Departement Wiskunde, Faculteit Bètawetenschappen, UU.
In elektronische vorm beschikbaar gemaakt door de $\mathcal{T}_{\!\mathcal{B}}\mathcal{C}$ van A–Eskwadraat.
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Eerste deeltentamen Statistiek (WISB361)
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Please write your name and student number on each page you turn in. You may use all your lecture notes, the course literature, the course handouts and a simple calculator. Each subproblem is worth 10 points. You can use the following identity: $\int_0^\infty u^{m-1}e^{-\lambda u}du = \frac{(m-1)!}{\lambda^m}$ for all $m \in \mathbb{N}$, $\lambda > 0$.

1. Let $X_1, \ldots, X_n \sim \text{Poisson}(\sqrt{\theta})$ (that is $P(X_1 = k) = \frac{\theta^{k/2}e^{-\sqrt{\theta}}}{k!}$, $k = 0, 1, \ldots$), with unknown $\theta > 0$.

   (i) Determine the moment estimator T_1 and the MLE T_2 for $\sqrt{\theta}$. Are they unbiased? Compute the $\text{MSE}(T_2, \sqrt{\theta}) = E_\theta(T_2 - \sqrt{\theta})^2$.

   (ii) Determine a sufficient and complete statistic in this model. Determine UMVU estimators for $\sqrt{\theta}$ and θ.

   (iii) Compute the Fisher information about θ in one observation and determine the Cramér-Rao lower bound for the estimation of $\sqrt{\theta}$. Is this lower bound sharp?

   (iv) Describe the posterior distribution and the corresponding Bayes estimator of θ with respect to the prior distribution $\pi(\theta)$: $\pi(\theta = 2) = \pi(\theta = 10) = \frac{1}{2}$.

2. Let $X_1, \ldots, X_n$ be a sample from a distribution with the density

$$f_\theta(x) = \frac{\theta 2^\theta}{x^{\theta+1}} I\{x \geq 2\},$$

   with unknown $\theta \geq 3$.

   (i) Find the moment estimator T_1 for $\frac{1}{\theta}$ and the MLE T_2 for $\frac{1}{\theta}$. Is T_2 unbiased?

   (ii) Compute $\text{MSE}(T_2) = E_\theta(T_2 - \frac{1}{\theta})^2$ and determine the MLE for $g(\theta) = P_\theta(1 \leq X_1 \leq 3)$.

   (iii) Determine the Fisher information about parameter θ and the Cramér-Rao lower bound for the estimation of $\frac{1}{\theta}$. Is this lower bound sharp?

   (iv) Find a sufficient and complete statistic. Determine an UMVU estimator for $42 + \frac{2}{\theta}$.

   (v) Determine the moment estimator T_3 for $\frac{\theta}{\theta-1}$ and check whether the Cramér-Rao lower bound for the estimation of $\frac{\theta}{\theta-1}$ is attained by T_3.

   (vi) Determine the posterior distribution with respect to the prior probability density $\pi(\theta) = 4\theta e^{-2\theta} I\{\theta > 0\}$ and the corresponding Bayes estimator of $1/\theta$.