Whittaker functions and Fourier inversion on real reductive Lie groups

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From E_6 to $\tilde{\tilde{E}}_{60}$

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Setting: G reductive Lie group / $\mathbb{R}$,

$G = KAN_0$ Iwasawa decomp

$\chi \in \widehat{N_0} = \mathrm{Hom}(N_0, U(1))$, regular

i.e. $\forall\, \alpha \in \Sigma^+ = \Sigma(\mathfrak{n}_0, \mathfrak{a})$ simple: $d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0$

Def $L^2(G/N_0;\chi) = \{ f : G \to \mathbb{C} \mid f$ measble, $f(gn_0) = \chi(n_0)^{-1} f(g)$, $|f| \in L^2(G/N_0) \}$

has invar. measure

$L = \mathrm{Ind}_{N_0}^G(\chi)$

Whittaker - Plancherel:

$$\mathrm{Ind}_{N_0}^G(\chi) \cong \int_{\hat{G}}^{\oplus} m_\pi \, \pi \, d\mu(\pi)$$

unitary dual

mult^y

WP measure

History:

- Harish-Chandra 1980: announced precise WP-decomposition. Proof appeared in Coll^d works vol 5, 2018 [eds Gangolli, Varadarajan, assistance of Kolk]
  Proof incomplete! (details will follow)
- Wallach RRG 2, important, but errors

Thm (HC, Wallach)

$$\pi \in \hat{G},\ \underset{\substack{\uparrow \\ \text{discretely}}}{\pi <} \mathrm{Ind}_{N_0}^{G}(\chi) \implies \pi \in \hat{G}_{ds}$$

HC's strategy: prove Plancherel for $f \in$

$$C_c^\infty(G/N_0 ; \chi)_K$$ (dense subspace)

mod N_0 K-finite for L.

Fix (V_τ, τ) fin dim^l unitary rep of K.

<u>Def</u> τ-spherical functions

$$C^\infty(\tau : G/N_0 ; \chi) := \left(C^\infty(G/N_0 ; \chi) \otimes V_\tau\right)^K$$
$$= \{ f : G \xrightarrow{C^\infty} V_\tau \mid f(kgn) = \tau(k) f(g) \chi(n)^{-1} \}$$

<u>Def</u> Whittaker functions

center $U(\mathfrak{g})$.

$$\mathcal{A}(\tau : G/N_0 ; \chi) := \{ f \in C^\infty(\tau : G/N_0 ; \chi) \mid \dim Z(\mathfrak{g}) f < \infty \}$$

Ex $G = SL(2,\mathbb{R})$, $\tau \in \widehat{SO(2)}$: essentially classical.
Whittaker functions in coordinate $x = e^{-t} = e^{-\rho\log a(\cdot)}$

Def $\mathcal{A}_2(\tau : G/N_0 : \chi) = \mathcal{A}(\tau : G/N_0 : \chi) \cap L^2(\tau : G/N_0 : \chi)$

Thm (HC, Wallach) $\dim \mathcal{A}_2(\tau : G/N_0 ; \chi) < \infty$.

Parabolic Induction

$P_0 := Z_G(\mathfrak{a}) N_0 = MAN_0$ minimal psgp of G

$\mathcal{P} := \{ P < G \mid P \text{ psgp}, P \supset A \}$ $\quad \# \mathcal{P} < \infty$

$\mathcal{P}_{st} := \{ P \in \mathcal{P} \mid P > P_0 \}$ standard psg's

Fix $P \in \mathcal{P}_{st}$, $P = M_P A_P N_P$ Langlands deco

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$\chi_P := \chi|_{M_P \cap N_0}$ (is regular), $\qquad \tau_P := \tau|_{M_P \cap K}$

$\mathcal{A}_{2,P} := \mathcal{A}_2(\tau_P : M_P/_{M_P \cap N_0} : \chi_P)$

$$\left.\begin{array}{l} \psi \in \mathcal{A}_{2,P} \\ \nu \in \mathfrak{a}^*_{P\mathbb{C}} \end{array}\right\} \xrightarrow[\mathrm{Ind}_P^G]{} \underset{\text{meromorphic}}{\overset{\text{norm}^n \qquad \text{Whittaker integral}}{Wh^\circ(P, \nu, \cdot)(\psi)}} \in \mathcal{A}(\tau; G/N_0; \chi)$$

Normalisation:

$$Wh^\circ(P, \nu, ma)\psi \sim a^{\nu - \rho_P}\psi(m) \quad (a \xrightarrow{A_P^+} \infty,\ m \in M_P)$$

for $\mathrm{Re}\,\nu >_P 0$.

Lemma $\nu \mapsto Wh^\circ(P,\nu)$ is meromorphic on $\mathfrak{a}^*_{P\mathbb{C}}$ with values in $\mathcal{A}^*_{2,P} \otimes \mathcal{A}(\tau, G/N_0 : \chi)$, regular on $i\mathfrak{a}^*_P$.

Singular set $=$ locally finite union of hyperplanes $H_{\alpha,c} : \langle \nu, \alpha \rangle = c$,
($\alpha \in \Sigma(\mathfrak{n}_P, \mathfrak{a}_P)$, $c \in \mathbb{R}$ unif^{ly} bdd from above).

Fourier

Def $\mathcal{F}_P : C^\infty_c(\tau : G/N_0 : \chi) \longrightarrow C^\infty(i\mathfrak{a}^*_P) \otimes \mathcal{A}_{2,P}$ by

$$\langle \mathcal{F}_P f, \psi \rangle = \int_{G/N_0} \langle f(x), Wh^\circ(P, \psi, \nu)(x) \rangle_{V_\tau} \, dx$$

HC's Plancherel identity

$$\|f\|^2_{L^2} = \sum_{P \in \mathcal{P}_{st}/\sim} \int_{i\mathfrak{a}^*_P} \underbrace{\| \mathcal{F}_P f(\nu) \|^2}_{\text{depends on } [P]_\sim} \, d\lambda_P(\nu)$$

$P \sim Q \iff \mathfrak{a}_P \overset{W(\mathfrak{a})}{\sim} \mathfrak{a}_Q$.

Lebesgue measure

To complete HC's proof of this, need:

Thm 1 $(\sim)$ $\mathcal{F}_P$ extends to cts linear map

$$\mathcal{C}(\tau : G/N_0 : \chi) \longrightarrow \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}.$$

HC's Schwartz space

Eucl^n Schwartz

Def Wavepacket transform of $\psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$

$$\mathcal{W}_P(\psi) := \int_{i\mathfrak{a}_P^*} \mathrm{Wh}^\circ(P,\nu)\psi(\nu)\, d\lambda_P(\nu).$$

Thm 2 $(\sim)$ $\mathcal{W}_P : \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P} \xrightarrow{\text{cts linear}} \mathcal{C}(\tau : G/N_0 : \chi)$

Rem $\langle f, \mathcal{W}_P \psi \rangle = \langle \mathcal{F}_P f, \psi \rangle$ $\quad \begin{pmatrix} f \in \mathcal{C}(\tau : G/N_0 : \chi) \\ \psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P} \end{pmatrix}$

By using HC's philosophy of cusp forms, get

Thm (Plancherel inversion)

$$I = \sum_{P \in \mathcal{P}_{st}/\sim} \mathcal{W}_P \circ \mathcal{F}_P \quad \text{on} \quad \mathcal{C}(\tau : G/N_0 ; \chi)$$

Rem $\mathcal{F}_{P_0}$ not injective on $\mathcal{C}(\tau : G/N_0 ; \chi)$ (in general)

Prop $\mathcal{F}_{P_0}$ is injective on $C_c^\infty(\tau : G/N_0 ; \chi)$.

Pbm invert $\mathcal{F}_{P_0} : C_c^\infty(\tau : G/N_0 : \chi) \longrightarrow \mathcal{M}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathcal{A}_{2,P_0}$.

will do, inspired by Heckman-Opdam, Ann '97

$\sim$ & Schlichtkrull Acta '99 (Opdam, Jussieu '04)

Pseudo Wave packets

- $Wh^\circ(P_o, \nu) \in \mathcal{A}^*_{2,P_o} \otimes \mathcal{A}(\tau : G/N_o ; \chi)$ is annihilated by cofinite ideal $I_\nu \triangleleft \mathcal{Z}(\mathfrak{g})$.

- radial components of $Z \in I_\nu$ according to $G = KAN_o$

  $\rightsquigarrow$ cofinite system on A with regular singularities at ∞ in $A^+(P_o)$. (<u>not</u> for remaining chambers)

- $\Rightarrow$ $\exists!$ family $Wh_1(\nu) \in \mathcal{A}^*_{2,P_o} \otimes \mathcal{A}(\tau : G/N_o ; \chi)$ mero in $\nu \in \mathfrak{a}^*_{\mathbb{C}}$:

  (1) $I_\nu \, Wh_1'(\nu) = 0 \quad (\forall \nu)$

  (2) $$Wh_1(\nu, ma)\psi = a^{\nu - \rho} \sum_{\xi \in \mathbb{N}\Sigma^+} a^{-\xi} \big(\Gamma_\xi(\nu)\psi\big)(m)$$

  $(\psi \in \mathcal{A}_{2,P_o},\ m \in M,\ a \in A)$.

$(\Gamma_\xi \in \mathcal{M}(\mathfrak{a}^*_{\mathbb{C}}) \otimes \mathrm{Hom}(\mathcal{A}_{2,P_o}, V_\tau))$.

Lemma
$$Wh^\circ(P_0,\nu) = \sum_{s\in W(\mathfrak{a})} Wh_1(s\nu)\, C^\circ(s:\nu)$$

where $C^\circ(s:\cdot) \in \mathcal{M}(\mathfrak{a}^*_{\mathbb{C}}) \otimes \mathrm{End}(\mathcal{A}_{2,P_0})$

Thm (HC) $C^\circ(s:-\bar{\nu})^* C^\circ(s:\nu) = I$ (Maass–Selbergs)

Cor For $f \in C_c^\infty(\tau : G/N_0 ; \chi)$:
$$Wh^\circ(P_0,\nu)\,\mathcal{F}_{P_0} f(\nu) = \sum_{s\in W(\mathfrak{a})} Wh_1(s\nu)\, \mathcal{F}_{P_0} f(s\nu)$$

Cor
$$\mathcal{W}_{P_0}\circ \mathcal{F}_{P_0} f = \sum_{s\in W(\mathfrak{a})} \int_{i\mathfrak{a}^* + s\varepsilon_0} Wh_1(\nu)\, \mathcal{F}_{P_0} f(\nu)\, d\lambda_{\mathfrak{a}}(\nu)$$

$\varepsilon_0 \in \mathfrak{a}^{*+}$, $\varepsilon_0 \to 0$

Fourier inversion

Thm for $f \in C_c^\infty(\tau : G/N_0 : \chi)$, $x \in G$:

pseudo wave packet

$$f(x) = \int_{i\mathfrak{a}^* + \eta} |W(\mathfrak{a})| \, Wh_1(\nu) \, \mathcal{F}_{P_0} f(\nu) \, d\lambda_0(\nu) \quad (=: T_\eta f)$$

$\eta \in \mathfrak{a}^*$, $\eta \ll_{P_0} 0$

Rem analogous to G/H (red. symm. sp.), affine Hecke alg

Rem previous formula: $\mathcal{W}_{P_0} \mathcal{F}_{P_0} f = \frac{1}{|W(\mathfrak{a})|} \sum_{s \in W(\mathfrak{a})} T_{s\varepsilon_0}(f)$

Comparison with Plancherel inversion:

- by residue shift in the spirit of Heckman – Opdam, '97
- using residue weights as in ~ – S, '99.

Def residue weight is map $t : \mathcal{P} \to [0,1]$ s.t. $\forall Q \in \mathcal{P}$:

$$\sum_{P \in \mathcal{P},\, \mathfrak{a}_P = \mathfrak{a}_Q} t(P) = 1, \quad \left(t(wQw^{-1}) = t(Q), \quad t(\bar{Q}) = Q \right)$$

Let $W_P = Z_W(\mathfrak{a}_P)$, let $W^P \leftrightarrow W/W_P$ (minimal length)

$$T_\eta f =$$

$$\sum_{P\in\mathcal{P}_{st}} \sum_{\xi\in\mathfrak{a}_P^{*\perp}} t(P)\,|W(\mathfrak{a}_P)| \int_{\xi+\varepsilon_P+i\mathfrak{a}_P^*} \mathrm{Res}^{P_0,t}_{\xi+i\mathfrak{a}_P^*}\left[\sum_{s\in W^P} Wh_1(s\cdot)\,\mathcal{F}_{P_0}f(s\cdot)\right] d\lambda_P$$

finite

$\in \mathfrak{a}_P^+$, $\varepsilon_P \to 0$

$\mathcal{F}f(s\cdot) = C^0(s:\nu)\,\mathcal{F}f(\cdot)$

Lebesgue

$$= \sum_{P\in\mathcal{P}_{st}} t(P)\,|W(\mathfrak{a}_P)|\,\mathcal{W}_P\circ\mathcal{F}_P(f)$$

$(W(\mathfrak{a}_P) = N_W(\mathfrak{a}_P)/W_P)$

Cor $f = \sum_{P \in \mathcal{P}_{st}} t(P)\,|W(\sigma_P)|\,\mathcal{W}_P \circ \mathcal{F}_P(f)$

Observe: $\sum_{Q \in \mathcal{P}_{st},\, Q \sim P} t(Q) = |W(\sigma_P)|^{-1}$!

$\Longrightarrow \quad I = \sum_{P \in \mathcal{P}_{st}/\sim} \mathcal{W}_P \circ \mathcal{F}_P$ (Plancherel inversion)

Residue approach fruitful for finding spectra:

1) Automorphic forms $L^2(G/\Gamma)$
2) Affine Hecke algebras
3) $L^2(G/H)$ (including $G \simeq G \times G/d(G)$)
4) $L^2(G/N_0 ; \chi)$ Whittaker

Dear Eric,

My warmest congratulations !