Solutions to home work problems for WISB324

1. Let, as usual, $\mathbb{C}G$ be the group algebra of a finite group G.

   (a) Show that for every $\mathbb{C}G$-homomorphism $\phi : \mathbb{C}G \to \mathbb{C}G$ there exists $w \in \mathbb{C}G$ such that $\phi(r) = rw$ (hint: take $w = \phi(e)$).
   Solution: Let us write $r = \sum_{g \in G} \lambda_g g$. Then, using the fact that ϕ is a linear map and a $\mathbb{C}G$-homomorphism,

   $$\phi(r) = \phi(re) = \phi\left(\left(\sum_{g\in G} \lambda_g g\right) e\right) = \left(\sum_{g \in G} \lambda_g g)\right) \phi(e) = rw.$$

   (b) Let $W \subset \mathbb{C}G$ be an irreducible $\mathbb{C}G$-submodule of $\mathbb{C}G$. Let $w \in W$ be a non-zero element. Show that $W = \{rw | r \in \mathbb{C}G\}$.
   Solution: The set $U := \{rw | r \in \mathbb{C}G\}$ is a $\mathbb{C}G$-submodule of W. It is also non-trivial since $w \in U$ and w is non-trivial. Because W is irreducible the only remaining possibility is $U = W$.

2. Define the group

   $$G = \left\langle a, b | a^5 = b^4 = e, b^{-1}ab = a^{-1} \right\rangle.$$

   (a) Show that b^2 commutes with all elements of G.
   Solution: Notice that

   $$b^{-2}ab^2 = b^{-1}(b^{-1}ab)b = b^{-1}a^{-1}b = a.$$

   Hence b^2 commutes with a. Since b^2 also commutes with b the result follows.

   (b) Determine all conjugacy classes of G.
   Solution: Two classes are given by $\{e\}$ and $\{b^2\}$. The conjugacy class of a^k is given by $\{a^k, b^{-1}a^k b\} = \{a^k, a^{-k}\}$ for all integers k. This gives us the conjugacy classes $\{a, a^4\}$ and $\{a^2, a^3\}$. Since b^2 is in the center of G we get the additional classes $\{b^2a, b^2a^4\}$ and $\{b^2a^2, b^2a^3\}$.
   Notice that $aba^{-1} = ba^{-1}a^{-1} = ba^{-2}$. By induction we get $a^k ba^{-k} = ba^{-2k}$. Since a has odd order 5 we see that the conjugation class of b is $\{ba^r | r = 0, 1, 2, 3, 4\}$. Similarly the class of b^{-1} is $\{b^{-1}a^r | r = 0, 1, 2, 3, 4\}$.

   (c) Determine all one-dimensional representations of G.
   Solution: The one-dimensional representations are obtained by assigning a complex number α to a and a complex number β to b such that $\alpha^5 = \beta^4 = 1$ and $\alpha = \alpha^{-1}$ (the group relations). We necessarily get $\alpha = 1$ and $\beta = i^k$ for some $k = 0, 1, 2, 3$.

(d) Determine the dimensions of all irreducible representations.

Solution: The order of the group is 20. Let A, B, C, D be the number of irreducible representations of dimensions 4,3,2,1 respectively. Then $20 = 4^2A + 3^2B + 2^2C + 1^2D$. Moreover, there are 8 irreducible representations, so $A+B+C+D = 8$. We have already seen that $D = 4$. The equations now reduce to $20 = 16A+9B+4C+4$ and $A+B+C+4 = 8$. One quickly sees that the only solution is $A = B = 0$ and $C = 4$.

(e) Determine all higher dimensional (i.e. dim>1) representations of G. Give the matrix images (up to conjugation) of a, b for these representations.

Solution: We must associate to a, b 2×2-matrices A, B such that $A^5 = B^4 = I_4$ and $B^{-1}AB = A^{-1}$. Let $\zeta = e^{2\pi i/5}$ and choose $A = \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix}$ (inspired by the dihedral case, e.g. Exercise 3.5) with $k = 1, 2, 3, 4$. Choose $B = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$. The condition $B^4 = I_4$ implies that $\beta^4 = 1$, hence $\beta = i^l$, $l = 0, 1, 2, 3$. We automatically get $B^{-1}AB = A^{-1}$. Hence these choices give us a representation. We need to find 4 inequivalent ones among them. The choices $k = 1, 2$ give us two different values of the trace of A. The choices $l = 0, 1$ gives us two different trace values of B^2. These choices give us the 4 representations.

3. Define the vector space

$$V = \left\{ \sum_{1\le i<j\le 4} a_{ij}x_ix_j \,\middle|\, a_{ij} \in \mathbb{C} \right\} \subset \mathbb{C}[x_1, \ldots, x_4].$$

Define the representation ρ of S_4 on V by $\sigma : x_ix_j \mapsto x_{\sigma(i)}x_{\sigma(j)}$ for all i, j.

(a) Determine the characters of ρ.

Solution: The conjugay classes of S_4 are the classes of the cycle types $(1), (12), (123), (1234)$ and $(12)(34)$. For any $\sigma \in S_4$ the trace of $\rho(\sigma)$ is the number of monomials x_ix_j that are fixed under σ. We get the table

class	(1)	(12)	(123)	(1234)	(12)(34)
χ	6	2	0	0	2

(b) Determine the irreducible representations that compose ρ.

Solution: Some linear algebra using the character table show that $\chi = \chi_{\text{triv}} + \chi_\Delta + \chi_{rmtetr}$.

(c) Determine a basis for each of the irreducible subrepresentations of ρ.

Solution: It is clear that the sum $\sum_{i<j} x_ix_j$ is a basis for the trivial representation. Using a variation of this idea we see that the polynomials $x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3$ (sums $x_ix_j + x_kx_l$ with all indices i, j, k, l distinct) are permuted by the action of S_4. So the space U_1 spanned by them is a $\mathbb{C}S_4$-submodule. The sum of the basiselements is again the trivial representation. Recall the permutation

representation of S_3 on the space spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The sum gives the trivial representation and $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3$ span the two-dimensional triangle representation. Analogously, replacing $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ by $x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3$ we see that $x_1x_2 + x_3x_4 - x_1x_3 - x_2x_4$ and $x_1x_2 + x_3x_4 - x_1x_4 - x_2x_3$ span the two-dimensional representation.

In a similar vein we see that the space generated by the polynomials $x_1x_2 - x_3x_4, x_1x_3 - x_2x_4, x_1x_4 - x_2x_3$ span a three-dimensional representation U_2. Notice also that $U_1 \oplus U_2 = V$, so U_2 must be the tetrahedral $\mathbb{C}S_4$-submodule that we found above.

4. Consider the representation ρ of S_5 on $\mathbb{C}^5$ given by

$$\sigma : \mathbf{e}_i \mapsto \mathbf{e}_{\sigma(i)}$$

for all i, where $\mathbf{e}_1, \ldots, \mathbf{e}_5$ is the standaard basis of $\mathbb{C}^5$. Show that ρ is a direct sum of the trivial representation and an irreducible one.

Solution: Clearly $\sum_{i=1}^{5} \mathbf{e}_i$ is a basis of the $\mathbb{C}S_5$-module with trivial character. Denote it by U_1. The submodule $\{x_1\mathbf{e}_1 + \cdots + x_5\mathbf{e}_5 | x_1 + \cdots + x_5 = 0\}$ is a dimension 4 $\mathbb{C}S_5$-submodule. Denote it by U_2. Then $\mathbb{C}^5 = U_1 \oplus U_2$. We must show that U_2 is irreducible. To do so we determine the character of U_2. It is the character of the permutation representation ρ minus the character of the trivial representation. The trace of $\rho(\sigma)$ equals the number of $\mathbf{e}_i$ fixed by σ. Let us put this in a table. The top row contains the cycle types. The second row the number of elements in the corresponding conjugacy class, then the character of ρ and in the last line the character of ρ minus the trivial character, which we denote by χ.

class	(1)	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
size	1	10	20	30	24	15	20
trace(ρ)	5	3	2	1	0	1	0
χ	4	2	1	0	-1	0	-1

Now compute

$$\langle \chi, \chi \rangle = \frac{1}{120} \left(1 \times 4^2 + 10 \times 2^2 + 20 \times 1^2 + 30 \times 0^2 + 24 \times (-1)^2 + 15 \times 0^2 + 20 \times (-1)^2 \right) = 1.$$

Hence χ is an irreducible character.