Exercises GQT School
Lie groups and homogeneous spaces

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Exercise 1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $\mathfrak{j}$ be a Cartan subalgebra, $R = R(\mathfrak{g}, \mathfrak{j})$ the associated root system, and R^+ a positive system. Let S denote the associated set of simple roots in R^+, i.e., the set of roots $\alpha \in R^+$ that cannot be written as a sum of two positive roots. Let $\mathfrak{n}$ be the sum of the positive root spaces, and $\mathfrak{b} = \mathfrak{j} \oplus \mathfrak{n}$ the associated Borel subalgebra.

(a) Show that $\mathfrak{b}$ is solvable

(b) Show that the normalizer $\mathcal{N}_{\mathfrak{g}}(\mathfrak{b})$ of $\mathfrak{b}$ in $\mathfrak{g}$ equals $\mathfrak{b}$.

(c) Assume that $\mathfrak{b} \subsetneq \mathfrak{q}$ with $\mathfrak{q}$ a subalgebra of $\mathfrak{g}$. Show that there exists a positive root $\beta \in R^+$ such that $\mathfrak{g}_{-\beta} \subset \mathfrak{q}$. Show that $\mathfrak{q}$ is not solvable.

Exercise 2. **Flag manifold.** Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let $n \geq 2$ and let $d = (d_1, \ldots, d_k)$ be a sequence of positive integers with $\sum_{j=1}^k d_j = n$. We define a *flag* of type d in $\mathbb{K}^n$ to be an ordered sequence $F = (F_0, F_1, \ldots, F_{k-1}, F_k)$ of linear subspaces of $\mathbb{K}^n$ with $0 = F_0 \subset F_1 \subset \cdots \subset F_k = \mathbb{K}^n$ and with $\dim(F_j/F_{j-1}) = d_j$, for all $1 \leq j \leq k$. The collection of all flags of type d, denoted by $\mathcal{F} = \mathcal{F}_d$, is called a flag manifold.

Let $G = \mathrm{GL}(n, \mathbb{K})$ and let $\alpha : G \times \mathcal{F} \to \mathcal{F}$ be defined by $\alpha(g, F) = g \cdot F := (g(F_j) \mid 0 \leq j \leq k)$.

(a) Show that α is a transitive action of G on $\mathcal{F}$.

Let the standard flag E of type d be defined by $E_0 = 0$ and $E_j = \mathrm{span}\,\{e_1, \ldots, e_{d_1 + \cdots + d_j}\}$, for $1 \leq j \leq k$. We define the map $\varphi : G \to \mathcal{F}$ by $\varphi(g) = g \cdot E$.

(b) Determine the stabilizer $P = P_d$ of E in G. Show that P is a closed subgroup of G.

(c) Show that $\varphi : G \to \mathcal{F}$ induces a bijection $\bar{\varphi} : G/P \to \mathcal{F}$. Accordingly, we equip $\mathcal{F}$ with the structure of a smooth manifold such that $\bar{\varphi}$ is a diffeomorphism.

(d) Put $K = \mathrm{O}(n)$ if $\mathbb{K} = \mathbb{R}$ and $K = \mathrm{U}(n)$ if $\mathbb{K} = \mathbb{C}$. In both cases show that $\varphi(K) = \mathcal{F}$. Put $H = K \cap P$ and show that $\mathcal{F}$ is diffeomorphic to K/H. Conclude that $\mathcal{F}$ is compact.

(e) With notation as in (d), show that $m : K \times P \mapsto G$, $(k, p) \mapsto kp$ is a surjective map. Hint: use (d) and (b). Moreover, show that m is a smooth submersion. Hint: use homogeneity.

(f) Determine d such that $\mathcal{F}_d \simeq \mathbb{P}^{n-1}(\mathbb{K})$. More generally, let $1 \leq k < n$. Determine d such that $\mathcal{F}_d \simeq G_{n,k}(\mathbb{K})$.

(g) Determine d such that P is a Borel subgroup (in case $\mathbb{K} = \mathbb{C}$) or a minimal parabolic subgroup (in case $\mathbb{K} = \mathbb{R}$).

Exercise 3. We assume that G is a connected real semisimple Lie group with finite center, that $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ the associated Cartan decomposition and K the connected Lie subgroup of G with algebra $\mathfrak{k}$. We use the derivative of the projection $\pi : G \to G/K$ at e to identify $\mathfrak{g}/\mathfrak{k} \simeq T_{[e]}(G/K)$ and we use the natural linear isomorphism $\mathfrak{s} \to \mathfrak{g}/\mathfrak{k}$ to identify these spaces. Accordingly,

$$\mathfrak{s} \simeq T_{[e]}(G/K).$$

By the Cartan decomposition, the map $\varphi : \mathfrak{s} \times K \to G$ given by

$$\varphi(X, k) = \exp(X)k, \qquad (X \in \mathfrak{s},\ k \in K)$$

is a diffeomorphism.

We denote by β the restriction of the Killing form B to $\mathfrak{s}$. Then β is a positive definite inner product on $\mathfrak{s}$ which is $\mathrm{Ad}\,(K)$-invariant. We view this as an inner product on $T_{[e]}(G/K)$. Given $x \in G$ we define the inner product β_x on $T_{[x]}(G/K)$ by

$$\beta_x = dl_x([e])^{-1*}\beta.$$

(a) Show that β_x depends on x through its image $[x]$ in G/K.

(b) Show that $[x] \mapsto \beta_x$ defines a Riemannian structure on G/K which is invariant for the natural left action by G. Thus, G acts by isometries.

(c) Show that $\mathrm{Exp} := \pi \circ \exp : \mathfrak{s} \to G/K$ is a diffeomorphism, whose tangent map at 0 can be identified with the identity on $\mathfrak{s}$. It can be shown that this map corresponds to the Riemannian exponential map.

On a Riemannian manifold M, the local geodesic reflection S_a at a point $a \in M$ is defined by $S_a(\mathrm{Exp}_a(X)) = \mathrm{Exp}_a(-X)$.

(d) The Cartan involution Θ on G is the unique involution with $d\Theta(e) = \theta$. It is usually denoted by θ instead of Θ. Let $\bar{\theta} : G/K \to G/K$ denote the map induced by θ. Show that $\bar{\theta}$ equals the local geodesic reflection of G/K at $[e]$.

(e) Show that $\bar{\theta}$ is an isometry.

(f) Show that G/K is a global Riemannian symmetric space.

Exercise 4. Let G be a connected real semisimple Lie group with finite center and let $\sigma : G \to G$ be an involution of G, i.e., an automorphism of order 2. Let $H = (G^\sigma)_e$. The purpose of this exercise is to show that H is spherical. In a crucial way we will make use of the assumption that there exists a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ that commutes with σ. Let $\mathfrak{s}$ be the -1 eigenspace of θ.

(a) Show that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{q}$ is the -1 eigenspace of $\sigma_* := d\sigma(e)$ in $\mathfrak{g}$. (From now on we write σ for σ_*.)

Let $\mathfrak{a}_{\mathrm{q}} \subset \mathfrak{q}$ be a maximal abelian subspace of $\mathfrak{s} \cap \mathfrak{q}$ (its elements will automatically be semisimple, with real eigenvalues).

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace which contains $\mathfrak{a}_{\mathrm{q}}$.

(b) Show that $\mathfrak{a}$ is σ-invariant, so that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{q}).$$

Show that the second summand equals $\mathfrak{a}_{\mathrm{q}}$.

We recall that since σ is an automorphism of $\mathfrak{g}$ which leaves $\mathfrak{a}$ invariant, it follows that the map $\sigma : \mathfrak{a}^* \to \mathfrak{a}^*$ given by $\lambda \mapsto \lambda \circ \sigma^{-1}$ leaves the set of roots $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ invariant. Furthermore, if $\alpha \in \Sigma$ then

$$\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma\alpha}.$$

(c) Show that there exists an element $X_0 \in \mathfrak{a}_i q$ such that for every $\alpha \in \Sigma$ we have $\alpha(X_0) = 0 \Rightarrow \alpha|_{\mathfrak{a}_{\mathrm{q}}} = 0$.

(d) Put $\Sigma_{\mathrm{q}} = \{\alpha \in \Sigma \mid \alpha(X_0) \neq 0\}$ and $\Sigma_{\mathrm{q}}^+ := \{\alpha \in \Sigma \mid \alpha(X_0) > 0\}$. Show that there exists $Y \in \mathfrak{a} \cap \mathfrak{h}$ such that for all $\alpha \in \Sigma$, $\alpha(X_0) = 0 \Rightarrow \alpha(Y) \neq 0$.

(e) Let X_0, Y be as above, take $t > 0$ sufficiently small, and put $X = X_0 + tY$. Show that $\Sigma^+ := \{\alpha \in \Sigma \mid \alpha(X) = 0\}$ is a positive system for Σ and that for all $\alpha \in \Sigma^+$ with $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \neq 0$ we have $\alpha \in \Sigma_{\mathrm{q}}^+$.

Let $\mathfrak{n}$ be the sum of the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Sigma^+$ and and let

$$\mathfrak{p} := \mathscr{Z}_{\mathfrak{g}}(\mathfrak{a}) + \mathfrak{n} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}.$$

(f) Let $\alpha \in \Sigma_{\mathrm{q}}^+$. Show that for $X \in \mathfrak{g}_{-\alpha}$ we have

$$X = X + \sigma(X) - \sigma(X) \in \mathfrak{h} + \mathfrak{n}$$

(g) Let $\alpha \in \Sigma^+ \setminus \Sigma_{\mathrm{q}}$ and let $X \in \mathfrak{g}_{-\alpha}$. Show that $U := X - \sigma(X) \in \mathfrak{g}_\alpha \cap \mathfrak{q}$. Show that $V := U - \theta U \in \mathfrak{p} \cap \mathfrak{q}$ commutes with $\mathfrak{a}_{\mathrm{q}}$. Show that $V = 0$, that $U = 0$ and conclude that

$$\mathfrak{g}_{-\alpha} \subset \mathfrak{h}.$$

(h) Show that $\bar{\mathfrak{n}} \subset \mathfrak{h} + \mathfrak{p}$

(i) Show that H is spherical.

Exercise 5. Let $\mathfrak{g}, \mathfrak{j}, R, R^+, S, \mathfrak{n}$ and $\mathfrak{b}$ be as in Exercise 1. The following two properties of root systems will be important for this exercise.

- Every root $\alpha \in R^+$ can be written as a sum of roots from S.
- $R^+ \setminus S \subseteq S + R^+$.

We assume that $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ (thus, $\mathfrak{q}$ is parabolic).

(a) Show that there exists a subset $T \subset R$ such that

$$\mathfrak{q} = \mathfrak{j} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha.$$

(b) Show that there exists a unique set $F \subset S$ such that $T = R_F \cup R^+$. Here R_F denotes the collection of roots from R that belong to the $\mathbb{Z}$-span of F.

(c) If $F \subset S$ show that

$$\mathfrak{q}_F := \mathfrak{j} \oplus \bigoplus_{\alpha \in R_F \cup R^+} \mathfrak{g}_\alpha$$

is a subalgebra of $\mathfrak{g}$, containing $\mathfrak{b}$.

(d) Show that nilpotent radical (i.e., the maximal nilpotent ideal) of $\mathfrak{q}_F$ equals

$$\mathfrak{n}_F := \sum_{\alpha \in R^+ \setminus R_F} \mathfrak{g}_\alpha.$$

(e) Show that $\mathfrak{n}_F \lhd \mathfrak{q}_F$ and that $\mathfrak{q}_F = \mathfrak{l}_F \oplus \mathfrak{n}_F$, where

$$\mathfrak{l}_F = \mathfrak{j} \oplus \bigoplus_{\alpha \in R_F} \mathfrak{g}_\alpha.$$

We recall that the real span of the elements $H_\alpha \in \mathfrak{j}$ is the real form $\mathfrak{a}$ of $\mathfrak{j}$ (sometimes written $\mathfrak{j}_\mathbb{R}$) given by

$$\mathfrak{a} := \{H \in \mathfrak{j} \mid \alpha(H) \in \mathbb{R} \;\; (\forall \alpha \in R)\}.$$

For $F \subset S$ we define

$$\mathfrak{a}_F = \mathfrak{a} \cap \bigcap_{\alpha \in F} \ker \alpha$$

and $\mathfrak{a}_F^+ = \{H \in \mathfrak{a}^+ \mid \forall \alpha \in R^+ \setminus R_F : \alpha(H) > 0\}$.

(f) Show that $\mathfrak{a}^+ := \mathfrak{a}_\emptyset^+$ is the open positive chamber in $\mathfrak{a}$ and that the closed positive chamber can be written as the following disjoint union:

$$\overline{\mathfrak{a}^+} = \dot{\bigcup}_{F \subset S} \mathfrak{a}_F^+.$$

(g) Show that for ever $X \in \mathfrak{a}$ the subspace

$$\mathfrak{q}_X := \bigoplus_{s \in \mathrm{spec}(\mathrm{ad}\,(X))} \ker(\mathrm{ad}\,(X) - sI_\mathfrak{g})$$

is a parabolic subalgebra of $\mathfrak{g}$.

We define the equivalence relation $\sim$ on $\mathfrak{a}$ by $X \sim Y \iff \mathfrak{q}_X = \mathfrak{q}_Y$

(h) Show that the $\mathfrak{a}_F^+$ defined above are equivalence classes for this relation.

Remark: Let $S(\mathfrak{a})$ be the unit sphere in $\mathfrak{a}$ for a choice of Weyl group invariant inner product. Then the closures of the equivalence classes for $\sim$ induce a simplicial complex on $S(\mathfrak{a})$, which is known as the Coxeter complex for the root system R.