Harish-Chandra's philosophy of cusp forms for Whittaker functions

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Whittaker functions

Setting

- G real reductive group
- K maximal compact, $G = KAN_0$ Iwasawa decomposition
- $\chi : N_0 \to \mathrm{U}(1)$ unitary character, regular (!)

i.e.: $\forall \alpha \in \Sigma(\mathfrak{n}_0, \mathfrak{a})$ simple: $d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0$.

Whittaker functions

$$L^2_{\mathrm{loc}}(G/N_0, \chi) := \{f \in L^2_{\mathrm{loc}}(G) \mid f(xn) = \chi(n)^{-1} f(x) \quad (x \in G, n \in N_0)\}$$

$$L^2(G/N_0, \chi) := \{f \in L^2_{\mathrm{loc}}(G/N_0, \chi) \mid |f| \in L^2(G/N_0)\}$$

- Left reg^r rep^n: $L = \mathrm{Ind}_{N_0}^G(\chi)$ is unitary

Whittaker Plancherel formula

Abstractly

- $\mathrm{Ind}_{N_0}^{G}(\chi) = \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$.

Concrete realization

- Harish-Chandra, Announcement 1982.

  Details in Collected Papers Vol 5 (posthumous), 141- 307, eds. R. Gangolli, V.S. Varadarajan, Springer 2018.

- HC's approach: philosophy of cusp forms, final step unclear.

- Today: HC's approach, and sketch of final step using results on Whittaker Fourier transform and Wave packets

- Important ref: Wallach, book RRG II: discrete part, cusp forms, and functional equation and holomorphic dependence of Whittaker vectors

Discrete part

$\pi \in \widehat{G}$ (unitary dual) is said to appear discretely in $L^2(G/N_0, \chi)$ if it can be realized as a closed subrepresentation.
The closed span of such π is denoted $L^2_d(G/N_0, \chi)$.

Theorem (HC, W)

If $\pi \in \widehat{G}$ appears in $L^2_d(G/N_0, \chi)$, then it appears in $L^2_d(G)$, i.e., π belongs to the discrete series of G.

Lemma $L^2_d(G/N_0, \chi)_K \subset \mathcal{C}(G/N_0, \chi)$.

Definition (Whittaker Schwartz space)

$\mathcal{C}(G/N_0, \chi) :=$ space of $f \in C^\infty(G/N_0, \chi)$ s.t. $\forall u \in U(\mathfrak{g}), N \in \mathbb{N}$,

$$|L_u f(kan)| \leq C_{u,N} \left(1 + |\log(a)|\right)^{-N} a^{-\rho} \qquad (kan \in KAN_0),$$

where $\rho \in \mathfrak{a}^*$ is defined by $\rho(X) := \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{N_0})$.

Cusp forms

Property: $\mathcal{C}(G/N_0, \chi)$ is left G-invariant and

$$C_c^\infty(G/N_0, \chi) \subset \mathcal{C}(G/N_0, \chi) \subset L^2(G/N_0, \chi).$$

- $P_0 := Z_K(A)A\,N_0$, minimal psg.
- $\mathcal{P}_{st}$: (finite) set of psg's $P < G$ with $P \supset P_0$ (standard psg's).
- For $P \in \mathcal{P}_{st}$, Langlands deco: $P = M_P A_P N_P$, $\quad M_{1P} := M_P A_P$.

Lemma (HC, W)

If $f \in \mathcal{C}(G/N_0; \chi)$ and $P \in \mathcal{P}_{st}$ then $\int_{\bar{N}_P} |f(\bar{n})|\, d\bar{n} < \infty$.

The map $f \mapsto \int_{\bar{N}_P} |f(\bar{n})|\, d\bar{n}$ is continuous.

Definition (Space of cusp forms)

${}^\circ\mathcal{C}(G/N_0, \chi)$:= space of $f \in \mathcal{C}(G/N_0, \chi)$ s.t. $\forall P \in \mathcal{P}_{st} \setminus \{G\}$,

$$\int_{\bar{N}_P} f(x\bar{n})\, d\bar{n} = 0, \qquad (\forall\, x \in G).$$

Cusp forms and discrete part

Spherical functions Let (τ, V_τ) be a finite dimensional unitary rep^n of K.

$$L^2(\tau, G/N_0, \chi) := (L^2(G/N_0, \chi) \otimes V_\tau)^K$$
$$\subset \{f \in L^2_{\text{loc}}(G, V_\tau) \mid f(kxn) = \chi(n)^{-1}\tau(k)f(x)\}$$

$${}^\circ\mathcal{C}(\tau, G/N_0, \chi) := ({}^\circ\mathcal{C}(G/N_0, \chi) \otimes V_\tau)^K.$$

Thm (HC,W)

Suppose G has compact center ($\iff A_G = \{e\}$). Then

$${}^\circ\mathcal{C}(\tau, G/N_0, \chi) = L^2_d(\tau, G/N_0, \chi).$$

The space is finite dimensional.

Harish-Chandra descent transform

For $P \in \mathcal{P}_{st}$ define $d_P : P \to \mathbb{R}^+$ by $d_P(p) := |\det \mathrm{Ad}(p)|_{\mathfrak{n}_P}|^{1/2}$.

Definition (HC transform)

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$ define $f^{(\bar{P})} : M_{1P} \to V_\tau$ by

$$f^{(\bar{P})}(m) := d_P(m)^{-1} \int_{\bar{N}_P} f(m\bar{n})\, d\bar{n}.$$

Property

$$f^{(\bar{P})} \in C^\infty(\tau_P, M_{1P}/M_{1P} \cap N_0, \chi_P),$$

where $\tau_P := \tau|_{M_{1P} \cap K}, \quad \chi_P := \chi|_{M_{1P} \cap N_0}.$

Thm (HC)

For $a \in A_P$ define $R_a(f^{(\bar{P})})|_{M_P} : M_P \to \mathbb{C}$, $m \mapsto f^{(\bar{P})}(ma)$. Then

$$R_a(f^{(\bar{P})})|_{M_P} \in \mathcal{C}(\tau_P, M_P/M_P \cap N_0, \chi_P).$$

Transitivity of descent

Let $Q \in \mathcal{P}_{st}$.

Fact

If $P \in \mathcal{P}_{st}$ and $P \subset Q$ then ${}^*P := P \cap M_{1Q}$ is a standard parabolic subgroup of M_{1Q}. The assignment $P \mapsto {}^*P$ is bijective

$$\{P \in \mathcal{P}_{st}(G) \mid P \subset Q\} \xrightarrow{1-1} \mathcal{P}_{st}(M_{1Q}).$$

Lemma (transitivity)

Let $P \in \mathcal{P}_{st}$, $P \subset Q$, then for $f \in \mathcal{C}(\tau, G/N_0, \chi)$,

$$f^{(\bar{P})} = (f^{(\bar{Q})})^{({}^*\bar{P})}.$$

Proof

Use $\bar{N}_P = \bar{N}_{{}^*P}\bar{N}_Q$ and Fubini.

Role of the descent transform

- For $P \in \mathcal{P}_{st}$ put ${}^\circ\mathcal{C}_{P,\tau} := {}^\circ\mathcal{C}(\tau_P, M_P/M_P \cap N_0, \chi_P)$.

Def (HC)

Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$. Then

$$f^{(\bar{P})} \sim 0 \quad :\Longleftrightarrow \quad R_a(f^{(\bar{P})})|_{M_P} \perp {}^\circ\mathcal{C}_{P,\tau} \ (\forall a \in A_P).$$

More explicitly, the assertion on the right means that for all $a \in A_P$ and all $\psi \in {}^\circ\mathcal{C}_{P,\tau}$,

$$\int_{M_P/M_P \cap N_0} \langle\, f^{(\bar{P})}(ma), \psi(m)\,\rangle_{V_\tau}\, dm = 0.$$

Thm (HC's completeness theorem)

Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$. If $f^{(\bar{P})} \sim 0$ for each $P \in \mathcal{P}_{st}$, then $f = 0$.

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Sketch of proof

Assume $f^{(\bar{P})} \sim 0$ for all $P \in \mathcal{P}_{st}$.

(1) Transitivity of descent Let $Q \in \mathcal{P}_{st}$, then for all ${}^*P \in \mathcal{P}_{st}(M_{1Q})$,

$$(f^{(\bar{Q})})^{({}^*\bar{P})} (= f^{(\bar{P})}) \sim 0.$$

(2) Induction on $\mathrm{rk}_{\mathbb{R}} G = \dim \mathfrak{a}$. If $Q \neq G$ then $\mathrm{rk}_{\mathbb{R}} M_Q < \mathrm{rk}_{\mathbb{R}} G$ hence by induction

$$f^{(\bar{Q})} = 0.$$

(3) Assertion (2) for all $Q \in \mathcal{P}_{st} \setminus \{G\}$ implies $\forall a \in A_G : (R_a f)|_{M_G} \in {}^\circ\mathcal{C}_{G,\tau}$.

(4) Note that $f^{(\bar{G})} = f$. Thus, $f^{(\bar{G})} \sim 0$ means $\forall a \in A_G : (R_a f)|_{M_G} \perp {}^\circ\mathcal{C}_{G,\tau}$.
From (3) it follows that $\forall a \in A_G : (R_a f)|_{M_G} = 0$.
Hence $f = 0$ on $M_G A_G = G$.

□

Parabolic induction and Whittaker integrals

Let $P = M_P A_P N_P \in \mathcal{P}_{st}$ and $\psi \in {}^\circ\mathcal{C}_{P,\tau}$. For $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ define $\psi_\nu : G \to V_\tau$ by

$$\psi_\nu(km\bar{n}) = a^{\nu+\rho_P}\tau(k)\psi(m).$$

For $\mathrm{Re}(\nu) >_P 0$, the integral

$$\mathrm{Wh}(P,\psi,\nu,x) := \int_{N_P} \chi(n)\psi_\nu(xn)\, dn \qquad (x \in G)$$

is abs^y conv^t and defines a function $\mathrm{Wh}(P,\psi,\nu) \in C^\infty(\tau, G/N_0, \chi)$ which depends holomorphically on ν in the indicated region.

Remark

The above Whittaker integral is essentially a finite sum of generalized matrix coefficients (defined by Jacquet integrals) of $\mathrm{Ind}_{\bar{P}}^G(\sigma \otimes -\nu \otimes 1)$, with $\sigma \in \widehat{M}_{P,\mathrm{ds}}$ appearing in ${}^\circ\mathcal{C}_{P,\tau}$. (Analogue of Eisenstein integral.)

Holomorphic extension

Theorem (W)

$\mathrm{Wh}(P, \psi, \nu)$, initially defined for $\mathrm{Re}\nu >_P 0$, extends to entire holom^c function of $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with values in $C^\infty(\tau, G/N_0, \chi)$.

Remark: HC: there exists a merom^c extension, regular on $i\mathfrak{a}^*_P$.

Theorem ($\sim$): Uniformly tempered estimates

Let $\varepsilon > 0$ be suff^{t}ly small. If $u \in U(\mathfrak{g})$ then $\exists C, N, r > 0$ s.t.

$$|\mathrm{Wh}(P, \psi, \nu, u; ka)| \leq C(1 + |\nu|)^N (1 + |\log a|)^N e^{r|\mathrm{Re}\nu||\log a|} a^{-\rho},$$

for all $k \in K$, $a \in A$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with $|\mathrm{Re}\nu| < \varepsilon$.

Ingredients of proof

- Bernstein-Sato type functional equation for Jacquet integrals.
- Uniformly moderate estimates.
- Wallach's method of improving estimates along max psg's, with parameters.

Fourier transform

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$, $P \in \mathcal{P}_{st}$, $\nu \in i\mathfrak{a}_P^*$, the Fourier transform ${}^\backprime\mathcal{F}_P f(\nu) \in {}^\circ\mathcal{C}_{P,\tau}$ is defined by

$$\langle\, {}^\backprime\mathcal{F}_P f(\nu), \psi \,\rangle := \int_{G/N_0} \langle\, f(x), \mathrm{Wh}(P, \psi, \nu, x) \,\rangle_{V_\tau} \; dx, \quad (\psi \in {}^\circ\mathcal{C}_{P,\tau}).$$

Theorem ($\sim$)

$${}^\backprime\mathcal{F}_P : \mathcal{C}(\tau, G/N_0, \chi) \to \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau},$$

continuous linearly.

Remark: HC proves this for ${}^\backprime\mathcal{F}_P$ restricted to $C_c^\infty(\tau, G/N_0, \chi)$.

Proof

This follows from the uniformly tempered estimates. □

Remark: Suppose G has compact center. Then:
${}^\backprime\mathcal{F}_G \; = L^2-$ orth$^\ell$ proj^n: $\quad \mathcal{C}(\tau, G/N_0, \chi) \to {}^\circ\mathcal{C}(\tau, G/N_0, \chi)$.

Relation Fourier transform and HC descent transform

Let $\mathcal{F}_e$ denote the Euclidean Fourier transform $\mathcal{S}(A_P) \to \mathcal{S}(i\mathfrak{a}_P^*)$.

Thm ($\sim$)

If $f \in \mathcal{C}(\tau, G/N_0, \chi)$ and $\psi \in {}^\circ\mathcal{C}_{P,\tau}$, define

$$f_\psi^{(\bar{P})} : a \mapsto \int_{M_P/M_P \cap N_0} \langle\, f^{(\bar{P})}(ma), \psi(m) \,\rangle_{V_\tau}\, dm.$$

Then $f_\psi^{(\bar{P})}$ belongs to $\mathcal{S}(A_P)$ and

$$\mathcal{F}_e(f_\psi^{(\bar{P})})(\nu) = \langle\, {}^\backprime\mathcal{F}_P f(\nu), \psi \,\rangle, \qquad (\nu \in i\mathfrak{a}_P^*).$$

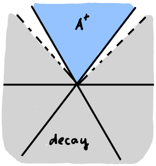
Corollary (injectivity FT)

Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$. If ${}^\backprime\mathcal{F}_P(f) = 0$ for all $P \in \mathcal{P}_{st}$ then $f = 0$.

Proof

1. ${}^\backprime\mathcal{F}_P f = 0$ implies that $f_\psi^{(\bar{P})} = 0$ for all $\psi \in {}^\circ\mathcal{C}_{P,\tau}$. Hence $f^{(\bar{P})} \sim 0$.
2. $f = 0$ by HC's completeness thm.

C-function, Normalized Whittaker integral



- $\mathrm{Wh}(P, \psi, \nu)$ is finite under $\mathfrak{Z} := \mathrm{center}(U(\mathfrak{g}))$,
- top order asymptotic behavior of exp^l type along $\mathrm{cl}(A^+)$,
- very rapid decay outside $\mathrm{cl}(A^+)$.

Lemma *Let $P \in \mathcal{P}_{st}$. For $\psi \in {}^\circ\mathcal{C}_{P,\tau}$, $\mathrm{Re}\,\nu \in \mathfrak{a}_P^{*+}$, $m \in M_P$,*

$$\mathrm{Wh}(P, \psi, \nu)(ma) \sim a^{\nu - \rho_P}[C_P(\nu)\psi](m), \qquad (a \to \infty \ \text{ in } A_P^+),$$

with $C_P(\nu) \in \mathrm{End}({}^\circ\mathcal{C}_{P,\tau})$, merom^c in $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ (reg^r for $\mathrm{Re}\nu \in \mathfrak{a}_P^{*+}$).

Definition (HC)

$$\mathrm{Wh}^\circ(P, \psi, \nu) := \mathrm{Wh}(P, C_P(\nu)^{-1}\psi, \nu) \quad (\text{mero}^c \text{ in } \nu)$$

- $P \sim Q :\iff \exists w \in W(\mathfrak{a}) : \ w(\mathfrak{a}_P) = \mathfrak{a}_Q$ (associated).
- $W(\mathfrak{a}_Q|\mathfrak{a}_P) := \{s \in \mathrm{Hom}(\mathfrak{a}_P, \mathfrak{a}_Q) \mid \exists w \in W(\mathfrak{a}) : \ s = w|_{\mathfrak{a}_P}\}$.

Functional equations, Maass-Selberg relations

Lemma (Functional equations: HC)

Let $P, Q \in \mathcal{P}_{st}$, $P \sim Q$. Then for all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$,

$$\mathrm{Wh}^\circ(Q, C^\circ_{Q|P}(s,\nu)\psi, s\nu) = \mathrm{Wh}^\circ(P, \psi, \nu), \quad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}),$$

with $C^\circ_{Q|P}(s,\nu) \in \mathrm{Hom}({}^\circ\mathcal{C}_{P,\tau}, {}^\circ\mathcal{C}_{Q,\tau})$ a uniquely determined merom^c function of $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$.

Thm (Maass-Selberg relations, HC)

For all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$C^\circ_{Q|P}(s, -\bar{\nu})^* \circ C^\circ_{Q|P}(s,\nu) = \mathrm{id}_{\circ\mathcal{C}_{P,\tau}}.$$

In particular, for $\nu \in i\mathfrak{a}^*_P$, the map $C^\circ_{Q|P}(s,\nu)$ is unitary.

Theorem (HC)

$\nu \mapsto \mathrm{Wh}^\circ(P, \psi, \nu)$ *is regular on $i\mathfrak{a}^*_P$.*

Wave packets

Definition

For $P \in \mathcal{P}_{st}$, $\psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$, $x \in G$,

$$\mathcal{W}_P(\psi)(x) := \int_{i\mathfrak{a}_P^*} \mathrm{Wh}^\circ(P, \psi(\nu), \nu, x)\, d\nu.$$

Theorem ($\sim$)

$$\mathcal{W}_P : \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau} \to \mathcal{C}(\tau, G/N_0, \chi)$$

is continuous linear.

Remark: HC proves this for $\mathcal{W}_P$ restricted to a subspace of $\mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$.

Proof requires

- the uniformly tempered estimates
- theory of constant term with parameter
- families of type $\mathrm{II}_{\mathrm{hol}}(\Lambda)$ (as in previous joint work with Carmona and Delorme for reductive symmetric space G/H).

Normalized Fourier transform

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$ define $\mathcal{F}_P f : i\mathfrak{a}_P^* \to {}^\circ\mathcal{C}_{P,\tau}$ as ${}^\backprime\mathcal{F}_P f$, but with $\mathrm{Wh}^\circ(P, \cdot)$ in place of $\mathrm{Wh}(P, \cdot)$.

Then $\mathcal{F}_P : \mathcal{C}(\tau, G/N_0, \chi) \to \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$ is continuous linear.

Lemma

$\mathcal{W}_P\mathcal{F}_P \in \mathrm{End}(\mathcal{C}(\tau, G/N_0, \chi))$ *only depends on* $[P] \in \mathcal{P}_{st}/\sim$.

Proof

This follows from the Maass-Selberg relations. □

Lemma (projection)

(a) *If* $Q \in \mathcal{P}_{st}$, $Q \not\sim P$ *then* $\mathcal{F}_Q\mathcal{W}_P = 0$.

(b) $(\exists! c_P > 0) : \Pi_P = c_P\mathcal{F}_P\mathcal{W}_P$ *is a projection operator in* $\mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$. *Moreover,*

$$\Pi_P \circ \mathcal{F}_P = \mathcal{F}_P.$$

Lemma *Let $P, Q \in \mathcal{P}_{st}$. Then*

$$\mathcal{F}_Q c_P \mathcal{W}_P \mathcal{F}_P = \delta_{[Q],[P]} \mathcal{F}_Q. \qquad (*)$$

Proof If $[P] \neq [Q]$, use Lemma (projection) (a). If $P \sim Q$, then by Lemma (projection) (b),

$$\mathcal{F}_Q c_P \mathcal{W}_P \mathcal{F}_P = \mathcal{F}_Q c_Q \mathcal{W}_Q \mathcal{F}_Q = \Pi_Q \circ \mathcal{F}_Q = \mathcal{F}_Q.$$

□

Plancherel theorem

If $f \in \mathcal{C}(\tau, G/N_0, \chi)$, then

$$f = \sum_{[P] \in \mathcal{P}_{st}/\sim} c_P \, \mathcal{W}_P \mathcal{F}_P f.$$

Proof

Put $g = f - \sum \mathcal{W}_P \mathcal{F}_P f$. Then $g \in \mathcal{C}(\tau, G/N_0, \chi)$ and by (*):

$$\mathcal{F}_Q g = \mathcal{F}_Q f - \mathcal{F}_Q f = 0.$$

From this, $\mathcal{F}_Q(g) = 0$ for all $Q \in \mathcal{P}_{st}$, hence $g = 0$ (injectivity FT).