# A Hybrid 2D Method for Sparse Matrix Partitioning 

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## Outline

1. Introduction

- Mondriaan 2D matrix partitioning
- Fine-grain 2D partitioning

2. New: hybrid method for 2D partitioning

- The difficulty of hybrids
- Combining the Mondriaan and fine-grain methods

3. Experimental results

- PageRank matrices: Stanford, Stanford-Berkeley
- Other sparse matrices: term-by-document, linear programming, polymers

4. Conclusions and future work

## Parallel sparse matrix-vector multiplication $u$ := $A v$

$A$ sparse $m \times n$ matrix, u dense $m$-vector, $\mathbf{v}$ dense $n$-vector

$$
u_{i}:=\sum_{j=0}^{n-1} a_{i j} v_{j}
$$


u

$$
A
$$

$$
p=2
$$

4 phases: communicate, compute, communicate, compute
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## Hypergraph



Hypergraph with 9 vertices and 6 hyperedges (nets), partitioned over 2 processors

## 1D matrix partitioning using hypergraphs


nets
Column bipartitioning of $m \times n$ matrix

- Hypergraph $\mathcal{H}=(\mathcal{V}, \mathcal{N}) \Rightarrow$ exact communication volume in sparse matrix-vector multiplication.
- Columns $\equiv$ Vertices: $0,1,2,3,4,5,6$. Rows $\equiv$ Hyperedges (nets, subsets of $\mathcal{V}$ ):

$$
\begin{array}{rll}
n_{0}=\{1,4,6\}, & n_{1}=\{0,3,6\}, & n_{2}=\{4,5,6\}, \\
n_{3}=\{0,2,3\}, & n_{4}=\{2,3,5\}, & n_{5}=\{1,4,6\} .
\end{array}
$$

## Minimising communication volume



- Broken nets: $n_{1}, n_{2}$ cause one horizontal communication
- Use Kernighan-Lin/Fiduccia-Mattheyses for hypergraph bipartitioning
- Multilevel scheme: merge similar columns first, refine bipartitioning afterwards
- Used in PaToH (Çatalyürek and Aykanat 1999) for 1D matrix partitioning.
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## Mondriaan 2D matrix partitioning



- Block distribution (without row/column permutations) of $59 \times 59$ matrix impcol_b with 312 nonzeros, for $p=4$
- Mondriaan package v1.0 (May 2002). Originally developed by Vastenhouw and Bisseling for partitioning term-by-document matrices for a parallel web search machine


## Mondriaan 2D partitioning



- Recursively split the matrix into 2 parts
- Try splits in row and column directions, allowing permutations. Each time, choose the best direction

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## Fine-grain 2D partitioning

- Assign each nonzero of $A$ individually to a part.
- Each nonzero becomes a vertex; each matrix row and column a hyperedge.
- Hence $n z(A)$ vertices and $m+n$ hyperedges.
- Proposed by Çatalyürek and Aykanat, 2001.


## Matrix view of fine-grain 2D partitioning



- $m \times n$ matrix $A$ with $n z(A)$ nonzeros
- $(m+n) \times n z(A)$ matrix $F=F_{A}$ with $2 \cdot n z(A)$ nonzeros
- $a_{i j}$ is $k$ th nonzero of $A \Leftrightarrow f_{i k}, f_{m+j, k}$ are nonzero in $F$


## Communication for fine-grain 2D partitioning



- Broken net in first $m$ nets of hypergraph of $F$ : nonzeros from row $a_{i *}$ are in different parts, hence horizontal communication in $A$.
- Broken net in last $n$ nets of hypergraph of $F$ : vertical communication in $A$.


## Fine-grain 2D partitioning



- Recursively split the matrix into 2 parts
- Assign individual nonzeros to parts


## The difficulty of hybrids - a story

The beautiful American dancer Isadora Duncan (1878-1927) suggested to the Irish writer George Bernard Shaw (1856-1950) that they should have a child together:
"Think of it! With your brains and my body, what a wonder it would be."
Shaw's reply:
"Yes, but what if it had my body and your brains?"

Source:
http://www.chiasmus.com/mastersofchiasmus/shaw.shtml Many different versions exist. Story may be apocryphal.

## Hybrid 2D partitioning



- Recursively split the matrix into 2 parts
- Try splits in row and column directions, and fine-grain
- Each time, choose the best of 3

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## Recursive, adaptive bipartitioning algorithm

MatrixPartition $(A, p, \epsilon)$
input: $\quad \epsilon=$ allowed load imbalance, $\epsilon>0$. output: $p$-way partitioning of $A$ with imbalance $\leq \epsilon$. if $p>1$ then

$$
\begin{aligned}
& q:=\log _{2} p ; \\
& \left(A_{0}^{\mathrm{r}}, A_{1}^{\mathrm{r}}\right):=h(A, \text { row }, \epsilon / q) ; \text { hypergraph splitting } \\
& \left(A_{0}^{\mathrm{c}}, A_{1 \mathrm{p}}^{\mathrm{p}}:=h(A, \text { col, } \epsilon / q) ;\right. \\
& \left(A_{0}^{\mathrm{f}}, A_{1}^{\mathrm{f}}:=h(A, \text { fine }, \epsilon / q) ;\right. \\
& \left(A_{0}, A_{1}\right):=\text { best of }\left(A_{0}^{\mathrm{r}}, A_{1}^{\mathrm{r}}\right),\left(A_{0}^{\mathrm{c}}, A_{1}^{\mathrm{c}}\right),\left(A_{0}^{\mathrm{f}}, A_{1}^{\mathrm{f}}\right) ;
\end{aligned}
$$

$$
\operatorname{maxn} z:=\frac{n z(A)}{p_{n}}(1+\epsilon) ;
$$

$$
\epsilon_{0}:=\frac{\operatorname{maxnz}}{n z\left(A_{0}\right)} \cdot \frac{p}{2}-1 \text {; MatrixPartition }\left(A_{0}, p / 2, \epsilon_{0}\right) \text {; }
$$

$$
\epsilon_{1}:=\frac{\operatorname{maxazn}}{n z\left(A_{1}\right)} \cdot \frac{p}{2}-1 \text {; MatrixPartition }\left(A_{1}, p / 2, \epsilon_{1}\right) \text {; }
$$

else output $A$;

## Similarity metric for column merging (coarsening)

Column-scaled inner product:

$$
M(u, v)=\frac{1}{\omega_{u v}} \sum_{i=0}^{m-1} u_{i} v_{i}
$$

- $\omega_{u v}=1$ measures overlap
- $\omega_{u v}=\sqrt{d_{u} d_{v}}$ measures cosine of angle
- $\omega_{u v}=\min \left\{d_{u}, d_{v}\right\}$ measures relative overlap

- $\omega_{u v}=\max \left\{d_{u}, d_{v}\right\}$

Here, $d_{u}$ is the number of nonzeros of column $u$.

## Speeding up the fine-grain method



- ip = standard inner product matching

| $\square$ | coarsening |
| :--- | :--- |
| $\square$ | init. part. |
| $\square$ | refimement |
| rest |  |

- ip1 = inner product matching using an upper bound on the overlap, e.g. $d_{u}$ to stop searching early. For fine-grain method, bound is sharper: 1 at first level.
- ip2 = alternate between matching with overlap in top and bottom rows.
- rnd = choose a random match with overlap $\geq 1$

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# Web searching: which page ranks first? 

c +
(3) http://www.google.com/search?client=safari\&rls=nl-ni\&q=san+francisco\&ie
san francisco - Google zoeken
[7 Apple (118) *
Amazon eBay Yahoo! Nieuws (312) T
Q. san francisco

CoOOR
Het Internet Afbeeldingen Discussiegroepen Gids NieuwsNieuw! meern
san francisco
OHet web doorzoeken $\bigcirc$ Zoeken in pagina's in het Nederlands

## Het Internet

Resultaten 1-10 van circa $\mathbf{3 2 6 . 0 0 0} \mathbf{0 0 0}$ voor san francisco ( $\mathbf{0 , 1 2}$ seconden)

San Francisco
www.hotels.com
Wij garanderen de laagste prijzen bij 120 hotels in San Francisco!

## Naar San Francisco?

www.ingcard.nl Een credit card is dan makkelijk 1 e jaar gratis +mp 3 speler!
Tip: Alleen in het Nederiands zoeken. U kunt uw zoektaal bepalen in Voorkeuren
Only in San Francisco - Official travel guide to hotels ...
A comprehensive source for visitor info with features for leisure and business travelers, convention planners, travel trade, travel media and Bureau ...
www.sfvisitor.org/ - 33k-15 feb 2006 - In cache - Gelijkwaardige pagina's
SF Gate: News and Information for the San Francisco Bay Area SFGate: The Bay Area's Home Page - online home of the San Francisco Chronicle, and much more.
www.sfgate.com/ - 83k - 15 feb 2006 - In cache - Geliikwaardige pagina's

## SF Gate: San Francisco Chronicle

News, sports, entertainment and business articles provided by online newspaper edition.

$$
\text { www.sfgate.com/chronicle/ - } 75 \mathrm{k} \text { - } 15 \text { feb } 2006 \text { - In cache - Gelijkwaardige pagina's }
$$

## :: San Francisco Hotels : San Francisco Real Estate : Tours ...

Guide to San Francisco travel, hotels, tourism, real estate, theater, arts and entertainment, restaurants, music, people.
www.sanfrancisco.com/-92k-15 feb 2006 - In cache - Gelijkwaardige pagina's

## Gesponsorde Koppelingen

San Francisco Hotel Deal
Official Site - CP Union Square in San Francisco. Book Online and Save www.CPUnionSquare.com
49 San Fran Attractions.
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## San francisco

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Local San Fran Guide
Hotels, Restaurants, Tours \& Shows Deals and Discounts - Book on-Line www.san-francisco.cc

## San Francisco

Welcome to
San Francisco!
flybirdy.com
San Francisco

## The link matrix A

- Given $n$ web pages with links between them. We can define the sparse $n \times n$ link matrix $A$ by

$$
a_{i j}= \begin{cases}1 & \text { if there is a link from page } j \text { to page } i \\ 0 & \text { otherwise }\end{cases}
$$

- Let $\mathbf{e}=(1,1, \ldots, 1)^{T}$, representing an initial uniform importance (rank) of all web pages. Then

$$
(A \mathbf{e})_{i}=\sum_{j} a_{i j} e_{j}=\sum_{j} a_{i j}
$$

is the total number of links pointing to page $i$.

- The vector $A$ e represents the importance of the pages; $A^{2} \mathrm{e}$ takes the importance of the pointing pages into account as well; and so on.
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## The Google matrix

- A web surfer chooses each of the outgoing $N_{j}$ links from page $j$ with equal probability. Define the $n \times n$ diagonal matrix $D$ with $d_{j j}=1 / N_{j}$.
- Let $\alpha$ be the probability that a surfer follows an outlink of the current page. Typically $\alpha=0.85$. The surfer jumps to a random page with probability $1-\alpha$.
- The Google matrix is defined by (Brin and Page 1998)

$$
G=\alpha A D+(1-\alpha) \mathbf{e e}^{T} / n .
$$

- The PageRank of a set of web pages is obtained by repeated multiplication by $G$, involving sparse matrix-vector multiplication by $A$, and some vector operations.


## Comparing 1D, 2D fine-grain, and 2D Mondriaan

- The following 1D and 2D fine-grain communication volumes for PageRank matrices are published results from the parallel program Parkway v2.1 (Bradley, de Jager, Knottenbelt, Trifunović 2005).
- The 2D Mondriaan volumes are results with all our improvements (to be incorporated in v2.0), but using only row/column partitioning, not the fine-grain option.


## Communication volume: PageRank matrix stanford



- $n=281,903$ (pages), $n z(A)=2,594,228$ nonzeros (links).
- Represents the Stanford WWW subdomain, obtained by a web crawl in September 2002 by Sep Kamvar.
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## Communication volume: Stanford_Berkeley



- $n=683,446, n z(A)=8,262,087$ nonzeros.
- Represents the Stanford and Berkeley subdomains, obtained by a web crawl in Dec. 2002 by Sep Kamvar.
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## Meaning of results

- Both 2D methods save an order of magnitude in communication volume compared to 1D.
- Parkway fine-grain is slightly better than Mondriaan, in terms of partitioning quality. This may be due to a better implementation, or due to the fine-grain method itself. Further investigation is needed.
- 2D Mondriaan is much faster than fine-grain, since the hypergraphs involved are much smaller:
$7 \times 10^{5}$ vs. $8 \times 10^{6}$ vertices for Stanford_Berkeley.


## Transition matrix cage 6 of Markov model



- Reduced transition matrix cage 6 with $n=93$, $n z(A)=785$ for polymer length $L=6$.
- Larger matrix cage10 is included in our test set of 18 matrices representing various applications: 3 linear programming matrices, 2 information retrieval, 2 chemical engineering, 2 circuit simulation, 1 polymer simulation, ...


## Average communication volume for 3 methods



- Test set of 18 matrices (smaller than PageRank matrices).
- Volume relative to original Mondriaan program, v1.02
- Implementation: Mondriaan's own hypergraph partitioner
- Fine-grained method has more freedom to find a good partitioning, but shows no gains on average

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## Average communication volume for 3 methods



- Test set of 18 matrices.
- Volume relative to original Mondriaan program, v1.02
- Implementation: PaToH hypergraph partitioner. Highly optimised, and it shows.
- Hybrid method shows a little gain over 2D Mondriaan

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## Conclusions and ...

- We have presented a new hybrid method which combines two different 2D matrix partitioning methods: Mondriaan and fine-grain. The hybrid improves upon both.
- With a highly optimised hypergraph partitioner such as PaToH as the partitioning engine, the Mondriaan 2D method achieves almost the same quality as the hybrid method, but much faster.
- PageRank is a wonderful non-PDE application:
- it affects our lives daily
- it has embedded mathematical high technology
- it uses the power method; only mathematicians and computer scientists know what this really means!
- it exposes the power of 2D matrix partitioning methods
- We keep on improving the Mondriaan and PaToH hypergraph partitioners.
- New release of Mondriaan, v2.0, will incorporate all improvements.
- Mondriaan and PaToH are sequential.
- Soon, the parallel hypergraph partitioner Zoltan will be released by Sandia National Laboratories (Devine, Boman, Heaphy, Bisseling, Çatalyürek 2006), with many features from Mondriaan and PaToH, and a lot more.
- First parallel partitioner Parkway 2.1 (Knottenbelt, Trifunović 2005) is also publicly available.
- Partition PageRank in paralle!!

