Recurrent sequences coming from Shimura curves

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On the occasion of Cam Stewart's 60th birthday

An example

Recall

$$(n+1)^2 u_{n+1} = (11n^2 + 11n + 3)u_n + n^2 u_{n-1}$$

Let a_n be the solution with starting values $a_0 = 0, a_1 = 5, \dots$ and b_n the solution with $b_0 = 1, b_1 = 3, b_2 = 19, b_3 = 147, \dots$.
Then $a_n/b_n \to \zeta(2)$ as $n \to \infty$ fast enough to prove irrationality.

Recurrences and ODE's

Consider the generating function

$$u(z) = \sum_{n \geq 0} b_n z^n.$$

Then $u(z) = 1 + 3z + 19z^2 + 147z^3 + \cdots$ satisfies

$$z(z^2 + 11z - 1)u'' + (3z^2 + 22z - 1)u' + (z + 3)u = 0.$$

This is a linear second order differential equation with a G-function solution (i.e. coefficients have denominators of at most exponential growth).

The modular connection

Basis of solutions of
$z(z^2+11z-1)u''+(3z^2+22z-1)u'+(z+3)u=0$ is

$$y_1 = u(z), \quad y_2 = u(z)\log(z) + v(z)$$

where $v(z) = 5z + 75z^2/2 + 5565z^3/18 + \cdots$
The map $z \mapsto \frac{1}{2\pi i} y_2/y_1$ maps $\mathbb{P}^1$ to complex upper half plane $\mathscr{H}$.
Its inverse is the map

$$\mathscr{H} \to \mathscr{H}/\Gamma_1(5) \hookrightarrow \mathbb{P}^1$$

where $\Gamma_1(5) \subset SL(2,\mathbb{Z})$ is congruence subgroup modulo 5.

The challenge

Find recurrences of the form

$$P(n)u_{n+1} = Q(n)u_n + R(n)u_{n-1}$$

where P, Q, R are polynomials of degree 2, which allow a solution u_n whose coefficients are at most exponential in n.

Alternatively, one can try to find second order linear differential equations of the form

$$z(z^2 + a_1 z + a_0)y'' + (b_2 z^2 + b_1 z + b_0)y' + (c_1 z + c_0)y = 0$$

which have a Siegel G-function solution.

An idea

Start with congruence subgroup Γ of $SL(2,\mathbb{Z})$ with four cusps and $X(\Gamma)$ genus zero. The map

$$\mathscr{H} \to \mathscr{H}/\Gamma$$

gives rise to a second order linear differential equation of the desired kind.

This gives us 5 more cases.

Chudnovsky's idea

Start with an arithmetic quaternion group $\Gamma \subset SL(2,\mathbb{R})$ and then consider $\mathscr{H} \to \mathscr{H}/\Gamma$.

Example of Lamé equation from Chudnovsky's *Theta functions*, 1989

$$P(z)u'' + \frac{1}{2}P'(z)u' + \left(-\frac{3}{128} - \frac{3}{64}z\right)u = 0$$

where $P(z) = z(z-1)(z-1/2)$.
Recurrence

$$(n+1)(n+1/2)u_{n+1} = (n^2+3/64)u_n - ((n-1)(2n-1)-3/32)u_{n-1}.$$

Quaternion groups

Let B be a quaternion algebra over totally real number field F. More concrete, take $a, b \in F^*$ and define $B = F \oplus Fi \oplus Fj \oplus Fk$ with

$$i^2 = a, \qquad j^2 = b, \qquad k = ij = -ji.$$

Let $\mathscr{O}$ be a maximal order of B and $\mathscr{O}^\times$ its units.

Any embedding $\iota : F \hookrightarrow \mathbb{R}$ induces an embedding of B into either $M_2(\mathbb{R})$ (2×2 real matrices) or $\mathbb{H}$ (Hamilton's quaternions).

Suppose that $B \hookrightarrow M_2(\mathbb{R})$ for exactly one place $\iota : F \hookrightarrow \mathbb{R}$. Then we call $\mathscr{O}^\times$, embedded in $M_2(\mathbb{R})$, an *arithmetic quaternion group*.

More generally, any subgroup $\Gamma \subset B$ commensurable with $\mathscr{O}^\times$ is called an arithmetic quaternion group.

Commensurable means that $\Gamma \cap \mathscr{O}^\times$ has finite index in both Γ and $\mathscr{O}^\times$.

Takeuchi's list

A discrete subgroup $\Gamma \subset SL(2, \mathbb{R})$ is said to be of type $(1; e)$ if $E_\Gamma := \mathscr{H}/\Gamma$ has genus one and the projection $\mathscr{H} \to \mathscr{H}/\Gamma$ ramifies above exactly one point of order e.

Such groups are generated by two elements A, B with the single relation $[A, B]^e = -\mathrm{Id}$. The group is determined by the traces of A, B and AB.

Theorem (Takeuchi, 1983)

There exist, up to conjugation, precisely 71 arithmetic quaternion groups of type $(1; e)$.

The problem

Let Γ be an arithmetic group of type $(1; e)$. The problem is twofold,

1. Determine a Weierstrass equation for $\mathscr{H}/\Gamma$ of the form $y^2 = P(x)$, (P cubic and monic).
2. Determine the constant C (accesory parameter) so that the covering $\mathscr{H} \to \mathscr{H}/\Gamma/\mathrm{inv}$ is determined by

$$P(z)y'' + \frac{1}{2}P'(z)y' + (C - n(n+1)z/4)y = 0$$

with $n = (-1 + 1/e)/2$.

Sijsling's thesis

In the recent PhD-thesis of Jeroen Sijsling he tackled the first problem and found almost all j-invariants in Takeuchi's list. Techniques used:

1. If Γ is commensurable with a triangle group there exists Belyi map E_Γ to $\mathbb{P}^1$.
2. According to Shimura-Deligne theory there exists a canonical model of E_Γ, defined over the narrow classfield of F, with good reduction outside a known set of primes.
3. Using explicit calculation of Hecke operators T_p on $H_1(E_\Gamma, \mathbb{Z})$ and the Eichler-Shimura theorem one determines the zeta-function of E_Γ at p for a large set of primes p.
4. To select a j-invariant in an isogeny class one determines the reduction mod p of E_Γ at the primes p of multiplicative reduction using a refinement of Cerednik-Drin'feld by Boutot-Zink.
5. Prove correctness for the candidate j-invariants.

A sample j-invariant

There are three arithmetic quaternion groups of type $(1;7)$ not commensurable with a triangle group. The j-invariants of the Shimura curve $E(\Gamma)$ are the conjugates of

$$\frac{-1448892\alpha^2 - 1930931\alpha + 1318350}{7 \cdot 13^2}$$

where α is a zero of $x^3 - x^2 - 2x + 1$.

The corresponding quaternion algebra is defined over the field $\mathbb{Q}(\alpha)$ and the discriminant is $\wp_7 \wp_{13} \infty_1 \infty_2$. Discriminant of $\mathbb{Q}(\alpha)$ is 49.

Determination of the accessory parameter

Recall, we must determine $P(z)$ and C in

$$P(z)y'' + \frac{1}{2}P'(z)y' + (C - n(n+1)z/4)y = 0$$

with $n = (-1 + 1/e)/2$. We know $P(z)$ from the j-invariant computation. As yet there is no systematic method to compute C.

Numerically, given $P(z)$ and n and C, one can compute generators of the monodromy group and their traces. By interpolation determine C as precise as possible to obtain the desired traces given by the quaternion group. Then guess an algebraic value of C.

Example of accessory parameter

We take the two $(1;4)$-groups Γ from Takeuchi's list corresponding to the quaternion algebra over $\mathbb{Q}(\sqrt{2})$ of discriminant $\wp_7\infty$. The curves $\mathscr{H}/\Gamma$ correspond to the conjugates of

$$y^2 = P(x) = x(x-1)(x-(3-2\sqrt{2})/4).$$

Numerical approximation (50 decimal places) indicates that $C = (2-\sqrt{2})/2^4$ in

$$P(z)y'' + \frac{1}{2}P'(z)y' + (C + 15z/256)y = 0$$