Analysis on real semisimple Lie groups 29/5-2013

Last time: G $\mathbb{R}$ semisimple, connected, $\# Z(G) < \infty$

θ Cartan, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{p}$ max abelian

$\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, Σ^+, $G \simeq K \times A \times N$

$P = MAN$ $\quad K/M \xrightarrow{\simeq} G/P$.

$Q < G$ closed, (ξ, V_ξ) fin. dim rep of Q

$\mathrm{ind}_Q^G(\xi)^\infty$ rep of G in $\Gamma^\infty(\mathcal{V}_\xi)$. $\mathcal{V}_\xi = G \times_Q V_\xi$.

- $\Gamma^\infty(\mathcal{V}_\xi) \cong \{ f : G \xrightarrow{C^\infty} V_\xi \mid f(xq) = \xi(q)^{-1} f(x),\ x \in G,\ q \in Q \}$

  ("induced picture")

Specialize: $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\xi_\lambda(man) = a^\lambda$ character of P

$`\pi_\lambda^\infty = \mathrm{ind}_P^G(\xi_\lambda)$ in

$\{ f : G \to \mathbb{C} \mid f(xman) = a^{-\lambda} f(x) \} =: `C^\infty(P:\lambda)$

The map $f \mapsto f|_K$ induces a topological

- linear iso $`C^\infty(P:\lambda) \xrightarrow{r} C^\infty(K)^M \cong C^\infty(K/M)$

Inverse: $\psi \mapsto \psi_\lambda$ where $\psi_\lambda(kan) = a^{-\lambda}\psi(k)$.

By transference under r:

$`\pi_\lambda^\infty$ on $C^\infty(K/M)$ (space indep^t of λ)

Given by

$$(`\pi_\lambda^\infty(x)\psi)(k) = \psi_\lambda(x^{-1}k)$$

$$= e^{-\lambda H(x^{-1}k)} \psi(\mathcal{K}(x^{-1}k)).$$

$G \to K$

Iwasawa map $G \to \mathfrak{a}$

Unitary induction

Assume $Q < G$ closed, V_ξ a Hilbert space $/\mathbb{C}$, and ξ a unitary rep of Q in V_ξ.

Then $\mathcal{V}_\xi := G \times_Q V_\xi$ is a Hilbert bundle, so we have a natural G-invariant sesquilinear pairing

$$\langle \cdot , \cdot \rangle_\xi : \Gamma_c^\infty(\mathcal{V}_\xi) \times \Gamma_c^\infty(\mathcal{V}_\xi) \longrightarrow C_c^\infty(G/Q)$$

given by

$$\langle f, g \rangle_\xi (x) = \langle f(x), g(x) \rangle_{\mathcal{V}_{\xi x}} \qquad (x \in G/Q).$$

In the induced picture, with $f, g \in {}'C^\infty(G : Q : \xi)$, this pairing is given by

$$\langle f, g \rangle_\xi (x) = \langle f(x), g(x) \rangle_{V_\xi}.$$

However, in general we have no natural integral $\int_{G/Q}$ mapping $C_c^\infty(G/Q) \to \mathbb{C}$. For this we need to build in densities.

Density bundle

For this notion, see the lecture notes "Analysis on Manifolds", by M. Crainic and myself.

The density bundle on a manifold M is given by

$$\mathcal{D}_M := \coprod_{x \in M} \mathcal{D}_{T_x M}$$

with appropriate bundle structure. Here

$\mathcal{D}_{T_xM}$ denotes the one-dimensional complex linear space of densities on T_xM, i.e. functions $\lambda: \overbrace{T_xM \times \cdots \times T_xM}^{n} \to \mathbb{C}$ ($n = \dim M$), transforming according to the rule

$$A^*\lambda \,(= \lambda \circ A^n) = |\det A|\,\lambda$$

for all $A \in \mathrm{End}(T_xM)$. An element of the space $\Gamma_c(\mathcal{D}_M)$ of continuous compactly supported densities on M may be integrated over M to produce a complex value. One has the following substitution of variables theorem.

<u>Thm</u> II.3.1 Let $\varphi: M \to N$ be a diffeo of mfds, $\omega \in \Gamma_c(N)$. Then

$$\int_M \varphi^*(\omega) = \int_N \omega.$$

We return to the setting of a Lie gr. G with a closed subgroup Q. Let $\omega \in \Gamma_c(\mathcal{D}_{G/Q})$. Then by substitution of variables

$$\int_{G/Q} \ell_x^* \omega = \int_{G/Q} \omega \qquad (\forall x \in G)$$

I.e., the integral defines a <u>G-invariant</u> continuous linear functional on $\Gamma_c(\mathcal{D}_{G/Q})$.

Let $\delta \in \widehat{Q}$ be a character (i.e. group homomorphism $Q \to \mathbb{C}^*$). Then we write $\mathbb{C}_\delta$ for $\mathbb{C}$ equipped with the Q-action given by the character δ.

<u>Lemma</u> Let $\delta : Q \to \mathbb{R}^+$ be defined by

$$\delta(q) = |\det \overline{Ad(q)}|_{\mathfrak{g}/\mathfrak{q}}|^{-1}.$$

Let $\omega \in \mathcal{D}_{\mathfrak{g}/\mathfrak{q}} \setminus \{0\}$. Then the map

$$G \times \mathbb{C} \xrightarrow{\varphi} \mathcal{D}_{G/Q}$$
$$(g, z) \longmapsto z \, dl_g(e)^{*-1} \omega$$

factors through an isomorphism $G \times_Q \mathbb{C}_\delta \to \mathcal{D}_{G/Q}$

(In other words $\mathcal{D}_{G/Q}$ is isomorphic to the associated line bundle determined by the character δ of Q).

<u>Proof</u>. First we note that φ maps $\{g\} \times \mathbb{C}$ linearly isomorphically onto $\mathcal{D}_{T_{gQ}(G/Q)}$.

Then we note that

$$(gq, \delta(q)^{-1} z) \longmapsto \delta(q)^{-1} z \, dl_g(\bar{e})^{*-1} dl_q(\bar{e})^{*-1} \omega$$

Now use the lemma below to conclude that

$dl_q(e)^{*-1} \omega = |\det \overline{Ad(q)}|_{\mathfrak{g}/\mathfrak{q}}|^{-1} \omega$ so that

$\varphi(gq, \delta(q)^{-1}z) = \varphi(g,z)$ Thus φ induces a smooth map $\bar{\varphi}: G\times_Q \mathbb{C}_\delta \to \mathcal{D}_{G/Q}$ which is readily seen to be linear bijective on the fibers. ∎

<u>Lemma</u> Let $l_q: G/Q \to G/Q$, $gQ \mapsto qgQ$. Then as a map $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{q}$, $dl(\bar{e})$ equals the map $\overline{Ad(q)}$ induced by $Ad(q): \mathfrak{g} \to \mathfrak{g}$.

<u>Proof</u> The following diagram commutes

$$\begin{tikzcd} G \arrow[r, "Ad_q"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/Q \arrow[r, "l_q"] & G/Q \end{tikzcd}.$$

Differentiating at e and $\bar{e}$ we find the commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad(q)"] \arrow[d, "T_e\pi"'] \arrow[dr, phantom, "G"] & \mathfrak{g} \arrow[d, "T_e\pi"] \\ T_{\bar{e}}(G/Q) \arrow[r, "dl_q(\bar{e})"] & T_{\bar{e}}(G/Q) \end{tikzcd}.$$

Now $\ker(T_e\pi) = T_e\pi^{-1}(\bar{e}) = T_eQ = \mathfrak{q}$, so that $T_e\pi$ induces an iso $\mathfrak{g}/\mathfrak{q} \xrightarrow{\overline{T_e\pi}} T_{\bar{e}}(G/Q)$ and the following diagram is commutative

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$$\begin{tikzcd}
\mathfrak{g}/\mathfrak{q} \arrow[r, "\overline{\mathrm{Ad}(q)}"] \arrow[d, "\overline{T_e\pi}"'] \arrow[dr, phantom, "G"] & \mathfrak{g}/\mathfrak{q} \arrow[d, "\overline{T_e\pi}"] \\
T_{\bar{e}}(G/Q) \arrow[r, "dl_q(e)"] & T_{\bar{e}}(G/Q)
\end{tikzcd}$$

The usual convention is to identify $T_{\bar{e}}(G/Q)$ with $\mathfrak{g}/\mathfrak{q}$ via the isomorphism $\overline{T_e\pi}$. The result now follows ☒.

We agree to write ${}'C_c^\infty(G:Q:\delta)$ for the space of smooth functions $G \xrightarrow{f} \mathbb{C}$ with

(1) $pr(\operatorname{supp} f)$ cpt in G/Q

(2) $f(gq) = \delta(q)^{-1} f(g)$.

Then

$${}'C_c^\infty(G:Q:\delta) \simeq \Gamma_c^\infty(G \times_Q \mathbb{C}_\delta) \overset{\omega}{\simeq} \Gamma_c^\infty(\mathcal{D}_{G/Q})$$

The second isomorphism is determined by a choice $\omega \in \mathcal{D}_{\mathfrak{g}/\mathfrak{q}}$. The isomorphism

$${}'C_c^\infty(G:Q:\delta) \longrightarrow \Gamma_c^\infty(\mathcal{D}_{G/Q}),$$
$$f \longmapsto f_\omega$$

is given by

$$f_\omega(gQ) = f(g)\, dl_g(\bar{e})^{-1*}\omega.$$

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<u>Definition</u> Given $\omega \in \mathcal{D}_{\mathfrak{g}/\mathfrak{q}} \setminus \{0\}$ we define

$$I_\omega : {}'C_c^\infty(G : Q : \delta) \longrightarrow \mathbb{C}$$

by

$$I_\omega(f) = \int_{G/Q} f\omega .$$

<u>Remark</u> Note that $I_\omega(L_x f) = I_\omega(f)$ for all $f \in {}'C_c^\infty(G : Q : \delta)$, $x \in G$. If $Q = P$, then G/Q is compact ($\simeq K/M$) and $I_\omega : C^\infty(G : Q : \delta) \longrightarrow \mathbb{C}$ defines a G-invariant continuous linear functional on $\operatorname{ind}_P^G(\delta)$.

<u>Normalized induction</u> Let $Q < G$ be a closed subgroup, and (ξ, V_ξ) a continuous representation of G (in Fréchet). We define the normalized induced representation $\operatorname{Ind}_Q^G(\xi)^\infty := \operatorname{ind}_Q^G(\xi \otimes \delta)^\infty$.

Likewise we define a version $\operatorname{Ind}_Q^G(\xi)_c^\infty$ with compact supports, $\operatorname{Ind}_Q^G(\xi)_c^\infty = \operatorname{ind}_Q^G(\xi \otimes \delta^{1/2})_c^\infty$, in $C_c^\infty(G : Q : \xi) := {}'C_c^\infty(G : Q : \xi \otimes \delta^{1/2}) \simeq$

$$\simeq \Gamma_c^\infty(G \times_Q (\xi \otimes \delta^{1/2})).$$

Let (η, V_η) be a second continuous rep of G (in a Fréchet space)

Assume that $b : V_\xi \times V_\eta \to \mathbb{C}$ is a sesquilinear pairing (anti-linear in second var) which is G-invariant, i.e. $\forall x \in G$:

$$b(\xi(x)v, \eta(x)w) = b(v, w)$$

(for $v \in V_\xi$, $w \in V_\eta$) Then b induces a sesquilinear pairing

$$\underline{b} : {}'C_c^\infty(G:Q:\xi \otimes \delta^{1/2}) \times {}'C_c^\infty(G:Q:\eta \otimes \delta^{1/2})$$
$$\longrightarrow {}'C_c^\infty(G:Q:\delta^{1/2} \otimes \delta^{1/2}).$$

We now note that $\mathbb{C}_{\delta^{1/2}} \otimes \mathbb{C}_{\delta^{1/2}} \simeq \mathbb{C}_\delta$ naturally and define (omitting G in notation) sesquil pairing

$$\langle \cdot , \cdot \rangle : C_c^\infty(Q:\xi) \times C_c^\infty(Q:\eta) \to \mathbb{C}$$

by

$$\langle f, g \rangle = I_\omega(\underline{b}(f,g))$$

(here we use a fixed $\omega \in \mathcal{D}_{\mathfrak{g}/\mathfrak{q}} \setminus \{0\}$, $\omega \geq 0$)

<u>Lemma</u> The sesquilinear pairing $\langle \cdot , \cdot \rangle$ is G-invariant for $\mathrm{Ind}_Q^G(\xi)_c^\infty$, $\mathrm{Ind}_Q^G(\eta)_c^\infty$.

<u>Special case</u> Assume $V_\xi = H_\xi$ a Hilbert

space, and ξ a unitary representation. Then we may take $\eta = \xi$ and $b = \langle \cdot , \cdot \rangle_\xi$, the inner product of $\mathcal{H}_\xi$.

<u>Corollary</u> Let $(\xi, \mathcal{H}_\xi)$ be a unitary rep of Q. Then $C_c^\infty(Q:\xi)$ has a G-invariant pre-Hilbert structure, induced by $\langle \cdot , \cdot \rangle_\xi$.

We denote the Hilbert completion of $C_c^\infty(Q:\xi)$ by

$$L^2(Q:\xi) := L^2(G:Q:\xi).$$

By general principles of functional analysis it follows that the induced representation $\mathrm{Ind}_Q^G(\xi)_c^\infty$ extends to a unitary representation of G on $L^2(Q:\xi)$.

<u>Def</u> The obtained unitary rep'n of G on $L^2(Q:\xi)$ is said to be obtained by <u>unitary</u> (or <u>normalized</u>) induction from ξ.

<u>Notation</u>: $\mathrm{Ind}_Q^G(\xi)$.

Application to $P = MAN$.

We apply preceding theory with $Q = P$.

Def We define $\rho = \rho_P \in \mathfrak{a}^*$ by

$$\rho(X) = \frac{1}{2} \operatorname{tr}(\operatorname{ad}(X)|_{\mathfrak{n}}), \quad X \in \mathfrak{a}.$$

Let $m_\alpha = \dim \mathfrak{g}_\alpha$. Then we note that

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

Lemma The character $\delta^{1/2}$ of P is given by $\delta^{1/2} = \xi_\rho$, i.e.,

$$\delta(man)^{1/2} = a^\rho \qquad ((m,a,n) \in M \times A \times N).$$

Proof We recall that

$$\delta(man) = |\det \overline{\operatorname{Ad}(man)}|_{\mathfrak{g}/\mathfrak{p}}|^{-1}$$

(where $\mathfrak{p} = \operatorname{Lie}(P)$).

Since M is compact, and $\mathfrak{n}$ nilpotent,

$$|\det \operatorname{Ad}(m)|_{\mathfrak{g}}| = |\det \operatorname{Ad}(m)|_{\mathfrak{p}} = 1$$
$$|\det \operatorname{Ad}(n)|_{\mathfrak{g}}| = |\det \operatorname{Ad}(n)|_{\mathfrak{p}} = 1$$

and we see that $\delta(man) = \delta(a)$. Now the decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{p}$ is $Ad(a)$-stable, hence

$$\delta(a) = |\det Ad(a)|_{\bar{\mathfrak{n}}}|^{-1}.$$

Using the decomposition $\bar{\mathfrak{n}} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$, which is again $Ad(a)$-stable, we find

$$\delta(a) = \prod_{\alpha \in \Sigma^+} |\det a^{-\alpha} I_{\mathfrak{g}_{-\alpha}}|^{-1}$$

$$= \prod_{\alpha \in \Sigma^+} a^{+m_\alpha \alpha} \qquad (\text{use } m_\alpha = m_{-\alpha})$$

so that

$$\delta(man) = \delta(a) = a^{2\rho} \quad \boxtimes.$$

In the following, we fix normalized invariant density $d\bar{k}$ on K/M (normalized: $\int_{K/M} d\bar{k} = 1$). Let $\omega \in \mathcal{D}_{\mathfrak{g}/\mathfrak{p}}$ be the positive density whose pull-back under $\mathfrak{k}/\mathfrak{m} \xrightarrow{\simeq} \mathfrak{g}/\mathfrak{p}$ equals $(d\bar{k})_{eM}$.

<u>Lemma</u> For $f \in C^\infty(G : P : \delta)$,

$$I_\omega(f) = \int_{K/M} f(k)\, d\bar{k}.$$

Proof. By definition, $I_\omega(f) = \int_{G/P} f_\omega$, where $f_\omega \in \Gamma^\omega(\mathcal{D}_{G/P})$ is defined by

$$f_\omega(gP) = f(g)\, dl_g(\bar{e})^{-1*}\omega.$$

(note that G/P is compact).

The inclusion $K \to G$ induces a diffeomorphism

$$j : K/M \longrightarrow G/P.$$

By substitution of variables

$$I_\omega(f) = \int_{K/M} j^*(f_\omega) \qquad (*)$$

We calculate $j^*(f_\omega)$ as follows.

Let $k \in K$. Then

$$\begin{aligned} j^*(f_\omega)_{kM} &= dj(kM)^*(f_\omega)_{kP} \\ &= f(k)\, dj(kM)^*\, dl_k(\bar{e})^{-1*}\omega \\ &= f(k)\, dl_k^{K/M}(\bar{e})^{-1*}\, dj(eM)^*\omega \\ &= f(k)\, dl_k^{K/M}(\bar{e})^{-1*}(d\bar{k})_{eM} \\ &= f(k)\,(d\bar{k})_{kM}. \end{aligned}$$

By substitution in $(*)$ complete proof ☒

<u>Corollary</u> $\mathrm{Ind}_P^G(\xi_\lambda)^\infty \cong \mathrm{ind}_P^G(\xi_{\lambda+\rho})^\infty$. $(\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$

<u>Pf</u>. We note that $\xi_\lambda \otimes \delta^{1/2} = \xi_\lambda \otimes \xi_\rho \simeq \xi_{\lambda+\rho}$ since the natural map $\mathbb{C}_\lambda \otimes \mathbb{C}_\rho \to \mathbb{C}_{\lambda+\rho}$ is a linear isomorphism.

Thus,

$$\mathrm{Ind}_P^G(\xi_\lambda)^\infty = \mathrm{ind}_P^G(\xi_\lambda \otimes \delta^{1/2})^\infty \cong \mathrm{ind}_P^G(\xi_{\lambda+\rho})^\infty \quad \boxtimes$$

We agree to write (for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$)

$$C^\infty(G:P:\lambda) := {}'C^\infty(G:P:\lambda+\rho)$$
$$\pi_\lambda^\infty := {}'\pi_{\lambda+\rho}^\infty$$

We now note that the map $(z, w) \mapsto z\bar{w}$ defines a P-invariant sesquilinear pairing

$$\mathbb{C}_\lambda \times \mathbb{C}_{-\bar{\lambda}} \longrightarrow \mathbb{C}.$$

<u>Cor.</u> The sesquilinear pairing

$$C^\infty(G:P:\lambda) \times C^\infty(G:P:-\bar{\lambda}) \longrightarrow \mathbb{C}$$
$$(f, g) \longmapsto \int_{K/M} f(k)\,\overline{g(k)}\, d\bar{k}$$

is G-equivariant.

<u>Corollary</u> If $\lambda \in i\mathfrak{a}^*$ then the induced rep'n $\pi_\lambda^\infty = {}^{`}\pi_{\lambda+\rho}^\infty$ on $C^\infty(G:P:\lambda)$ comes equipped with the G-invariant pre-Hilbert structure

$$\langle f, g \rangle := \int_{K/M} f(k)\overline{g(k)}\, d\bar{k}.$$

In the compact picture the unitary representation $\pi_\lambda = \mathrm{Ind}_P^G(\xi_\lambda)$ is realized on $L^2(K/M)$ and given by

$$(\pi_\lambda(x) f)(k) = e^{(-\lambda-\rho)H(x^{-1}k)} f(\mathcal{K}(x^{-1}k))$$

$(f \in L^2(K/M),\ x \in G,\ k \in K/M)$

<u>Poisson transform</u> We define the Poisson transform

$$\mathcal{P}_\lambda : C^\infty(P:-\lambda) \longrightarrow C^\infty(G/K),$$

by

$$\mathcal{P}_\lambda f(xK) = \int_{K/M} f(xk)\, d\bar{k}$$

<u>Remark</u> $\mathcal{P}_\lambda$ intertwines $\pi_{-\lambda}^\infty$ with the left regular rep L of G on $C^\infty(G/K)$

Indeed,

$$\mathcal{P}_\lambda(\pi_{-\lambda}(x)f)(y) = \int_{K/M} (\pi_{-\lambda}(x)f)(yk)\, d\bar{k}$$

$$= \int_{K/M} f(x^{-1}yk)\, dk$$

$$= \mathcal{P}_\lambda f(x^{-1}y)$$

$$= L_x(\mathcal{P}_\lambda f)(y).$$

Using the equivariance of the pairing $C^\infty(P:-\lambda) \times C^\infty(P:\bar{\lambda}) \to \mathbb{C}$ we may rewrite the Poisson transform as a matrix coefficient.

Let $\mathbb{1}_\lambda : G \to \mathbb{C}$ denote the unique function in $C^\infty(P:\lambda)$ which equals 1 on K. Thus

$$\mathbb{1}_\lambda(kan) = a^{-\lambda-\rho}, \quad \text{or}$$

$$\mathbb{1}_\lambda(x) = e^{(-\lambda-\rho)H(x)}.$$

<u>Lemma</u> Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $f \in C^\infty(P:-\lambda)$, $x \in G$. Then

$$\mathcal{P}_\lambda f(x) = \langle f, \pi_{\bar{\lambda}}(x)\mathbb{1}_{\bar{\lambda}} \rangle$$

$$= \int_{K/M} f(k)\, e^{(-\lambda-\rho)H(x^{-1}k)}\, dk$$

Proof.

$$\mathcal{P}_\lambda f(x) = \int_{K/M} f(xk)\, d\bar{k}$$
$$= \int_{K/M} f(xk)\, \overline{\mathbb{1}_{\bar{\lambda}}(k)}\, d\bar{k}$$
$$= \langle \pi_{-\lambda}(x)^{-1} f, \mathbb{1}_{\bar{\lambda}} \rangle$$
$$= \langle f, \pi_{\bar{\lambda}}(x) \mathbb{1}_{\bar{\lambda}} \rangle$$
$$= \int_{K/M} f(k)\, \overline{\mathbb{1}_{\bar{\lambda}}(x^{-1}k)}\, d\bar{k}$$
$$= \int_{K/M} f(k)\, e^{(-\lambda-\rho)H(x^{-1}k)}\, d\bar{k}.$$

It follows from the above that $\mathcal{P}_\lambda$ may be viewed as an integral operator $C^\infty(K/M) \to C^\infty(G/K)$ with integral kernel

$$P_\lambda(gK, kM) = e^{(-\lambda-\rho)H(g^{-1}k)}$$
$$= \mathbb{1}_\lambda(g^{-1}k),$$

Define
$$P_\lambda(g) := \mathbb{1}_\lambda(g^{-1}).$$

Then

$$P_\lambda(gK, kM) = P_\lambda(k^{-1}g). \qquad (16\text{-}1)$$

Special case: The upper half plane

We recall that $GL(2,\mathbb{C})$ acts on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ $\simeq \mathbb{P}^1(\mathbb{C})$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

In particular, so does the subgroup $G = SL(2,\mathbb{R})$. Let $K = SO(2)$,

$$A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$N = \left\{ n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}.$$

Then $G = NAK$ is an Iwasawa decomposition for G. We note that

$$\begin{aligned} n_s a_t k \cdot i &= n_s \cdot (a_t \cdot i) \\ &= e^{2t} i + s. \end{aligned}$$

Thus,

$$\varphi : gK \longmapsto g \cdot i$$

defines a diffeomorphism $G/K \xrightarrow{\simeq} H_+$, where

$$H_+ = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}.$$

In particular, we see that $SL(2,\mathbb{R})$ acts transitively on H_+.

We will calculate the function

$$\tilde{\mathbb{P}}_\rho := \varphi^{-1*}\mathbb{P}_\rho \in C^\infty(H_+).$$

First of all, consider the standard $\mathfrak{sl}(2,\mathbb{R})$ basis

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$.

We note $\mathfrak{a} = \mathbb{R}H$, $\mathfrak{n} = \mathbb{R}X$, $a_t = \exp tH$ and $n_s = \exp sX$. The root system equals $\Sigma = \{\alpha, -\alpha\}$, where $\alpha(H) = 2$. The positive system corresponding to N equals $\Sigma^+ := \{\alpha\}$. Now $\rho = 1/2\,\alpha$, so $2\rho = \alpha$, and we see that

$$\mathbb{1}_\rho(k a_t n_s) = a_t^{-2\rho} = e^{-2t}$$

It follows that

$$\tilde{\mathbb{P}}_\rho(n_s a_t \cdot i) = \mathbb{P}_\rho(n_s a_t)$$

$$= \mathbb{1}_\rho(a_t^{-1} n_s^{-1}) = e^{2t}.$$

On the other hand,

$$n_s a_t \cdot i = e^{2t} i + s$$

and we see that

$$\tilde{P}_p(z) = \mathrm{Im}(z) = y. \quad (19.1)$$

Special case: the Poincaré disk.

We will now do a similar calculation for the Poincaré disk, and relate $\mathcal{P}_p$ to the classical Poisson transform.

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. There exists $\gamma \in GL(2, \mathbb{C})$ s.t. $\gamma \cdot H_+ = D$.

To find γ, we require that the fractional linear transformation $\gamma\cdot$ maps $0 \mapsto -1$, $i \mapsto 0$, $\infty \mapsto 1$. Put

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

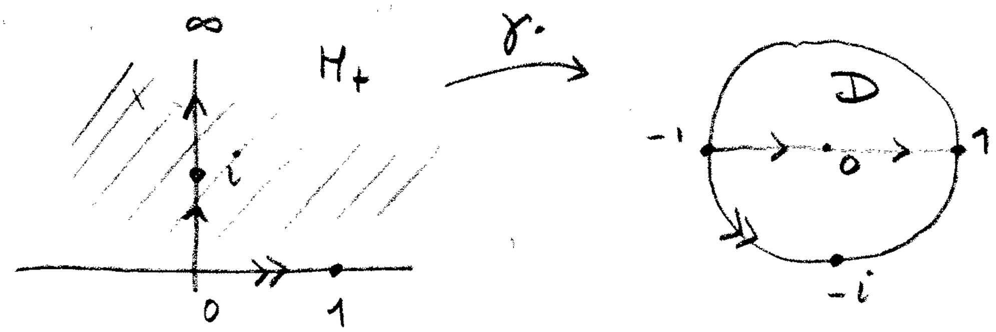
Then
$$\gamma(0) = -1 \iff b = -d,$$
$$\gamma(i) = 0 \iff ai + b = 0,$$
$$\gamma(\infty) = 1 \iff a = c.$$

This determines γ up to a scalar. We choose

$$\gamma = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

Then $\gamma \cdot 1 = -i$ and we see that

$\gamma \cdot$ maps ∂H_+ to ∂D and hence H_+ to D.



Put ${}^{\backprime}G = \gamma \, SL(2,\mathbb{R}) \, \gamma^{-1}$. Then

$$ {}^{\backprime}G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \,\middle|\, |\alpha|^2 - |\beta|^2 = 1 \right\} $$

and we see that ${}^{\backprime}G$ acts transitively on D. The conjugation $\mathcal{C}_\gamma : g \mapsto \gamma g \gamma^{-1}$ maps $SO(2)$ onto ${}^{\backprime}K = S(U(1) \times U(1))$,

so

$$ {}^{\backprime}K = \left\{ k_\varphi = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \,\middle|\, \varphi \in \mathbb{R} \right\}. $$

Since $\gamma(i) = 0$, ${}^{\backprime}K$ is the stabilizer of 0 in ${}^{\backprime}G$, and we see that

$$ {}^{\backprime}\varphi : {}^{\backprime}G \setminus {}^{\backprime}K \to D, \quad {}^{\backprime}g\,{}^{\backprime}K \mapsto {}^{\backprime}g \cdot 0 $$

is a diffeomorphism.

Moreover, the following diagram commutes

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$$\begin{tikzcd}
G/K \arrow[r, "\overline{\varphi}_\gamma"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & {}'G/{}'K \arrow[d, "{}'\varphi"] \\
H_+ \arrow[r, "\gamma\cdot"] & D
\end{tikzcd} \qquad (21.1)$$

We will use this to calculate the Poisson transform ${}'\mathcal{P}_{'\rho}$ for ${}'G$ with the Iwasawa decomposition ${}'G = {}'K\,{}'A\,{}'N$, where ${}'A = \gamma A \gamma^{-1}$, ${}'N = \gamma N \gamma^{-1}$.
We note that

$$\gamma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\gamma^{-1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\gamma^{-1} = \begin{pmatrix}i & -i\\ i & -i\end{pmatrix}$$

Hence

$${}'a_t = \gamma a_t \gamma^{-1} = \begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}$$

$${}'n_s = \gamma n_s \gamma^{-1} = \begin{pmatrix}1+si & -si\\ si & 1-si\end{pmatrix}.$$

The ${}'\rho$ associated with ${}'N$ is given by

$$({}'a_t)^{'\rho} = (a_t)^{\rho} = e^{-2t}.$$

Define

$${}'\widetilde{\mathcal{P}}_{'\rho} \in C^\infty(D)$$

by

$$`\widetilde{P}_{`p} = `\varphi^{-1*}\, `\mathcal{P}_{`p}.$$

Then by commutativity of the diagram (21.1) we find that, for $z \in H_+$,

$$`\widetilde{P}_{`p}(\gamma \cdot z) = \widetilde{P}_p(z).$$

Hence, for $w \in \mathbb{D}$,

$$`\widetilde{P}_{`p}(w) = \widetilde{P}_p(\gamma^{-1} \cdot w)$$

$$= \operatorname{Im}(\gamma^{-1} \cdot w)$$

Now

$$\gamma^{-1} = \frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix},$$

so

$$`P_{`p}(w) = \operatorname{Im}\left(\frac{iw + i}{-w + 1}\right)$$

$$= \operatorname{Re}\left(\frac{1 + w}{1 - w}\right)$$

$$= \operatorname{Re}\left(\frac{(1 + w)(1 - \overline{w})}{|1 - w|^2}\right)$$

$$= \frac{1 - |w|^2}{|1 - w|^2}.$$

We note that the centralizer $`M$ of $`A$ in $`K$ equals $`M = \{\pm I\}$. Note that

$$k_\varphi \cdot 1 = e^{2i\varphi}$$

Hence $kM \longmapsto k\cdot 1$ defines a diffeom.

$$`\psi : `K/`M \longrightarrow \partial D.$$

- We define the Poisson kernel $`\tilde{P}_{`\rho} \in C^\infty(D \times \partial D)$ by

$$`\tilde{P}_{`\rho} = (\varphi^{-1} \times \psi^{-1})^* `P_{`\rho}.$$

Then

$$`\tilde{P}_{`\rho}(g\cdot 0, k\cdot 1) = `P_{`\rho}(gK, kM).$$

Hence

- $$`\tilde{P}_{`\rho}(g\cdot 0, k\cdot 1) = `P_{`\rho}((k^{-1}g)^{-1})$$

$$= `\tilde{P}_{`\rho}((k^{-1}g)\cdot 0)$$

It follows that

$$`\tilde{P}_{`\rho}(w, e^{i\varphi}) = `\tilde{P}_{`\rho}(w, k_{\varphi/2}\cdot 1)$$

$$= `\tilde{P}_{-\rho}(k_{\varphi/2}^{-1} w)$$

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Now $k_{\varphi/2}^{-1} w = k_{-\varphi/2} w = e^{-i\varphi} w.$

Hence

$$`\tilde{P}_{`p}(w, e^{i\varphi}) = \frac{1 - |w|^2}{|1 - e^{-i\varphi} w|^2}.$$

This is the classical Poisson transform given by the formula

$$\sum_{n \in \mathbb{Z}} (e^{-i\varphi} w)^n.$$

Thus, $`\mathcal{P}_{`p}$ may be viewed as the classical Poisson transform

$$C'(\partial D) \to C^\infty(D)$$

mapping functions on the boundary ∂D to harmonic functions on D.

The invariant Riemannian metric on $`G/`K$ corresponds to the hyperbolic metric on D, which is given by

$$\langle \cdot , \cdot \rangle_z = (1-|z|^2)^{-2} \langle \cdot , \cdot \rangle_{\mathrm{eucl}}$$

on $T_z D \simeq \mathbb{R}^2$. Hence, the hyperbolic Laplace operator Δ_D on D is given by

$$\Delta_D = (1-|z|^2)^2 \, \Delta$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. It follows that $\mathcal{P}_{\rho}$ maps $C(\partial D)$ to the kernel of Δ_D in $C^\infty(D)$. This is a special case of a very general theorem which we will describe in the next lecture.