Exploring E-functions

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On the occasion of
Dale Brownawell's 60-th birthday

Definition: An entire function $f(z)$ given by a powerseries

$$\sum_{n=0}^{\infty} \frac{a_k}{k!} z^k$$

is called an E-function if

1. $a_0, a_1, a_2, \ldots \in \overline{\mathbb{Q}}$

2. $f(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

3. $h(a_0, a_1, \ldots, a_N) = O(N)$ for all N.

Here, $h(\alpha_1, \ldots, \alpha_n)$ denotes the logarithmic absolute height of the vector $(\alpha_1, \ldots, \alpha_n) \in \overline{\mathbb{Q}}^n$.

In Siegel's original definition condition 3) reads

$$h(a_0, a_1, \ldots, a_N) = o(N \log N)$$

Examples:

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$J_0(-z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!k!}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

where $a_0 = 1, a_1 = 3, a_2 = 19, a_3 = 147, \ldots$ are the Apéry numbers corresponding to Apéry's irrationality proof of $\zeta(2)$.

Differential equations

$$y' - y = 0$$

$$zy'' + y' - 4zy = 0$$

$$z^2y''' - (11z^2 - 3z)y'' - (z^2 + 22z - 1)y' - (z+3)y = 0$$

Let $f_1(z), \ldots, f_n(z)$ be E-functions satisfying a system of n differential equations

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where A is an $n \times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. We assume that the common denominator of the entries is $T(z)$.

Theorem (Siegel-Shidlovskii, 1929, 1956). Let $\alpha \in \overline{\mathbb{Q}}$ and $\alpha T(\alpha) \neq 0$. Then

$$\text{degtr}_{\overline{\mathbb{Q}}}(f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)) =$$

$$\text{degtr}_{\mathbb{C}(z)}(f_1(z), f_2(z), \ldots, f_n(z))$$

The differential galois group

Let $Y(z)$ be an $n \times n$ invertible matrix with functional entries $y_{ij}(z)$ for $i,j = 1,\ldots n$ such that

$$\frac{d}{dz}Y = AY.$$

Consider the ring $R = \mathbb{C}(z)[X_{i,j}]_{i,j=1,\ldots,n}$ and the *ideal of relations* I defined by the kernel of the natural evaluation map

$$P(X_{ij}) \mapsto P(y_{ij}(z)).$$

The group $GL(n,\mathbb{C})$ acts on R via

$$(X_{ij}) \mapsto (X_{ij})g$$

for any $g \in GL(n,\mathbb{C})$.

The *differential galois group* G of the differential equation is the subgroup of $GL(n,\mathbb{C})$ given by

$$G = \{g \in GL(n,\mathbb{C}) \mid g : I \to I\}$$

As a result any $g \in G$ acts also on the y_{ij} via $Y \mapsto Yg$.

Remark: When f is a solution of an n-th order equation, the vector of functions $f, f', \ldots, f^{(n-1)}$ satisfies a system of n first order equations.

Algorithms to compute G by Kovacic for $n = 2$ (1986) and by Singer, Ulmer for $n = 3$ (1990's).

Theoretical algorithm for general n by Compoint,Singer (1999) for reductive G and Hrushovski (2003) in complete generality.

Theorem Let G be the differential galois group of a linear system of n first order differential equations. Then,

1. G is a linear algebraic group.

2. For any solution $(f_1(z), \ldots, f_n(z))$ the dimension of its orbit under G equals the transcendence degree of $f_1(z), \ldots, f_n(z)$ over $\overline{\mathbb{Q}}(z)$.

Example

$$f(z) = \sum_{k=0}^{\infty} \frac{((2k)!)^2}{(k!)^2(6k)!} z^k$$

and $f(z^4)$ is an E-function satisfying a differential equation of order 5. The differential galois group is $SO(5, \mathbb{C})$. Dimension of its orbits is 4 and we have a quadratic form Q with coefficients in $\mathbb{Q}(z)$ such that

$$Q(f, f', f'', f''', f'''') = 1$$

Explicitly,

$$f(z) = \sum_{k=0}^{\infty} \frac{((2k)!)^2}{(k!)^2(6k)!}(2916z)^k$$

satisfies

$$\mathbf{F}^t \mathcal{Q} \mathbf{F} = (z)$$

where

$$\mathbf{F} = \begin{pmatrix} f(z) \\ Df(z) \\ D^2 f(z) \\ D^3 f(z) \\ D^4 f(z) \end{pmatrix}, \qquad D = z\frac{d}{dz}$$

and

$$\mathcal{Q} = \begin{pmatrix} z - 324z^2 & -18z & 198z & -486z & 324z \\ -18z & -\frac{10}{9} & \frac{23}{2} & -28 & 18 \\ 198z & \frac{23}{2} & -120 & 297 & -198 \\ -486z & -28 & 297 & -729 & 486 \\ 324z & 18 & -198 & 486 & -324 \end{pmatrix}$$

Theorem (Nesterenko-Shidlovskii, 1996). Let $f_1(z), \ldots, f_n(z)$ be E-functions which satisfy a system of n first order equations. Then there is a finite set S such that for every $\xi \in \overline{\mathbb{Q}}, \xi \notin S$ the following statement holds. To any relation of the form $P(f_1(\xi), \ldots, f_n(\xi)) = 0$ where $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ is homogeneous, there exists a $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_n]$, homogeneous in X_i, such that $Q(z, f_1(z), \ldots, f_n(z)) \equiv 0$ and

$$P(X_1, \ldots, X_n) = Q(\xi, X_1, \ldots, X_n)$$

Roughly speaking, any algebraic relation over $\overline{\mathbb{Q}}$ between $f_1(\xi), \ldots, f_n(\xi)$ at some point $\xi \in \overline{\mathbb{Q}} - S$ comes from specialisation at $z = \xi$ of some functional algebraic relation between $f_1(z), \ldots,$ over $\overline{\mathbb{Q}}(z)$.

The exceptional set S can be computed from the polynomial relations over $\overline{\mathbb{Q}}(z)$ between the f_i.

Theorem (Y.André, 2000) Let $f(z)$ be an E-function. Then $f(z)$ satisfies a differential equation of the form

$$z^m y^{(m)} + \sum_{k=0}^{m-1} z^k q_k(z) y^{(k)} = 0$$

where $q_k(z) \in \overline{\mathbb{Q}}[z]$ has degree $\le m-k$.

Corollary Let $f(z)$ be an E-function *with coefficients in* $\mathbb{Q}$ and suppose that $f(1) = 0$. Then 1 is an apparent singularity of the minimal differential equation satisfied by f.

Proof Consider $f(z)/(1-z)$. This is again E-function. So its minimal differential equation has a basis of analytic solutions at $z = 1$. This means that the original differential equation for $f(z)$ has a basis of analytic solutions all vanishing at $z = 1$. So $z = 1$ is apparent singularity.

Corollary: π is transcendental.

Suppose $\alpha := 2\pi i$ algebraic. Then the E-function $e^{\alpha z}-1$ vanishes at $z=1$. The product over all conjugate E-functions is an E-function with rational coefficients vanishing at $z=1$. So the above corollary applies. However linear forms in exponential functions satisfy differential equations with constant coefficients, contradicting existence of a singularity at $z=1$.

By a combination of André's Theorem and differential galois theory one can show more.

Theorem (FB, 2004) Let $f(z)$ be an E-function and suppose that $f(\xi)=0$ for some $\xi \in \overline{\mathbb{Q}}^*$. Then ξ is an apparent singularity of the minimal differential equation satisfied by f.

Corollary The Nesterenko-Shidlovskii theorem holds with $S=$ singularities $\cup\, 0$.

Relations between values at singular points. Example $f(z) = (z-1)e^z$. It satisfies $(z-1)f' = zf$ and $f(1) = 0$.

More generally,

$$(z-\xi)^k \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}$$

where $A(\xi) \neq O$. Then,

$$A(\xi)\frac{d}{dz} \begin{pmatrix} f_1(\xi) \\ \vdots \\ f_n(\xi) \end{pmatrix} = \mathbf{0}.$$

Theorem Let $\mathbf{f}(z) = (f_1(z), \ldots, f_n(z))$ be E-function solution of system of n first order equations and suppose they are $\overline{\mathbb{Q}}(z)$-linear independent. Then there exists an $n \times n$- matrix B with entries in $\overline{\mathbb{Q}}[z]$ and $\det(B) \neq 0$ and E-functions $\mathbf{e}(z) = (e_1(z), \ldots, e_n(z)$ such that $\mathbf{f}(z) = B\mathbf{e}(z)$ and $\mathbf{e}(z)$ satisfies system of equations with singularities in the set $\{0, \infty\}$.