Irrationality of some p-adic L-values

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Abstract

We give a proof of the irrationality of the p-adic zeta-values $\zeta_p(k)$ for $p = 2, 3$ and $k = 2, 3$. Such results were recently obtained by F.Calegari as an application of overconvergent p-adic modular forms. In this paper we present an approach using classical continued fractions discovered by Stieltjes. In addition we show irrationality of some other p-adic L-series values, and values of the p-adic Hurwitz zeta-function.

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1 Introduction

The arithmetic nature of values of Dirichlet L-series at integer points $r > 1$ is still a subject with many unanswered questions. It is classically known that if a Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ has the same parity as r, the number $L(r, \chi)$ is an algebraic multiple of π^r, hence transcendental. When χ has parity opposite from r, the matter is quite different. The only such value known to be irrational is $\zeta(3)$ as R.Apéry first proved in 1978. From later work by Rivoal and Ball [1] it follows that $\zeta(2n + 1)$ is irrational for infinitely many n, and W.Zudilin [2] showed recently that at least one among $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. We also recall analogous statements for L-values with the odd character modulo 4 in [3].
Although there have been many attempts to generalise Apéry's original irrationality proof to higher zeta-values, all have failed due to the absence of convenient miracles which did occur in the case of $\zeta(3)$. One such attempt was made by the present author [4] through the use of elementary modular forms. Although the approach looked elegant it provided no new significant results. Ever since then the method has lain dormant with no new applications.
In a recent, very remarkable and beautiful paper, Frank Calegari [5] managed to establish further irrationality results using modular forms. However, the numbers involved are values of Leopoldt-Kubota p-adic L-series. For example, Calegari managed to prove irrationality of 2-adic $\zeta(2)$ and 2- and 3-adic $\zeta(3)$. The underlying mechanism is the overconvergence of certain p-adic modular forms, a subject which has recently attracted renewed attention in connection with deformation theory of Galois representations.
Since overconvergent modular form theory is an advanced subject I tried to reverse engineer the results of Calegari in order to toss out the use of modular forms and find a more classical

approach. This turns out to be possible. We show irrationality of a large family of p-adic numbers, some of which turn out to be values of p-adic L-series at the points 2 or 3. In Theorems 7.2, 9.2 and 11.2 one finds the main results of this paper. Incidently we note that in Calegari's paper irrationality of the 2-adic Catalan constant is mentioned. Although the term 'Catalan constant' is perfectly reasonable, it does not correspond with the Kubota-Leopoldt $L_2(2,\chi_4)$ where χ_4 is the odd Dirichlet character modulo 4. It is well-known that Kubota-Leopoldt L-series with odd character vanish identically. The work of Calegari actually entails irrationality of the Kubota-Leopoldt $\zeta_2(2)$. The difference is due to the extra Teichmüller character which occurs in the definition of the Kubota-Leopoldt L-functions. In [5] the irrationality of 2- and 3-adic $\zeta(3)$ is also shown.

In Sections 6, 8 and 10 we shall discuss p-adic irrationality results proved using Padé approximations to the infinite Laurent series

$$\begin{aligned}\Theta(x) &= \sum_{n\ge 0} t_n(-1/x)^{n+1}\\ R(x) &= \sum_{n\ge 0} B_n(-1/x)^{n+1}\\ T(x) &= \sum_{n\ge 0}(n+1)B_n(-1/x)^{n+2}\end{aligned}$$

where the B_n are the Bernoulli numbers and $t_n=(2^{n+1}-2)B_n$. Continued fraction expansions to $R(x)$ and $T(x)$ were already known to T.J.Stieltjes in 1890. In a first version of this paper I worked out the corresponding Padé approximations using hypergeometric functions, which can be found in this paper. However, it was pointed out to me by T.Rivoal that the Padé approximations for $R(x)$ and $T(x)$ were also described in a different way in a paper by M.Prévost [6]. In this paper the author gives an alternative irrationality proof of Apéry's result $\zeta(2),\zeta(3)\notin\mathbb{Q}$. In a paper by Rivoal [7] the author makes a similar attempt at proving irrationality of Catalan's constant. The Padé approximations involved in there are precisely the approximations to $\Theta(x)$! Ironically in both [6] and [7] the implications for proving p-adic irrationality results are not noted.

The irrationality results of Calegari are contained in the irrationality results that were found in the Padé approximation approach sketched above.

We collected the definition and basic properties of p-adic L-series in the Appendix of this paper. Throughout we use the conventions made in Washington's book [8, Ch.5] on cyclotomic fields.

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Thanks are also due to the authors of the number theory package PARI which enabled me to numerically verify instances of p-adic identities. I also thank the authors P.Paule, M.Schorn, A.Riese of the Fast Zeilberger package for Mathematica. Their implementation of Gosper and Zeilberger summation turned out to be extremely useful.

Finally I like to thank T.Rivoal for pointing out the connections with existing irrationality results.

2 Arithmetic considerations

The principle of proving irrationality of a p-adic number α is to construct a sequence of rational approximations p_n/q_n which converges p-adically to α sufficiently fast. To be more precise,

Proposition 2.1 *Let α be a p-adic number and let $p_n, q_n,\ n = 0, 1, 2, \ldots$ be two sequences of integers such that*

$$\lim_{n\to\infty} \max(|p_n|, |q_n|)|p_n - \alpha q_n|_p = 0$$

and $p_n - \alpha q_n \neq 0$ infinitely often. Then α is irrational.

Proof. Suppose α is rational, say A/B with $A, B \in \mathbb{Z}$ and $B > 0$. Whenever $p_n - (A/B)q_n$ is non-zero we have trivially, $\max(|p_n|, |q_n|)|p_n - (A/B)q_n|_p \geq 1/\max(|A|, B)$. Hence the limit as $n \to \infty$ cannot be zero. Thus we conclude that α is irrational. □

We will also need some arithmetic statements about hypergeometric coefficients.

Lemma 2.2 *Let β be a rational number with the integer $F \in \mathbb{Z}_{>1}$ as denominator. Then $(\beta)_n/n!$ is a rational number whose denominator divides $\mu_n(F)$, where*

$$\mu_n(F) = F^n \prod_{q|F} q^{[n/(q-1)]}$$

where the product is over all primes q dividing F. Moreover, the number of primes p in the denominator of $(\beta)_n/n!$ is at least $n(r + 1/(p-1)) - \log n/\log p - 1$, where r is defined by the relation $|F|_p = p^{-r}$.

Proof. Let us write $\beta = b/F$ with $b \in \mathbb{Z}$. Then

$$\frac{(\beta)_n}{n!} = \frac{\prod_{k=0}^{n-1}(b + Fk)}{F^n n!}.$$

Let q be a prime. Suppose q divides F. Then q does not divide the product $\prod_k(b + Fk)$ and the number of primes q in the denominator is the number of primes q in $F^n n!$. The number of primes q in $n!$ equals

$$[n/q] + [n/q^2] + [n/q^3] + \cdots$$

which is bounded above by

$$n/q + n/q^2 + n/q^3 + \cdots = n/(q-1).$$

This explains the factor $q^{[n/(q-1)]}$ in our assertion. Morever, we also have the lower bound

$$\sum_{k=1}^{[\log n/\log q]} [n/q^k] \geq \sum_{k=1}^{[\log n/\log q]} (n/q^k - 1) \geq \frac{n-1}{q-1} - \frac{\log n}{\log q}.$$

This lower bound accounts for the second assertion when $q = p$.
To finish the proof of the first assertion we must show that $(n!)^{-1}\prod_{k=0}^{n-1}(b + Fk)$ is q-adically integral if q does not divide F. This follows easily from the fact that the number of $0 \leq k \leq n-1$ for which $b + Fk$ is divisible by a power q^s is always larger or equal than the number of $1 \leq k \leq n$ for which k is divisible by q^s. □

3 Differential equations

In the next sections we shall consider solutions of linear differential equations of orders 2 and 3. Here we derive some generalities on the arithmetic of the coefficients of the solutions in Taylor series.

Let R be a domain of characteristic zero with quotient field $Q(R)$. Consider a differential operator L_2 defined by

$$L_2(y) := zp(z)y'' + q(z)y' + r(z)y$$

where $p(z), q(z), r(z) \in R[z]$, $p(0) = 1$. Suppose there exists $W_0 \in R[[z]]$ such that $W_0(0) = 1$ and the logarithmic derivative of W_0/z equals $-q(z)/zp(z)$. We call W_0/z the Wronskian determinant of L_2. Suppose in addition that the equation $L_2(y) = 0$ has a formal power series $y_0 \in R[[z]]$ with $y_0(0) = 1$ as solution. Such a solution is determined uniquely since the space of solutions in $Q(R)[[z]]$ has dimension one.

The operator L_2 has a symmetric square L_3 which we write as

$$L_3(y) := z^2P(z)y''' + Q(z)y'' + R(z)y' + S(z)y$$

with $P, Q, R, S \in R[z]$, $P(0) = 1$. This symmetric square is characterised by the property that the solution space of $L_3(y)$ is spanned by the squares of the solutions of $L_2(y) = 0$. The equation $L_3(y) = 0$ has a unique formal power series solution with constant term 1, which is y_0^2.

Proposition 3.1 *Let notations and assumptions be as above. Then the inhomogeneous equation $L_2(y) = 1$ has a unique solution $y_{\mathrm{inhom},2} \in Q(R)[[z]]$ starting with $z + O(z^2)$. Moreover, the n-th coefficient of $y_{\mathrm{inhom},2}$ has denominator dividing* $\mathrm{lcm}(1, 2, \ldots, n)^2$.

The inhomogeneous equation $L_3(y) = 1$ has a unique solution $y_{\mathrm{inhom},3} \in Q(R)[[z]]$ starting with $z+O(z^2)$. Moreover, the n-th coefficient of $y_{\mathrm{inhom},3}$ has denominator dividing $\mathrm{lcm}(1, 2 \ldots, n)^3$.

Proof. In this proof we shall use the following identities, which hold for any $f \in Q(R)[[z]]$,

$$\int_0^z f \log z \; dz = \log z \int_0^z f dz - \int_0^z \frac{1}{z} \int_0^z f dz.$$

and

$$\int_0^z f(\log z)^2 \; dz = (\log z)^2 \int_0^z f dz - 2 \log z \int_0^z \frac{1}{z} \int_0^z f dz + 2 \int_0^z \frac{1}{z} \int_0^z \frac{1}{z} \int_0^z f dz.$$

These identities can be shown by (repeated) partial integration.

One easily verifies that a second, independent solution of $L_2(y) = 0$ is given by $y_1 = y_0 \int (W_0/zy_0^2) dz$. The quotient W_0/zy_0^2 equals $1/z$ plus a Taylor series in $R[[z]]$. Integration and multiplication by y_0 then shows that $y_1 = y_0 \log z + \tilde{y}_0$ where $\tilde{y}_0 \in Q(R)[[z]]$ and whose n-th coefficient has denominator dividing $\mathrm{lcm}(1, 2, \ldots, n)$. We choose the constant of integration in such a way that $\tilde{y}_0(0) = 0$.

Note by the way that $y_1'y_0 - y_1y_0' = W_0/z$ which is precisely how the Wronskian should be defined. A straightforward verification shows that

$$y_1 \int_0^z \frac{y_0}{pW_0} \; dz - y_0 \int_0^z \frac{y_1}{pW_0} \; dz$$

is solution of the inhomogeneous equation $L_2(y)=1$. Now substitute $y_1=y_0\log z+\tilde{y}_0$. We obtain, using the identity for $\int_0^z f\log z\,dz$,

$$\tilde{y}_0\int_0^z \frac{y_0}{pW_0}\,dz-y_0\int_0^z\frac{\tilde{y}_0}{pW_0}\,dz+y_0\int_0^z\frac{1}{z}\int_0^z\frac{y_0}{pW_0}\,dz\,dz.$$

We have thus obtained a power series solution of $L_2(y)=1$ and the assertion about the denominators of the coefficients readily follows.
Another straightforward calculation shows that

$$y_0^2\int_0^z\frac{y_1^2}{W_0^2P}\,dz-2y_0y_1\int_0^z\frac{y_0y_1}{W_0^2P}dz+y_1^2\int_0^z\frac{y_0^2}{W_0^2P}\,dz$$

is a solution of $L_3(y)=2$. Continuation of our straightforward calculation using $y_1=y_0\log z+\tilde{y}_0$ shows that this solution equals

$$2y_0^2\int_0^z\frac{1}{z}\int_0^z Y\,dz\,dz+2y_0\tilde{y}_0\int_0^z Y\,dz+\tilde{y}_0^2\int_0^z\frac{y_0^2}{W_0^2P}\,dz$$

where

$$Y=-\frac{y_0\tilde{y}_0}{W_0^2P}+\frac{1}{z}\int_0^z\frac{y_0^2}{W_0^2P}\,dz.$$

The last statement of our Proposition follows in a straightforward manner.

□

4 Some identities

Consider the field of rational functions $\mathbb{Q}(x)$ with a discrete valuation such that $|x|>1$. Denote its completion with respect to that valuation by K. We see that K is the field of formal Laurent series in $1/x$. Our considerations will take place within this field. Later we shall substitute $x=a/F$ where a/F is a rational number with $|a/F|_p>1$ and perform an evaluation in $\mathbb{Q}_p$.
Define, following J.Diamond in [9],

$$\begin{bmatrix} n \\ x\end{bmatrix}=\frac{n!}{x(x+1)\cdots(x+n)}.$$

Proposition 4.1 *Let $\Theta(x),R(x),T(x)\in K$ be the Taylor series in $1/x$ which we defined in the introduction. Then we have the following identities in K,*

$$\Theta(x)=-\sum_{n=0}^{\infty}\begin{bmatrix} n \\ x\end{bmatrix}\begin{bmatrix} n \\ 1-x\end{bmatrix},$$

$$R(x)=-\sum_{k=0}^{\infty}\frac{1}{k+1}\begin{bmatrix} k \\ x\end{bmatrix},$$

$$T(x)=-\sum_{k=0}^{\infty}\frac{1}{k+1}\begin{bmatrix} k \\ x\end{bmatrix}\begin{bmatrix} k \\ 1-x\end{bmatrix}.$$

It will be the purpose of this section to prove these equalities. Let us first record the following relations between $\Theta(x), R(x)$ and $T(x)$ which follow directly from their definition. Namely

$$T(x) = R'(x) \qquad \Theta(x) = R(x/2) - 2R(x).$$

For any $A(x) \in K$ we can also consider $A(x+\lambda)$ for any $\lambda \in \mathbb{Q}$ as element of K if we expand $1/(x+\lambda)$ formally in a power series in $1/x$ again. We use the following important observation.

Lemma 4.2 *Suppose $A(x) \in K$ and suppose there exists a non-zero $\lambda \in \mathbb{Q}$ such that $A(x+\lambda) = A(x)$. Then $A(x)$ is a constant.*

Proof. The equality $A(x+\lambda) = A(x)$ remains true if we subtract the constant coefficient a_0 from A. Let us now assume that $A(x) - a_0$ is not identically zero. Then there exists a non-zero integer n and non-zero a_n such that $A(x) - a_0 = a_n(1/x)^n +$ higher order terms in $1/x$. It is straightforward to verify that $A(x+\lambda) - A(x) = -n\lambda a_n(1/x)^{n+1} +$ higher order terms in $1/x$. This contradicts $A(x+\lambda) = A(x)$. Hence $A(x) - a_0$ is identically zero.

□

We require the following property of Bernoulli-numbers.

Lemma 4.3 *For any $n \neq 1$ we have*

$$\sum_{k=0}^{n} B_k \binom{n}{k} = B_n.$$

When $n = 1$ we have $B_0 + B_1 = 1 + B_1$.

Proof. Recall the definition

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n t^n / n!.$$

Multiplication by e^t gives

$$t + \frac{t}{e^t - 1} = \sum_{n \geq 0} \left(\sum_{k=0}^{n} B_k \binom{n}{k} \right) t^n / n!$$

Our Lemma follows by comparison of coefficients of t^n.

□

We are now ready to prove the following functional equations.

Proposition 4.4 *We have the identities*

1. $R(x+1) - R(x) = 1/x^2$
2. $R(x) + R(1-x) = 0$
3. $R(x) + R(x+1/2) = 4R(2x)$

Proof. The first statement follows from

$$\begin{aligned} R(x+1) &= \sum_{k\geq 0} B_k(-1/(x+1))^{k+1} \\ &= \sum_{k\geq 0} B_k \sum_{n=0}^{\infty} \binom{n}{k} (-1/x)^{n+1} \end{aligned}$$

Now interchange the summations to get

$$\begin{aligned} R(x+1) &= \sum_{n\geq 0}\sum_{k=0}^{n} B_k \binom{n}{k} (-1/x)^{n+1} \\ &= 1/x^2 + \sum_{n\geq 0} B_n(-1/x)^{n+1} \end{aligned}$$

where the last equality follows from Lemma 4.3.
To show the second statement we use the identity $R(-x) + R(x) = 1/x^2$ which follows from the fact that the only odd index n for which $B_n \neq 0$ is $n = 1$. Combining this with the first statement yields the second statement.
To show the third statement we use Lemma 4.2. Write $A(x) = 4R(2x) - R(x) - R(x + 1/2)$. Notice that $A(x)$ has constant term zero and from our first two results we deduce

$$\begin{aligned} A(x+1/2) - A(x) &= 4R(2x+1) - 4R(2x) - R(x) + R(x+1) \\ &= 4/(4x^2) - 1/x^2 = 0. \end{aligned}$$

Hence our Lemma implies that $A(x)$ is identically zero.

□

For $T(x), \Theta(x)$ there are a few immediate corollaries.

Corollary 4.5 *We have*

1. $T(x+1) - T(x) = -2/x^3$
2. $T(x) = T(1-x)$
3. $\Theta(x+1) + \Theta(x) = -2/x^2$
4. $\Theta(x) = R(x/2) - R(x/2 + 1/2)$.

Proof. The first two statement follow from the first two statements of Proposition 4.4 because $T(x) = R'(x)$.
For the third statement we use $\Theta(x) = R(x/2) - 2R(x)$ and the third statement of Proposition 4.4. We get

$$\begin{aligned} \Theta(x+1) + \Theta(x) &= R(x/2) + R(x/2+1/2) - 2R(x) - 2R(x+1) \\ &= 4R(x) - 2R(x) - 2R(x+1) = -2/x^2. \end{aligned}$$

The last statement follows from $\Theta(x) = R(x/2) - 2R(x)$ and $4R(x) = R(x/2) + R(x/2 + 1/2)$.
□

We are now ready to prove Proposition 4.1. To prove the first identity we set

$$S(x) = \sum \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}$$

and show that it satisfies $S(x+1) + S(x) = -2/x^2$. Our assertion then follows from $S(x+1) - \Theta(x+1) + S(x) - \Theta(x) = 0$, hence $S(x) - \Theta(x)$ is periodic with period 2. Application of Lemma 4.2 then shows that $S(x) - \Theta(x)$ is identically zero.
By straightforward calculation we find

$$\begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} + \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ -x \end{bmatrix} = 2 \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.$$

Using Gosper summation we get

$$\begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} = \Delta_n \left(\frac{(n+1-x)(n+1+x)}{x^2} \begin{bmatrix} n \\ 1-x \end{bmatrix} \begin{bmatrix} n \\ 1+x \end{bmatrix} \right)$$

where Δ_n is the forward difference operator $\Delta_n(g)(n) = g(n+1) - g(n)$. Now carry out the summation and use telescoping of series to find that $S(x+1) + S(x) = -2/x^2$.
To prove the second assertion of Proposition 4.1 we denote the summation on the right again by $S(x)$. Observe that

$$\begin{bmatrix} k \\ x+1 \end{bmatrix} - \begin{bmatrix} k \\ x \end{bmatrix} = -\frac{k+1}{x} \begin{bmatrix} k \\ x+1 \end{bmatrix}.$$

Hence

$$S(x+1) - S(x) = \sum_{k=0}^{\infty} \frac{1}{x} \begin{bmatrix} k \\ x+1 \end{bmatrix}.$$

Using Gosper summation one quickly finds that

$$\begin{bmatrix} k \\ x+1 \end{bmatrix} = -\frac{1}{x} \Delta_k \left((1+k+x) \begin{bmatrix} k \\ x+1 \end{bmatrix} \right).$$

Summation over k then yields

$$S(x+1) - S(x) = \frac{1}{x^2}.$$

Hence $R(x) - S(x)$ is periodic with period 1 and thus identically 0 according to Lemma 4.2.
To prove the third assertion of Proposition 4.1 we again denote the righthand side by $S(x)$. Observe that

$$\begin{bmatrix} k \\ x+1 \end{bmatrix} \begin{bmatrix} k \\ -x \end{bmatrix} - \begin{bmatrix} k \\ x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix} = -2\frac{k+1}{x} \begin{bmatrix} k \\ 1+x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix}.$$

Hence

$$S(x+1) - S(x) = \sum_{k=0}^{\infty} \frac{-2}{x} \begin{bmatrix} k \\ 1+x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix}.$$

Using Gosper summation one easily finds that

$$\begin{bmatrix} k \\ 1+x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix} = \frac{-1}{x^2} \Delta_k \left((x^2 - (k+1)^2) \begin{bmatrix} k \\ x+1 \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix} \right).$$

Summation over k then yields

$$S(x+1) - S(x) = -\frac{2}{x^3}.$$

Hence $T(x) - S(x)$ is periodic with period 1 and thus identically 0 according to Lemma 4.2. □

5 Some p-adic identities

In the following results we relate p-adic values of $R(x), T(x), \Theta(x)$ with some p-adic L-series. Let a/F be a rational number whose denominator F is divisible by p. The series obtained from $\Theta(x), R(x), T(x)$ by the substitution $x = a/F$ are p-adically convergent. We denote the p-adic values of these series by $\Theta_p(a/F)$, $R_p(a/F)$ and $T_p(a/F)$. First of all, it follows in a straightforward manner from the Appendix that

$$R_p(a/F) = -F^2\omega(a)^{-1}H_p(2, a, F)$$

where H_p is the p-adic Hurwitz zeta-function and ω the Teichmüller character modulo p. As a Corollary we get expressions for $\Theta_p(a/F)$ in terms of p-adic Hurwitz zeta-function values. As application we now have,

Proposition 5.1 *Let χ_d be the primitive even Dirichlet character modulo d. Then,*

1. $\Theta_2(1/2) = -8\zeta_2(2)$
2. $\Theta_2(1/6) = -40\zeta_2(2)$
3. $\Theta_2(1/4) = -16L_2(2, \chi_8)$
4. $\Theta_3(1/3) = -27\zeta_3(2)/2$
5. $\Theta_3(1/6) = -36L_3(2, \chi_{12})$

We show how to prove the first assertion. Using Corollary 4.5 (4) we get

$$\Theta_2(1/2) = (R_2(1/4) - R_2(3/4))/2.$$

Using the relation between $R_2(a/F)$ and $H_2(2, a, F)$ sketched above we find

$$\Theta_2(1/2) = -8(H_2(2, 1, 4) + H_2(2, 3, 4)) = 8\zeta_2(2),$$

where the last equality follows from the last formula in the Appendix.
Similarly we can find p-adic values of $T(x)$ as p-adic zeta-values. We use the fact that

$$T_p(a/F) = 2F^3\omega(a)^{-2}H_p(3, a, F).$$

As a consequence we get the following evaluations.

Proposition 5.2 *Let χ_d be the even primitive Dirichlet character modulo d. Then*

1. $T_2(1/4) = 4^3\zeta_2(3)$
2. $T_3(1/3) = 3^3\zeta_3(3)$
3. $T_5(1/5) = (5^3/2)(\zeta_5(3) - L_5(3,\chi_5))$
4. $T_5(2/5) = (5^3/2)(\zeta_5(3) + L_5(3,\chi_5))$
5. $T_2(1/8) = 2^8(\zeta_2(3) - L_2(3,\chi_8))$
6. $T_2(3/8) = 2^8(\zeta_2(3) + L_2(3,\chi_8))$

As illustration we prove the first equality. First use $T(x) = T(1-x)$ to get $T_2(1/4) = T_2(3/4)$. Then observe,

$$T_2(1/4) = (T_2(1/4) + T_2(3/4))/2 = 4^3(H_2(3,1,4) + H_2(3,3,4)) = 4^3\zeta_2(3).$$

6 Padé approximations I

In this section we will prove that $\Theta(x)$ has the following continued fraction expansion,

$$\Theta(x) = \cfrac{1}{x^2 - x + a_1 - \cfrac{b_1}{x^2 - x + a_2 - \cfrac{b_2}{\ddots}}}$$

where $a_n = 2n^2 - 2n + 1, b_n = n^4$. We shall study its convergents and use these to derive irrationality results. For example, if we substitute $x = 1/2$ we obtain a continued fraction expansion which converges 2-adically very fast to the 2-adic evaluation $\Theta_2(1/2) = -8\zeta_2(2)$. From the theory of continued fractions it follows that the convergents are of the form V_n/U_n $n = 0, 1, 2, \ldots$ where V_n, U_n are polynomials of degrees $2n-2, 2n$ respectively. Moreover U_n, V_n satisfy the recurrence relation

$$U_{n+1} = (2n^2 + 2n + 1 - x + x^2)U_n - n^4 U_{n-1}.$$

Now substitute $U_n = (n!)^2 u_n$ and we get a new recurrence relation

$$(n+1)^2 u_{n+1} = (2n(n+1) + 1 - x + x^2)u_n - n^2 u_{n-1} \tag{1}$$

Consider the solutions $q_n(x)$ and $p_n(x)$ given by

$$\begin{aligned} q_0(x) &= 1 \\ q_1(x) &= x^2 - x + 1 \\ q_2(x) &= (x^4 - 2x^3 + 7x^2 - 6x + 4)/4 \\ q_3(x) &= (x^6 - 3x^5 + 22x^4 - 39x^3 + 85x^2 - 66x + 36)/36 \\ &\ldots \end{aligned}$$

and

$$\begin{array}{rcl} p_0(x) & = & 0 \\ p_1(x) & = & 1 \\ p_2(x) & = & (x^2 - x + 5)/4 \\ p_3(x) & = & (x^4 - 2x^3 + 19x^2 - 18x + 49)/36 \\ & & \ldots \end{array}$$

Then the sequence of rational functions $p_n(x)/q_n(x)$ are the convergents of our continued fraction. To determine the $q_n(x)$ we consider the generating function

$$y_0(z) = \sum_{n=0}^{\infty} q_n(x) z^n.$$

Due to the recursion relation it is straightforward to see that $y_0(z)$ is a power series solution of the second order linear differential equation

$$L_2(y) = z(z-1)^2 y'' + (3z-1)(z-1)y' + (z - 1 + x(1-x))y = 0$$

where the $'$ denotes differentation with respect to z. Power series solutions in z are uniquely determined up to a scalar factor. Since it is also straightforward to see that $(1-z)^{x-1}\ {}_2F_1(x, x, 1, z)$ is another such solution we conclude that

$$y_0(z) = (1-z)^{x-1}\ {}_2F_1(x, x, 1, z).$$

Comparison of coefficients gives us

$$q_n(x) = \sum_{k=0}^{n} \frac{(1-x)_{n-k}(x)_k^2}{(n-k)!(k!)^2}.$$

Let us also consider the generating function for the $p_n(x)$,

$$y_1(z) = \sum_{n=0}^{\infty} p_n(x) z^n.$$

A straightforward calculation using the recurrence shows that $L_2(y_1) = 1$.
The problem is now to show that the rational functions $p_n(x)/q_n(x)$ approximate $\Theta(x)$ in K. To that end we define for each n,

$$\Theta(n, x) = (-1)^n \sum_{k=0}^{\infty} \binom{k}{n} \begin{bmatrix} k \\ x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix}.$$

Notice that $\Theta(0, x) = -\Theta(x)$ via Proposition 4.1.

Proposition 6.1 *Letting notations be as above, we have for each n,*

$$p_n(x) - q_n(x)\Theta(x) = \Theta(n, x) = O(1/x^{2n+2})$$

as Laurent series in $1/x$.

Proof. Letting

$$F(k,n)=(-1)^n\binom{k}{n}\begin{bmatrix}k\\x\end{bmatrix}\begin{bmatrix}k\\1-x\end{bmatrix}$$

the Zeilberger algorithm shows that

$$n^2F(k,n-1)-(-x+x^2+2n^2+2n+1)F(k,n)+(n+1)^2F(k,n+1)=\Delta_k(F(k,n)(x+k)(k+1-x)).$$

When $n\geq 1$ summation over k yields

$$n^2\Theta(n-1,x)-(-x+x^2+2n^2+2n+1)\Theta(n,x)+(n+1)^2\Theta(n+1,x)=0.$$

When $n=0$ we get

$$-(-x+x^2+1)\Theta(0,x)+\Theta(1,x)=1.$$

From this, and the fact that $\Theta(0,x)=-\Theta(x)$ we conclude that

$$p_n(x)-q_n(x)\Theta(x)=\Theta(n,x)$$

for all $n\geq 0$.

□

It was remarked to me by T.Rivoal that these approximations can also be found in [7] as $P_{2n}(z)$. When we replace the z there by $1-2x$ we obtain the alternative expression,

$$q_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{-x}{k}\binom{k-x}{k},$$

Notice that by taking $x=-n$ we recover Apéry's numbers for the irrationality of $\zeta(2)$ again. By taking $x=-n+1/2$ one obtains numbers which play a role in approximations of Catalan's constant (see [7]).
From [7] we find an explicit formula for p_n (known as Q_{2n} in [7]),

$$p_n(x)=\sum_{k=1}^{n}\binom{n}{k}\sum_{j=1}^{k}\binom{k-x}{k-j}\binom{-x-j}{k-j}\frac{(-1)^{j-1}}{j^2\binom{k}{j}^2}.$$

7 Application I

Proposition 7.1 *Let p_n,q_n be as in the previous section and let $\mu_F(n)$ be as in Lemma 2.2. Then,*

1. *For every n the number $q_n(a/F)$ is rational with denominator dividing $\mu_F(n)^2$.*
2. *For every n the number $p_n(a/F)$ is rational with denominator dividing $\mathrm{lcm}(1,\ldots,n)^2\mu_F(n)^2$.*
3. *For every $\epsilon>0$ we have that $|q_n(a/F)|,|p_n(a/F)|<e^{\epsilon n}$ for sufficiently large n.*
4. *Suppose $p^r||F$ where $r>0$ and a is not divisible by p. Then*

$$|p_n(a/F)-\Theta_p(a/F)q_n(a/F)|_p\leq p^2n^2p^{-2n(r+1/(p-1))}$$

for every n.

Proof. The numbers $q_n(a/F)$ are given by

$$\sum_{k=0}^{n} \frac{(1-a/F)_{n-k}(a/F)_k^2}{(n-k)!(k!)^2}.$$

The first assertion follows from Lemma 2.2.
The generating function of the $p_n(a/F)$ is the series y_1 with $x = a/F$. To apply Proposition 3.1 we replace z by $F^2\lambda^2 z$ and x by a/F in the equation $L_2(y) = 0$ where $\lambda = \prod_{q|F} q^{1/(q-1)}$. When we take the ring $R = \mathbb{Z}[q^{1/(q-1)}]_{q|F}$, the conditions of Proposition 3.1 are still satisfied with $y_0(F^2\lambda^2 z) \in R[[z]]$ as power series solution. From this Proposition it follows that the n-th coefficient of $y_1(F^2\lambda^2 z)$ has denominator dividing $\mathrm{lcm}(1,\ldots,n)^2$. Thus, our second statement follows.
The third statement on the Archimedean size of $q_n(a/F)$ and $p_n(a/F)$ follows from the fact that y_0 and y_1 have radius of convergence 1.
The fourth statement follows from Proposition 6.1. It is a consequence of Lemma 2.2 that

$$\left|\begin{bmatrix} k \\ a/F \end{bmatrix}\begin{bmatrix} k \\ 1-a/F \end{bmatrix}\right|_p < k^2 p^{2-2k(r+1/(p-1))}$$

for all k. Hence

$$|\Theta_p(n, a/F)|_p \le \max_{k\ge n} < k^2 p^{2-2k(r+1/(p-1))}$$

from which our assertion follows.

□

We are now ready to state our irrationality results for $\Theta_p(a/F)$.

Theorem 7.2 *Let a be an integer not divisible by p and F a natural number divisible by p. Define r by $|F|_p = p^{-r}$. Suppose that*

$$\log F + \sum_{q|F} \frac{\log q}{q-1} + 1 < 2r\log p + 2\frac{\log p}{p-1}. \tag{A}$$

Then the p-adic number $\Theta_p(a/F)$ is irrational.

Proof. Let $\epsilon > 0$. According to Proposition 7.1, $q_n(a/F), p_n(a/F)$ have a common denominator dividing $Q_n := \mathrm{lcm}(1,2,\ldots,n)^2\mu_F(n)^2$. We also have, for n large enough, $|q_n(a/F)|, |p_n(a/F)| < e^{n\epsilon}$. Furthermore $p_n(a/F) - q_n(a/F)\Theta_p(a/F)$ is non-zero for infinitely many n. This follows from the fact that

$$p_{n+1}(x)q_n(x) - p_n(x)q_{n+1}(x) = 1/(n+1)^2,$$

which can be shown by induction using recurrence (1). We get

$$|Q_n p_n(a/F) - Q_n q_n(a/F)\Theta_p(a/F)|_p < p^{(-2r-2/(p-1)+\epsilon)n}|Q_n|_p$$

when n is large enough. We now apply Proposition 2.1 with $\alpha = \Theta_p(a/F), q_n = Q_n q_n(a/F), p_n = Q_n p_n(a/F)$. Notice that, for n large enough,

$$\begin{aligned} |q_n|, |p_n| &< e^{\epsilon n}\mathrm{lcm}(1,2,\ldots,n)^2\mu_F(n)^2 \\ &< \exp\left(n\epsilon + 2n(1+\epsilon) + 2n\log F + 2n\sum_{q|F}\frac{\log q}{q-1}\right) \end{aligned}$$

In the latter we used the estimate $\mathrm{lcm}(1,\ldots,n) < e^{(1+\epsilon)n}$ which follows from the prime number theorem. Since

$$|Q_n|_p < p^{-2rn}p^{-2[n/(p-1)]} \le p^{2-2n(r+1/(p-1))}$$

we get the estimate

$$|p_n - q_n\Theta_p(a/F)|_p < \exp\left(-4rn\log p - 4n\frac{\log p}{p-1} + \epsilon n\right).$$

From Proposition 2.1 we can conclude irrationality of $\Theta_p(a/F)$ if

$$2 + 3\epsilon + 2\log F + 2\sum_{q|F}\frac{\log q}{q-1} - 4r\log p - 4\frac{\log p}{p-1} + \epsilon < 0.$$

From assumption (A) in our Theorem this certainly follows if ϵ is chosen sufficiently small.

□

Corollary 7.3 *Let χ_8 be the primitive even character modulo 8. Then $\zeta_2(2), \zeta_3(2)$ and $L_2(2,\chi_8)$ are irrational.*

Proof. This is a direct consequence of Theorem 7.2 and Proposition 5.1.

□

From Corollary 2.1 and its proof we see that irrationality of $\zeta_2(2)$ follows from the 2-adically converging

$$\zeta_2(2) = \cfrac{-1/2}{a_1 - \cfrac{b_1}{a_2 - \cfrac{b_2}{\ddots}}}$$

where $a_n = 8n^2 - 8n + 3, b_n = 16n^4$. The original irrationality proof of Calegari [5] uses the related

$$\zeta_2(2) = 1/2 + \cfrac{4}{a_1 - \cfrac{b_1}{a_2 - \cfrac{b_2}{\ddots}}}$$

where $a_n = 1 - 8n^2, b_n = 16n^2(n+1)^2$.
Finally we remark that unfortunately condition (A) in Theorem 7.2 is not good enough to provide irrationality of $\Theta_3(1/6)$ which is related to $L_3(2,\chi_{12})$.

8 Padé approximations II

In [10] Stieltjes discovered the following continued fraction expansion

$$R(x) = \cfrac{-2}{2x - 1 + \cfrac{a_1}{2x - 1 + \cfrac{a_2}{\ddots}}}$$

with $a_n = n^4/(4n^2-1)$.
The convergents to this continued fraction were explicitly determined by Touchard [11] and Carlitz [12]. They were also used by Prévost [6]

in his alternative irrationality proofs for $\zeta(2)$ and $\zeta(3)$. In this section we give a self-contained derivation of the properties of these convergents.
The numerator and denominator of the convergents satisfy the recurrence relation

$$U_{n+1} = (2x-1)U_n + \frac{n^4}{(4n^2-1)}U_{n-1}.$$

If we set $U_n = (n!)^2 u_n/(1\cdot 3\cdot 5\cdots 2n-1)$ we get

$$(n+1)^2 u_{n+1} = (2n+1)(2x-1)u_n + n^2 u_{n-1} \qquad (2)$$

Consider the solutions $q_n(x)$ and $p_n(x)$ of this recurrence given by

$$\begin{aligned} q_0(x) &= 1 \\ q_1(x) &= 2x-1 \\ q_2(x) &= 3x^2-3x+1 \\ q_3(x) &= 10x^3/3-5x^2+11x/3-1 \\ &\dots \end{aligned}$$

and

$$\begin{aligned} p_0(x) &= 0 \\ p_1(x) &= -2 \\ p_2(x) &= -(6x-3)/2 \\ p_3(x) &= -(60x^2-60x+31)/18 \\ &\dots \end{aligned}$$

Then $p_n(x)/q_n(x)$ are the convergents of our continued fraction. To determine $q_n(x)$ we consider the generating function

$$y_0(z) = \sum_{n=0}^{\infty} q_n(x) z^n$$

and note that it satisfies the second order linear differential equation

$$L_2(y) = (z^3-z)y'' + (3z^2+(4x-2)z-1)y' + (z+2x-1)y$$

where the $'$ denotes differentation with respect to z. At $z=0$ the equation $L_2(y)=0$ has a unique (up to a scalar factor) holomorphic solution. In a straightforward manner one can thus verify that

$$y_0(z) = (1+z)^{2x-1}\ {}_2F_1(x,x,1,z^2).$$

Comparison of coefficients of z^n gives us the following explicit formula,

$$q_n(x) = \sum_{k\le n/2} \binom{2x-1}{n-2k}\binom{-x}{k}^2.$$

Consider the generating function of the $p_n(x)$,

$$y_1(z) = \sum p_n(x) z^n.$$

It is straightforward, using the recurrence relation, to see that y_1 satisfies the inhomogeneous equation $L_2(y_1) = 2$.
We must now show that the rational functions $p_n(x)/q_n(x)$ approximate $R(x)$ in K. To that end we define for each n,

$$R(n,x) = (-1)^n \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{(k+1)(k+2)\cdots(k+n+1)} \begin{bmatrix} k \\ x \end{bmatrix}.$$

Notice that it follows from Proposition 4.1 that $R(0,x) = -R(x)$.

Proposition 8.1 *Letting notations be as above, we have for each n,*

$$p_n(x) - q_n(x)R(x) = R(n,x) = O(1/x^{n+1})$$

as Laurent series in $1/x$.

Proof. Letting

$$F(k,n) = (-1)^n \frac{n!}{(k+1)\cdots(k+n+1)} \binom{k}{n} \begin{bmatrix} k \\ x \end{bmatrix}$$

the Zeilberger algorithm shows that

$$-n^2F(k,n-1) - (2n+1)(2x-1)F(k,n) + (n+1)^2F(k,n+1) = \Delta_k(2F(k,n)(x+k)(2n+1)).$$

When $n \ge 1$ summation over k yields

$$-n^2R(n-1,x) - (2n+1)(2x-1)R(n,x) + (n+1)^2R(n+1,x) = 0.$$

When $n = 0$ we get

$$-(2x-1)R(0,x) + R(1,x) = 2.$$

From this, and the fact that $R(0,x) = -R(x)$ we conclude that

$$p_n(x) - q_n(x)R(x) = R(n,x)$$

for all $n \ge 0$.

□

There exist several interesting ways to write $q_n(x)$ as a binomial sum, see for example [6]. One of them is

$$q_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\binom{-x}{k}.$$

Taking $x = -n$ one recovers the Apéry numbers for $\zeta(2)$ again.
Furthermore, in [6] we find the explicit expression

$$p_n(x) = (-1)^n \sum_{k=1}^{n} \binom{n}{k}\binom{n+k}{k}\binom{-x}{k} \sum_{j=1}^{k} \frac{(-1)^j}{j^2\binom{-x}{j}}.$$

9 Application II

In this section we prove irrationality for a large class of p-adic numbers of the form $R_p(a/F)$ where $|a/F|_p > 1$.

Proposition 9.1 *Let $p_n(x), q_n(x)$ be as in the previous section and let $\mu_F(n)$ be as in Lemma 2.2. Then,*

1. *For every n the number $q_n(a/F)$ is rational with denominator dividing $\mu_F(n)$.*
2. *For every n the number $p_n(a/F)$ is rational with denominator dividing $\mathrm{lcm}(1,\ldots,n)^2\mu_F(n)$.*
3. *For every $\epsilon > 0$ we have that $|q_n(a/F)|, |p_n(a/F)| < e^{\epsilon n}$ for sufficiently large n.*
4. *Suppose $p^r||F$ where $r > 0$ and a is not divisble by p, we have*

$$|p_n(a/F) - R_p(a/F)q_n(a/F)|_p \le (2n+1)\cdot p^{1-n(r+1/(p-1))}$$

for every n.

Proof. The numbers $q_n(a/F)$ are given by

$$\sum_{k\le n/2} \binom{2a/F-1}{n-2k}\frac{(a/F)_k^2}{(k!)^2}.$$

The first assertion follows from Lemma 2.2.
The generating function of the $p_n(a/F)$ is the series y_1 with $x = a/F$. To apply Proposition 3.1 we replace z by $F\lambda z$ and x by a/F in the equation $L_2(y) = 0$ where $\lambda = \prod_{q|F} q^{1/(q-1)}$. When we take the ring $R = \mathbb{Z}[q^{1/(q-1)}]_{q|F}$, the conditions of Proposition 3.1 are still satisfied with $y_0(F\lambda z) \in R[[z]]$ as power series solution. From this Proposition it follows that the n-th coefficient of $y_1(F\lambda z)$ has denominator dividing $\mathrm{lcm}(1,\ldots,n)^2$. Thus, our second statement follows.
The third statement on the Archimedean size of $q_n(a/F)$ and $p_n(a/F)$ follows from the fact that y_0 and y_1 have radius of convergence 1.
The fourth statement follows from Proposition 8.1. It is a consequence of Lemma 2.2 that

$$\left|\begin{bmatrix} k \\ a/F \end{bmatrix}\right|_p < kp^{1-k(r+1/(p-1))}$$

for all k. Moreover,

$$\left|\frac{n!}{(k+1)(k+2)\cdots(k+n+1)}\right|_p = \left|\sum_{l=0}^{n}(-1)^l\binom{n}{l}\frac{1}{k+l+1}\right|_p \le k+n+1.$$

Hence

$$|R_p(n, a/F)|_p \leq \max_{k \geq n} < (k+n+1)p^{1-k(r+1/(p-1))}$$

from which our assertion follows.

□

We are now ready to state our irrationality results for $R_p(a/F)$.

Theorem 9.2 *Let a be an integer not divisible by p and F a natural number divisible by p. Define r by $|F|_p = p^{-r}$. Suppose that*

$$\log F + \sum_{q|F} \frac{\log q}{q-1} + 2 < 2r \log p + 2\frac{\log p}{p-1}. \qquad \text{(B)}$$

Then the p-adic number $R_p(a/F)$ is irrational.

Proof Let $\epsilon > 0$. According to Proposition 9.1, $q_n(a/F), p_n(a/F)$ have a common denominator dividing $Q_n := \mathrm{lcm}(1, 2, \ldots, n)^2 \mu_F(n)$. We also have, for n large enough, $|q_n(a/F)|, |p_n(a/F)| < e^{n\epsilon}$. Furthermore $p_n(a/F) - q_n(a/F)R_p(a/F)$ is non-zero for infinitely many n. This follows from the fact that

$$p_{n+1}(x)q_n(x) - p_n(x)q_{n+1}(x) = (-1)^{n-1} \cdot 2/(n+1)^2.$$

This can be shown by induction using recurrence (2). We get

$$|Q_n p_n(a/F) - Q_n q_n(a/F)R_p(a/F)|_p < p^{(-r-1/(p-1)+\epsilon)n}|Q_n|_p$$

when n is large enough. We now apply Proposition 2.1 with $\alpha = R_p(a/F), q_n = Q_n q_n(a/F), p_n = Q_n p_n(a/F)$. Notice that, for n large enough,

$$\begin{aligned} |q_n|, |p_n| &< e^{\epsilon n} \mathrm{lcm}(1, 2, \ldots, n)^2 \mu_F(n) \\ &< \exp\left(n\epsilon + 2n(1+\epsilon) + n \log F + n \sum_{q|F} \frac{\log q}{q-1}\right) \end{aligned}$$

In the latter we used the estimate $\mathrm{lcm}(1, \ldots, n) < e^{(1+\epsilon)n}$ which follows from the prime number theorem. Since

$$|Q_n|_p < p^{-rn} p^{-[n/(p-1)]} \leq p^{1-n(r+1/(p-1))}$$

we get the estimate

$$|p_n - q_n \Theta_p(a/F)|_p < \exp\left(-2rn \log p - 2n\frac{\log p}{p-1} + \epsilon n\right).$$

From Proposition 2.1 we can conclude irrationality of $R_p(a/F)$ if

$$2 + 3\epsilon + \log F + \sum_{q|F} \frac{\log q}{q-1} - 2r \log p - 2\frac{\log p}{p-1} + \epsilon < 0.$$

From assumption (B) in our Theorem this certainly follows if ϵ is chosen sufficiently small.

□

Corollary 9.3 *Let p be a prime and F a power of p with $F \neq 2$. Let a be an integer not divisible by p. Then $\omega(a)^{-1}H_p(2,a,F)$ is irrational. For $F=2$ we have that $H_2(2,1,2)=0$.*

Proof Verify that condition (B) of Theorem 9.2 holds for every prime power $F>3$. The vanishing of $H_2(2,1,2)$ follows from $R(x)+R(1-x)=0$ which implies $2R_2(1/2)=0$. Finally, irrationality of $H_3(2,1,3)=H_3(2,2,3)$ follows from the irrationality of $\zeta_3(2)$ proved in Corollary 7.3.

□

10 Padé approximations III

In this section we prove the following continued fraction expansion of $T(x)=\sum_{n=0}^{\infty}(n+1)B_n(-1/x)^{n+2}$, namely

$$T(x)=\cfrac{1}{a_1-\cfrac{1^6}{a_2-\cfrac{2^6}{a_3-\ddots}}}$$

where $a_n=(2n-1)(2x^2-2x+n^2-n+1)$. In [10](23)] we find a related continued fraction for $x^2T(x)-1-1/x$, but we prefer the one presented here because it has simpler properties. Moreover, by substituting $x=1/4$, we obtain a continued fraction expansion which converges rapidly 2-adically to $T_2(1/4)=64\zeta_2(3)$. Without proof we note that Calegari's approximations a_n/b_n to $\zeta_2(3)$ (see proof of [5, Thm 3.4] coincide with the fractions $-\frac{p_n(1/4)-p_{n-1}(1/4)}{64(q_n(1/4)-q_{n-1}(1/4))}$. Here $p_n(x)/q_n(x)$ are the convergents of the continued fraction for $T(x)$, to be specified below. A similar remark holds for Calegari's approximations to $\zeta_3(3)$.
Our study of the convergents of the continued fraction expansion begins with the observation that the numerators and demoninators of the convergents can be normalised in such a way that they are solutions of the recurrence

$$U_{n+1}=(2n+1)(2x^2-2x+n^2+n+1)U_n-n^6U_{n-1}.$$

Replace U_n by n^3u_n to find

$$(n+1)^3u_{n+1}=(2n+1)(2x^2-2x+n^2+n+1)u_n-n^3u_{n-1} \tag{3}$$

Two independent solutions $q_n(x),p_n(x)$ are given by

$$\begin{aligned}
q_0(x) &= 1\\
q_1(x) &= 2x^2-2x+1\\
q_2(x) &= (3x^4-6x^3+9x^2-6x+2)/2\\
q_3(x) &= (10x^6-35x^5+85x^4-120x^3+121x^2-66x+18)/18\\
&\cdots
\end{aligned}$$

and

$$\begin{aligned}
p_0(x) &= 0 \\
p_1(x) &= 2 \\
p_2(x) &= 3(2x^2 - 2x + 3)/4 \\
p_3(x) &= (60x^4 - 120x^3 + 360x^2 - 300x + 251)/108 \\
&\dots
\end{aligned}$$

Consider the generating function

$$Y_0(z) = \sum_{n=0}^{\infty} q_n(x) z^n.$$

Using the recurrence we see that $Y_0(z)$ is solution of the linear differential equation

$$\begin{aligned}
L_3(y) &= z^2(z-1)^2 y''' + 3z(2z-1)(z-1)y'' + \\
&+(7z^2 - (4x^2 - 4x + 8)z + 1)y' + (z - 2x^2 + 2x - 1)y
\end{aligned}$$

One can verify in a straightforward manner that this equation is the symmetric square of the second order equation

$$L_2(y) = z(z-1)^2 y'' + (z-1)(2z-1)y' + (z/4 + x - x^2 - 1/2)y.$$

The unique power series solution in z of $L_2(y) = 0$ reads

$$y_0(z) = (1-z)^{x-1/2}\ {}_2F_1(x, x, 1, z).$$

As a consequence the function $Y_0(z)$ equals

$$Y_0(z) = y_0(z)^2 = (1-z)^{2x-1}\ {}_2F_1(x, x, 1, z)^2.$$

By comparison of coefficients we would be able to compute an explicit expression for $q_n(x)$. But this would be a double summation. A much nicer expression for $q_n(x)$ can be found from [6]. It reads

$$q_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{-x}{k} \binom{-x+k}{k}.$$

Notice that $x = -n$ recovers the Apéry numbers for $\zeta(3)$.
Let Y_1 be the generating function of the $p_n(x)$. Then Y_1 satisfies the inhomogeneous equation $L_3(Y_1) = 1$.
An explicit formula from [6] reads

$$p_n(x) = \sum_{k=1}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j^3} \frac{\binom{k-x}{k-j}\binom{-x-j}{k-j}}{\binom{k}{j}^2}.$$

We must now show that the rational functions $p_n(x)/q_n(x)$ approximate $T(x)$ in K. To that end we define for each n,

$$T(n,x) = (-1)^n \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{(k+1)(k+2)\cdots(k+n+1)} \begin{bmatrix} k \\ x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix}.$$

Notice, using Proposition 4.1, that $T(0,x) = -T(x)$.

Proposition 10.1 *Letting notations be as above, we have for each n,*

$$p_n(x) - q_n(x)T(x) = T(n,x) = O(1/x^{2n+2})$$

as Laurent series in $1/x$.

Proof. Letting

$$F(k,n) = (-1)^n \frac{n!}{(k+1)\cdots(k+n+1)} \binom{k}{n} \begin{bmatrix} k \\ x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix}$$

the Zeilberger algorithm shows that

$$n^3 F(k,n-1) - (2n+1)(2x^2 - 2x + n^2 + n + 1)F(k,n) + (n+1)^3 F(k,n+1)$$

$$= \Delta_k(2F(k,n)(x+k)(1-x+k)(2n+1)).$$

When $n \geq 1$ summation over k yields

$$n^3 T(n-1,x) - (2n+1)(2x^2 - 2x + n^2 + n + 1)T(n,x) + (n+1)^3 T(n+1,x) = 0.$$

When $n = 0$ we get

$$-(2x^2 - 2x + 1)T(0,x) + T(1,x) = 2.$$

From this, and the fact that $T(0,x) = -T(x)$ we conclude that

$$p_n(x) - q_n(x)T(x) = T(n,x)$$

for all $n \geq 0$.

□

11 Application III

In this section we prove irrationality for a large class of p-adic numbers of the form $T_p(a/F)$ where $|a/F|_p > 1$.

Proposition 11.1 *Let notations be as above and let $\mu_F(n)$ be as in Lemma 2.2. Then,*

1. *For every n the number $q_n(a/F)$ is rational with denominator dividing $\mu_F(n)^2$.*
2. *For every n the number $p_n(a/F)$ is rational with denominator dividing $\mathrm{lcm}(1,\ldots,n)^3 \mu_F(n)^2$.*
3. *For every $\epsilon > 0$ we have that $|q_n(a/F)|, |p_n(a/F)| < e^{\epsilon n}$ for sufficiently large n.*
4. *Suppose $p^r || F$ where $r > 0$ and a is not divisible by p, we have*

   $$|p_n(a/F) - T_p(a/F)q_n(a/F)|_p \leq (2n+1)n^2 \cdot p^{2-2n(r+1/(p-1))}$$

   for every n.

Proof. The numbers $q_n(a/F)$ are the coefficients of $(1-z)^{2a/F-1}\ {}_2F_1(a/F,a/F,1,z)^2$. Let again, $\lambda=\prod_{q|F} q^{1/(q-1)}$ and let R be the ring of integers in $\mathbb{Q}(\lambda)$. Then, by Lemma 2.2, we have $(1-\lambda^2 z)^{2a/F-1},\ {}_2F_1(a/F,a/F,1,\lambda^2 z)\in R[[z]]$. Hence part i) follows.
The generating function of the $p_n(a/F)$ is the series Y_1 with $x=a/F$. To apply Proposition 3.1 we replace z by $F\lambda z$ and x by a/F in the equation $L_2(y)=0$. The conditions of Proposition 3.1 are still satisfied with $y_0(F\lambda^2 z)\in R[[z]]$ as power series solution. From this Proposition it follows that the n-th coefficient of $Y_1(F\lambda^2 z)$ has denominator dividing $\mathrm{lcm}(1,\ldots,n)^3$. Thus, our second statement follows.
The third statement on the Archimedean size of $q_n(a/F)$ and $p_n(a/F)$ follows from the fact that y_0 and y_1 have radius of convergence 1.
The fourth statement follows from Proposition 10.1. It is a consequence of Lemma 2.2 that

$$\left|\begin{bmatrix}k\\a/F\end{bmatrix}\begin{bmatrix}k\\1-a/F\end{bmatrix}\right|_p < k^2 p^{2-2k(r+1/(p-1))}$$

for all k. Moreover,

$$\left|\frac{n!}{(k+1)(k+2)\cdots(k+n+1)}\right|_p = \left|\sum_{l=0}^{n}(-1)^l\binom{n}{l}\frac{1}{k+l+1}\right|_p \le k+n+1.$$

Hence

$$|T_p(n,a/F)|_p \le \max_{k\ge n} < (k+n+1)k^2 p^{2-2k(r+1/(p-1))}$$

from which our assertion follows.

□

We are now ready to state our irrationality results for $T_p(a/F)$.

Theorem 11.2 *Let a be an integer not divisible by p and F a natural number divisible by p. Define r by $|F|_p=p^{-r}$. Suppose that*

$$\log F+\sum_{q|F}\frac{\log q}{q-1}+3/2 < 2r\log p+2\frac{\log p}{p-1}. \tag{C}$$

Then the p-adic number $T_p(a/F)$ is irrational.

Proof Let $\epsilon>0$. According to Proposition 11.1, the rational numbers $q_n(a/F),p_n(a/F)$ have a common denominator dividing $Q_n:=\mathrm{lcm}(1,2,\ldots,n)^3\mu_F(n)^2$. We also have, for n large enough, $|q_n(a/F)|,|p_n(a/F)|<e^{n\epsilon}$. Furthermore $p_n(a/F)-q_n(a/F)T_p(a/F)$ is non-zero for infinitely many n. This follows from the fact that

$$p_{n+1}(x)q_n(x)-p_n(x)q_{n+1}(x)=1/(n+1)^3.$$

This can be shown by induction using recurrence (2). We get

$$|Q_np_n(a/F)-Q_nq_n(a/F)T_p(a/F)|_p < p^{2n(-r-1/(p-1)+\epsilon)}|Q_n|_p$$

when n is large enough. We now apply Proposition 2.1 with $\alpha = T_p(a/F), q_n = Q_n q_n(a/F), p_n = Q_n p_n(a/F)$. Notice that, for n large enough,

$$\begin{aligned} |q_n|, |p_n| &< e^{\epsilon n}\mathrm{lcm}(1,2,\ldots,n)^3 \mu_F(n)^2 \\ &< \exp\left(n\epsilon + 3n(1+\epsilon) + 2n\log F + 2n\sum_{q|F}\frac{\log q}{q-1}\right) \end{aligned}$$

In the latter we used the estimate $\mathrm{lcm}(1,\ldots,n) < e^{(1+\epsilon)n}$ which follows from the prime number theorem. Since

$$|Q_n|_p < p^{-2rn}p^{-2[n/(p-1)]} \le p^{2-2n(r+1/(p-1))}$$

we get the estimate

$$|p_n - q_n T_p(a/F)|_p < \exp\left(-4rn\log p - 4n\frac{\log p}{p-1} + \epsilon n\right).$$

From Proposition 2.1 we can conclude irrationality of $T_p(a/F)$ if

$$3 + 4\epsilon + 2\log F + 2\sum_{q|F}\frac{\log q}{q-1} - 4r\log p - 4\frac{\log p}{p-1} + \epsilon < 0.$$

From assumption (C) in our Theorem this certainly follows if ϵ is chosen sufficiently small.

□

Corollary 11.3 *Let p be a prime and F a power of p with $F > 2$. Let a be an integer not divisible by p. Then $\omega(a)^{-2}H_p(3,a,F)$ is irrational.*

Proof Verify that condition (C) of Theorem 11.2 holds for every prime power $F > 2$.

□

Corollary 11.4 *Let χ_d be a primitive even character modulo d. Then the following numbers are irrational: $\zeta_2(3), \zeta_3(3), \zeta_5(3) - L_3(3,\chi_5), \zeta_5(3) + L(3,\chi_5), \zeta_2(3) - L_2(3,\chi_8), \zeta_2(3) + L_2(3,\chi_8)$.*

Proof Use the previous Corollary and Proposition 5.2.

□

12 Padé approximations IV

So far we have studied continued fraction expansions of the functions $\Theta(x), R(x)$ and $T(x)$. Clearly $R(x), T(x)$ are the generator series of the Bermoulli numbers and its derivatives. We like to remark here that the coefficients $(2^{n+1}-2)B_n$ of $\Theta(x)$ are actually $(n-1)T_{n-1}$ for $n > 1$ where T_n is the hyperbolic tangent number defined by

$$\tanh(t/2) = \frac{e^t - 1}{e^t + 1} = \sum_{n=0}^{\infty}\frac{T_n}{n!}t^n.$$

This follows from the observation that

$$\sum_{n=0}^{\infty}(2^{n+1}-2)\frac{B_n}{n!}t^n = \frac{4t}{e^{2t}-1} - \frac{2t}{e^t-1} = t\tanh(t/2) - t.$$

The series $\Theta(x)$ is also related to the Euler numbers E_n via

$$\Theta((1-z)/2) = -4\sum_{n=0}^{\infty}(n+1)E_n(1/z)^{n+2}.$$

The Euler numbers are defined by
The only interesting additional continued fraction (S-fraction in the sense of Stieltjes) we have been able to find is one for

$$\theta(x) = \sum_{n=1}^{\infty}(2^{n+1}-2)\frac{B_n}{n}(-1/x)^n.$$

It reads

$$\theta(x) = \cfrac{2}{2x-1+\cfrac{1}{2x-1+\cfrac{4}{2x-1+\cfrac{9}{2x-1+\cfrac{16}{2x-1+\ddots}}}}}.$$

We will not give any proofs here (they are parallel to the previous sections), but only quote some formulas. The recurrence relation involved with this continued fraction is

$$U_{n+1} = (2x-1)U_n + n^2U_{n-1}$$

Substitute $U_n = n!u_n$. Then,

$$(n+1)u_{n+1} = (2x-1)u_n + nu_{n-1}.$$

Consider the solutions

$$\begin{aligned} q_0(x) &= 1 \\ q_1(x) &= 2x-1 \\ q_2(x) &= 2x^2-2x+1 \\ q_3(x) &= (2x-1)(2x^2-2x+3)/3 \\ &\ldots \end{aligned}$$

and

$$\begin{aligned} p_0(x) &= 0 \\ p_1(x) &= 2 \\ p_2(x) &= 2x-1 \\ p_3(x) &= (4x^2-4x+5)/3 \\ &\ldots \end{aligned}$$

The generating function

$$y_0(z) = \sum_{n=0}^{\infty} q_n(x) z^n$$

satisfies the differential equation

$$(1 - z^2)y' - (2x - 1 + z)y = 0.$$

One easily recovers that

$$y_0(z) = (1 - z)^{-x}(1 + z)^{x-1}.$$

From this we infer with a bit of effort

$$q_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{-x}{k} 2^k.$$

The generating function

$$y_1(z) = \sum_{n=0}^{\infty} p_n(x) z^n$$

satisfies

$$(1 - z^2)y' - (2x - 1 + z)y = 2.$$

We also have the identity

$$\theta(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \begin{bmatrix} k \\ x \end{bmatrix}.$$

Let us define

$$\theta(n, x) = (-1)^n \sum_{k=0}^{\infty} \frac{\binom{k}{n}}{2^k} \begin{bmatrix} k \\ x \end{bmatrix}.$$

Then we have the Padé approximation property

$$p_n(x) - q_n(x)\theta(x) = -\theta(n, x) = O(1/x^n).$$

Just as in the previous sections we could apply this to p-adic irrationality proofs, but will not pursue this here. We only remark that $L_2(1, \chi_8)$ and $L_3(1, \chi_{12})$ can be proven irrational.

13 Appendix: p-Adic Hurwitz series

Let p be a prime. Let F be a positive integer and a an integer not divisible by F. Later, when we define p-adic functions, we shall assume in addition that p divides F. Define the Hurwitz zeta-function

$$H(s, a, F) = \sum_{n=0}^{\infty} \frac{1}{(a + nF)^s}.$$

This series converges for all $s \in \mathbb{C}$ with real part > 1. As is well-known $H(s, a, F)$ can be continued analytically to the entire complex s-plane, with the exception of a pole at $s = 1$.

Let n be an integer ≥ 1. To determine the value $H(1-n,a,F)$ we expand

$$\frac{te^{at}}{e^{Ft}-1} = \sum_{n\geq 0} \frac{B_n(a,F)}{n!} t^n.$$

Then we have

Proposition 13.1 *For any F, a, n we have*

$$H(1-n,a,F) = -B_n(a,F)/n.$$

We can express $B_n(a,F)$ in terms of the ordinary Bernoulli numbers B_k which are given by

$$\frac{t}{e^t-1} = \sum_{k\geq 0} \frac{B_k}{k!} t^k.$$

We get

Lemma 13.2 *For any positive integer n,*

$$B_n(a,F) = \frac{a^n}{F} \sum_{j=0}^{n} \binom{n}{j} B_j \left(\frac{F}{a}\right)^j.$$

Proof. We expand in powers of t,

$$\begin{aligned}
\frac{te^{at}}{e^{Ft}-1} &= \frac{1}{F} e^{at} \sum_{j\geq 0} \frac{B_j}{j!} (Ft)^j \\
&= \frac{1}{F} \sum_{n\geq 0} \sum_{i+j=n} \left(\frac{a^i}{i!} \frac{B_j}{j!} F^j \right) t^n \\
&= \frac{1}{F} \sum_{n\geq 0} \left(\sum_{j=0}^{n} \binom{n}{j} B_j \left(\frac{F}{a}\right)^j \right) \frac{a^n}{n!} t^n
\end{aligned}$$

The proof of our Lemma now follows by comparison of the coefficient of t^n.

$\square$

From now on we assume that F is divisible by p and a is not divisible by p. Then $|B_j(F/a)^j|_p \to 0$ as $j \to \infty$ and we can think of p-adic interpolation. We would like to interpolate the values $H(1-n,a,F)$ p-adically in n. Strictly speaking this is impossible, but we can interpolate $H(1-n,a,F)\omega(a)^{-n}$ where ω is the Teichmüller character $\mathbb{Z} \to \mathbb{Z}_p$ given as follows. When $p|m$ we define $\omega(m) = 0$. When $\gcd(p,m) = 1$ and p is odd, we define $\omega(m)^{p-1} = 1$ and $\omega(m) \equiv m (\text{mod } p)$. When $p = 2$ and m odd, we define $\omega(m) = (-1)^{(m-1)/2}$. We also define $< x >= \omega(x)^{-1}x$ for all integers x not divisible by p. Notice that $s \mapsto < x >^s$ is p-adically analytic on $\mathbb{Z}_p$.

We define the p-adic function H_p by

$$H_p(s,a,F) = \frac{1}{F(s-1)} < a >^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} B_j \left(\frac{F}{a}\right)^j$$

for all $s \in \mathbb{Z}_p$.
Note in particular the value at $s = 2$. This equals

$$H_p(2, a, F) = -\frac{\omega(a)}{F^2} \sum_{j=0}^{\infty} B_j \left(-\frac{F}{a}\right)^{j+1}.$$

The latter summation is precisely the series $R(x)$, defined in the text in which we have substituted $x = a/F$.
Finally we define the p-adic Kubota-Leopoldt L-series. Let $\phi : \mathbb{Z} \to \overline{\mathbb{Q}}$ be a periodic function with period f. Let $F = \text{lcm}(f, p)$ if p is odd and $F = \text{lcm}(f, 4)$ if $p = 2$. We now define the p-adic L-series associated to ϕ by

$$L_p(s, \phi) = \sum_{a=1, a\not\equiv 0 (\text{mod } p)}^{F} \phi(a) H_p(s, a, F).$$

We remark that the value of $L_p(s, \phi)$ remains the same if we choose instead of F a multiple period mF. To see this it suffices to show that for all integers $n \geq 0$,

$$\sum_{a=1, a\not\equiv 0 (\text{mod } p)}^{F} \phi(a) H(1 - n, a, F) = \sum_{a=1, a\not\equiv 0 (\text{mod } p)}^{mF} \phi(a) H(1 - n, a, mF).$$

This follows from Proposition 13.1 and the identity

$$\sum_{a=1, a\not\equiv 0 (\text{mod } p)}^{mF} \frac{t\phi(a)e^{at}}{e^{mF} - 1} = \sum_{a=1, a\not\equiv 0 (\text{mod } p)}^{F} \frac{t\phi(a)e^{at}}{e^{F} - 1}$$

The latter follows from the periodicity of ϕ with period F and $p|F$.
When $\phi(n) = 1$ for all n we get the p-adic zeta-function

$$\zeta_p(s) = \sum_{a=1}^{p-1} H_p(s, a, p)$$

when p is odd and when $p = 2$,

$$\zeta_2(s) = H_2(s, 1, 4) + H_2(s, 3, 4).$$

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