Analysis on real semisimple Lie groups, 1

Setting: G connected real Lie group semisimple, i.e. $\mathfrak{g}$ semisimple
$\Leftrightarrow$ $\mathfrak{g}$ direct sum of simple ideals
$\Leftrightarrow$ $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X,Y) \mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ (Killing form) is non-degenerate.

<u>Lemma</u>: $\mathfrak{g}$ has a <u>Cartan</u> involution θ, i.e.
$\theta \in \mathrm{Aut}(\mathfrak{g})$, $\theta^2 = I_{\mathfrak{g}}$, $\langle \cdot, \cdot \rangle = -B(\cdot, \theta(\cdot))$ is a positive definite inner product.

<u>Remark</u> All Cartan involutions are conjugate under $\mathrm{Ad}(G)$.

<u>Lemma</u>: $\exists! \; \tilde{\theta} \in \mathrm{Aut}(G)$: $d\tilde{\theta}(e) = \theta$. Moreover, $\tilde{\theta}$ is an involution (we write $\theta = \tilde{\theta}$).

<u>Define</u> $K = G^{\theta} = \{ g \in G \mid \theta(g) = g \}$

<u>Lemma</u>: T.F.A.E. (The following are equivalent)
(a) K cpt
(b) center(G) finite

From now on: <u>assume</u> center(G) finite.

<u>Lemma</u> K is a maximal compact subgroup of G. All such are conjugate to K.

22/5 – 2013

Define $\mathfrak{k} := \ker(\Theta - I)$, $\mathfrak{p} := \ker(\Theta + I)$.
Then
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{(Killing orthogonal)}$$
Moreover: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$
(from defi's).

<u>Note</u>: $\mathfrak{k} = \mathrm{Lie}(K)$

<u>Example</u>: $G = SL(n, \mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R}) =$
$= \{X \in M_n(\mathbb{R}) \mid \operatorname{tr} X = 0\}$

$\Theta : \mathfrak{g} \to \mathfrak{g}$, $X \longmapsto -X^T$

$\Theta : G \to G$, $x \longmapsto (x^T)^{-1}$

$K = SO(n)$, $\mathfrak{k} = \mathfrak{so}(n)$, $\mathfrak{p} = \mathcal{S}_n = \{X \in M_n(\mathbb{R}) \mid \operatorname{tr} X = 0 \ \& \ X^T = X\}$

<u>Fact</u> (from defi's) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant.
$\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}(K)$-invariant.

Hence $T_{eK}(G/K) \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$ carries a
K-invariant positive definite inner product,
(corresponding to $\langle \cdot, \cdot \rangle|_{\mathfrak{p}}$).

<u>Fact</u>. This extends to a G-invariant Riemannian metric on G/K.

22/5 - 2013

Prop. (Cartan deco) The map $\mathfrak{p}\times K \to G$, $(X, k) \mapsto \exp X \cdot k$ is a diffeomorphism. The manifold G/K is geodesically complete and the following diagram commutes:

$$\begin{tikzcd} \mathfrak{p} \arrow[d] \arrow[dr, "\pi\circ\exp"] & \\ \mathfrak{g}/\mathfrak{k} \simeq T_{eK}(G/K) \arrow[r, "\mathrm{Exp}_{eK}"'] & G/K \end{tikzcd}$$

($\pi : G \to G/K$ canonical, Exp_{eK} exponential map of Riemannian manifold G/K at eK).

Def. If M is a Riemannian mfd, $a \in M$, then the local geodesic reflection s_a in the point a is defined by

$$s_a(\mathrm{Exp}_a X) = \mathrm{Exp}_a(-X)$$

for X in a suff'y small neighborhood of 0 in $T_a M$.

Def M (connected) is said to be Riemannian (globally) symmetric if $\forall_{a\in M}$ s_a extends to a global isometry $M \to M$. ($\Longrightarrow$ extension unique)

22/5 - 2013

Lemma G/K is Riemannian (globally) symmetric

Pf. $\theta: G \to G$ induces $\mathcal{S} = \mathcal{S}_{eK} : G/K \to G/K$, since $\theta(\exp X\, k) = \exp \theta X \cdot k = \exp(-X) k$.

Let γ denote Riemannian metric on G/K. Then $\mathcal{S}^*\gamma$ is a Riemannian metric. It is G-inv't, since

$$l_g^* \mathcal{S}^* \gamma = \mathcal{S}^* l_{\theta(g)}^* \gamma = \mathcal{S}^* \gamma.$$

Also, $(\mathcal{S}^*\gamma)_{eK} = (-I)^* \gamma_{eK} = \gamma_{eK}$ hence $\mathcal{S}^*\gamma = \gamma$. It follows that $\mathcal{S}$ is an isometry. Now use that G acts transitively on G/K by isometries.

Remark If M is Riemannian (globally) symmetric of non-positive sectional curvature, then

$$M \simeq (\mathbb{R}/\mathbb{Z})^p \times \mathbb{R}^q \times (G/K)$$

with G, K of above type

(part of E. Cartan's classification of Riemannian symmetric spaces)

Example $SL(2,\mathbb{R}) \curvearrowright H_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by fractional linear transformations :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

The action is transitive:

$$H_+ = SL(2,\mathbb{R}) \cdot i \simeq SL(2,\mathbb{R})/SO(2)$$

since $SO(2)$ is the stabilizer of i.

$$\langle \cdot , \cdot \rangle_z = y^{-2} \langle \cdot , \cdot \rangle_{eucl} \quad \text{on } T_z H_+ \simeq \mathbb{R}^2$$

is $SL(2,\mathbb{R})$-invariant: hyperbolic metric.

G/K has G-invariant positive density $d\bar{x}$
(volume density)

Fourier analysis: decomposition of $L^2(G/K, d\bar{x})$ requires unitary representations of G.

Def unitary repⁿ π of G in Hilbert space $\mathcal{H}_\pi$ is group homomorphism $\pi : G \to U(\mathcal{H}_\pi)$ which is strongly continuous ($\Leftrightarrow$ $G \times \mathcal{H}_\pi \to \mathcal{H}_\pi$ continuous)

Def G is called of non-compact type if $\forall_{I \triangleleft \mathfrak{g} \text{ simple}} : I \not\subset \mathfrak{k}$

22/5 – 2013

Thm Assume G of non-compact type.
If $(\pi, \mathcal{H}_\pi)$ is a non-trivial unitary representation of G then $\dim_{\mathbb{C}} \mathcal{H}_\pi = \infty$

Proof Assume $\dim_{\mathbb{C}} \mathcal{H}_\pi < \infty$. Then $G \xrightarrow{\pi} U(\mathcal{H}_\pi)$ is a (smooth) Lie group homomorphism
Then
$$\pi_* = d\pi(e) : \quad \mathfrak{g} \longrightarrow \mathfrak{u}(\mathcal{H}_\pi)$$
is a Lie algebra homomorphism Let $\langle \cdot , \cdot \rangle$ denote the positive definite inner product on $\mathcal{H}_\pi$.
Then $(\pi_*)^* \langle \cdot , \cdot \rangle = \beta$ is a $Ad(G)$-invariant positive semi-definite bilinear form on every simple ideal $\mathfrak{y}$ of $\mathfrak{g}$. By simpleness and Schur's lemma, $\beta|_{\mathfrak{y}} = c_y \cdot B_y$ for a scalar $c_y \in \mathbb{C}$. Since $\mathfrak{y}$ is non-compact, the Killing form B_y is non-degenerate and not definite. Hence $c_y = 0$, hence $\pi_* = 0$, and by connectedness of G, π is the trivial representations ☒

Thus: infinite dimensional unitary representations have to be considered

Some structure thy

We retain assumptions: G connected, semisimple, finite center.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan deco

Fix $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace, i.e. a maximal linear subspace s.t. $[\mathfrak{a}, \mathfrak{a}] = 0$.

As $\mathfrak{p} \times K \to G$, $(X, k) \mapsto \exp X \cdot k$ is diffeo, $A = \exp \mathfrak{a}$ is a closed subgroup of G. Its image in G/K is a closed submanifold.

Fact. $A \cdot eK$ is a maximally flat totally geodesic submanifold of G/K.
$\dim \mathfrak{a}$ is called the rank of G/K.

View $\mathfrak{a}$ as analogous to a maximal torus in a compact Lie algebra.

From the definitions one checks that

$$\mathrm{ad}(X)^T = -\mathrm{ad}(X), \quad \mathrm{ad}(Y)^T = \mathrm{ad}(Y)$$

for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$, relative to $\langle \cdot, \cdot \rangle$. Hence $\mathrm{ad}(Y) : \mathfrak{g} \to \mathfrak{g}$ diagonalizes over an orthon. basis of $\mathfrak{g}$. The elements $\mathrm{ad}(H)$, $H \in \mathfrak{a}$

commute in $\mathrm{End}(\mathfrak{g})$ hence allow simultaneous diagonalization.

Given $\lambda \in \mathfrak{a}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$, define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \ \forall H \in \mathfrak{a}\}$$
$$= \bigcap_{H \in \mathfrak{a}} \ker(\mathrm{ad}H - \lambda(H)I)$$

Define $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\lambda \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\lambda \neq 0\}$
Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha . \qquad (8\text{-}1).$$

<u>Lemma</u> $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is a (possibly non-reduced) root system in $\mathfrak{a}^*$.

By this we mean that the set

$$\Sigma_0(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \Sigma \mid [0,1]\alpha \cap \Sigma = \{\alpha\}\}$$

of indivisible roots is a genuine root system in $\mathfrak{a}^*$ and that $\forall \alpha \in \Sigma_0(\mathfrak{g}, \mathfrak{a})$

$$\mathbb{R}\alpha \cap \Sigma \subset \{\pm\alpha, \pm 2\alpha\}$$

Clearly the sets of root hyperplanes

$\mathcal{H}_0 = \{\ker\alpha \mid \alpha\in\Sigma_0\}$, $\mathcal{H} = \{\ker\beta \mid \beta\in\Sigma\}$ are equal, so that Σ_0 and Σ generate the same reflection groups:

$$W(\Sigma(\mathfrak{g},\mathfrak{a})) = W(\Sigma_0(\mathfrak{g},\mathfrak{a})) \quad \text{(Weyl group)}$$

Lemma The natural embedding

$$N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \longrightarrow GL(\mathfrak{a}),\ k\mapsto Ad(k)|_{\mathfrak{a}}$$

gives an isomorphism

$$N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \xrightarrow{\simeq} W(\Sigma(\mathfrak{g},\mathfrak{a})).$$

We will briefly write $W = W(\Sigma(\mathfrak{g},\mathfrak{a}))$

Remark be aware that $m_\alpha = \dim\mathfrak{g}_\alpha$ may be greater than 1 for $\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})$.
(in fact, $\alpha, 2\alpha\in\Sigma \Rightarrow m_\alpha \geq 2$)

Let $\mathfrak{a}^+$ be a choice of connected component of $\mathfrak{a}\setminus\bigcup\mathcal{H}$ (positive Weyl chamber).
Put

$$\Sigma^+ := \{\alpha\in\Sigma \mid \alpha>0 \text{ on } \mathcal{C}\}$$

Then

$$\Sigma = \Sigma^+ \amalg (-\Sigma^+)$$

(Σ^+: positive system)

<u>Lemma</u> $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ (from defi's)

Put
$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$
$$\bar{\mathfrak{n}} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$$

Then $\bar{\mathfrak{n}} = \theta \mathfrak{n}$. Moreover (8-1) leads to

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{g}_0 \oplus \mathfrak{n} \qquad (10-1)$$

Since $\theta = -I$ on $\mathfrak{a}$, $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$ and accordingly

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}).$$

The second summand is $\mathfrak{a}$, since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Put $\mathfrak{m} := \mathfrak{g}_0 \cap \mathfrak{k}$. Then

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}. \qquad (10-2).$$

Note that:

$$\mathfrak{k} = \mathfrak{g}^\theta = \mathfrak{m} \oplus (\bar{\mathfrak{n}} \oplus \mathfrak{n})^\theta$$
$$= \mathfrak{m} \oplus \{ X + \theta X \mid X \in \bar{\mathfrak{n}} \}.$$
$$(10-3)$$

From (10-2) en (10-3) we find

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \text{ induces } \mathfrak{k} \xrightarrow{\simeq} \mathfrak{g}/(\mathfrak{a}\oplus\mathfrak{n})$$

(linear iso) Hence:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \qquad (11-1)$$

which is known as the infinitesimal <u>Iwasawa</u> <u>decomposition</u>.

<u>Example</u> $G = SL(n, \mathbb{R})$, $K = SO(n)$

We may take

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \middle|\ \sum t_j = 0 \right\}$$

$$\Sigma^+ = \{ \varepsilon_i - \varepsilon_j \mid j > i \}$$

$$\left(\varepsilon_i : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i \right).$$

Then

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

$$\mathfrak{k} = \{ X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T = -X \}$$

and the Iwasawa decomposition is readily checked.

<u>Lemma</u> $\mathfrak{n}$ is nilpotent, $N := \exp(\mathfrak{n})$ is a closed subgroup of G and $\exp : \mathfrak{n} \to N$ is a diffeomorphism.

<u>Example</u> $G = SL(n, \mathbb{R})$, $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$,
$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} = I + \mathfrak{n}$.

$\exp : \mathfrak{n} \to N$ is given by $\exp X = \sum_{k=0}^{n} \frac{X^k}{k!}$ and its inverse by $\log(1+X) = \sum_{k=1}^{n} (-1)^k \frac{X^k}{k}$ for $X \in \mathfrak{n}$.

<u>Theorem</u> $K \times A \times N \to G$, $(k, a, n) \mapsto kan$ is a diffeomorphism. (global Iwasawa decomposition)

Let $M = Z_K(\mathfrak{a})$ (then $\mathrm{Lie}(M) = \mathfrak{m}$), then one readily checks that $MA = Z_G(\mathfrak{a}) \simeq M \times A$.
All root spaces are normalized by MA, hence $\mathfrak{n}$ and N are normalized by MA.
It follows that

$$P := MAN \quad (\simeq M \times A \times N \text{ by Iwasawa})$$

is a subgroup of G. Its Lie algebra $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is its own normalizer in $\mathfrak{g}$, and using the Iwasawa decomposition one finds

$$N_G(\mathfrak{m} + \mathfrak{a} + \mathfrak{n}) = P.$$

Note that, again by the Iwasawa deco,

$$K \cap P = M,$$

and the inclusion $K \longrightarrow G$ induces a diffeomorphism

$$K/M \xrightarrow{\simeq} G/P. \qquad (13$$

<u>Example</u> $G = SL(n, \mathbb{R})$, then

$$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \begin{array}{l} a_j > 0 \\ \prod a_j = 1 \end{array} \right\}$$

$$M = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \middle| \det = 1 \right\}$$

so

$$P = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in SL(n, \mathbb{R}) \right\}$$

$$\simeq M \times A \times N.$$

A full flag in $\mathbb{R}^n$ is a sequence $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n$ of linear subspaces. The set of these is denoted $\mathcal{F}$. The group $SL(n, \mathbb{R})$ naturally acts on $\mathcal{F}$

by $g \cdot (V_j) = [g(V_j)]$.

14-1 __Lemma__ The action $SL(n,\mathbb{R}) \circlearrowright \mathcal{F}$ is transitive.

__Proof__ Let $\mathbb{E} = (E_j)$ be the standard flag i.e. $E_j = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_j$, (e_j) standard base. Let $(V_j) \in \mathcal{F}$. Fix a basis $v_1 \dots v_n$ of $\mathbb{R}^n$ s.t. $V_j \oplus \mathbb{R}v_{j+1} = V_{j+1}$ $(0 \leq j < n)$.
There exists $g \in SL(n,\mathbb{R})$ s.t. $g e_j = v_j$ for $j<n$. Then $g \cdot \mathbb{E} = (V_j)$. $\boxtimes$.

__Lemma__ The stabilizer of $\mathbb{E}$ in $SL(n,\mathbb{R}) = G$ equals P.
Pf Easy

__Cor__ $g \mapsto g \cdot \mathbb{E}$ induces a bijection $G/P \xrightarrow{\simeq} \mathcal{F}$ (thus $\mathcal{F}$ becomes equipped with the structure of a manifold).

__Lemma__ $SO(n) \circlearrowright \mathcal{F}$ is transitive as well.
Pf. In proof of Lemma 14-1 we may

take $v_{j+1} \perp V_j$ and $\|v_j\| = 1$. There exists $k \in SO(n)$ s.t. $k(e_j) = v_j$ $(j<n)$. Now $k \cdot \mathbb{E} = V$. $\boxtimes$

<u>Cor</u> $k \mapsto k \cdot \mathbb{E}$ induces $K/M \xrightarrow{\simeq} \mathcal{F}$.

<u>Cor</u> $K \to G$ induces $K/M \xrightarrow{\simeq} G/MAN$

<u>Cor</u> $G \simeq K \times A \times N$.

——— End example.

We define $\mathcal{H} = \mathcal{H}_P$: to be the composition of the projection $G = K \times A \times N \to A$ and $\log : A \xrightarrow{\simeq} \mathfrak{a}$. Thus

$$x \in K \exp(\mathcal{H}(x)) N, \qquad (x \in G).$$

The Iwasawa projection will appear in the induced representations which we will now discuss.

Let $Q < G$ be a closed subgroup. Let (ξ, V_ξ) be a finite dimensional continuous representation of Q. We consider the

right action of Q on $G \times V_\xi$ given by
$$(x, v) \cdot q = (xq, \xi(q)^{-1} v).$$
This action is proper and free. Let
$$\mathcal{V}_\xi = G \times_Q V_\xi := (G \times V_\xi)/Q$$
be the quotient manifold. Then the projection $pr_1 : G \times V_\xi \to G$ induces a map
$$p : G \times_Q V_\xi \to G/Q$$
which turns $G \times_Q V_\xi$ into a vector bundle over G/Q with fiber $\simeq V_\xi$. This construction is known as the <u>associated vector bundle construction</u> for the principal fiber bundle $G \to G/Q$.

Note that $\mathcal{V}_\xi$ is G-equivariant. Indeed, the natural left action of G on $G \times V_\xi$ commutes with action of Q, hence induces an action on $\mathcal{V}_\xi^g = G \times_Q V_\xi$, by vector bundle isomorphisms such that
$$p : \mathcal{V}_\xi^g \to G/Q$$
is G-equivariant. This action induces a representation π_ξ of G in the space $\Gamma^\infty(\mathcal{V}_\xi^g)$

of smooth sections of $\mathcal{V}_\xi$. Let $s \in \Gamma^\infty(\mathcal{V}_\xi)$, $g \in G$ then $\pi_\xi(g)s$ is given by

$$[\pi_\xi(g)s](x) = g \cdot s(g^{-1}x).$$

The representation $\pi_\xi : G \to GL(\Gamma^\infty(\mathcal{V}_\xi))$, which is readily seen to be strongly continuous, i.e. $G \times \Gamma^\infty(\mathcal{V}_\xi) \to \Gamma^\infty(\mathcal{V}_\xi)$ is continuous (for the Fréchet topology on $\Gamma^\infty(\mathcal{V}_\xi)$.).

The representation π_ξ is called the (smoothly) induced representation of ξ from Q to G, notation

$$\mathrm{ind}_Q^G(\xi)^\infty = \pi_\xi.$$

The above model is called the "<u>Vector bundle picture</u>" of the induced representation. We will now discuss another model, the so called "<u>induced picture</u>". For this we note that there are natural isomorphisms

$$\begin{aligned}\Gamma^\infty(\mathcal{V}_\xi) &\simeq \Gamma^\infty(G \times V_\xi)^Q \quad \text{trivial bundle} \\ &\simeq C^\infty(G, V_\xi)^Q \\ &= \{\varphi : G \xrightarrow{C^\infty} V_\xi \mid \varphi(gq) = \xi(q)^{-1}\varphi(g)\} \\ &\qquad\qquad \forall g \in G, q \in Q\end{aligned}$$

The final space will be denoted

$$C^\infty(G : Q : \xi)$$

By transfer under the iso's we may realize $\pi_\xi = \mathrm{ind}_Q^G(\xi)$ on it by the formula

$$\pi_\xi(g)\varphi(x) = \varphi(g^{-1}x)$$

(the restriction of the left regular rep of G on $C^\infty(G, V_\xi)$).

The return to our original setting with our minimal parabolic subgroup $P = MAN$.

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \cong \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$. Then λ defines the character $\xi_\lambda : A \to \mathbb{C}^*$ given by

$$\xi_\lambda(a) := a^\lambda := e^{\lambda(\log a)}, \quad (a \in A).$$

I.e. the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\xi_\lambda"] & \mathbb{C}^* \\ \mathfrak{a} \arrow[u, "\exp"', "\simeq"] \arrow[r, "\lambda"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{C} \arrow[u, "e^{\cdot}"'] \end{tikzcd}$$

Since $P \cong (M \times A) \ltimes N$ (MA normalizes all root spaces) ξ_λ extends to a character of P by trivial extension on MN, i.e.

$$\xi_\lambda(man) = a^\lambda$$

The character ξ_λ determines the rep^n of P in $\mathbb{C}$ given by

$$\xi_\lambda(p)(z) = \xi_\lambda(p)\cdot z \quad (z \in \mathbb{C},\ p \in P).$$

This representation is often denoted

$$\xi_\lambda = 1 \otimes \lambda \otimes 1$$

in the literature. We agree to write

$$'C^\infty(G:P:\lambda) := 'C^\infty(G:P:\xi_\lambda)$$

for the associated space on which the induced representation $'\pi_\lambda = '\pi_{\xi_\lambda} = \mathrm{ind}_P^G(\xi_\lambda)^\infty$ is realized. Thus,

$$'C^\infty(G:P:\lambda) = \{\varphi: G \xrightarrow{C^\infty} \mathbb{C} \mid \varphi(gman) = a^{-\lambda}\varphi(g)\}$$

$$(\pi_\lambda(g)\varphi)(x) = \varphi(g^{-1}x).$$

If $\varphi \in {}'C^\infty(G:P:\lambda)$ then

$$\varphi(kan) = a^{-\lambda}\varphi(k)$$

so that φ is completely determined by $\varphi|_K$. Moreover, $\varphi|_K \in C^\infty(K)$ is right M-invariant. The space $C^\infty(K)^M$ of right M-invariant functions on K may be identified with $C^\infty(K/M)$. We note that for $\psi \in C^\infty(K/M)$ the function $\varphi : G \to \mathbb{C}$ defined by

$$\varphi(kan) = a^{-\lambda}\psi(k) = e^{-\lambda \mathcal{H}(kan)}\psi(k).$$

belongs to ${}'C^\infty(G:P:\lambda)$.

<u>Lemma</u> Restriction to K induces a topological linear isomorphism

$$r : {}'C^\infty(G:P:\lambda) \xrightarrow{\simeq} C^\infty(K/M)$$

which intertwines $\mathrm{ind}_P^G(\xi_\lambda)|_K$ with the left regular representation L of K in $C^\infty(K/M)$.

By transfer under r, the representation ${}'\pi_\lambda = \mathrm{ind}_P^G(\xi_\lambda)$ may be realized on the λ-independent space $C^\infty(K/M)$. This realization, the so-called <u>compact picture</u> of π_λ is given by

$$[\pi_\lambda(g)\psi](k) = r^{-1}(\psi)(g^{-1}k)$$

$$= e^{-\lambda \mathcal{H}(g^{-1}k)}\psi(\mathcal{K}(g^{-1}k)), \qquad (20-1)$$

where $\mathcal{K} : G \simeq K \times A \times N \to K$.