the coefficient of $\Psi^{\lambda_k}_{\mu_0}[T_{(m_k, \lambda_k)} f]$ at λ_k equals $T_{(m_k, \lambda_k)} f$, for all $f \in C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$. From this and the definition of $T_{(m_k, \lambda_k)}$ it follows that λ_k does not occur as an exponent of $[(\mu-\mu_0)^{m_k} \Psi'_\mu f]_{\mu=\mu_0}$. We now see that either the order of the pole of Ψ' is strictly smaller than m_k, or it equals m_k, but then the singular exponents at μ_0 are strictly smaller than λ_k or have real part less than zero. It follows from this that the pair associated to $\mathscr{S}[\Psi']$ is strictly smaller than (m_k, λ_k) and therefore so is the pair (m_{k+1}, λ_{k+1}) associated to $\mathscr{S}[\Psi'] \circ \pi_\delta$.

We claim that

$$\Psi^\delta(m_{k+1}, \lambda_{k+1}) = \mathscr{S}[\Psi'] \circ \pi_\delta$$

satisfies all the requirements of the recursion. Indeed, (A) and (C) are straightforward consequences of the definitions. For (B) we compute for $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$, $\mu \neq \mu_0$, and $k \in K$,

$$\begin{aligned} p_\mu^{\lambda_\delta(\mu)}(\mathscr{S}[\Psi']_\mu \circ \pi_\delta f)(k) &= p_\mu^{\lambda_\delta(\mu)}(\mathscr{S}[\Psi']_\mu f)(k) \\ &= p_\mu^{\lambda_\delta(\mu)}(\Psi'_\mu f)(k) \\ &= f(k), \end{aligned}$$

where we use that $\lambda_\delta(\mu) \in X^+(\mu)$ for the second equality and our previous remark that the coefficient at $\lambda_\delta(\mu)$ of Ψ' equals that of $\Psi_\mu^\delta(m_k, \lambda_k)$ for the last equality. So (B) is satisfied. This completes the description of the recursion process.

We are left to prove that the sequence terminates at $(0, \lambda_0)$. As long as $m_k > 0$ we can continue with the recursion process, so eventually we arrive at a pair $(0, \lambda')$, with $\lambda' \preccurlyeq \lambda_0$. The family $\Psi^\delta(0, \lambda')$ satisfies requirement (B). Since it has a pole of order zero it belongs to $\mathscr{T}_{\lambda_0} \cap \mathscr{H}(\Omega, r, \mu_0)$, and applying Lemma 5.11, we see that (B) is satisfied for $\mu = \mu_0$ as well. Hence $\lambda_\delta(\mu_0) = \lambda_0$ is an exponent for $\Psi_\mu^\delta(0, \lambda')$. But λ' is leading, so $\lambda' = \lambda_0$.

This ends the proof of the induction step and the theorem is proved. ∎

THEOREM 5.15. *There are an open neighbourhood Ω of μ_0, $r \in \mathbb{R}$, and $\Psi \in \mathscr{H}(\Omega, r, \mu_0)$ such that for all $\mu \in \Omega$, $\delta \in \sigma$, and $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ we have*

(1) *$\lambda_\delta(\mu)$ is a leading exponent of $\Psi_\mu f$,*

(2) *the coefficient at $\lambda_\delta(\mu)$ of $\Psi_\mu f$ is equal to f.*

Proof. Let $\lambda_1, ..., \lambda_s$ be the maximal elements in $X^+(\mu_0)$, then $X^+(\mu_0) = \coprod_{i=1}^s X^{\lambda_i}(\mu_0)$ is the decomposition of $X^+(\mu_0)$ into mutually incomparable, totally ordered subsets. Let π_i denote the projection from W onto $\bigoplus_{\delta \in \Delta^{\lambda_i}} W(\delta)$. Then

$$\Psi = \sum_{i=1}^s \Psi^{\lambda_i} \circ \pi_i,$$

where the Ψ^{λ_i} are constructed as in Theorem 5.14, satisfies all the requirements of the theorem. ∎

DEFINITION 5.16. Let Ω be an open subset of $\mathbb{C}$. By $\mathscr{K}(\Omega)$ we denote the set of continuous maps T from $\Omega \times C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ satisfying the following properties for some $q \in \mathbb{N}_0$:

(1) for all $\mu \in \Omega$ the map T_μ, defined by $T_\mu f = T(\mu, f)$, is continuous, linear, and K-equivariant from $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to itself,

(2) for all $\mu \in \Omega$ the map T_μ extends to a continuous linear map from $C^q \operatorname{Ind}_M^K(\sigma_{|M})$ to $C \operatorname{Ind}_M^K(\sigma_{|M})$,

(3) the map $\mu \mapsto T_\mu$ is holomorphic from Ω to the Banach space of continuous linear maps from $C^q \operatorname{Ind}_M^K(\sigma_{|M})$ to $C \operatorname{Ind}_M^K(\sigma_{|M})$.

Remark 5.17. The K-equivariance implies that the restriction $\bar{T}_\mu$ to $C^{q+s} \operatorname{Ind}_M^K(\sigma_{|M})$ of each T_μ maps into $C^s \operatorname{Ind}_M^K(\sigma_{|M})$ for each $s \in \mathbb{N}_0$. It is easy to see that $\bar{T}_\mu$ is continuous for each s. Let $B_K(C^{s_1}, C^{s_2})$ denote the Banach space of K-invariant, continuous, and linear operators from $C^{s_1} \operatorname{Ind}_M^K(\sigma_{|M})$ to $C^{s_2} \operatorname{Ind}_M^K(\sigma_{|M})$. The restriction operator $T_\mu \mapsto \bar{T}_\mu$ from $B_K(C^q, C)$ to $B_K(C^{q+s}, C^s)$ is continuous and hence we conclude that for each $s \in \mathbb{N}_0$ the map $\mu \mapsto \bar{T}_\mu$ is holomorphic from Ω to the Banach space of continuous linear maps from $C^{q+s} \operatorname{Ind}_M^K(\sigma_{|M})$ to $C^s \operatorname{Ind}_M^K(\sigma_{|M})$.

It is clear from the definition of $\mathscr{K}(\Omega)$ and the above remark that for every $S, T \in \mathscr{K}(\Omega)$ the product $S \cdot T$, defined by $(S \cdot T)_\mu = S_\mu \circ T_\mu$, is an element of $\mathscr{K}(\Omega)$. Likewise we have that for each $T \in \mathscr{K}(\Omega)$ and $\Phi \in \mathscr{H}(\Omega, r, \mu_0)$ the product $\Phi \cdot T$, defined by $(\Phi \cdot T)_\mu = \Phi_\mu \circ T_\mu$, is an element of $\mathscr{H}(\Omega, r, \mu_0)$.

EXAMPLE 5.18. For $\Phi \in \mathscr{H}(\Omega, r, \mu_0)$ consider the map $(\mu, f) \mapsto \beta_{\mu, \mu_0}[\Phi(\mu, f)]$. We claim it is an element of $\mathscr{K}(\Omega)$. As in [2, p. 129], we can prove that there is a $q \in \mathbb{N}_0$ such that

$$(\mu, H) \mapsto \sum_{\xi \in \Xi_\delta(\mu)} p_\mu^\xi(\cdot)(H)\, e^{\xi(H)} \tag{29}$$

is continuous from $\Omega \times U$ to the Banach space of continuous linear maps from $C_r^q(G, W)$ to $C_{r'}(G, W)$. Here U is open in $\mathfrak{a}_\mathbb{C}^*$ and we view p_μ^ξ as an operator on $\mathscr{E}_{\mu, r}^{\mu_0}$. This proves that $(\mu, f) \mapsto \beta_{\mu, \mu_0}[\Phi_\mu(f)]$ is continuous on $\Omega \times C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ and that there is a q' such that $\beta_{\mu, \mu_0} \circ \Phi_\mu$ is a continuous linear transform from $C^{q'} \operatorname{Ind}_M^K(\sigma_{|M})$ to $C \operatorname{Ind}_M^K(\sigma_{|M})$. Since (29) is holomorphic in the first variable and Φ is holomorphic we see that $\mu \mapsto \beta_{\mu, \mu_0} \circ \Phi_\mu$ is holomorphic on Ω. Hence $(\beta_{\mu, \mu_0} \circ \Phi_\mu)_{\mu \in \Omega} \in \mathscr{K}(\Omega)$.

Define the following order on the set $\{\delta \mid \delta \in \sigma\}$:

$$\delta \preccurlyeq \delta' \Leftrightarrow \lambda_\delta(\mu_0) \preccurlyeq \lambda_{\delta'}(\mu_0)$$

(since μ_0 is fixed we ignore the dependence on μ_0 in the notation).

LEMMA 5.19. *The Ψ_μ constructed in Theorem* 5.15 *are of the following form*:

$$\Psi_\mu = \mathfrak{P}_\mu \circ T_\mu + \Phi_\mu.$$

Here $(\Phi_\mu)_{\mu\in\Omega} \in \mathcal{M}(\Omega, r, \mu_0)$, *the transforms* Φ_μ *map into* $L^2_{\mu_0} \operatorname{Ind}_K^G(\sigma)$ *for all* $\mu \neq \mu_0$ *and* $T_\mu\colon C^\infty \operatorname{Ind}_M^K(\sigma_{|M}) \to C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ *satisfies*

(1) *there is an* $m \in \mathbb{N}_0$ *such that* $([\mu-\mu_0]^m T_\mu)_{\mu\in\Omega} \in \mathcal{K}(\Omega)$,

(2) *for all* $\delta \in \sigma$ *and* $\mu \neq \mu_0$ *the transform* T_μ *maps* $C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ *to* $\bigoplus_{\delta' \preccurlyeq \delta} C^\infty \operatorname{Ind}_M^K(\sigma_{\delta'})$.

Proof. This follows from the proof of Theorem 5.14 where the Ψ^λ, $\lambda \in X^+(\mu_0)$, were constructed. Following the induction and the recursion we see that each $\Psi_\delta(m_k, \lambda_k)$ is of the above form. The terms which yield Φ_μ arise from the definition of $\mathcal{S}$ (see Lemma 5.12). The transforms T_μ are built from the maps $f \mapsto T_{(m_k, \lambda_k)}(f)$ occurring in (28). Since Ψ is the sum of some of the Ψ^λ the lemma follows. ∎

LEMMA 5.20. *Define* $U_\mu = \beta_{\mu,\mu_0} \circ \Psi_\mu$. *Then* $(U_\mu)_{\mu\in\Omega} \in \mathcal{K}(\Omega)$ *and there is an* $N \in \mathbb{N}_0$ *such that for all* $\mu \in \Omega$ *we have* $(U_\mu - I)^N = 0$.

Proof. The first statement has been proved in Example 5.18.

Since β_{μ,μ_0} is zero on $L^2_{\mu_0} \operatorname{Ind}_K^G(\sigma)$ we have $\beta_{\mu,\mu_0} \circ \Phi_\mu = 0$. Hence for all $\delta \in \sigma$ we have

$$\begin{aligned}\pi_\delta \circ U_\mu \circ \pi_\delta &= \pi_\delta \circ \beta_{\mu,\mu_0} \circ \mathfrak{P}_\mu \circ T_\mu \circ \pi_\delta\\ &= \beta^\delta_{\mu,\mu_0} \circ \sum_{\delta' \preccurlyeq \delta} \mathfrak{P}^{\delta'}_\mu \circ (T_\mu \circ \pi_\delta),\end{aligned}$$

where we use Property 2 of Lemma 5.19. So we need to know the coefficients of $\sum_{\delta' \preccurlyeq \delta} \mathfrak{P}^{\delta'}_\mu \circ (T_\mu \circ \pi_\delta)(f)$, $f \in C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$, at the exponents in $\Xi_\delta(\mu)$ (defined in (10)). In view of Proposition 4.10 and Lemma 4.24 the intersection of $\Xi_\delta(\mu)$ with the set of exponents of $\sum_{\delta' \preccurlyeq \delta} \mathfrak{P}^{\delta'}_\mu$ is equal to $\{\lambda_\delta(\mu)\}$. By Theorem 5.15 the corresponding coefficient of $\Psi_\mu \circ \pi_\delta(f)$ is $\pi_\delta(f)$. Hence $\pi_\delta \circ U_\mu \circ \pi_\delta = \pi_\delta$.

If $\delta, \delta' \in \sigma$ are such that either $\lambda_\delta(\mu_0)$ and $\lambda_{\delta'}(\mu_0)$ are incomparable or $\lambda_{\delta'}(\mu_0) \prec \lambda_\delta(\mu_0)$ we have that the intersection of $\Xi_\delta(\mu)$ with the exponents of $\Sigma_{\delta'' \preccurlyeq \delta'} \mathfrak{P}^{\delta''}_\mu$ is empty, and hence $\pi_\delta \circ U_\mu \circ \pi_{\delta'} = 0$.

This proves that $U_\mu - I$ is an operator which maps $C^\infty \operatorname{Ind}_M^K(\sigma_\delta)$ to $\bigoplus_{\delta' \prec \delta} C^\infty \operatorname{Ind}_M^K(\sigma_{\delta'})$. Since the partial order on $\{\delta \mid \delta \in \sigma\}$ only depends on the order on $X^+(\mu_0)$ there is a fixed N, namely $\max_{\lambda \in X^+(\mu_0)} |X^\lambda(\mu_0)|$, for which $(U_\mu - I)^N = 0$ for all $\mu \in \Omega$. ∎

Proof of the Main Theorem. By the above lemma we have $U_\mu^{-1} = \sum_{i=0}^{N-1} (-1)^i (U_\mu - I)^i$. By the discussion following Remark 5.17 this implies that $U^{-1} = (U_\mu^{-1})_{\mu\in\Omega}$ is an element of $\mathcal{K}(\Omega)$. Now $T'_\mu := T_\mu \circ U_\mu^{-1}$ satisfies $\beta_{\mu,\mu_0} \circ \mathfrak{P}_\mu \circ T'_\mu = \beta_{\mu,\mu_0} \circ \mathfrak{P}_\mu \circ T_\mu \circ U_\mu^{-1} = \beta_{\mu,\mu_0} \circ \Psi_\mu \circ U_\mu^{-1} = \mathrm{Id}$ for all μ. Hence the family $(\beta_{\mu,\mu_0} \circ \mathfrak{P}_\mu = S_{\mu,\mu_0})_{\mu \in \Omega}$ has a meromorphic inverse $(T'_\mu)_{\mu \in \Omega}$. Because Ψ and U^{-1} are holomorphic we see that $\Psi_\mu \circ U_\mu^{-1} = \mathfrak{P}_\mu \circ T_\mu \circ U_\mu^{-1} + \Phi_\mu \circ U_\mu^{-1} = \mathfrak{P}_\mu \circ S_{\mu,\mu_0}^{-1} + \Phi_\mu \circ U_\mu^{-1}$ is an element of $\mathscr{H}(\Omega, r, \mu_0)$. Redefine $\Psi_\mu := \Psi_\mu \circ U_\mu^{-1}$ and $\Phi'_\mu := \Phi_\mu \circ U_\mu^{-1}$. We have seen that

$$(\Psi_\mu = \mathfrak{P}_\mu \circ S_{\mu,\mu_0}^{-1} + \Phi'_\mu)_{\mu \in \Omega}$$

is an element of $\mathscr{H}(\Omega, r, \mu_0)$.

Consider the identities

$$(\mathscr{C} - \mu)\, \Psi_\mu = (\mu_0 - \mu)\, \Phi'_\mu,$$
$$(\mathscr{C} - \mu_0)\, \Psi_\mu = (\mu - \mu_0)\, \mathfrak{P}_\mu \circ S_{\mu,\mu_0}^{-1},$$

both for $\mu \neq \mu_0$, which follow from the facts that $\mathfrak{P}_\mu$ maps into $\ker(\mathscr{C} - \mu)$ and Φ' into $\ker(\mathscr{C} - \mu_0)$. The left-hand sides of both equations are holomorphic, hence the right-hand sides must also be. Both left-hand sides tend to $(\mathscr{C} - \mu_0)\, \Psi_{\mu_0}$ as $\mu \to \mu_0$, hence the right-hand sides are equal at $\mu = \mu_0$:

$$-[(\mu - \mu_0)\, \Phi'_\mu]_{\mu = \mu_0} = [(\mu - \mu_0)\, \mathfrak{P}_\mu \circ S_{\mu,\mu_0}^{-1}]_{\mu = \mu_0}.$$

So the residue of Φ'_μ at $\mu = \mu_0$ is equal to $-[(\mu - \mu_0)\, \mathfrak{P}_\mu \circ S_{\mu,\mu_0}^{-1}]_{\mu = \mu_0} = -\Phi_{\mu_0}$, which proves that

$$\mu \mapsto \mathfrak{P}_\mu \circ S_{\mu,\mu_0}^{-1} - (\mu - \mu_0)^{-1}\, \Phi_{\mu_0}$$

is an element of $\mathscr{H}(\Omega, r, \mu_0)$.

This proves the Main Theorem.

6. Applications

In this section we prove some applications of the Main Theorem. Throughout this section we fix $\mu_0 \in \mathbb{C}$ with $0 \notin X_R(\mu_0)$.

Theorem 6.1. *The space* $\mathscr{E}_{\mu_0}^{\mu_0} / L^2_{\mu_0} \operatorname{Ind}_K^G(\sigma)$ *has the same composition factors as*

$$\bigoplus_{\delta \in \sigma} C^\infty \operatorname{Ind}_P^G(\sigma_\delta \otimes -\lambda_\delta(\mu_0) \otimes 1).$$

Proof. For $\lambda \in X^+(\mu_0)$ let π_λ be the sum of the projections π_δ, $\delta \in \Delta^\lambda$. Define $V(\lambda) = \operatorname{Im} \Psi_{\mu_0} \circ \pi_\lambda$. Let $\overline{V(\lambda)}$ be the image of $V(\lambda)$ under the canonical projection $\mathscr{E}_{\mu_0}^{\mu_0} \to \mathscr{E}_{\mu_0}^{\mu_0} / L^2_{\mu_0} \operatorname{Ind}_K^G(\sigma)$. The spaces $\overline{V(\lambda)}$ consist of functions in $\mathscr{E}^{\mu_0}_{\mu_0}$ with leading exponents in $X^\lambda(\mu_0)$ modulo square integrable eigenfunctions. This is clear from the construction of Ψ^λ in the proof of Theorem 5.14. From this description the G-invariance of the spaces $\overline{V(\lambda)}$ follows. Moreover, if λ and λ' are not comparable then $\overline{V(\lambda)} \cap \overline{V(\lambda')} = 0$.

Let $\lambda_1, ..., \lambda_s$ be the maximal elements in $X^+(\mu_0)$. Then

$$\mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma) = \bigoplus_{i=1}^{s} \overline{V(\lambda_i)}. \tag{30}$$

This follows from Proposition 5.6, Theorem 5.15, and the fact that the λ_i are not comparable. For each $\overline{V(\lambda_i)}$ we describe the structure of its composition series.

Recall that for $\lambda \in X^+(\mu_0)$ the set $X^\lambda(\mu_0)$ is totally ordered. In this proof for $\lambda \in X^+(\mu_0)$ let ${}'\lambda$ be the maximal element in $X^\lambda(\mu_0) \backslash \{\lambda\}$. Obviously we have $\overline{V('\lambda)} \subseteq \overline{V(\lambda)}$.

For every function g in $V(\lambda)$ the exponent λ is a leading exponent. By Proposition 4.8 this implies that $p^\lambda_{\mu_0}(g)$ is an element of $\sum_{\delta \in \Delta(\lambda)} C^\infty \operatorname{Ind}^G_P(\sigma_\delta \otimes -\lambda \otimes 1)$, in particular it is polynomially constant. By Theorem 4.1 the map $p^\lambda_{\mu_0}: g \mapsto p^\lambda_{\mu_0}(g)$ is G-equivariant into $C^\infty(G, W)$. So $p^\lambda_{\mu_0}: V(\lambda) \to \sum_{\delta \in \Delta(\lambda)} C^\infty \operatorname{Ind}^G_P(\sigma_\delta \otimes -\lambda \otimes 1)$ is G-equivariant. Since square integrable eigenfunctions do not have exponents with real part greater than zero $p^\lambda_{\mu_0}$ factorizes over the square integrable functions in $V(\lambda)$. Hence we have a G-equivariant map

$$\bar{p}^\lambda_{\mu_0}: \overline{V(\lambda)} \to \sum_{\delta \in \Delta(\lambda)} C^\infty \operatorname{Ind}^G_P(\sigma_\delta \otimes -\lambda \otimes 1),$$

induced by $p^\lambda_{\mu_0}$.

By Theorem 5.14 we know that for all $f \in C^\infty \operatorname{Ind}^K_M(\sigma_\delta)$, $\delta \in \Delta(\lambda)$, the coefficient of $\Psi^\lambda_{\mu_0} f$ at λ restricted to K is equal to f. Since λ is a leading exponent this completely determines the coefficient: $p^\lambda_{\mu_0} \circ \Psi^\lambda_{\mu_0}(f) = \mathscr{L}_{\mu_0} f$. Let $\bar{\Psi}^\lambda_{\mu_0}$ be the map $\Psi^\lambda_{\mu_0}$ composed with the canonical projection $\mathscr{E}^{\mu_0}_{\mu_0} \to \mathscr{E}^{\mu_0}_{\mu_0}/L^2_{\mu_0} \operatorname{Ind}^G_K(\sigma)$. Then obviously we have $\bar{p}^\lambda_{\mu_0} \circ \bar{\Psi}^\lambda_{\mu_0}(f) = \mathscr{L}_{\mu_0} f$, which proves that $\bar{p}^\lambda_{\mu_0}$ is surjective.

If $[g] \in \ker \bar{p}^\lambda_{\mu_0}$ then g has leading exponents strictly smaller than λ, so $[g]$ is an element of $\overline{V('\lambda)}$.

We have proved that $\bar{p}^\lambda_{\mu_0}$ induces a G-isomorphism

$$\overline{V(\lambda)}/\overline{V('\lambda)} \simeq \bigoplus_{\delta \in \Delta(\lambda)} C^\infty \operatorname{Ind}^G_P(\sigma_\delta \otimes -\lambda \otimes 1).$$

This together with (30) proves the theorem. ∎

Remark 6.2. The exact sequence in Proposition 5.6 together with Theorem 6.1 gives a partial result on the composition factors of

$C^{\infty}_{\mu_0,*}\operatorname{Ind}_K^G(\sigma)$: we do not describe the orthogonal complement of $\operatorname{Im}\Phi_{\mu_0}$ in $L^2_{\mu_0}\operatorname{Ind}_K^G(\sigma)$. If there are no square integrable functions in $C^{\infty}_{\mu_0,*}\operatorname{Ind}_K^G(\sigma)$ we see that the composition factors of $C^{\infty}_{\mu_0,*}\operatorname{Ind}_K^G(\sigma)$ are of the same kind as for generic μ_0 (cf. Theorem 4.11).

Let π_μ be the G-action on $C^\infty\operatorname{Ind}_M^K(\sigma_{|M})$ defined by

$$\pi_\mu(x)=\beta_{\mu,\mu_0}\circ L_x\circ\Psi_\mu.$$

From the properties of β_{μ,μ_0} and Ψ_μ the following easily follows. For every $x\in G$ and $\mu\in\Omega$ the map $\pi_\mu(x)$ from $C^\infty\operatorname{Ind}_M^K(\sigma_{|M})$ to itself is linear and continuous, and there is a $q\in\mathbb{N}_0$ such that for all $s\in\mathbb{N}_0$ the map $\pi_\mu(x)$ extends to a map from $C^{q+s}\operatorname{Ind}_M^K(\sigma_{|M})$ to $C^s\operatorname{Ind}_M^K(\sigma_{|M})$. Moreover, for all $x\in G$ and $s\in\mathbb{N}_0$ the map $\mu\mapsto\pi_\mu(x)$ is holomorphic from Ω to the Banach space of continuous linear operators from $C^{q+s}\operatorname{Ind}_M^K(\sigma_{|M})$ to $C^s\operatorname{Ind}_M^K(\sigma_{|M})$. Furthermore, by the K-equivariance of Ψ_μ we have $\pi_\mu(k)=L_k$ for all $k\in K$.

Let σ_μ be the P-representation on W defined by

$$\sigma_\mu(man)=\sigma(m)\circ\sum_{\delta\in\sigma}a^{-\lambda_\delta(\mu)}\pi_\delta$$

for $man\in MAN$. Denote the left regular representation on $C^\infty\operatorname{Ind}_P^G(\sigma_\mu)$ by L_μ. Let $(L_{\mu,K}, C^\infty\operatorname{Ind}_M^K(\sigma_{|M})$ be the compact picture of $(L_\mu, C^\infty\operatorname{Ind}_P^G(\sigma_\mu))$; that is, $L_{\mu,K}(x)f(k)=(\mathscr{L}_\mu f)(x^{-1}k)$. The Poisson transform $\mathscr{P}_\mu$ is a G-equivariant map from $C^\infty\operatorname{Ind}_P^G(\sigma_\mu)$ to $C^\infty_{\mu,*}\operatorname{Ind}_K^G(\sigma)$. Hence

$$\begin{aligned}\pi_\mu(x)&=\beta_{\mu,\mu_0}\circ L(x)\circ[\mathscr{P}_\mu\circ\mathscr{L}_\mu\circ S^{-1}_{\mu,\mu_0}-(\mu-\mu_0)^{-1}\Phi_{\mu_0}]\\&=\beta_{\mu,\mu_0}\circ\mathscr{P}_\mu\circ L_\mu(x)\circ\mathscr{L}_\mu\circ S^{-1}_{\mu,\mu_0}\\&=\beta_{\mu,\mu_0}\circ\mathscr{P}_\mu\circ\mathscr{L}_\mu\circ L_{\mu,K}(x)\circ S^{-1}_{\mu,\mu_0}\\&=S_{\mu,\mu_0}\circ L_{\mu,K}(x)\circ S^{-1}_{\mu,\mu_0}.\end{aligned}$$

So if S_{μ,μ_0} is invertible, π_μ is equivalent to $L_{\mu,K}$.

Define for $k\in\mathbb{N}_0$ the vector space

$$\mathscr{V}^k=W\otimes[\mathscr{U}(\mathfrak{n})/\mathfrak{n}^k\mathscr{U}(\mathfrak{n})]^*\otimes C^\infty(A).$$

Define an MA-representation τ_1^k on $\mathscr{V}^k$ by

$$\tau_1^k(ma)=a^{-\rho}\sigma(m)\otimes\operatorname{Ad}^\vee(ma)\otimes L_a.$$

Here $\operatorname{Ad}^\vee$ is the contragredient of Ad and L is the left regular representation of A on $C^\infty(A)$. For $X\in\mathfrak{n}$ let $\lambda(X)\colon\mathscr{U}(\mathfrak{n})\to\mathscr{U}(\mathfrak{n})$ denote the left multiplication with X. Let $\lambda^\vee$ be the contragredient and define $\bar{\tau}_2^k = 1 \otimes \lambda^\vee \otimes 1$, a representation of $\mathfrak{n}$ on $\mathscr{V}^k$. The representation $\bar{\tau}_2^k$ integrates to a representation τ_2^k of N.

Let $\tau^k: P \to \mathrm{End}(\mathscr{V}^k)$ be defined by

$$\tau^k(man) = \tau_1^k(ma)\, \tau_2^k(n).$$

One easily checks that τ^k is a P-representation.

Let $\mathbb{D}(A)$ denote the space of translation invariant differential operators on A. For $D \in \mathbb{D}(A) \setminus \{0\}$ let $\mathscr{V}^k_D$ denote the finite dimensional subspace of $\mathscr{V}^k$ defined by $\mathscr{V}^k_D = \{v \in \mathscr{V}^k \mid (1 \otimes 1 \otimes D)\, v = 0\}$. Since D commutes with L_a for all $a \in A$ the representation τ^k restricts to a P-representation τ^k_D. Define $\langle \mathrm{ev}_1, \varphi \rangle = \varphi(1 + \mathfrak{n}^k \mathscr{U}(\mathfrak{n}))$ for $\varphi \in (\mathscr{U}(\mathfrak{n})/\mathfrak{n}^k \mathscr{U}(\mathfrak{n}))^*$ and $\langle \mathrm{ev}_e, h \rangle = h(e)$ for $h \in C^\infty(A)$. Define

$$\mathrm{pr} = 1 \otimes \mathrm{ev}_1 \otimes \mathrm{ev}_e : \mathscr{V}^k \to W,$$

an M-equivariant surjective map. For $D \in \mathbb{D}(A)$ let pr_D denote the restriction of pr to $\mathscr{V}^k_D$. Let $\mathscr{R}: C^\infty \operatorname{Ind}_P^G(\tau^k) \to C^\infty \operatorname{Ind}_M^K(\tau^k_{\mid M})$ denote the restriction to K.

THEOREM 6.3. *Let $\Omega \subset \mathbb{C}$ be as in the Main Theorem.*

There are k, $m \in \mathbb{N}_0$ and for each $\mu \in \Omega$ a differential operator $D_\mu \in \mathbb{D}(A)$ of degree m (independent of μ) such that

(i) *the D_μ depend holomorphically on μ,*

(ii) *for each $\mu \in \Omega$ there is an embedding $I_K^G(\mu)$ which maps $(\pi_\mu, C^\infty \operatorname{Ind}_M^K(\sigma_{\mid M}))$ G-equivariantly into $C_*^\infty \operatorname{Ind}_P^G(\tau^k_{D_\mu})$ and is unique with respect to the property*

$$I_G^K(\mu) \circ I_K^G(\mu) = \mathrm{Id},$$

where $I_G^K(\mu) = \mathrm{pr}_{D_\mu} \circ \mathscr{R}$.

Before proving the theorem we study certain G-representations on $C^\infty \operatorname{Ind}_M^K(\delta)$, where (δ, V_δ) is a finite dimensional M-representation. Let π be a G-representation on $C^\infty \operatorname{Ind}_M^K(\delta)$ whose restriction to K is equal to the left regular representation of K on $C^\infty \operatorname{Ind}_M^K(\delta)$.

DEFINITION 6.4. The representation $(C^\infty \operatorname{Ind}_M^K(\delta), \pi)$ is called uniformly AN-finite if there are k, $d \in \mathbb{N}_0$ and $\nu_1, ..., \nu_n \in \mathfrak{a}_\mathbb{C}^*$ such that for all $f \in C^\infty \operatorname{Ind}_M^K(\delta)$

(1) $[\pi(u) f](e) = 0$ for all $u \in \mathfrak{n}^k$,

(2) $[\pi(\prod_{i=1}^n (H - \nu_i(H))^d) f](e) = 0$ for all $H \in \mathfrak{a}$.

Define

$$\mathscr{V}' = V_\delta \otimes [\mathscr{U}(\mathfrak{n})/\mathfrak{n}^k\mathscr{U}(\mathfrak{n})]^* \otimes C^\infty(A)$$

and let τ' be the P-representation on $\mathscr{V}'$ defined analogous to $(\tau^k, \mathscr{V}^k)$ (replace $\sigma_{|M}$ by δ). Let $(\pi, C^\infty \operatorname{Ind}_M^K(\delta))$ be uniformly AN-finite with parameters k, $d \in \mathbb{N}_0$ and $\nu_i \in \mathfrak{a}_\mathbb{C}^*$. Define $D = \prod_{i=1}^n (H_0 - \nu_i(H_0))^d$ for a fixed nonzero $H_0 \in \mathfrak{a}$. With the identification $\mathbb{D}(A) \simeq \mathscr{U}(\mathfrak{a})$ we view D as an element of $\mathbb{D}(A)$. Let $\mathscr{V}'_D$ be the finite dimensional subspace of $\mathscr{V}'$ defined by $\mathscr{V}'_D = \{v \in \mathscr{V}' \mid (1 \otimes 1 \otimes D)\, v = 0\}$. The representation τ' restricts to a representation τ'_D on $\mathscr{V}'_D$. Define $\mathrm{pr}_D = 1 \otimes \mathrm{ev}_1 \otimes \mathrm{ev}_e \colon \mathscr{V}'_D \to V_\delta$.

LEMMA 6.5. *There is a unique G-equivariant embedding I_D from $(C^\infty \operatorname{Ind}_M^K(\delta), \pi)$ to $C^\infty \operatorname{Ind}_P^G(\mathscr{V}'_D)$ such that* $\mathrm{pr}_D \circ \mathscr{R} \circ I_D = \mathrm{Id}$.

Proof. For $f \in C^\infty \operatorname{Ind}_M^K(\delta)$ define $I_D f \colon G \to \mathscr{V}'$ by

$$I_D f(x)(u + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), a) = [\pi(a)^{-1}\, \pi(u^\vee)\, \pi(x)^{-1} f](e)$$

for $x \in G$, $a \in A$, and $u \in \mathscr{U}(\mathfrak{n})$. Here $u \mapsto u^\vee$ is the anti-automorphism of $\mathscr{U}(\mathfrak{g})$ induced by $X \mapsto -X$ on $\mathfrak{g}$. By Definition 6.4(1) the map I_D is well defined. By Definition 6.4(2) the function $I_D f$ takes values in $\mathscr{V}'_D$. Clearly I_D is G-equivariant. One easily checks that

$$I_D f(xma) = a^{-\rho}\tau'_D(ma)^{-1}\, I_D f(x), \qquad ma \in MA,$$
$$I_D f(x; X) = -\lambda^\vee(X)\, I_D f(x), \qquad X \in \mathfrak{n},$$

where we use the notation $f(x; u) = R_u f(x)$. This implies that $I_D f \in C^\infty \operatorname{Ind}_P^G(\tau'_D)$. So I_D is a G-equivariant operator from $(C^\infty \operatorname{Ind}_M^K(\delta), \pi)$ to $C^\infty \operatorname{Ind}_P^G(\tau'_D)$.

One computes $[\mathrm{pr}_D \circ \mathscr{R} \circ I_D] f(k) = I_D f(k)(1 + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), e) = [\pi(e)^{-1}\, \pi(1^\vee)\, \pi(k)^{-1} f](e) = f(k)$. So $\mathrm{pr}_D \circ \mathscr{R} \circ I_D = \mathrm{Id}$.

On the other hand, suppose I is a G-equivariant operator from $(C^\infty \operatorname{Ind}_M^K(\delta), \pi)$ to $C^\infty \operatorname{Ind}_P^G(\tau'_D)$ such that $\mathrm{pr}_D \circ \mathscr{R} \circ I = \mathrm{Id}$. For $f \in C^\infty \operatorname{Ind}_M^K(\delta)$, $u \in \mathscr{U}(\mathfrak{n})$, and $a \in A$ we compute

$$\begin{aligned}
If(e)(u + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), a) &= a^{-\rho}\tau'(a)^{-1}\, \tau'(u^\vee)\, If(e)(1 + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), e) \\
&= If(u; a)(1 + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), e) \\
&= L_a^{-1} L_{u^\vee}\, If(e)(1 + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), e) \\
&= I[\pi(a)^{-1}\, \pi(u^\vee) f](e)(1 + \mathfrak{n}^k\mathscr{U}(\mathfrak{n}), e) \\
&= \mathrm{pr}_D \circ \mathscr{R} \circ I[\pi(a)^{-1}\, \pi(u^\vee) f](e) \\
&= [\pi(a)^{-1}\, \pi(u^\vee) f](e).
\end{aligned}$$

By the G-equivariance of I this implies $I = I_D$. So I_D is unique. ∎

We now start with the proof of Theorem 6.3. We prove that $(C^\infty \operatorname{Ind}_M^K(\sigma_{|M}), \pi_\mu)$ is uniformly AN-finite for each $\mu \in \Omega$, with parameters $k_\mu \in \mathbb{N}_0$ and $\nu_i(\mu) \in \mathfrak{a}_\mathbb{C}^*$. Lemma 6.5 then implies (ii) of the theorem. We prove that there is a k such that $k_\mu \leqslant k$ for all $\mu \in \Omega$ and that the $\nu_i(\mu)$, and therefore the corresponding differential operators in $\mathbb{D}(A)$, depend holomorphically on μ, which proves the remaining part of the theorem.

Let $f \in C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ and $x \in G$. Then $[\pi_\mu(x) f](e) = [\beta_{\mu,\mu_0} L_x \Psi_\mu f](e)$. Hence if we want to check the requirements of Definition 6.4 for π_μ we have to check that for an arbitrary $g \in \mathscr{E}_\mu^{\mu_0}$ we have

$$\beta_{\mu,\mu_0}(L(u^\vee) g)(e) = 0$$

for $u \in \mathfrak{n}^k$ and an analogous formula for the A-behaviour. By the definition of β_{μ,μ_0} we have to study the coefficients of left translated eigenfunctions at exponents in

$$\Xi(\mu) = \bigcup_{\delta \in \sigma} \Xi_\delta(\mu).$$

Let $g \in \mathscr{E}_\mu^{\mu_0}$ and $X \in \mathfrak{g}_\alpha$, the root space corresponding to α. Then

$$\begin{aligned} g(X; a) &= a^{-\alpha} g(a; X) \\ &= -a^{-\alpha}\sigma(X + \theta X)\, g(a) - a^{-2\alpha} g(\theta X; a) \\ &\sim -\sum_\xi [\sigma(X + \theta X)\, p_\mu^{\xi+\alpha}(g)(e)(\log a) \\ &\qquad + p_\mu^{\xi+2\alpha}(g)(e; \theta X)(\log a)]\, a^{\xi-\rho} \qquad (a \to \infty). \end{aligned} \tag{31}$$

Now there is a $k \in \mathbb{N}_0$ such that for all μ and for all $\xi \in \Xi(\mu)$ we have $\xi + k\alpha \notin \Xi(\mu)$. From the expansion (31) for $L_{X^\vee} g(a)$ it follows that $\beta_{\mu,\mu_0}[\mathfrak{n}^k g](e) = 0$ for all $g \in \mathscr{E}_\mu^{\mu_0}$. So $(C^\infty \operatorname{Ind}_M^K(\sigma_{|M}), \pi_\mu)$ is uniformly N-finite with a fixed k for all μ.

Concerning the A-action we find that

$$g(a_1 a) \sim \sum_\xi p_\mu^\xi(g)(e)(\log[a_1 a])[a_1 a]^{\xi-\rho} \qquad (a \to \infty).$$

So

$$\beta_{\mu,\mu_0}\left(\prod_{\xi \in \Xi(\mu)} [H - (\xi - \rho)(H)]^d f\right)(e) = 0.$$

So the finite set of $\mathfrak{a}$-weights in Definition 6.4 is equal to $\{\xi - \rho \mid \xi \in \Xi(\mu)\}$.

For generic μ we know that the coefficients in the asymptotic expansions of eigenfunctions have constant polynomial part, so $d=1$. We also know that the elements of $\Xi(\mu)$ depend holomorphically on μ.

The fact that the injection $I_K^G(\mu)$ is continuous from $C^\infty \operatorname{Ind}_M^K(\sigma_{|M})$ to $C_*^\infty \operatorname{Ind}_P^G(\tau_{D_\mu}^k)$ follows easily from the definition of π_μ.

This completes the proof of Theorem 6.3.

Theorem 6.3 enables us to extend the Main Theorem to eigenfunctions which satisfy less restrictive growth conditions and have distribution valued boundary values. Let

$$C^*(G, W)$$

be the space of functions $f\colon G \to W$ for which there is an $r \in \mathbb{R}$ such that $\|f\|_r < \infty$ (so we do not impose a restriction on the derivatives of f). Let $\mu_0 \in \mathbb{C}$ be such that $0 \notin X_R(\mu_0)$ and such that there are no square integrable eigenfunctions with eigenvalue μ_0. Define

$$C_{\mu_0}^* \operatorname{Ind}_K^G(\sigma) = \{f \in C^\infty \operatorname{Ind}_K^G(\sigma) \cap C^*(G, W) \mid (\mathscr{C} - \mu_0) f = 0\}.$$

By $C^{-\infty} \operatorname{Ind}_M^K(\sigma_{|M})$ we denote the space of generalized functions from K to W with the obvious transformation property under M.

THEOREM 6.6. *Let μ_0 be as above. The transform Ψ_{μ_0} extends to an isomorphism from $C^{-\infty} \operatorname{Ind}_M^K(\sigma_{|M})$ to $C_{\mu_0}^* \operatorname{Ind}_K^G(\sigma)$.*

For the proof we refer the reader to [18]. The proof uses the same methods as the proof of Theorem 12.2 in [2]. Of importance is Theorem 6.3 which gives us G-equivariant operators for which the convolution trick in [2] works. The thesis [18] also contains a dual analogue of Theorem 6.3: there is a finite dimensional representation τ'_μ of P for which there is a surjective G-equivariant map from $C^\infty \operatorname{Ind}_P^G(\tau'_\mu)$ onto $(C^\infty \operatorname{Ind}_M^K(\sigma_{|M}), \pi_\mu)$. Composition with Ψ_μ yields a Poisson transform on $C^\infty \operatorname{Ind}_P^G(\tau'_\mu)$.

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