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Harmonic lifts of modular forms

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Abstract It is shown that each complex conjugate of a meromorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$ of any complex weight p occurs as the image of a harmonic modular form under the operator $2iy^p\partial_{\bar z}$. These harmonic lifts occur in holomorphic families with the weight as the parameter.

Keywords Modular form · Harmonic lift

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1 Introduction

In the theory of mock modular forms, see Sect. 3 of [3] and also Sect. 5 of [13], one meets the exact sequence

$$0 \longrightarrow M_p^! \longrightarrow H_p^! \xrightarrow{\xi_p} \bar{M}_{2-p}^!, \tag{1.1}$$

and gives conditions under which the last map is surjective. Here $M_p^!$ denotes the space of holomorphic modular forms of weight p with at most exponential growth at the cusps (also called the space of weakly holomorphic modular forms), and $H_p^!$ denotes the corresponding space of p-harmonic modular forms, defined by replacing the condition of holomorphy by the condition of p-harmonicity, which means being in the kernel of the operator

$$\Delta_p = -4\,(\mathrm{Im}\,z)^2\,\partial_z\partial_{\bar z} + 2ip\,(\mathrm{Im}\,z)\,\partial_{\bar z}. \tag{1.2}$$

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The operator $\xi_p = 2i\,(\operatorname{Im} z)^p\,\partial_{\bar z}$ maps $H^!_p$ into the space $\bar M^!_{2-p}$ of antiholomorphic modular forms of weight $2-p$ with at most exponential growth at the cusps. (Actually, the operator ξ_p in [3] also conjugates the function, and ends up in a space of holomorphic modular forms.) The elements of $\bar M^!_{2-p}$ are complex conjugates of elements of an appropriate space $M^!_{2-p}$. (In Sect. 2 we will give a more precise discussion of these spaces of modular forms.)

A *p-harmonic lift* of an element F in $\bar M^!_{2-p}$ is an element H of $H^!_p$ such that $\xi_p H = F$. The concept stems from the study of mock modular forms. Zwegers started in [14] with mock theta functions M, which are holomorphic functions on the upper half-plane given by a q-series, and added a simpler but non-holomorphic function C to it such that $M + C$ has modular transformation behavior. The function $M + C$ is no longer holomorphic, but p-harmonic for some weight p. Applying the operator ξ_p to C, or to $M + C$, gives an antiholomorphic cusp form of weight $2-p$, from which C can be reconstructed. Conversely, we may ask for a given antiholomorphic automorphic form F of weight $2-p$ whether it occurs as the image under ξ_p of a p-harmonic form H.

Poincaré series form a convenient tool to construct harmonic lifts. See Theorem 1.1 in the paper [1] of Bringmann and Ono, or Sect. 6 in [4] by Bruinier, Ono and Rhoades. If the parameters of the Poincaré series are in the domain of absolute convergence this gives a description of harmonic lifts by absolutely convergent series. For other values of the parameters one has to use the meromorphic continuation of the Poincaré series. An alternative approach is the use of Hodge theory. See Corollary 3.8 in [3] of Bruinier and Funke. The method of holomorphic projection can be used to construct harmonic lifts. See Sects. 3 and 5 in the preprint [2] of Bringmann, Kane and Zwegers.

My purpose in this paper is to show that the approach with Poincaré series can be modified to work for arbitrary complex weights. I will use results from perturbation theory of automorphic forms as investigated in [5]. To avoid complications I consider only the full modular group.

Theorem 1.1 *Let F be an antiholomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $2-p \in \mathbb{C}$ with multiplier system v on $\mathrm{SL}_2(\mathbb{Z})$ suitable for the weight p, and assume that F has at most exponential growth at the cusps. Then there exists a p-harmonic modular form H on $\mathrm{SL}_2(\mathbb{Z})$ of weight p with the same multiplier system v and at most exponential growth at the cusps, such that $\xi_p H = F$.*

This is a mere existence result. The construction of H is based on the resolvents of self-adjoint families of operators in Hilbert spaces, and does not give the p-harmonic lift H explicitly.

Let us denote by, respectively, $M^!_p(v)$, $\bar M^!_p(v)$ and $H^!_p(v)$ the spaces of respectively holomorphic, antiholomorphic and harmonic modular forms, with at most exponential growth, weight p, and multiplier system v.

Holomorphic and antiholomorphic modular forms occur in families, for instance the powers of the Dedekind eta-function $r \mapsto \eta^{2r}$ form a family holomorphic in the weight $r \in \mathbb{C}$, with a multiplier system that we denote by v_r. We have $\eta^{2r} \in M^!_r(v_r)$, and $\bar\eta^{2r} \in \bar M^!_r(v_{-r})$. All antiholomorphic modular forms with at most exponential growth are of the form $F\,\bar\eta^{-2r}$, where $r \in \mathbb{C}$, and $F \in \bar{M}^{!}_{2-\ell}(1)$ for some $\ell \in 2\mathbb{Z}$. Such a family $r \mapsto F\,\bar\eta^{-2r}$ is a holomorphic family on $\mathbb{C}$. It turns out that harmonic lifts also occur in families, which, however, are not defined on all of $\mathbb{C}$, due to a branching phenomenon. We work with domains of the form

$$U_M = \mathbb{C} \setminus [12M, \infty) \tag{1.3}$$

with $M \in \mathbb{Z}$.

Theorem 1.2 *Let $F \in \bar{M}^{!}_{2-\ell}(1)$ with $\ell \in 2\mathbb{Z}$. There is $\mu_F \in \mathbb{Z}$ such that for all integers $M \geq \mu_F$ there are holomorphic families $r \mapsto \mathsf{H}_{M,r}$ on U_M for which $\mathsf{H}_{M,r} \in H^{!}_{\ell+r}(v_r)$ and $\xi_{\ell+r}\mathsf{H}_{M,r} = F\,\bar\eta^{-2r}$ for all $r \in U_M$.*

This result implies Theorem 1.1.

Meromorphic modular forms may have singularities in points of the upper half-plane $\mathfrak{H}$. The space $M^{!!}_p(v_r)$ of meromorphic modular forms of weight p with the multiplier system v_r is contained in the space $H^{!!}_p(v_r)$ of harmonic functions F on $\mathfrak{H} \setminus S$ that are invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ of weight p with the multiplier system v_r, where $S \subset \mathfrak{H}$ consists of finitely many Γ-orbits and where F satisfies near each $\zeta \in S$ an estimate $F(z) = \mathrm{O}((\frac{z-\zeta}{z-\bar\zeta})^{-a})$ as $z \to \zeta$ for some $a > 0$. The space $\bar{M}^{!!}_p(v_r)$ consists of the complex conjugates of the functions in $M^{!!}_{\bar p}(v_{-\bar r})$.

Theorem 1.3 *Let $p, r \in \mathbb{C}$ with $p \equiv r$ mod 2. For each $F \in \bar{M}^{!!}_{2-p}(v_r)$ there exists a harmonic lift $H \in H^{!!}_p(v_r)$ such that $\xi_p H = F$.*

This lifting can also be done in holomorphic families. Theorems 1.2 and 1.3 follow from the more general Theorem 4.5 in Sect. 4.4.

To obtain these results we start in Sect. 2 with a more precise discussion of the spaces of holomorphic, antiholomorphic and harmonic modular forms. Section 3 reformulates the equation $\xi_{\ell+r}H = \bar\eta^{-2r}F$ in terms of the more general class of real-analytic modular forms. In this way we can embed the family $r \mapsto \bar\eta^{-2r}F$ in a family with two parameters, the weight and a "spectral parameter". This makes it possible to use analytic perturbation theory to arrive at meromorphic families $r \mapsto H_{N,r}$ of modular solutions of the equation $\xi_{\ell+r}H_{N,r} = \bar\eta^{-2r}F$. Section 4 removes the singularities of these families, and leads to Theorem 4.5, from which Theorems 1.1–1.3 follow.

Section 4.3 gives a normalization that determines the families of harmonic lifts uniquely. That does not mean that we obtain them explicitly. The theorems in this paper are existence results only. It is far from obvious how to write $h_{N,r}$ as the sum of a "mock modular form" and a "harmonic correction", especially if $h_{N,r}$ has singularities in the upper half-plane. See Sect. 4.6.

Section 4.5.4 discusses the possibilities and difficulties of extension to other discrete groups. Finally, Sect. 5 discusses, as an example, a lift of $r \mapsto \bar\eta^{-2r}$, and states an explicit formula for the first derivative of this lift at $r = 0$.

2 Modular forms

This section serves to define the concepts more precisely than in the introduction. The discrete group is $\varGamma := \mathrm{SL}_2(\mathbb{Z})$.

2.1 Holomorphic modular forms

The Dedekind eta-function

$$\eta(z) = e^{\pi i z/12} \prod_{n\geq 1} \bigl(1 - e^{2\pi i n z}\bigr)$$

has no zeros in the upper half-plane $\mathfrak{H} = \{z = x + iy \in \mathbb{C} \;:\; y > 0\}$. One chooses a branch of its logarithm

$$\log \eta(z) = \frac{\pi i z}{12} - \sum_{n\geq 1} \sigma_{-1}(n)\, q^n, \tag{2.1}$$

with $q = e^{2\pi i z}$ and $\sigma_u(n) = \sum_{d|n} d^u$, and then defines $\eta^{2r}(z) = e^{2r \log \eta(z)}$. The transformation behavior of $\log \eta$ is studied by R. Dedekind in the appendix [8] to the collected works of B. Riemann. One may also consult Chap. IX in [11]. This leads to the modular transformation behavior

$$\eta^{2r}(\gamma z) = v_r(\gamma)\,(cz+d)^r\,\eta(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \varGamma, \tag{2.2}$$

with the *multiplier system* v_r. A multiplier system suitable for the weight $p \in \mathbb{C}$ is a map $v : \varGamma \to \mathbb{C}^*$ such that

$$\left(F|_{v,r} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(z) = v \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} (cz+d)^{-p}\, F\left(\frac{az+b}{cz+d} \right) \tag{2.3}$$

defines a representation of $\varGamma/\{\pm I\} = \mathrm{PSL}_2(\mathbb{Z})$ in the functions on $\mathfrak{H}$. We use the convention of computing complex powers of $cz+d$ with $\arg(cz+d) \in (-\pi, \pi]$.

For the modular group all multiplier systems occur in one family $r \mapsto v_r$ with parameter $r \in \mathbb{C} \bmod 12\mathbb{Z}$. The multiplier system v_r is suitable for weights $p \equiv r \bmod 2$. It is determined on the two standard generators of $\mathrm{SL}_2(\mathbb{Z})$ by

$$v_r \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = e^{\pi i r/12}, \qquad v_r \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = e^{-\pi i r/2}. \tag{2.4}$$

Definition 2.1 Let $p, r \in \mathbb{C}$, $p \equiv r \bmod 2$. The space $M^!_p(r) = M^!_p(\varGamma, v_r)$ consists of the holomorphic functions F on $\mathfrak{H}$ that satisfy $F|_{v_r,p}\gamma = F$ for all $\gamma \in \varGamma$ and

$$F(z) = \mathrm{O}\bigl(e^{ay}\bigr) \quad \text{as } y \to \infty \text{ for some } A > 0, \tag{2.5}$$

uniformly for x in compact sets. By $M^{!!}_p(r)$ we denote the space of meromorphic modular forms of weight p with multiplier system v_r.

Here and in the sequel we use the standard convention $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$ for $z \in \mathfrak{H}$. If p and r are not real we cannot impose in (2.5) uniformity in $x \in \mathbb{R}$. The condition (2.5) is the condition of *exponential growth*.

Since η has no zeros in $\mathfrak{H}$, multiplication by η^{2r_1} gives a bijection between $M^!_p(r)$ and $M^!_{p+r_1}(r+r_1)$. This implies that all spaces in Definition 2.1 can be uniquely described as $M^!_{\ell+r}(r)$ with $r \in \mathbb{C}$ and $\ell \in L := \{0, 4, 6, 8, 10, 14\}$. The general form of an element of $M^!_{\ell+r}(r)$ is $p(J)\, E_\ell\, \eta^{2r}$, where $p(J)$ is a polynomial in the elliptic invariant $J \in M^!_0(0)$, and E_ℓ is the holomorphic Eisenstein series in weight $\ell \in L \setminus \{0\}$, and where we put $E_0 = 1$. The general form of an element of $M^{!!}_{\ell+r}(r)$ is also $p(J)\, E_\ell\, \eta^{2r}$, where now $p(J)$ is a rational function in J. (See, e.g., Sect. 4.1 of [12].)

We can formulate the meromorphy of F at $\zeta \in \mathfrak{H}$ by holomorphy in z on a pointed neighborhood of ζ in $\mathfrak{H}$ and the growth condition

$$F(z) = \mathrm{O}\big(\big((z-\zeta)/(z-\bar{\zeta})\big)^{-a}\big) \quad \text{as } z \to \zeta, \text{ for some } a > 0. \tag{2.6}$$

2.2 Harmonic modular forms and antiholomorphic modular forms

Definition 2.2 We say that a function F is *p-harmonic* on some subset of $\mathfrak{H}$ if $\Delta_p F = 0$ on that subset, where Δ_p is the operator given in (1.2).

The holomorphic action $|_{v_r,p}$ of Γ in (2.3) preserves p-harmonicity and commutes with Δ_p. Holomorphic functions are p-harmonic for each $p \in \mathbb{C}$.

Antiholomorphy is not preserved by the action $|_{v_r,q}$ of Γ, but by the following action:

Definition 2.3 For $p \equiv -r \bmod 2$ the antiholomorphic action $|^a_{v_r,p}$ of Γ in the functions on $\mathfrak{H}$ is given by

$$\left(F|^a_{v_r,p}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = v_r\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} (c\bar{z}+d)^{-p}\, F\left(\frac{az+b}{cz+d}\right) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{2.7}$$

where powers of $c\bar{z}+d$ are computed with $-\pi \le \arg(c\bar{z}+d) < \pi$.

The operator

$$\xi_p = 2iy^p\, \partial_{\bar{z}} \tag{2.8}$$

vanishes precisely on the holomorphic functions, and a function F is p-harmonic if and only if $\xi_p F$ is antiholomorphic. This operator ξ_p intertwines holomorphic and antiholomorphic actions:

$$\xi_p(F|_{v_r,p}\gamma) = (\xi_p F)|^a_{v_r,2-p}\gamma. \tag{2.9}$$

Let us define $M^S_p(r)$ as the space of $F \in M^{!!}_p(r)$ with singularities contained in the set S. Then $M^!_p(r) = M^{\emptyset}_p(r)$ and $M^{!!}_p(r) = \bigcup_S M^S_p(r)$ where S runs over the collection of unions of finitely many Γ-orbits in $\mathfrak{H}$. This suggests the following definition.

Definition 2.4 Let $S \subset \mathfrak{H}$ consist of finitely many (possibly zero) Γ-orbits in $\mathfrak{H}$. Let $p \equiv r \bmod 2$. We define $H_p^S(r)$ as the space of r-harmonic functions on $\mathfrak{H} \setminus S$ that are invariant under the action $|_{p,r}$ of Γ, satisfy the condition (2.5) of exponential growth at the cusp and the growth condition (2.6) at the points $\zeta \in S$.

The space $\bar{M}_{-p}^S(r)$ consists of the antiholomorphic functions on $\mathfrak{H} \setminus S$ that are invariant under the action $|_{-p,v_r}^a$ of Γ and satisfy (2.5), and condition (2.6) for all $\zeta \in S$.

We put $H_p^!(r) = H_p^{\emptyset}(r)$, $\bar{M}_{-p}^!(r) = \bar{M}_{-p}^{\emptyset}(r)$, $H_p^{!!}(r) = \bigcup_S H_p^S(r)$, and $\bar{M}_{-p}^{!!}(r) = \bigcup_S \bar{M}_{-p}^S(r)$, where S runs over the collection of unions of finitely many Γ-orbits in $\mathfrak{H}$.

	$M_p^S(r)$	$H_p^S(r)$	$\bar{M}_{-p}^S(r)$
Growth at ∞	Condition (2.5) is satisfied		
Near $\zeta \in S$	Condition (2.6) is satisfied		
For all $\gamma \in \Gamma$	Invariant under $\|_{v_r,p}\gamma$		Invariant under $\|_{v_r,-p}\gamma$
On $\mathfrak{H} \setminus S$	Holomorphic	p-harmonic	Antiholomorphic

For each S consisting of finitely many Γ-orbits in $\mathfrak{H}$ we have an exact sequence

$$0 \to M_p^S(r) \to H_p^S(r) \xrightarrow{\xi_p} \bar{M}_{2-p}^S(r). \tag{2.10}$$

This is not immediately clear. The question is whether the operator ξ_p preserves the growth conditions (2.5) and (2.6). This can be shown by looking at the growth of the terms in the expansions in the next subsection, and the effect of the operator ξ_p. It is a special case of an analogous result for Maass forms, which we will mention in the next section. Near the end of Sect. 3.2 we will derive the statement from Proposition 4.5.3 in [5].

The central question in this paper is whether $\xi_p H_p^S(r) \to \bar{M}_{2-p}^S(r)$ is surjective.

2.3 Expansions

We fix a union S of finitely many Γ-orbits in $\mathfrak{H}$. Let $\mathcal{P}$ be the set consisting of ∞ and of representatives of the Γ-orbits in S, for instance representatives in the standard fundamental domain.

A meromorphic modular form $F \in M_p^S(r)$ has a Fourier expansion at ∞ of the form

$$F(z) = \sum_{\nu \geq \mu} a_\mu\, q^{\nu + r/12}, \tag{2.11}$$

with $q^\alpha = e^{2\pi i \alpha z}$. The integer μ may be negative. If F has singularities at points of S then this expansion converges only on a region $y > A$ not intersecting S.

Near each $\zeta \in \mathcal{P} \cap \mathfrak{H}$ the function has an expansion of the form

$$F(z) = (z - \bar{\zeta})^{-p} \sum_{\nu \geq \mu} a_\nu\, w^\nu, \tag{2.12}$$

with $w = \frac{z-\zeta}{z-\bar{\zeta}}$. If F has a singularity at ζ then $\mu < 0$. If $\zeta \in \Gamma i$ then $a_\nu = 0$ if $\nu \not\equiv \frac{p-r}{2} \bmod 2$, and for $\zeta \in \Gamma e^{\pi i/3}$ there is a similar condition modulo 3. The expansion at ζ will represent F only on some open hyperbolic disk around ζ that does not contain other singularities.

An antiholomorphic modular form $F \in \bar{M}^S_{-p}(r)$ has similar expansions:

$$\begin{aligned} \text{near } \infty\text{:} \quad & F(z) = \sum_{\nu \geq \mu} a_\nu \bar{q}^{\nu + r/12}, \\ \text{near } \zeta\text{:} \quad & F(z) = (\bar{z} - \zeta)^p \sum_{\nu \geq \mu} a_\nu \, \bar{w}^\nu. \end{aligned} \tag{2.13}$$

A p-harmonic modular form $F \in H^S_p(r)$ has also expansions at points of $\mathcal{P}$, the terms of which inherit the p-parabolicity:

$$\begin{aligned} \text{near } \infty\text{:} \quad & F(z) = \sum_{\nu \in \mathbb{Z}} f_{\infty,\nu}(y) \, e^{2\pi i (\nu + r/12) x}, \\ \text{near } \zeta\text{:} \quad & F(z) = (z - \bar{\zeta})^{-p} \sum_{\nu \in \mathbb{Z}} f_{\zeta,\nu}\big(|w|\big) \left(\frac{w}{|w|} \right)^\nu. \end{aligned}$$

The harmonicity induces second order differential equations for the coefficients, which then are elements of a two-dimensional space, with a one-dimensional subspace corresponding to holomorphic terms. We note that the operator ξ_p sends the term with $(w/|w|)^\nu$ to the term with $(w/|w|)^{\nu+1} = \bar{w}^{-\nu-1} \, |w|^{\nu+1}$ in the expansion (2.13), with p replaced by $p - 2$.

3 Real-analytic modular forms

The task to find a p-harmonic modular form H such that $\xi_p H = F$ for a given antiholomorphic modular form becomes easier if we embed F in a family of modular forms of a more general type. For this purpose one may use Poincaré series. Here we modify that approach in such a way that it works for complex weights.

We recall the definition of Maass forms, which are real-analytic modular forms that satisfy more general conditions than just (anti)holomorphy or harmonicity. The surjectivity of $\xi_p : H^S_p(r) \to \bar{M}^S_{2-p}(r)$ can be reformulated in terms of Maass forms. To this reformulated problem we will apply results in [5] that lead to meromorphic families of lifts.

3.1 Maass forms

We define a third action of $\Gamma/\{\pm I\} = \mathrm{PSL}_2(\mathbb{Z})$ on the functions on $\mathfrak{H}$:

Definition 3.1 For $p, r \in \mathbb{C}$, $p \equiv r \bmod 2$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$:

$$\big(f|^{\mathrm{an}}_{v_r,p}\gamma\big)(z) = v_r(\gamma)^{-1} \, e^{-ip \arg(cz+d)} \, f(\gamma z).$$

As before, $\arg(cz + d) \in (-\pi, \pi]$.

This action is intermediate between the actions $|_{v_r,p}$ and $|^a_{v_r,p}$, and does not favor either holomorphy or antiholomorphy. The relation between these actions is given by the following simple multiplication operators:

$$\bigl(R^h_p F\bigr)(z) = y^{p/2}\, F(z), \qquad \bigl(R^a_p F\bigr)(z) = y^{-p/2}\, F(z). \tag{3.1}$$

The operator R^h_p intertwines the holomorphic action and the analytic action:

$$\bigl(R^h_p F\bigr)\big|^{\mathrm{an}}_{v_r,p}\gamma = R^h_p(F|_{v_r,p}\gamma) \quad (\gamma \in \Gamma), \tag{3.2}$$

and R^a_p intertwines the antiholomorphic and the analytic action:

$$\bigl(R^a_p F\bigr)\big|^{\mathrm{an}}_{v_r,p}\gamma = R^a_p\bigl(F\big|^a_{v_r,-p}\gamma\bigr) \quad (\gamma \in \Gamma). \tag{3.3}$$

The action $|^{\mathrm{an}}_{v_r,p}$ of Γ commutes with the *Casimir operator in weight* p:

$$\omega_p = -y^2\partial_y^2 - y^2\partial_x^2 + ipy\partial_x. \tag{3.4}$$

We define Maass forms with singularities in a fixed set S, which is a union of finitely many Γ-orbits in $\mathfrak{H}$. We choose a system of representatives $\mathcal{P}_Y$ of $\Gamma\backslash S$.

Definition 3.2 Let $p, r \in \mathbb{C}$ with $p \equiv r \bmod 2$. A *modular Maass form* of weight p for the multiplier system v_r with *spectral parameter* s is a twice differentiable function on $\mathfrak{H} \setminus S$, such that

(1) $f|^{\mathrm{an}}_{v_r,p}\gamma = f$ for all $\gamma \in \Gamma$,
(2) $\omega_p f = (\frac{1}{4} - s^2)\, f$.

By $\mathfrak{M}^S_p(r, s) = \mathfrak{M}^S_p(r, -s)$ we denote the space of such Maass forms.

This is a very large space. The definition does not impose growth conditions. The definition is invariant under $s \mapsto -s$, and we could work with the *eigenvalue* $\frac{1}{4} - s^2$. However, in practice the spectral parameter s is more convenient. As parametrizations both $s \mapsto \frac{1}{4} - s^2$ and $s \mapsto s(1-s)$ are in use. Here I choose $s \mapsto \frac{1}{4} - s^2$ for easy reference to [5]. In [5] Maass forms are considered as functions on the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. Here we stay on the upper half-plane, and mention only that to $f \in \mathfrak{M}^S_p(r, s)$ corresponds the function $p(z)k(\vartheta) \mapsto f(z)\, e^{ip\vartheta}$ in the notations of Sect. 2.2 of [5].

The operators

$$\mathbf{E}^+_p = 2iy\,\partial_x + 2y\,\partial_y + p, \qquad \mathbf{E}^-_p = -2iy\,\partial_x + 2y\,\partial_y - p, \tag{3.5}$$

satisfy the relations

$$\begin{aligned} \mathbf{E}^\pm_p \circ \omega_p = \omega_{p\pm 2} \circ \mathbf{E}^\pm_p, \qquad & \mathbf{E}^\pm_p\bigl(F\big|^{\mathrm{an}}_{v_r,p}\gamma\bigr) = \bigl(\mathbf{E}^\pm_p F\bigr)\big|^{\mathrm{an}}_{v_r,p\pm 2}\gamma, \\ \mathbf{E}^\pm_{p\mp 2} \circ \mathbf{E}^\mp_p = -4\,\omega_p - p^2 \pm 2p, & \end{aligned} \tag{3.6}$$

and give linear maps

$$\mathbf{E}_p^{\pm} : \mathfrak{M}_p^S(r,s) \longrightarrow \mathfrak{M}_{p\pm 2}^S(r,s). \tag{3.7}$$

For general combinations of the weight p and the spectral parameter s these weight shifting operators are bijections between spaces of Maass forms. Those values of (p,s) where this is not the case are related to the spaces of modular forms discussed in Sect. 2. The operators in (3.1) lead to the following commuting diagram:

$$\begin{tikzcd}
M_p^S(r) \arrow[r, hook, "R_p^h"] \arrow[d, hook] & \mathfrak{M}_p^S(r, \frac{p-1}{2}) \arrow[d, equal] \\
H_p^S(r) \arrow[r, hook, "R_p^h"] \arrow[d, "\xi_p"'] & \mathfrak{M}_p^S(r, \frac{p-1}{2}) \arrow[d, "-\frac{1}{2}\mathbf{E}_p^-"'] \\
\bar{M}_{2-p}^S(r) \arrow[r, hook, "R_{p-2}^a"] & \mathfrak{M}_{p-2}^S(r, \frac{p-1}{2})
\end{tikzcd} \tag{3.8}$$

The spaces on the right are much larger than those on the left, since we imposed growth conditions in Definition 2.4 and did not in Definition 3.1.

3.2 Expansions and growth conditions

Any $f \in \mathfrak{M}_p^S(r,s)$ has a Fourier expansion on a neighborhood $y > A_\infty$ of ∞ for a suitable $A_\infty > 0$, and at each $\zeta \in \mathcal{P}_Y$ a polar expansion on $0 < |\frac{z-\zeta}{z-\bar\zeta}| < A_\zeta$ for suitable A_ζ. The individual terms of these expansions are also eigenfunctions of ω_q with eigenvalue $\frac{1}{4} - s^2$. This leads to second order differential equations, the solutions of which can be described in special functions. Here we mention the results needed for this paper. Section 4.2 in [5] gives more information.

In the Fourier expansion at ∞

$$F(z) = \sum_{n \equiv r/12 \bmod 1} (\mathcal{F}_{\infty,n} f)(z), \qquad (\mathcal{F}_{\infty,n} f)(z) = e^{2\pi i n x}\, (\mathrm{F}_{\infty,n} f)(y), \tag{3.9}$$

the Fourier coefficients $\mathrm{F}_{\infty,n} f$ satisfy a second order differential equation, defining a two-dimensional space of solutions. If $\mathrm{Re}\, n \neq 0$ this space has a one-dimensional subspace of elements with rapid decay as $y \to \infty$. As a basis vector of this subspace we use

$$\omega_p(\infty; n, s; z) = e^{2\pi i n x}\, W_{p\mathrm{Sign}(\mathrm{Re}\, n)/2, s}\big(4\pi n \mathrm{Sign}(\mathrm{Re}\, n) y\big). \tag{3.10}$$

It satisfies

$$\begin{aligned}\omega_p\Big(\infty; n, \pm\frac{p-1}{2}; z\Big) &= (4\pi n)^{p/2}\, y^{p/2}\, q^n && \text{if } \operatorname{Re} n > 0,\\ \omega_p\Big(\infty; n; \pm\frac{p+1}{2}; z\Big) &= (-4\pi n)^{-p/2} y^{-p/2} \bar{q}^{-n} && \text{if } \operatorname{Re} n < 0.\end{aligned} \tag{3.11}$$

The other elements in the space are asymptotic to a multiple of $z \mapsto e^{2\pi i n x}$ $y^{-p\,\mathrm{Sign}(\operatorname{Re} n)/2}\, e^{2\pi n \mathrm{Sign}(\operatorname{Re} n) y}$ as $y \to \infty$. So these terms have exponential growth, of larger order if $\operatorname{Re} n$ gets larger. If $\operatorname{Re} n = 0$ all element of the solution space have less than exponential growth.

Near $\zeta \in \mathcal{P}_Y$ we have an analogous situation. We have an expansion

$$\begin{aligned}F(z) &= \sum_{\nu \in \mathbb{Z}} (\mathcal{F}_{\zeta,\nu} f)(z),\\ (\mathcal{F}_{\zeta,\nu} f)(z) &= e^{ip \arg(1-w)}\, e^{i\nu \arg w}\, (\mathrm{F}_{\zeta,\nu} f)(u),\end{aligned} \tag{3.12}$$

with $w = (z-\zeta)/(z-\bar{\zeta})$ and $u = \frac{|w|^2}{1-|w|^2} = \frac{|z-\zeta|^2}{4y \operatorname{Im} \zeta}$.

To see that this is the right type of expansion, we may compare it with (2.12) and (2.13), and use the operators in (3.1) to obtain

$$\begin{aligned}R_p^h\big((z-\bar{\zeta})^{-p}\,(w/|w|)^\nu\big) &= e^{ip\arg(1-w)}\, e^{i\nu \arg w}\, e^{-\pi i p/2}\, 2^{-p}\\ &\quad \times (\operatorname{Im}\zeta)^{-p/2}\,\big(1-|w|^2\big)^{p/2},\\ R_p^a\big((\bar{z}-\zeta)^{p}\,(\bar{w}/|w|)^\nu\big) &= e^{ip\arg(1-w)}\, e^{-i\nu \arg w}\, e^{-\pi i p/2}\, 2^{p}\\ &\quad \times (\operatorname{Im}\zeta)^{p/2}\,\big(1-|w|^2\big)^{-p/2}.\end{aligned}$$

Thus, we obtain Fourier terms of the form $e^{ip\arg(1-w)\pm i\nu \arg w}$ times a function of u.

The Fourier coefficients $\mathrm{F}_{\zeta,n}$ are elements of a two-dimensional space, with a one-dimensional subspace corresponding to functions without a singularity at ζ. This subspace is spanned by

$$\begin{aligned}\omega_p(\zeta; p+2\nu, s; z) &= e^{ip\arg(1-w)}\, e^{i\nu \arg w} \left(\frac{u}{u+1}\right)^{\varepsilon\nu/2} (u+1)^{-s-1/2}\\ &\quad \cdot {}_2F_1\Big(\frac{1}{2}+s+\frac{\varepsilon p}{2}+\varepsilon\nu, \frac{1}{2}+s-\frac{\varepsilon p}{2}; 1+\varepsilon\nu; \frac{u}{u+1}\Big),\end{aligned} \tag{3.13}$$

with $\varepsilon \in \{1,-1\}$ chosen such that $\varepsilon\nu \geq 0$. (I use the notations of Sect. 4.2 in [5]. A confusing point is that in [5] it made good sense to parametrize the order of the Fourier terms at $\zeta \in \mathcal{P}_Y$ by $p+2\nu$, while here parametrization by ν itself is more convenient.) We have

$$\begin{aligned}
&\omega_p\left(\zeta; p+2\nu, \pm\frac{p-1}{2}; z\right)\\
&\quad = 2^p\, e^{\pi i p/2}\, (\operatorname{Im}\zeta)^{p/2}\, y^{p/2}\, (z-\bar\zeta)^{-p}\, w^\nu \quad \text{if } \nu \ge 0,\\
&\omega_p\left(\zeta; p+2\nu, \pm\frac{p+1}{2}; z\right)\\
&\quad = 2^{-p}\, e^{\pi i p/2}\, (\operatorname{Im}\zeta)^{-p/2}\, y^{-p/2} (\bar z-\zeta)^p\, \bar w^{-\nu} \quad \text{if } \nu \le 0.
\end{aligned} \tag{3.14}$$

The other elements that can occur in the term of order n in the expansion have a singularity at ζ. This singularity is logarithmic if $n = p$ and behaves near $w = 0$ like $w^{(p-n)/2}$ if $n - p > 0$ and like $\bar w^{(n-p)/2}$ if $n - p < 0$.

These results show that the growth of Maass forms can be controlled by the Fourier expansion. Suppose that $f \in \mathfrak{M}_p^S(r,s)$ satisfies $f(z) = \mathrm{O}(e^{ay})$ as $y \to \infty$ for a given $a > 0$. The Fourier terms $\mathcal{F}_{\infty,n} f$ can be given by a Fourier integral, and hence satisfy the same estimate. So if $|\operatorname{Re} n| > a/2\pi$ then $\mathcal{F}_{\infty,n}$ has to be a multiple of $\omega_p(\infty; n, s)$. Similarly, if F satisfies near ζ the estimate in (2.6) for a certain $a > 0$, then all but finitely many terms in the expansion at ζ are multiples of $\omega_p(\zeta; n, s)$. Conversely, the contribution of the terms in the expansion at $\xi \in \mathcal{P} = \{\infty\} \cup \mathcal{P}_Y$ that are a multiple of $\omega_p(\xi; n, s)$ cannot give a large growth.

Definition 3.3 A *growth condition* $\mathbf{c}$ for $\mathfrak{M}_p^S(r,s)$ is a finite set of pairs (ξ, n) with $\xi \in \mathcal{P}$ and $n \equiv \frac{r}{12} \bmod 1$ if $\xi = \infty$, and $n \in \mathbb{Z}$ if $\xi \in \mathcal{P}_Y$. If $\operatorname{Re} r \in 12\mathbb{Z}$ we require that $\mathbf{c}$ contains (∞, it) for $t = -i\,\frac{r - \operatorname{Re} r}{12}$.

Notation: $\mathbf{c}(\xi) = \{n \,:\, (\xi, n) \in \mathbf{c}\}$.

A growth condition singles out finitely many terms from the expansions of Maass forms at points of $\mathcal{P}$. The additional condition can be understood from the fact that for $\operatorname{Re} n = 0$ there are no quickly decreasing non-zero Fourier terms at ∞.

Definition 3.4 Let $\mathbf{c}$ be a growth condition. By $\mathfrak{M}_p^{\mathbf{c}}(r,s)$ we denote the space of $f \in \mathfrak{M}_p^S(r,s)$ that satisfy

$$\begin{aligned}
&\mathcal{F}_{\infty,n} f \;\in\; \mathbb{C}\,\omega_p(\infty; n, s) \quad \text{if } n \equiv \frac{r}{12} \bmod 1 \text{ and } n \notin \mathbf{c}(\infty),\\
&\mathcal{F}_{\zeta,\nu} f \;\in\; \mathbb{C}\,\omega_p(\zeta; p+2\nu, s) \quad \text{if } \nu \in \mathbb{Z},\ \nu \notin \mathbf{c}(\zeta) \text{ for } \zeta \in \mathcal{P}_Y.
\end{aligned}$$

The weight shifting operators $\mathbf{E}_p^\pm$ in (3.5) behave nicely with respect to the Fourier expansion at ∞:

$$\mathbf{E}_p^\pm \mathcal{F}_{\infty,n} f = \mathcal{F}_{\infty,n} \mathbf{E}_p^\pm f. \tag{3.15}$$

For $\zeta \in \mathcal{P}_Y$ we have the more complicated relation

$$\mathbf{E}_p^\pm \mathcal{F}_{\zeta,\nu} f = \mathcal{F}_{\zeta,\nu\mp 1} \mathbf{E}_p^\pm f. \tag{3.16}$$

We define for a given growth condition $\mathbf{c}$ the growth conditions $\mathbf{c}^+$ and $\mathbf{c}^-$ by

$$\mathbf{c}^{\pm}(\infty)=\mathbf{c}(\infty) \quad \text{and} \quad \mathbf{c}^{\pm}(\zeta)=\left\{\nu \mp 1 \,:\, \nu \in \mathbf{c}(\zeta)\right\} \quad \text{if } \zeta \in \mathcal{P}_Y. \tag{3.17}$$

The differentiation relations in Table 4.1 on p. 63 of [5] imply that $\mathbf{E}_p^{\pm}$ sends $\omega_p(\xi;*,s)$ to $\omega_{p\pm 2}(\xi;*,s)$ for all $\xi \in \mathcal{P}_Y$. Hence

$$\mathbf{E}_p^{\pm} : \mathfrak{M}_p^{\mathbf{c}}(r,s) \to \mathfrak{M}_{p\pm 2}^{\mathbf{c}^{\pm}}(r,s). \tag{3.18}$$

See Proposition 4.5.3 in [5]. (The change in the growth condition is absent in [5]. This is a consequence of the difference in the parametrization of the order of terms in the expansions at points of $\mathcal{P}_Y$.)

If $F \in \mathfrak{M}_p^S(r,s)$ satisfies (2.5) at ∞ and (2.6) at the points in $\mathcal{P}_Y$, then it is in $\mathfrak{M}_p^{\mathbf{c}}(r,s)$ for some growth condition $\mathbf{c}$, and conversely each element of a given $\mathfrak{M}_p^{\mathbf{c}}(r,s)$ satisfies those growth conditions at the points of $\mathcal{P}$. Thus, the diagram (3.8) can be replaced:

$$\begin{tikzcd}
M_p^S(r) \arrow[r, hook, "R_p^h"] \arrow[d, hook] & \bigcup_{\mathbf{c}} \mathfrak{M}_p^{\mathbf{c}}(r, \frac{p-1}{2}) \arrow[d, equal] \\
H_p^S(r) \arrow[r, "R_p^h", "\cong"'] \arrow[d, "\xi_p"'] & \bigcup_{\mathbf{c}} \mathfrak{M}_p^{\mathbf{c}}(r, \frac{p-1}{2}) \arrow[d, "-\frac{1}{2}\mathbf{E}_p^-"'] \\
\bar{M}_{2-p}^S(r) \arrow[r, hook, "R_{p-2}^a"] & \bigcup_{\mathbf{c}} \mathfrak{M}_{p-2}^{\mathbf{c}^-}(r, \frac{p-1}{2})
\end{tikzcd} \tag{3.19}$$

Here $\mathbf{c}$ runs over all growth conditions for $\mathfrak{M}_p^S(r,s)$. Moreover,

$$\begin{aligned}
R_p^h M_p^S(r) &= \ker\left(\mathbf{E}_p^- : \bigcup_{\mathbf{c}} \mathfrak{M}_p^{\mathbf{c}}\left(r, \frac{p-1}{2}\right) \to \bigcup_{\mathbf{c}} \mathfrak{M}_{p-2}^{\mathbf{c}^-}\left(r, \frac{p-1}{2}\right)\right),\\
R_{p-2}^a \bar{M}_{2-p}^S(r) &= \ker\left(\mathbf{E}_{p-2}^+ : \bigcup_{\mathbf{c}} \mathfrak{M}_{p-2}^{\mathbf{c}}\left(r, \frac{p-1}{2}\right) \to \bigcup_{\mathbf{c}} \mathfrak{M}_p^{\mathbf{c}^+}\left(r, \frac{p-1}{2}\right)\right).
\end{aligned} \tag{3.20}$$

Relation (3.18) shows that the differential operators $\mathbf{E}_p^+$ and $\mathbf{E}_p^-$ transform Maass forms satisfying a given growth condition into Maass forms satisfying a slightly changed growth condition. So indeed $\xi_p H_p^S(r) \subset \bar{M}_{2-p}^S(r)$.

In the next subsections we will not work with individual Maass forms, but with families of Maass forms $(r,s) \mapsto f(r,s)$ for (r,s) in some domain $\Omega \subset \mathbb{C}^2$, such that $f(r,s) \in \mathfrak{M}_{\ell+r}^S(r,s)$ for a given $\ell \in 2\mathbb{Z}$. Then we will use growth conditions $\mathbf{c} = \{(\xi, \nu_0)\}$ in which the integers ν_0 determine functions of r. All $\mathbf{c}(\xi)$ are finite subsets of $\mathbb{Z}$:

$$\begin{aligned}
\nu_0 \in \mathbf{c}(\infty), &\quad \text{corresponds to} \quad r \mapsto \nu_0 + \frac{r}{12},\\
\nu_0 \in \mathbf{c}(\zeta),\ \zeta \in \mathcal{P}_Y, &\quad \text{corresponds to} \quad r \mapsto \nu_0.
\end{aligned} \tag{3.21}$$

The variable r should run over a set $U \subset \mathbb{C}$ such that $\nu + \frac{\operatorname{Re} r}{12} \neq 0$ for all $\nu \in \mathbf{c}(\infty)$ and $r \in U$. In this context we interpret $\mathrm{F}_{\xi,\nu}$ as $\mathrm{F}_{\zeta,\nu+r/12}$ if $\xi = \infty$ and as $\mathrm{F}_{\xi,\nu}$ if $\xi \in \mathcal{P}_Y$.

3.3 Perturbation theory

The basis for our proof of Theorems 1.1–1.3 is Theorem 9.4.1 in [5]. It gives meromorphic families of Maass forms with a prescribed behavior of the terms in the expansions at points of $\mathcal{P}$ given by a growth condition. In [5] it is a step in obtaining the meromorphic continuation of Poincaré series in (r,s) jointly. In this paper it is convenient to use this intermediate result and not the continued Poincaré series.

For our given sets S and $\mathcal{P}$ we have expansions of Maass forms on regions $y > A_\infty$ near ∞ and on regions $0 < |\frac{z-\zeta}{z-\bar{\zeta}}| < A_\zeta$ near $\zeta \in \mathcal{P}_Y$. These regions may overlap. To be able to apply the results in Chaps. 7–9 of [5] we shrink these regions such that their images in the quotient $\Gamma\backslash\mathfrak{H}$ are pairwise disjoint.

We choose for each $\xi \in \mathcal{P}$ a *truncation point* a_ξ such that $a_\infty > A_\infty$ and $a_\zeta \in (0, A_\zeta^2/(1-A_\zeta^2))$ for $\zeta \in \mathcal{P}_Y$. Hence $(\mathrm{F}_{\infty,n} f)(a_\infty)$ and $(\mathrm{F}_{\zeta,\nu} f)(a_\zeta)$ are well defined. The precise choice of the truncation points does not matter.

We use a sequence $(\mathbf{c}_N)_{N\geq 1}$ of growth conditions as in (3.21), with

$$\mathbf{c}_N(\infty) = \left\{\nu \in \mathbb{Z} \,:\, |\nu| < N\right\}, \tag{3.22}$$

and with finite sets $\mathbf{c}(\zeta)$ for $\zeta \in \mathcal{P}_Y$ that do not depend on N. We formulate part of the statement of Theorem 9.4.1 in [5]:

Theorem 3.5 *There is an open disk $V_{0,N} = V_0(\mathbf{c}_N)$ around 0 in $\mathbb{C}$ such that for each collection of holomorphic functions $\rho = (\rho_{\xi,\nu})_{(\xi,\nu)\in\mathbf{c}_N}$ on $V_{0,N} \times \mathbb{C}$ there is a unique meromorphic family e_ρ of Maass forms on $V_{0,N} \times \mathbb{C}$ with values in $\mathfrak{M}_\ell^{\mathbf{c}_N}$ satisfying*

$$\mathrm{F}_{\xi,\nu} e_\rho(r,s;a_\xi) = \rho_{\xi,\nu}(r,s) \quad \textit{for all } (\xi,n) \in \mathbf{c}_N. \tag{3.23}$$

A family $(r,s) \mapsto f(r,s)$ of Maass forms is holomorphic if it is pointwise holomorphic and also all terms $\mathrm{F}_{\xi,n} f(r,s)$ in the expansions at ∞ and ζ are pointwise holomorphic. It has values in $\mathfrak{M}_\ell^{\mathbf{c}}$ if its value at (r,s) is in $\mathfrak{M}_{\ell+r}^{\mathbf{c}}(r,s)$ for each (r,s) in its domain.

A meromorphic family f on $V_{0,N} \times \mathbb{C}$ with values in $\mathfrak{M}_\ell^{\mathbf{c}_N}$ is not just a family that is pointwise meromorphic on $\mathfrak{H} \setminus S$. We require that locally on its domain the family can be written as $\frac{1}{\psi} h$, where h is a holomorphic family with values in $\mathfrak{M}_\ell^{\mathbf{c}_N}$ and ψ is a non-zero holomorphic function. The idea is that "denominators should not depend on z".

We call (r_0,s_0) a *singularity* of the family if the family is not a holomorphic family on a neighborhood of (r_0,s_0). So $\psi(r_0,s_0)$ should vanish for the representation $f = \frac{1}{\psi} h$ that is valid on a neighborhood of (r_0,s_0).

It should be noted that although the functions $\rho_{\xi,n}$ are holomorphic, the families $(r,s) \mapsto \mathrm{F}_{\xi,\nu} e_\rho(r,s)$ are meromorphic. Their singularities are not visible in the functions $(r,s) \mapsto \mathrm{F}_{\xi,\nu} e_\rho(r,s;a_\xi)$.

The theorem is based on the existence of a holomorphic family $r \mapsto A^a(r)$ on a disk in $\mathbb{C}$ centered at 0 of self-adjoint operators in a Hilbert space. The Hilbert space and the family depend on the growth condition $\mathbf{c}_N$ and the truncation points a_ξ for $\xi \in \mathcal{P}$. After preparations in earlier chapters it is defined (in a more general context) in Sect. 9.2 of [5]. It is a generalization of the pseudo Laplace operator of Colin de Verdière in [7]. The family A^a can be studied with the methods of analytic perturbation theory in Kato's book [10]. Eigenvectors of $A^a(r)$ with eigenvalue $\frac{1}{4}-s^2$ correspond to Maass forms $F \in \mathfrak{M}^{\mathbf{c}_N}_{\ell+r}(r,s)$ for which $(\mathrm{F}_{\xi,n} f)(a_\xi)=0$ for all $(\xi,n) \in \mathbf{c}_N(r)$.

The resolvent gives a meromorphic family $(r,s) \mapsto R^a(r,s)$ of bounded operators. This resolvent is used in the construction of e_ρ in the theorem. We do not know much about this resolvent, except that it is meromorphic, and we have some eigenvalue estimates that give information on its singularities for $r \in \mathbb{R} \cap V_0$. This gives the following additional information:

Lemma 3.6 *Let e_ρ be as in Theorem 3.5. For each $r \in V_{0,N}$ the set of $s \in \mathbb{C}$ such that e_ρ has a singularity at (r,s) is discrete in $\mathbb{C}$. If (r,s) is a singularity of e_ρ with $r \in V_0 \cap \mathbb{R}$ then $\frac{1}{4}-s^2 \geq -\frac{1}{4}(\ell+r)^2$.*

Proof Theorem 9.4.1 in [5] states that each singularity of e_ρ is a singularity of the resolvent R^a. This gives the first assertion. The eigenvalue estimate follows from 9.2.1 in [5]. □

Lemma 3.7 *The $V_{0,N} := V_0(\mathbf{c}_N)$ can be chosen to form an increasing collection of open neighborhoods of 0 in $\mathbb{C}$ satisfying $V_{0,N} \subset \{r \in \mathbb{C} : |\mathrm{Re}\, r| < 12N\}$ and*

$$\bigcup_{N \geq 0} V_{0,N} = \mathbb{C}. \tag{3.24}$$

Proof If $r \in V_{0,N}$ would not satisfy $|\mathrm{Re}\, r| < 12(N+1)$ then $\mathbf{c}_N$ would not be a suitable growth condition. To see that the union of the sets $V_{0,N}$ equals $\mathbb{C}$ we have to go into some details of the reasoning in [5].

We start at the proof of Lemma 9.1.6 in [5]. There it is indicated that the set V_0 should consist of $r \in \mathbb{C}$ such that

$$b_{1,b}\, d_b\, |r| + b_{1,c}\, d_c\, |r| + b_{2,c}\, d_c^2\, |r|^2 \ < \ 1, \tag{3.25}$$

with positive factors that we have to trace back through the lemmas in [5].

The linear form $\varphi = \varphi_r$ in Lemma 9.1.6 is of the form $r\alpha$, with α as explained in 13.4.7. We define the factors d_b and d_c as $\|\alpha\|_b = \|\varphi_r\|_b/|r|$ and $\|\alpha\|_c = \|\varphi\|_c/|r|$. From 8.4.10 we see that $d_c > 0$ depends only on the group. For d_b we go from Lemma 8.4.11 via the definition of $b^{\pm}_\varphi$ in 8.2.3 to the function t_φ in Lemma 8.2.1. There we see that t_φ depends on the set $\mathcal{P}$, but not on the actual growth condition. Hence $d_b > 0$ is also $\mathrm{O}(1)$, independent of N and the finite sets $\mathbf{c}_N(\zeta) \subset \mathbb{Z}$ for $\zeta \in \mathcal{P}_Y$.

The constants $b_{*,*}$ are given in Lemma 9.1.5, and expressed in a large quantity ξ, which depends on N, on an arbitrary small quantity ε, and on a positive quantity n_1. In the proof of Lemma 8.4.11 the quantity n_1 is defined depending on the group only. This means that Lemma 9.1.5 gives

$$b_{1,b} = \mathrm{O}\big(\varepsilon^2\big), \qquad b_{1,c} = \mathrm{O}\big(\xi^{-1}\big) + \mathrm{O}(\varepsilon) + \mathrm{O}\big(\varepsilon^{7/4}\xi^{-1/4}\big), \qquad b_{2,c} = \mathrm{O}\big(\xi^{-2}\big).$$

The dependence on N is via ξ, and possibly via our choice of ε.

The definition of ξ in Lemma 8.4.11 gives for the present situation

$$\xi = 2\pi N - \frac{\ell}{2a_\infty} \geq 2\pi N.$$

(We used $\ell \leq 0$.)

Taking $\varepsilon = \frac{1}{N}$ we see that there is $C > 0$, not depending on N, such that

$$\frac{|r|}{N} + \left(\frac{|r|}{N}\right)^2 < C$$

implies that (3.25) is satisfied. Determining $V_{0,N}$ by

$$|r| < N \min\left(12, \frac{\sqrt{1+4C}-1}{2}\right) \tag{3.26}$$

we satisfy all conditions. □

3.4 Family of antiholomorphic modular forms with singularities

In this subsection we consider families $r \mapsto \bar{\eta}^{-2r} F$ of antiholomorphic forms, and show that such a family is the restriction of a family of Maass forms depending meromorphically on r and the spectral parameter s jointly. This will allow us to construct in the next subsection a meromorphic family of harmonic lifts of $\bar{\eta}^{-2r} F$.

Any element of $\bar{M}^{!!}_{2-p_1}(r_1)$ with $p_1 \equiv r_1 \bmod 2$ can be written as $\bar{\eta}^{-2r} F$ with $r \in \mathbb{C}$ and $F \in \bar{M}^{!!}_{2-\ell}(0)$ with $\ell \in 2\mathbb{Z}_{\leq 0}$; so F is the conjugate of a meromorphic automorphic form of even weight $2-\ell$ for the trivial multiplier system. We consider the holomorphic family $r \mapsto \bar{\eta}^{-2r} F$. For each $r \in \mathbb{C}$ we have $\bar{\eta}^{-2r} F \in \bar{M}^{!!}_{2-\ell-r}(r)$. There is freedom in the choice of ℓ and r; we use it to take $\ell \leq 0$.

Let $\mathcal{P}_Y$ be a set of representatives of the Γ-orbits of points in $\mathfrak{H}$ where F, and hence all $\bar{\eta}^{-2r} F$, have a singularity. Let μ_∞ denote the order of F at ∞, and μ_ζ the order at $\zeta \in \mathcal{P}_Y$. For each $N \in \mathbb{Z}_{\geq 1}$ that satisfies $N \geq 1 - \mu_\infty$ we form the growth condition $\mathbf{c}_N$ given by

$$\begin{aligned} \mathbf{c}_N(\infty) &= \{\nu \in \mathbb{Z} : |\nu| < N\}, \\ \mathbf{c}_N(\zeta) &= \{\nu \in \mathbb{Z} : 1 \leq \nu \leq -\mu_\zeta\} \quad \text{for } \zeta \in \mathcal{P}_Y. \end{aligned} \tag{3.27}$$

Lemma 3.8 *With the notations and assumptions given above, let $V_{0,N}$ be the disk in $\mathbb{C}$ in Theorem 3.5.*

There exists a meromorphic family of Maass forms e_ρ on $V_{0,N} \times \mathbb{C}$ that satisfies the growth condition $\mathbf{c}_N$, and for which the restriction to the complex line $s = \frac{\ell+r-1}{2}$ exists and satisfies

$$e_\rho\left(r, \frac{\ell+r-1}{2}\right) = R^a_{2-\ell-r}\left(\bar{\eta}^{-2r} F\right), \tag{3.28}$$

as an identity of meromorphic functions on $V_{0,N}$.

Proof Let S be the Γ-invariant set of points in $\mathfrak{H}$ at which F has a singularity. Then also $\bar\eta^{-2r}F$ has its singularities in S. We have chosen $\mathcal{P}_Y$ as a set of representatives of $\Gamma\backslash S$.

In (2.13) we have seen that the function $\bar\eta^{-2r}F$ has expansions of the following form:

$$\begin{aligned} \text{at } \infty\colon \quad & \big(\bar\eta^{-2r}F\big)(z) = \sum_{\nu\geq\mu_\infty} a_\nu^\infty(r)\,\bar q^{\,\nu-r/12}, \\ \text{at } \zeta\in\mathcal{P}_Y\colon \quad & \big(\bar\eta^{-2r}F\big)(z) = (\bar z-\zeta)^{\ell+r-2}\sum_{\nu\geq\mu_\zeta} a_\nu^\zeta(r)\,\bar w^{\nu}, \end{aligned} \tag{3.29}$$

with $\bar q = e^{-2\pi i\bar z}$ and $\bar w = \frac{\bar z-\bar\zeta}{\bar z-\zeta}$. The coefficients a_ν^ξ are holomorphic functions on $\mathbb{C}$. (The a_ν^∞ are actually polynomials in r of degree at most $\nu-\mu_\infty$.) We have $\mu_\zeta\leq -1$ for all $\zeta\in\mathcal{P}_Y$.

We denote

$$f_r = R^a_{2-\ell-r}\big(\bar\eta^{-2r}F\big), \qquad f_r(z) = y^{1-(\ell+r)/2}\,\overline{\eta(z)}^{\,-2r}\,F(z). \tag{3.30}$$

This is a holomorphic family on $\mathbb{C}$ of Maass forms, with $f_r\in\mathfrak{M}^S_{\ell+r-2}(r,\frac{\ell+r-1}{2})$. A comparison with (3.11) and (3.14) shows that

$$\begin{aligned} (\mathcal{F}_{\infty,-\nu+r/12}f_r)(z) &= a_\nu^\infty(r)\,y^{1-(\ell+r)/2}\bar q^{\,\nu-r/12} \quad \text{for all } \nu\geq\mu_\infty, \\ &= a_\nu^\infty(r)\left(4\pi\left(\nu-\frac{r}{12}\right)\right)^{(\ell+r)/2-1} \\ &\quad\times\omega_{\ell+r-2}\left(\infty;-\nu+\frac{r}{12},\frac{\ell+r-1}{2}\right) \\ &\quad \text{if } -\nu+\frac{\operatorname{Re} r}{12}<0, \\ (\mathcal{F}_{\zeta,-\nu}f_r)(z) &= a_\nu^\zeta(r)\,y^{1-(\ell+r)/2}\,(\bar z-\zeta)^{\ell+r-2}\,\bar w^{\nu} \quad \text{for all } \nu\geq\mu_\zeta, \\ &= -a_\nu^\zeta(r)\,e^{-\pi i(\ell+r)/2}\,(4\operatorname{Im}\zeta)^{(\ell+r)/2-1} \\ &\quad\times\omega_{\ell+r-2}\left(\zeta;-\nu;\frac{\ell+r-1}{2};z\right) \\ &\quad \text{for } \zeta\in\mathcal{P}_Y,\ \text{if } \nu\geq 0. \end{aligned}$$

So $f_r\in\mathfrak{M}^{\mathbf{c}_N}_{\ell+r-2}(r,\frac{\ell+r-1}{2})$ for all $r\in V_{0,N}$, for the disk $V_{0,N}$ in Theorem 3.5.

For $(\xi,\nu)\in\mathbf{c}_N(r)$ the families $r\mapsto(\mathrm{F}_{\xi,\nu}f_r)(a_\xi)$ are holomorphic multiples of $a^\xi_{-\nu}(r)$. This means that the functions

$$\rho_{\xi,\nu}(r,s) = (\mathrm{F}_{\xi,\nu}f_r)(a_\xi) \tag{3.31}$$

are holomorphic on $\mathbb{C}$ for all $(\xi,\nu)\in\mathbf{c}_N$. We apply Theorem 3.5 and obtain a meromorphic family e_ρ of Maass forms of $V_{0,N}\times\mathbb{C}$ with Fourier coefficients determined by $\mathbf{c}_N$ satisfying (3.23).

Now we ask whether the restriction of e_ρ to the complex line $s = \frac{\ell+r-1}{2}$ has anything to do with the family f_r, apart from relation (3.23). The first worry is that e_ρ might have a singularity along $s = \frac{\ell+r-1}{2}$, which would mean that there is no meromorphic restriction to this line at all.

Suppose that there were such a singularity carried along the line $s = \frac{\ell+r-1}{2}$. Take the minimal integer $k \geq 1$ such that

$$p(r) = \lim_{s \to (\ell+r-1)/2} \left(s - \frac{\ell+r-1}{2}\right)^k e_\rho(r,s)$$

exists for a dense set of $r \in V_{0,N}$. Since k is minimal, $p(r)$ is non-zero for some r, and hence the meromorphic family $r \mapsto p(r)$ of Maass forms is non-zero. For each $(\xi, \nu) \in \mathbf{c}_N$ we have $(\mathrm{F}_{\xi,\nu}\, p(r))(a_\xi) = 0$. So p is a meromorphic family of eigenfunctions of the family of operators $A^a(\cdot)$. Lemma 3.6 implies that for those $r \in V_{0,N} \cap \mathbb{R}$ at which it has no singularity we have

$$\frac{1}{4} - \left(\frac{\ell+r-1}{2}\right)^2 \geq -\frac{1}{4}(\ell+r)^2.$$

This cannot be true for $r \in V_{0,N} \cap (-\infty, 0)$, since we have taken $\ell \leq 0$. Hence $k = 0$ and e_ρ has a restriction to the line $s = \frac{\ell+r-1}{2}$. This restriction may be meromorphic; the good thing is that it exists at all.

Next, we check that the restriction is equal to f_r. We consider the meromorphic family $p_1 : r \mapsto e_\rho(r, \frac{\ell+r-1}{2}) - f_r$. It might be a non-zero family. We know from Theorem 3.5 that $r \mapsto (\mathrm{F}_{\xi,\nu}\, p_1(r))(a_\xi)$ is the zero function for all $(\xi, \nu) \in \mathbf{c}_N$. Again by the eigenvalue estimate in Lemma 3.6 this is impossible. Hence $f_r = R^1_{2-\ell-r}(\bar{\eta}^{-2r} F)$ is equal to the restriction of the family e_ρ to the line $s = \frac{\ell+r-1}{2}$. □

3.5 Lift of the family

The advantage of describing $r \mapsto f_r$ as the restriction of a family of Maass forms in two variables, is that it is easier to lift such a family.

In diagram (3.8) we see that we want to find h_r such that $-\frac{1}{2}\mathbf{E}^-_{\ell+r} h_r = f_r$. The differential operator $\frac{1}{4}\mathbf{E}^-_{\ell+r}\mathbf{E}^+_{\ell+r-2}$ acts on the space $\mathfrak{M}^{\mathbf{c}_N}_{\ell+r-2}(r,s)$ as multiplication by

$$\left(s - \frac{\ell+r-1}{2}\right)\left(s + \frac{\ell+r-1}{2}\right).$$

So let us consider

$$h(r,s) = -\frac{1}{2}\left(s - \frac{\ell+r-1}{2}\right)^{-1}\left(s + \frac{\ell+r-1}{2}\right)^{-1} \mathbf{E}^+_{\ell+r-2}\, e_\rho(r,s). \qquad (3.32)$$

This is well defined as a meromorphic family on $V_{0,N} \times \mathbb{C}$. Since the family $\bar{\eta}^{-2r} F$ that we started with is antiholomorphic, we have $\mathbf{E}^+_{\ell+r-2} f_r = 0$. Hence $\mathbf{E}^+_{\ell+r-2} e_\rho(r,s)$ has a zero along $s = \frac{\ell+r-1}{2}$, which cancels the factor $(s - \frac{\ell+r-1}{2})^{-1}$, and hence h_r has no singularity carried by the line $s = \frac{\ell+r-1}{2}$. So the restriction

$h_r = h(r, \frac{\ell+r-1}{2})$ exists as a meromorphic family of Maass forms on $V_{0,N}$ and satisfies $-\frac{1}{2}\mathbf{E}^-_{\ell+r} h_r = f_r$. It is a family that satisfies $h_r \in \mathfrak{M}^{\mathbf{c}^+_N}_{\ell+r}(r, \frac{\ell+r-1}{2})$.

Going back by the operator $R^h_{\ell+r}$ in diagram (3.19) we get a meromorphic family $H_r = (R^h_{\ell+r})^{-1} h_r$ of harmonic modular forms. We obtain the following intermediate result:

Proposition 3.9 *Let F be an antiholomorphic modular form of weight $2-\ell \in 2\mathbb{Z}_{\geq 1}$ with singularities in the set $S \subset \mathfrak{H}$. Then there is a collection $\{V_{0,N} : N \geq N_F\}$, for some $N_F \geq 1$, of open neighborhoods of 0 in $\mathbb{C}$ with the properties in Lemma 3.7, such that on each $V_{0,N}$ there is a meromorphic family $H_{N,r}$ of harmonic automorphic forms such that*

$$H_{N,r} \in H^S_{\ell+r}(r) \quad \text{and} \quad \xi_{\ell+r} H_{N,r} = \bar{\eta}^{-2r}\, F$$

for each $r \in V_{0,N}$ at which $H_{N,r}$ is defined.

The construction of $H_{N,r}$ is far from unique. It depends on the choice of the truncation parameters a_ξ, and in (3.31) we could have taken other holomorphic functions with the same restrictions to the line $s = \frac{\ell+r-1}{2}$.

Let $\mathcal{P} = \{\infty\} \cup \mathcal{P}_Y$ as before. The family $H_{r,N}$ that we have constructed satisfies the growth condition $\mathbf{c}^+_N$. The growth condition depends on the order $-\mu_\zeta$ of the singularities of F at $\zeta \in \mathcal{P}_Y$. (Here the order is determined by the lowest power of $\bar{w}$ occurring in the expansion, even if ζ is in an elliptic orbit.) The order of the term $\bar{q}^{\mu_\infty}$ at which the expansion of F at ∞ starts determines the lower bound $N_F = \max(1, 1-\mu_\infty)$ in the proposition.

We summarize the information that we now have. All terms in the Fourier expansion

$$H_{N,r}(z) = \sum_{\nu \in \mathbb{Z}} H^\infty_{N,\nu}(r;z), \qquad H^\infty_{N,\nu}\big(r; z+x'\big) = e^{2\pi i(\nu+r/12)x'}\, H^\infty_{N,\nu}(r;z), \tag{3.33}$$

are meromorphic in $r \in V_{0,N}$ and satisfy

$$\begin{aligned}
\xi_{\ell+r} H^\infty_{N,\nu}(r;z) &= a^\infty_{-\nu}(r)\,\bar{q}^{-\nu-r/12}, \\
H^\infty_{N,\nu}(r;z) &= b^\infty_{N,\nu}(r)\, q^{\nu+r/12} \quad \text{if } \nu \geq N, \\
&= a^\infty_{-\nu}(r)\left(-4\pi\left(\nu+\frac{r}{12}\right)\right)^{\ell+r-1} q^{\nu+r/12} \\
&\quad \cdot \Gamma\left(1-\ell-r, -4\pi\left(\nu+\frac{r}{12}\right)y\right) \quad \text{if } \nu \leq -N,
\end{aligned} \tag{3.34}$$

with the convention that $a^\infty_\mu = 0$ if $\mu < \mu_\infty$. The meromorphic functions $b^\infty_{N,\nu}$ are unknown, the functions $a^\infty_{-\nu}$ are holomorphic on $\mathbb{C}$ and occur in the Fourier expansion (3.29) of $\bar{\eta}^{-2r} F$. The incomplete gamma-function $\Gamma(p,t) = \int_{u=t}^\infty u^{p-1}\, e^{-u}\, du$

is obtained from specialization of the Whittaker function in (3.11). Anyhow, the expression for $\nu < -N$ gives the unique quickly decreasing function with the right image under $\xi_{\ell+r}$.

Now let $\zeta \in \mathcal{P}_Y$. The expansion at ζ has the form

$$\begin{aligned} H_{N,r}(z) &= (z-\bar\zeta)^{-\ell-r} \sum_{\nu\in\mathbb{Z}} H^\zeta_{N,\nu}\left(r; \frac{z-\zeta}{z-\bar\zeta}\right), \\ H^\zeta_{N,\nu}(tw) &= t^\nu\, H^\zeta_{N,\nu}(w) \quad \text{for } |t|=1. \end{aligned} \tag{3.35}$$

The $H^\zeta_{\nu,r}$ are meromorphic families on $V_{0,N}$ with $(\ell+r)$-harmonic values, and satisfy

$$\begin{aligned} \xi_{\ell+r}\left((z-\bar\zeta)^{-\ell-r}\, H^\zeta_{N,\nu}\left(r; \frac{z-\zeta}{z-\bar\zeta}\right)\right) &= (\bar z-\zeta)^{\ell+r-2}\, a^\zeta_{-\nu-1}(r) \left(\frac{\bar z-\bar\zeta}{\bar z-\zeta}\right)^{-\nu-1}, \\ H^\zeta_{N,\nu}(w) &= b^\zeta_{N,\nu}(r)\, w^\nu \quad \text{if } \nu \geq -\mu_\zeta, \\ &= a^\zeta_{-\nu-1}(r)\,(4\,\mathrm{Im}\,\zeta)^{\ell+r-1}\, w^\nu \\ &\quad \times \mathsf{B}\big(|w|^2, -\nu, 1-\ell-r\big) \quad \text{if } \nu \leq -1. \end{aligned} \tag{3.36}$$

The meromorphic functions $b^\zeta_{N,\nu}$ are unknown. The coefficients $a^\zeta_{-\nu-1}$ from the expansion of $\bar\eta^{-2r} F$ are holomorphic on $\mathbb{C}$. Here we meet the incomplete beta-function

$$\mathsf{B}(t,a,b) = \int_0^t u^{a-1}\,(1-u)^{b-1}\,du \quad (0 \leq t < 1,\ \mathrm{Re}\,a > 0). \tag{3.37}$$

It arises from specialization of the hypergeometric function in (3.13).

At this point the Maass forms have served their purpose. We have obtained a meromorphic family of harmonic lifts of $\bar\eta^{-2r}\,F$.

4 Holomorphic families of harmonic forms

In the previous section we have used the analytic perturbation theory of automorphic forms to construct meromorphic families of harmonic lifts of families $r \mapsto \bar\eta^{-2r} F$. In this section we modify these families to obtain holomorphic families of lifts, and extend them to larger domains than the disks $V_{0,N}$ in Theorem 3.5. This will bring us to the main result, Theorem 4.5.

4.1 Freedom in the choice of the lifts

The families $r \mapsto H_{N,r}$ may be far from unique. We have the freedom to add a meromorphic family of holomorphic modular forms.

Adding such a family $r \mapsto A_r$ on $V_{0,N}$ should not change the description of the Fourier terms of $H_{N,r}$ in (3.34) and (3.36). At ∞ the family $r \mapsto A_r$ should have a Fourier expansion of the form $\sum_{\nu\geq 1-N} c_\nu(r)\, q^{\nu+r/12}$, and at each $\zeta \in \mathcal{P}_Y$ an expansion $(z-\bar\zeta)^{-\ell-r} \sum_{\nu\geq 0} c_\nu(r)\, w^\nu$.

So A_r should not have singularities at points of $S \subset \mathfrak{H}$, and hence should be in $M^!_{\ell+r}(r)$. Each such A_r has the form $A_r = \eta^{2r-24m_\ell} E_k\, p(J)$ where $\ell = k - 12m_\ell$, $m_\ell \in \mathbb{Z}$, $k \in \{0,4,6,8,10,14\}$, with $E_0 = 1$ and E_k is the holomorphic Eisenstein series in weight k otherwise, and where $p(J)$ is a polynomial in the modular invariant J. The Fourier expansion of $\eta^{2r-24m_\ell} E_k$ starts with $q^{r/12-m_\ell}$, so the polynomial p should have degree at most $N-1-m_\ell$. This determines the dimension $N-m_\ell$ of the space of holomorphic modular forms that we can add.

It may happen that $N-m_\ell \le 0$. Then $r \mapsto H_{N,r}$ is the unique meromorphic family on $V_{0,N}$ of $(\ell+r)$-harmonic lifts of $r \mapsto \bar{\eta}^{-2r} F$ with expansions of the required type.

Let $N - m_\ell \ge 1$. We start with the holomorphic families $r \mapsto \eta^{2r-24m_\ell} E_k\, J^{a-m_\ell}$ on $\mathbb{C}$ of holomorphic modular forms, with $a \ge m_\ell$, and form successively linear combinations $r \mapsto j_{\ell,a,r}$, such that $j_{\ell,a,r}$ has a Fourier expansion of the form

$$j_{\ell,a,r}(z) = q^{r/12-a} + \sum_{\nu \ge 1-m_\ell} c_{\ell,a,\nu}(r)\, q^{\nu+r/12}. \tag{4.1}$$

These families and the coefficients $c_{\ell,a,\nu}$ are holomorphic on $\mathbb{C}$. We have the freedom to add to the family $r \mapsto H_{N,r}$ a meromorphic linear combination of the $j_{\ell,a,r}$ with $m_\ell \le a \le N-1$.

4.2 Removal of singularities

Proposition 4.1 *For each family $r \mapsto H_{N,r}$ as in Proposition 3.9 with expansions as described in (3.34) and (3.36) there is a meromorphic family $r \mapsto A_r$ on $V_{0,N}$ of meromorphic modular forms such that $r \mapsto H_{N,r} - A_r$ is a holomorphic family of harmonic forms that has expansions of the form indicated in (3.34) and (3.36).*

Proof We describe the Fourier terms $H^\infty_{N,\nu}$ with $-N < \nu < N$ in terms of explicit special functions. Specializing the family of functions in 4.2.5 in [5], we arrive at the following terms in the expansion at ∞:

$$\begin{aligned}&\text{for } -N < \nu < N:\\ &H^\infty_{N,\nu}(r;z) = \frac{a^\infty_{-\nu}(r)}{\ell+r-1}\, y^{1-\ell-r}\, q^{\nu+r/12}\, {}_1F_1\left(1-\ell-r; 2-\ell-r; 4\pi\left(\nu+\frac{r}{12}\right)y\right)\\ &\qquad + b^\infty_{N,\nu}(r)\, q^{\nu+r/12},\end{aligned} \tag{4.2}$$

with meromorphic functions $b^\infty_{N,\nu}$ on $V_{0,N}$. The basis functions that we use may have singularities at points of $V_{0,N} \cap \mathbb{Z}$, and the identities for $H^\infty_{N,\nu}$ are understood as identities of meromorphic functions of r.

Now we form the meromorphic family

$$\tilde{H}_{N,r}(z) = H_{N,r}(z) - \sum_{a=m_\ell}^{N-1} b^\infty_{N,-a}(r)\, j_{\ell,a,r}(z),$$

with $j_{\ell,a,r}$ as in (4.1). The new family $\tilde{H}_{N,r}$ has the same properties as $H_{N,r}$, with Fourier terms as in (4.2), but now with $b^\infty_{N,\nu} = 0$ for $1-N \le \nu \le -m_\ell$. If $m_\ell \ge N$, then $\tilde{H}_{N,r} = H_{N,r}$.

Suppose that $\tilde{H}_{N,r}$ has a singularity at $r_0 \in V_{0,N}$. Then there is an integer $k \geq 1$ such that the meromorphic family

$$q(r) = (r - r_0)^k \, \tilde{H}_{N,r} \tag{4.3}$$

is holomorphic on a neighborhood of r_0 in $V_{0,N}$, with non-zero value $q(r_0)$. The $(\ell + r_0)$-harmonic modular form $q(r_0)$ satisfies $\xi_{\ell+r_0} q(r_0) = (r_0 - r_0)^k \, \bar{\eta}^{-2r_0} \, F = 0$. So $q(r_0) \in M^{!!}_{\ell+r_0}(r_0)$. In its Fourier expansion there are non-zero multiples of $q^{\nu+r_0/12}$ with $\nu \leq -m_\ell$. On the other hand, the Fourier expansion of $\tilde{H}_{N,r}$ shows that $q(r_0)$ can have only multiples of $q^{\nu+r_0/12}$ with $\nu > -m_\ell$, unless the factor

$$\frac{a^{\infty}_{-\nu}(r)}{\ell + r - 1} \, {}_1F_1\left(1 - \ell - r; 2 - \ell - r; 4\pi\left(\nu + \frac{r}{12}\right)\right)$$

has a singularity at $r = r_0$. So if $r_0 \notin V_{0,N} \cap \mathbb{Z}$, then $\tilde{H}_{N,r}$ cannot have a singularity at r_0.

We have arrived at the knowledge that the family $r \mapsto \tilde{H}_{N,r}$ has at most finitely many singularities in $V_{0,N}$, which we can attack one by one. Suppose that $H_{N,r}$ has a singularity of order $k \geq 1$ at $r_0 \in V_{0,N}$. (It does not matter anymore that then r_0 is an integer.) Define q as in (4.3). Then $q(r_0) \in M^{!!}_{\ell+r_0}(r_0)$. Its Fourier expansion has no terms $q^{\nu+r_0/12}$ with $\nu \leq -N$ and at points $\zeta \in \mathcal{P}_Y$ it has a pole of order at most $-\mu_\zeta$. We subtract from $\tilde{H}_{N,r}$ the family

$$p_{r_0} : r \mapsto \frac{1}{(r - r_0)^k} \, \eta^{2(r-r_0)} \, q(r_0).$$

This is a meromorphic family on $\mathbb{C}$ of meromorphic modular forms. The family $r \mapsto \tilde{H}_{N,r} - p_{r_0}(r)$ satisfies the properties as $r \mapsto H_{N,r}$, and has at r_0 a singularity of order strictly less than k.

Proceeding in this way we remove all remaining singularities of the family $r \mapsto \tilde{H}_{N,r}$ in finitely many steps, thus completing the proof of the proposition. $\square$

4.3 Normalization

Now we can suppose that the family $r \mapsto H_{N,r}$ is holomorphic. If $m_\ell \leq N - 1$ there is still the freedom of adding holomorphic multiples of the families $j_{\ell,a,r}$ in (4.1). We use this freedom to normalize the Fourier expansion further, in order to compare the families for different values of N. To do this the confluent hypergeometric function in (4.2) is inconvenient, since it has singularities as a function of $\ell + r$. Instead we use the following function:

$$\begin{aligned} M_p(n; y) &= \frac{y^{1-p}}{p-1} \, {}_1F_1(1 - p; 2 - p; ny) + \sum_{k=0}^{\infty} \frac{n^k}{k! \, (1 + k - p)} \\ &= \int_{t=y}^{1} t^{-p} \, e^{nt} \, dt. \end{aligned} \tag{4.4}$$

This is holomorphic as a function of $p \in \mathbb{C}$. The sum on the first line converges absolutely for all $p \in \mathbb{C}$, defining a meromorphic function on $\mathbb{C}$ with the opposite principal parts as the term with the confluent hypergeometric function.

With these slightly more complicated basis functions we write (4.2), with $-N < \nu < N$ as

$$H^{\infty}_{N,\nu}(r;z) = \left(a^{\infty}_{-\nu}(r)\, M_{\ell+r}\left(4\pi\left(\nu+\frac{r}{12}\right);y\right) + \hat{b}^{\infty}_{N,\nu}(r)\right) q^{\nu+r/12)}, \tag{4.5}$$

with the convention that $a^{\infty}_{\nu} = 0$ for $\nu < \mu_\infty$. Subtracting suitable multiples of $j_{\ell,-\nu,r}$ with $1-N \le \nu \le -m_\ell$ we arrange that $\hat{b}^{\infty}_{N,\nu}(r) = 0$ for $1-N \le \nu \le -m_\ell$. Thus we arrive at the following normalization:

Proposition 4.2 *Let $\ell \in 2\mathbb{Z}_{\le 0}$, and let $F \in M^{!!}_{2-\ell}(0)$, with singularities in the set S of the form $\Gamma\mathcal{P}_Y$. Let $\mu_\infty \in \mathbb{Z}$ and $\mu_\zeta \le -1$ for $\zeta \in \mathcal{P}_Y$ be as in (3.29). For each $N \ge \max(1, 1-\mu_\infty)$ there is a neighborhood $V_{0,N}$ of 0 in $\mathbb{C}$ with the properties in Lemma 3.7, and on $V_{0,N}$ a holomorphic family $r \mapsto H_{N,r}$ of $(\ell+r)$-harmonic modular forms in $H^{!!}_{\ell+r}(r)$ such that $\xi_{\ell+r}H_{N,r} = \bar{\eta}^{-2r}F$, uniquely determined by having near ∞ a Fourier expansion of the form*

$$\begin{aligned} H_{N,r}(z) = & \sum_{\nu \le -\max(N,\mu_\infty)} a^{\infty}_{-\nu}(r)\left(-4\pi\left(\nu+\frac{r}{12}\right)\right)^{\ell+r-1} q^{\nu+r/12} \\ & \cdot \Gamma\left(1-\ell-r, -4\pi\left(\nu+\frac{r}{12}\right)y\right) \\ & + \sum_{\nu=1-N}^{-\mu_\infty} a^{\infty}_{-\nu}(r)\, q^{\nu+r/12}\, M_{\ell+r}\left(4\pi\left(\nu+\frac{r}{12}\right);y\right) \\ & + \sum_{\nu \ge 1-m_\ell} b^{\infty}_{N,\nu}(r)\, q^{\nu+r/12}. \end{aligned} \tag{4.6}$$

The functions $b_{N,\nu}$ are holomorphic on $V_{0,N}$.

The holomorphic functions a_ν on $\mathbb{C}$ depend on F by (3.29). The integer $m_\ell \ge 0$ is defined in Sect. 4.1. See (4.4) for the functions $M_{\ell+r}$.

We note that there may be an overlap in the ranges of the variable ν in the sums in (4.6), and that the sum over $1-N \le \nu \le -\mu_\infty$ may be empty.

From this point on we use this normalization of $H_{N,r}$ and the functions $b^{\infty}_{N,\nu}$. The functions $b^{\infty}_{N,\nu}$ are not known explicitly. The choice of $M_{\ell,\nu}$ is not canonical. So this normalization is non-canonical as well.

Since we deal with real-analytic functions on $\mathfrak{H} \setminus S$, the expansion near ∞ determines the family completely. At points $\zeta \in \mathcal{P}_Y$ we have expansions like in (3.35), with terms that are holomorphic on $V_{0,N}$.

4.4 Extension

The normalization in Proposition 4.2 is convenient for the comparison of $H_{N,r}$ and $H_{N+1,r}$. The difference $H_{N+1,r} - H_{N,r}$ is a holomorphic family of holomorphic modular forms on $V_{0,N}$:

Lemma 4.3 *Let* $N \geq \max(1, 1 - \mu_\infty)$. *If* $N < \max(m_\ell, \mu_\infty)$ *then* $H_{N+1,r} = H_{N,r}$ *for* $r \in V_{0,N}$. *If* $N \geq \max(m_\ell, \mu_\infty)$, *then we have for* $r \in V_{0,N}$:

$$H_{N+1,r} = H_{N,r} - a^\infty_{-N}(r) \left(4\pi\left(N - \frac{r}{12}\right)\right)^{\ell+r-1} \Gamma\left(1-\ell-r, 4\pi\left(N-\frac{r}{12}\right)\right) j_{\ell,N,r}. \tag{4.7}$$

Proof The difference

$$\begin{aligned} &H_{N+1,r}(z) - H_{N,r}(z) \\ &\quad = \sum_{\nu \geq 1-m_\ell} \left(b^\infty_{N+1,\nu}(r) - b^\infty_{N,\nu}(r)\right) q^{\nu+r/12} \\ &\quad\quad + \begin{cases} a^\infty_{-N}(r)\, q^{-N+r/12}\, (M_{\ell+r}(4\pi(-N+\frac{r}{12}); y) \\ \quad - (-4\pi(-N+\frac{r}{12}))^{\ell+r-1} \\ \quad \times \Gamma(1-\ell-r, -4\pi(-N+\frac{r}{12})y)) & \text{if } N \geq \mu_\infty, \\ 0 & \text{if } N < \mu_\infty \end{cases} \end{aligned}$$

is a holomorphic family on $V_{0,N}$ of holomorphic modular forms. If $N < m_\ell$ or if $N < \mu_\infty$, then it has non-zero Fourier terms only of order $\nu \geq 1 - m_\ell$, hence it vanishes. If $N \geq m_\ell$ and $N \geq \mu_\infty$, then a computation shows that the starting term in the Fourier expansion is equal to

$$-\left(-4\pi\left(-N+\frac{r}{12}\right)\right)^{\ell+r-1} \Gamma\left(1-\ell-r, -4\pi\left(-N+\frac{r}{12}\right)\right) a^\infty_{-N}(r)\, q^{-N+r/12},$$

and the other terms have order $\nu \geq 1 - m_\ell$. So $H_{N+1,r} - H_{N,r}$ is equal to the multiple of $j_{\ell,N,r}$ indicated in the lemma. □

Lemma 4.4 *The family* $r \mapsto H_{N,r}$ *in Proposition* 4.2 *extends as a holomorphic family on* $\mathbb{C} \setminus [12M, \infty)$, *where* $M = \max(N, m_\ell, \mu_\infty)$.

Proof All $H_{N,r}$ with $N < \max(m_\ell, \mu_\infty)$ have a holomorphic extension to V_{0,N_1}, with $N_1 = \max(m_\ell, \mu_\infty)$. The function $w \mapsto w^{\ell+r-1}\, \Gamma(1-\ell-r, w)$ can be extended as a single-valued holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$, but not further for general values of $\ell + r$. Lemma 4.3 implies that for $N \geq N_1$ the family $H_{N,r}$ has a holomorphic extension to $V_{0,N+1} \setminus [12N, \infty)$. Applying this successively, we get for $N \geq N_1$ the holomorphic extension of $H_{N,r}$ to $\mathbb{C} \setminus [12N, \infty)$. □

Theorem 4.5 *Let F be an antiholomorphic form in $\bar{M}^{!!}_{2-\ell}(\mathrm{SL}_2(\mathbb{Z}), v_0)$ for the trivial multiplier system $v_0 = 1$, with weight $\ell \in 2\mathbb{Z}$. Let $M \in \mathbb{Z}$ be such that $M > -\mu_\infty$, where $F(z) = \sum_{\nu \geq \mu_\infty} a_\nu \bar{q}^\nu$ is the Fourier expansion of F near ∞. Then there is a holomorphic family $r \mapsto \mathsf{H}_{M,r}$ on $\mathbb{C} \setminus [12M, \infty)$ of $(\ell + r)$-harmonic modular forms satisfying $\xi_{\ell+r}\mathsf{H}_{M,r} = \bar{\eta}^{-2r} F$ for all $r \in \mathbb{C} \setminus [12M, \infty)$.*

The family $\mathsf{H}_{M,r}$ can be chosen uniquely by prescribing a Fourier expansion of the form

$$\begin{aligned} \mathsf{H}_{M,r}(z) = \sum_{\nu \leq -\max(M,\mu_\infty)} & a^\infty_{-\nu}(r) \left(-4\pi\left(\nu + \frac{r}{12}\right)\right)^{\ell+r-1} q^{\nu+r/12} \\ & \cdot \Gamma\left(1-\ell-r, -4\pi\left(\nu+\frac{r}{12}\right)y\right) \\ & + \sum_{\nu=1-M}^{-\mu_\infty} a^\infty_{-\nu}(r)\, q^{\nu+r/12}\, M_{\ell+r}\left(4\pi\left(\nu+\frac{r}{12}\right); y\right) \\ & + \sum_{\nu \geq 1-m_\ell} b^\infty_{M,\nu}(r)\, q^{\nu+r/12}. \end{aligned} \tag{4.8}$$

The function $M_{\ell+r}$ is defined in (4.4). The coefficients $a^\infty_{-\nu}$ are holomorphic functions on $\mathbb{C}$ occurring in the Fourier expansion $(\bar{\eta}^{-2r} F)(z) = \sum_{\nu \geq \mu_\infty} a^\infty_\nu(r)\, \bar{q}^{\nu - r/12}$. The quantity $m_\ell \in \mathbb{Z}$ is defined by $\ell + 12 m_\ell \in \{0, 4, 6, 8, 10, 14\}$.

The holomorphic functions $b_{M,\nu}$ on $\mathbb{C} \setminus [12M, \infty)$ are not known explicitly. The middle sum in (4.8) may be empty. We note that $a^\infty_{\mu_\infty}(r) = a_{\mu_\infty}$, which we can assume to be non-zero.

Proof We denote by $\tilde{F} \in \bar{M}^{!!}_{2-\tilde{\ell}}(v_0)$ the antiholomorphic modular form in the theorem, and will apply the earlier result to

$$F = \tilde{F}\, \bar{\Delta}^{-p} \in \bar{M}^{!!}_{2-\ell}(v_0),$$

with $p \in \mathbb{Z}$ not yet fixed. Denoting the quantities related to $\tilde{F}$ by a tilde we have

$$\begin{aligned} \ell &= \tilde{\ell} + 12p, & m_\ell &= m_{\tilde{\ell}} - p, \\ \mu_\infty &= \tilde{\mu}_\infty - p, & a_\nu(r) &= \tilde{a}_{\nu+p}(r + 12p). \end{aligned}$$

We take $N = M - p$ and choose

$$p \leq \min\left(-\frac{\tilde{\ell}}{12}, M-1, \frac{M-1+\tilde{\mu}_\infty}{2}\right).$$

Then $\ell \leq 0$ and $N \geq \max(1, 1-\mu_\infty)$. We apply Proposition 4.2 and Lemma 4.4 to F, and take $\mathsf{H}_{M,r} = H_{N,r-12p}$. Then $\xi_{\tilde{\ell}+r}\mathsf{H}_{M,r} = \xi_{\ell+r-12p} H_{N,r-12p} = \bar{\eta}^{-2r}\tilde{F}$ for $r \in \mathbb{C} \setminus [12M, \infty)$. □

4.5 Remarks

4.5.1 *Comparison: use of Poincaré series and use of perturbation theory*

The existence of a harmonic lift of a single antiholomorphic modular form can be proved with Poincaré series in the case of a real weight and a unitary multiplier system. If the weight is larger than 2 the Poincaré series converge absolutely, and the construction gives an explicit expression for the lift (with Kloosterman sums and Bessel functions). Outside the region of absolute convergence analytic extension is needed anyhow, and requires a careful analysis of the properties of this continuation. The approach in Sect. 3 uses a more general result, and works generally. I do not know another method that allows the handling of complex weights.

4.5.2 *Use of Hodge theory*

The approach in Sect. 3 of [3] is not restricted to the case of $\bar{M}^!_{2-p}(\mathrm{SL}_2(\mathbb{Z}), v_1)$. Jan Bruinier notes that singularities at points of $\mathfrak{H}$ can be accommodated in the divisor D used in the proof of Theorem 3.7 in [3], and that one may be able to handle multiplier systems v_r with rational values of r.

4.5.3 *Existence only*

Theorem 4.5 is an existence result. It gives an overview of harmonic lifts and organizes them in families. It does not give explicit knowledge of the lifts.

Sometimes we know explicitly a harmonic function by other means, and may be able to identify it as a member of a family. See Sect. 5 for some examples.

4.5.4 *Generalization*

Theorem 4.5 is stated only for the discrete group $\mathrm{SL}_2(\mathbb{Z})$, since for that case I have checked the details. I expect that a similar theorem can be proved for any cofinite discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ with cusps. For cocompact groups generalization seems much harder.

For cofinite groups Γ with cusps the group of multiplier systems is a commutative complex Lie group, with finite dimension; its dimension is 1 for $\mathrm{SL}_2(\mathbb{Z})$. The parameter r in this paper is essentially an element of the Lie algebra of the group of multiplier systems. The parameter φ used in [5] can be viewed as running through the Lie algebra of the group of multiplier systems. The results in Sect. 3 probably go through with open sets $V_{0,N}$ in that Lie algebra as parameter space. To transform meromorphic families into holomorphic families with the method of Sect. 4 is probably very hard if the dimension of the parameter space is larger than 1. For that purpose I think it might be wise to work with one-dimensional subvarieties of the parameter space.

4.6 Mock modular forms

In the Fourier expansion (4.6) of the normalized family $\mathsf{H}_{M,r}$ it seems natural to put

$$\begin{aligned} \mathsf{C}_{M,r}(z) = \sum_{\nu \le -\max(M,\mu_\infty)} & a_{-\nu}^{\infty}(r) \left(-4\pi\left(\nu+\frac{r}{12}\right)\right)^{\ell+r-1} q^{\nu+r/12} \\ & \cdot \Gamma\left(1-\ell-r, -4\pi\left(\nu+\frac{r}{12}\right)y\right) \\ & + \sum_{\nu=1-M}^{-\mu_\infty} a_{-\nu}^{\infty}(r)\, M_{\ell+r}\left(4\pi\left(\nu+\frac{r}{12}\right); y\right) q^{\nu+r/12} \end{aligned} \tag{4.9}$$

as the part of the expansion arising from $\bar{\eta}^{-2r} F$, and the remaining part

$$\mathsf{M}_{M,r}(z) = \sum_{\nu \ge 1-m_\ell} b_{N,\nu}^{\infty}(r)\, q^{\nu+r/12} \tag{4.10}$$

as the corresponding family of *mock modular forms*. This splitting depends on the choice of the basis vector $M_{\ell+r}(\cdot\,;\cdot)$ in the $(\ell+r)$-harmonic terms with $\nu \ge 1-m_\ell$. Another choice leads to another splitting.

If the set S of singularities in $\mathfrak{H}$ of F is non-empty, the series in (4.10) for $\mathsf{M}_{M,r}(z)$ defines a holomorphic function only on the region $\operatorname{Im} z > y_S$ where y_S is the maximum value om $\operatorname{Im}\zeta$ as ζ runs through S. It seems to be unknown whether the functions $\mathsf{M}_{N,r}$ and $\mathsf{C}_{N,r}$ have an analytic extension to a larger region in $\mathfrak{H}$.

The expansion of $\mathsf{H}_{M,r}$ at ζ in the system of representatives $\mathcal{P}_Y$ of $\Gamma\backslash S$ gives rise to a splitting $\mathsf{H}_{M,r} = M + C$ on a pointed neighborhood of ζ, and seems not to have a relation to the splitting $\mathsf{H}_{M,r} = \mathsf{M}_{M,r} + \mathsf{C}_{M,r}$.

My conclusion is that the concept of mock modular forms is still unclear in the generality of families of modular forms considered in this note.

5 Harmonic lift of eta-powers

As an example we look at $\bar{\eta}^{-2r}$ for r close to 0 in $\mathbb{C}$. Theorem 4.5, with the choice $1 \in \bar{M}_{2-2}(\mathrm{SL}_2(\mathbb{Z}), v_0)$ provides us with the family $r \mapsto \mathsf{H}_r := \mathsf{H}_{1,r}$ on $\mathbb{C}\setminus[12,\infty)$ of $(r+2)$-harmonic lifts of $\bar{\eta}^{-2r}$. We identify it with known harmonic lifts for certain values of r.

We note that

$$\bar{\eta}^{-2r} = \sum_{\nu\ge 0} p_\nu(-r)\, \bar{q}^{\nu-r/12}, \tag{5.1}$$

with polynomials p_ν of degree ν with rational coefficients. A first order expansion of $e^{2r\log\eta}$ at $r=0$ shows, with use of (2.1)

$$p_0 = 1, \quad \text{and} \quad \text{for } \nu \ge 1\text{:} \quad p_\nu(0) = 0 \quad \text{and} \quad p_\nu'(0) = -2\sigma_{-1}(\nu). \tag{5.2}$$

Theorem 4.5 gives the following Fourier expansion:

$$\mathsf{H}_r(z) = \sum_{\nu\le -1} p_{-\nu}(-r)\left(-4\pi\left(\nu+\frac{r}{12}\right)\right)^{r+1} q^{\nu+r/12}\,\Gamma\left(-1-r,-4\pi\left(\nu+\frac{r}{12}\right)y\right)$$
$$+\left(M_{2+r}\left(\frac{\pi r}{3};y\right)+b_0(r)\right)q^{r/12}+\sum_{\nu\ge 1} b_\nu(r)\,q^{\nu+r/12}. \tag{5.3}$$

The holomorphic functions b_ν on $\mathbb{C}\setminus[12,\infty)$ are unknown.

For three values of r we mention constructions of $(r+2)$-harmonic modular lifts of $\bar\eta^{-2r}$. If $r\in(-12,12)$ the sole term of H_r that q is not exponentially decreasing at ∞ is the term of order $\frac{r}{12}$. If the other lift also has this term as the only non-decreasing one, that lift coincides with H_r.

Lift of 1 A well known 2-harmonic lift of $1=\bar\eta^0$ is the non-holomorphic Eisenstein series

$$E_2^{\mathrm{nh}}(z) = y^{-1}-\frac{\pi}{3}+8\pi\sum_{\nu\ge 1}\sigma_1(\nu)\,q^\nu. \tag{5.4}$$

We have $M_2(0;y)=y^{-1}-1$. The terms with q^ν, $\nu\ge 1$ are quickly decreasing. We conclude that $\mathsf{H}_0=E_2^{\mathrm{nh}}(z)$, and find

$$b_0(0)=1-\frac{\pi}{3},\qquad b_\nu(0)=8\pi\,\sigma_1(\nu)\quad\text{for }\nu\ge 1. \tag{5.5}$$

So we have identified the value of the family at $r=0$ with a known 2-harmonic modular form.

In this case we can proceed a bit further. The computation used in Sect. 6.4 of [6] to produce an explicit example of a second order Maass form can be modified to get information on the derivative $\frac{d}{dr}\mathsf{H}_r|_{r=0}$. In this way one can arrive at the following result:

$$b_\nu'(0) = \begin{cases} -16\pi\sum_{\mu=1}^{\nu-1}\sigma_{-1}(\mu)\,\sigma_1(\nu-\mu)-8\pi\sum_{d|\nu}\frac{\nu}{d}\log\frac{d^2}{\nu} & \\ \quad -8\pi\,(1+\gamma-\log 4\pi)\,\sigma_1(\nu)+\frac{2\pi}{3}\,\sigma_{-1}(\nu), & \text{if } \nu\ge 1,\\ -1+\frac{\pi}{3}(2\gamma-\log 4)-\frac{4}{\pi}\zeta'(2) & \text{if } \nu=0, \end{cases} \tag{5.6}$$

where γ denotes Euler's constant.

Lift of η^3 This case can be related to an example of a mock modular form in [14]. The unary theta function $g_{a,b}$ in Proposition 1.15 in [14] with $a=b=\frac12$ gives $g_{1/2,1/2}=\eta^3$. (This follows from the transformation behavior of $g_{a,b}$ and inspection of its Fourier expansion.) The completed Lerch sum $\tilde\mu$ in Theorem 1.11 in [14] gives a $\frac12$-harmonic lift

$$z\mapsto\frac{\sqrt2}{3i}\left(\tilde\mu\left(\frac12,\frac12;z\right)+\tilde\mu\left(\frac z2,\frac z2;z\right)+\tilde\mu\left(\frac{z+1}{2},\frac{z+1}{2};z\right)\right) \tag{5.7}$$

of $\bar{\eta}^3$. Since the term with $q^{-1/8}$ is the sole increasing term (as $y \to \infty$), this lift is equal to $\mathsf{H}_{-3/2}(z)$.

Lift of η^4 The fourth power η^4 spans the space of holomorphic cusp forms for the commutator subgroup $\Gamma_{\text{com}} = [\mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z})]$.

The holomorphic function H on $\mathfrak{H}$ given by

$$H(\tau) = -2\pi i \int_{\infty}^{z} \eta^4(\tau)\, d\tau \tag{5.8}$$

has the transformation behavior $H(\gamma\tau) = H(\tau) + \lambda(\gamma)$ for some group homomorphism $\lambda : \Gamma_{\text{com}} \to \mathbb{C}$. The function $C(z) = \frac{-1}{4\pi}\bar{H}$ satisfies $\xi_0 C = \bar{\eta}^4$. In Sect. 4.3.1 in [6] we see that there can be found a linear combination M of the holomorphic functions H and $z \mapsto \zeta(H(z))$, where ζ is the Weierstrass zeta-function for an appropriate lattice, such that $M + C$ is a Γ_{com}-invariant harmonic lift of $\bar{\eta}^4$. The average $\sum_{n \bmod 6} e^{\pi i n/3}\,(M + C)|_0 T^n$ has the desired transformation behavior under $\mathrm{SL}_2(\mathbb{Z})$. Inspection of the growth of the Fourier terms of $M(z) + C(z)$ as $\operatorname{Im} z \to \infty$ shows that $M + C$ is equal to H_{-4}.

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