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ESTIMATES OF KLOOSTERMAN SUMS FOR GROUPS OF REAL RANK ONE

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1. The sum formula of Kuznetsov

1.1. Introduction. In [10] Kuznetsov gave a sum formula in which Fourier coefficients of real analytic modular forms on the upper half-plane are related to Kloosterman sums; see also [11]. This formula has been used in various ways. In [11] it is applied to the classical Kloosterman sums $S(n, m; k) = \sum_x^* e^{2\pi i(nx+m\bar{x})/k}$, where $n, m, k \in \mathbb{Z}$, $n, m \neq 0$, $k > 0$; in the sum, x runs over the integers $0 < x \leqslant k$ that are coprime to k, and satisfy $x\bar{x} \equiv 1 \bmod k$. Kuznetsov shows that

$$\sum_{1 \leqslant k \leqslant X} \frac{1}{k} S(n, m; k) = O(X^{1/6}(\log X)^{1/3}) \qquad (X \to \infty); \tag{1}$$

see Theorem 3 in [11]. This type of result is the main theme of this paper.

Kuznetsov's sum formula is concerned with automorphic forms on the group $SL_2(\mathbb{R})$. It has been extended in various ways. Its extension in [17] treats automorphic forms on Lie groups of real rank one. We use it to study sums of Kloosterman sums for this class of groups.

The main structure of the sum formula is

$$\int_{\mathscr{S}} h(\nu)\, d\sigma(\nu) = \int_{i\mathbb{R}} h(\nu)\, d\delta(\nu) + \sum_{\gamma} S(\gamma)\tilde{h}(\xi_\gamma). \tag{2}$$

Here $d\sigma$ is a measure with support $\mathscr{S} \subset i[0, \infty) \cup (0, \infty)$; this measure can be described in terms of Fourier coefficients of automorphic forms for a discrete subgroup Γ of the Lie group G under consideration. The γ run over a subset of Γ, the $S(\gamma)$ are generalized Kloosterman sums, and $\xi_\gamma \in G$ is determined by γ. The measure $d\delta$ is supported on the line $\operatorname{Re} \nu = 0$.

Before describing the sum formula more precisely, we remark that it can be used in two directions. One way is to focus on the spectral term $\int_{\mathscr{S}} h(\nu)\, d\sigma(\nu)$. One can get information concerning the measure $d\sigma$ by taking a suitable test function and estimating the two other terms in the sum formula; see Section 2. On the other hand, for another choice of h, it may be possible to give an estimate of the spectral term and the delta term $\delta(h)$ and end up with an estimate of sums of

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Kloosterman sums. So it is essential to know how h and $\tilde{h}$ depend on each other. This dependence is complicated. Ideally, one would like the transformation $h \mapsto \tilde{h}$ to be a bijection between two well-defined, not too small classes of functions. An idea of Good [6] is not to try to invert $h \mapsto \tilde{h}$ explicitly, but make do with an approximate inversion. The purpose of this paper is to develop this approach for groups of real rank one.

We obtain an estimate of sums of generalized Kloosterman sums; see Theorem 1 in 4.3. The bounds between which the sum is taken are smoothed with help of a test function. In the absence of exceptional eigenvalues, this leads to a good bound. In the case considered by Kuznetsov, $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the bound is $O((\log X)^2)$. It is $O(X^\varepsilon)$ for $G = \mathrm{SL}_2(\mathbb{C})$. To obtain an estimate of sums with sharp bounds we need an assumption on the growth of the Kloosterman sums; see Proposition 4 in Section 4.7. A variation of this assumption is satisfied in the case considered by Kuznetsov, and leads to the estimate (1). We study in some detail the case $G = \mathrm{SL}_2(\mathbb{C})$ and $\Gamma = \mathrm{SL}_2(\mathcal{O})$, with $\mathcal{O}$ the ring of integers in an imaginary quadratic number field, to show that this assumption is satisfied. To do that we extend classical estimates of Kloosterman sums to those defined over a number field; see Section 5.

The estimate with smooth bounds shows clearly that there is much cancellation in or between Kloosterman sums. The step from smooth to sharp bounds is done by estimating individual Kloosterman sums, so it does not take cancellation between Kloosterman sums into account. This leads to an error term that is considerably larger.

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1.2. Description of the sum formula for real rank one. In the description of the sum formula in [17], we restrict ourselves to the aspects that we need for the purpose of this paper.

We work with a real, connected semisimple Lie group G of $\mathbb{R}$-rank 1 and a discrete subgroup Γ of G such that $\Gamma\backslash G$ has finite volume but is not compact. Moreover, Γ is supposed to satisfy the assumption on p. 16 of [13].

The test function h has to be even and holomorphic on some strip $|\mathrm{Re}\, \nu| \leqslant \sigma$, on which

$$h(\nu) = O\left(\frac{e^{-\pi|\mathrm{Im}\, \nu|}}{(1 + |\mathrm{Im}\, \nu|)^a}\right), \tag{3}$$

with $a > 2$. In some versions of the sum formula, but not in the version in [17], the support of $d\sigma$ protrudes out of the strip. In that case the domain of h should be larger, and one may need an additional growth condition. The number σ is slightly larger than a positive quantity ρ depending on the group; see §1 of [17]. We take $\sigma \notin (1/2)\mathbb{Z}$. In the version of the sum formula under consideration, $\mathscr{S}$, the support of $d\sigma$, intersects $(0, \infty)$ in a finite number of points contained in $(0, \rho)$. Note that $\rho \notin \mathscr{S}$.

On the spectral side we have to know that $d\sigma$ depends on the choice of two percuspidal parabolic groups P and P', and nontrivial unitary characters χ and χ' of the associated unipotent groups N and N'. So $d\sigma = d\sigma_{P,P'}^{\chi,\chi'}$. If $P \neq P'$, they are assumed to be not Γ-conjugate. We shall use that $d\sigma_{P,P}^{\chi,\chi}$ is nonnegative, and that

$$\left|\int_{\mathscr{S}} f(\nu)\, d\sigma_{P,P'}^{\chi,\chi'}(\nu)\right| \leqslant \left(\int_{\mathscr{S}} |f(\nu)|\, d\sigma_{P,P}^{\chi,\chi}(\nu)\right)^{1/2}\left(\int_{\mathscr{S}} |f(\nu)|\, d\sigma_{P',P'}^{\chi',\chi'}(\nu)\right)^{1/2} \tag{4}$$

if f is integrable for both $d\sigma_{P,P}^{\chi,\chi}$ and $d\sigma_{P',P'}^{\chi',\chi'}$.

The measure $d\delta$ vanishes, unless $P = P'$ and $\chi = \chi'$. In that case it is of the form $\int h(\nu)\, d\delta(\nu) = iA \int_{\mathrm{Re}\, \nu = 0} h(\nu)\nu \sin \pi\nu\, d\nu$. The constant A depends on χ and P, but there is a bound valid for all P and χ; see part (i) in Proposition 1.2 of [17]. We shall see that $A \geqslant 0$.

Let K be a maximal compact subgroup of G such that $P = NAM$ is the decomposition of P in unipotent subgroup N, $\mathbb{R}$-split torus A, and centralizer M of A in K. Let α be the corresponding simple root. We write half the sum of the positive roots as $\rho\alpha$, with $\rho \in \mathbb{R}$. All characters of A are of the form $a \mapsto a^{\nu\alpha}$ with $\nu \in \mathbb{C}$. Often we write a^{ν} instead of $a^{\nu\alpha}$.

The big cell of the Bruhat decomposition is Ps^*N, with s^* a representative of the nontrivial Weyl group element. Each $g \in Ps^*N$ has a decomposition $g = n_1(g)a_g m_g s^* n_2(g)$, with $n_j(g) \in N$, $a_g \in A$, and $m_g \in M$. Denote $\Gamma_N = N \cap \Gamma$ and $\Gamma_{N'} = N' \cap \Gamma$. The unitary character χ of N is trivial on Γ_N, and χ' is trivial on Γ'_N. Write $P' = kPk^{-1}$, with $k \in K$. Then $(\chi')^k\colon n \mapsto \chi'(knk^{-1})$ is a unitary character χ_1 of N.

The γ occurring in the Kloosterman term of the sum formula are elements of the set $\Gamma \cap Ps^*k^{-1}N' = \Gamma \cap Ps^*Nk^{-1}$. The corresponding $m_{\gamma k}a_{\gamma k}$ form a discrete set Ξ in MA. For a given $\gamma \in \Gamma \cap Ps^*k^{-1}N'$ the set $\{\gamma' \in \Gamma \cap Ps^*k^{-1}N' \colon m_{\gamma' k}a_{\gamma' k} = m_{\gamma k}a_{\gamma k}\}$ consists of finitely many double cosets $\Gamma_N \gamma' \Gamma_{N'}$. The *generalized Kloosterman sum* is

$$S(\gamma) = S_{P,P'}(\chi, \chi'; \gamma) = \sum_{\delta} \chi(n_1(\delta k))\chi'(kn_2(\delta k)k^{-1}), \tag{5}$$

where δ runs through representatives of those double cosets. $S(\gamma)$ depends only on the corresponding $\xi_\gamma = m_{\gamma k}a_{\gamma k} \in \Xi$. So we may as well write $S(\xi_\gamma)$. The *Kloosterman term* in the sum formula is $\sum_{\xi \in \Xi} S(\xi)\tilde{h}(\xi)$.

Write $\xi = m_\xi a_\xi$. For a given $\xi \in \Xi$, let $S_0(\xi)$ be the number of terms occurring in the definition of the Kloosterman sum, (5). The convergence of the Eisenstein series implies that the series $\sum_{\xi \in \Xi} S_0(\xi)(a_\xi)^{\rho+\sigma}$ converges. This implies that the a_ξ^{α} stay under some positive bound B. Note that $|S(\xi)| \leqslant S_0(\xi)$ for all characters χ, χ'.

1.3. Examples. Take $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $P = P' = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, $\chi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{2\pi i m_1 x}$, $\chi'\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{2\pi i m_2 x}$, with $m_1, m_2 \in \mathbb{Z}\setminus\{0\}$. We have $A = \left\{\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t > 0\right\}$, $\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}^{\alpha} = t^2$, $\rho = 1/2$, and $k = I$. Take $K = \mathrm{SO}(2)$, then $M = \{\pm I\}$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \cap P\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}N$, we have $n_1(\gamma) = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix}$, $n_2(\gamma) = \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$, $m_\gamma = \mathrm{sign}(c)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $a_\gamma = \begin{pmatrix} 1/|c| & 0 \\ 0 & |c| \end{pmatrix}$. For $\xi = \pm\begin{pmatrix} 1/k & 0 \\ 0 & k \end{pmatrix}$ with $k \in \mathbb{N}$, we find the classical Kloosterman sum $S(\xi) = S(m_1, m_2; k)$. Note that $a_\xi^\alpha = k^{-2} \leqslant 1$; so the a_ξ^α form a bounded discrete set of positive numbers. Instead of estimating $\sum_{1 \leqslant k \leqslant X} S(n, m; k)/k$ as $X \to \infty$, we shall consider $\sum_{a_\xi^\alpha \geqslant x} a_\xi^\rho S(\xi)$ as $x \downarrow 0$.

Another example is $G = \mathrm{SL}_2(\mathbb{C})$. We take $\Gamma = \mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in some imaginary quadratic field extension F of $\mathbb{Q}$. We take $P = P' = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, $\chi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{2\pi i \mathrm{Tr}(m_1 x)}$, $\chi'\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{2\pi i \mathrm{Tr}(m_2 x)}$, with $m_1, m_2 \in L\setminus\{0\}$. Let Tr be the trace of F over $\mathbb{Q}$; to have the characters trivial on $\Gamma \cap N$, the condition $\mathrm{Tr}(m_j x) \in \mathbb{Z}$ for all $x \in \mathcal{O}$ is needed. We have $k = I$, $A = \left\{\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t > 0\right\}$, and $\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}^{\alpha} = t^2$. As α occurs in N with multiplicity 2, we have $\rho = 1$ in this case. Take $K = \mathrm{SU}(2)$; then $M = \left\{\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : |t| = 1\right\}$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \cap P\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}N$, we have $n_1(\gamma) = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix}$, $n_2(\gamma) = \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$, $m_\gamma = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$ with $u = \dfrac{c}{|c|}$, and $a_\gamma = \begin{pmatrix} 1/|c| & 0 \\ 0 & |c| \end{pmatrix}$. For $\xi = \begin{pmatrix} 1/u & 0 \\ 0 & u \end{pmatrix}$ with $u \in \mathcal{O}\setminus\{0\}$, we find $S(\xi) = \sum_v e(\mathrm{Tr}((m_1 \bar{v} + m_2 v)/u))$, where v runs over representatives of $\mathcal{O}/(u)$ for which $\bar{v} \in \mathcal{O}$ can be found such that $\bar{v}v \in 1 + (u)$; here (u) denotes the ideal $u\mathcal{O}$ and $e(y) = e^{2\pi i y}$.

We shall return to these examples. We indicate them by (R), respectively (C).

Kloosterman sums for groups locally isomorphic to $\mathrm{SO}(1, n)^0$, $n \geqslant 3$, are discussed in [3] and [1]. For $\mathrm{SU}(1, n)$ see [14].

1.4. Transformation. The test function $\tilde{h}$ depends on h in the following way:

$$\tilde{h}(ma) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,\nu=0} h(\nu)g(\nu)a^{\nu+\rho}\tau(\chi_1, \chi, ma, \nu)\,d\nu, \tag{6}$$

where $g(\nu)$ is a product of exponentials and gamma functions without zeros on $\mathrm{Re}\,\nu \geqslant 0$, and $\tau(\chi_1, \chi, ma, \nu)$ is a complicated function. Of τ and g we need to know that g is holomorphic on $|\mathrm{Re}\,\nu| \leqslant \sigma$, and τ is so on $-1/2 < \mathrm{Re}\,\nu \leqslant \sigma$, that they satisfy the estimates (8) and (7), and that g has a finite number of zeros in the strip $|\mathrm{Re}\, \nu| \leqslant \sigma$, all situated in $[-\rho, 0]$. There is a simple zero at $\nu = 0$. The other zeros may have an order larger than one. Always, $g(-\rho) = 0$.

The function g satisfies

$$|g(\nu)| \sim (\text{constant}) \frac{M^{\mathrm{Re}\,\nu} e^{\pi|\mathrm{Im}\,\nu|}}{(1 + |\mathrm{Im}\,\nu|)^{2\,\mathrm{Re}\,\nu + \kappa}} \qquad (|\mathrm{Re}\,\nu| \leqslant \sigma) \tag{7}$$

for some $\kappa \geqslant -1/2$ depending on the group and some $M > 0$ depending on P, P', χ, and χ'.

In Proposition 12, in Section 6, we shall give an asymptotic expansion for τ. As a special case of it we obtain (38), which implies

$$\tau(\chi_1, \chi, ma, \nu) = 1 + O\left(\frac{a^\alpha}{1 + |\mathrm{Im}\,\nu|}\right) \qquad (a^\alpha \downarrow 0) \tag{8}$$

on $-\varepsilon \leqslant \mathrm{Re}\,\nu \leqslant \sigma$ for each $\varepsilon \in (0, 1/2)$. The implicit constant depends on the choice of ε and σ.

The constant M depends on the characters in a simple way, but the dependence of τ on the characters is complicated. If one wants to make explicit the dependence on the characters for all estimates in the paper, then the study of τ is the hardest part.

For case (R) in the *examples* of 1.3 we have $\rho = 1/2$, and

$$\tau(\chi_1, \chi, ma, \nu) = \sum_{k=0}^{\infty} \frac{(-4\pi^2 m_1 m_2)^k a^k}{k!(2\nu + 1)_k}$$

$$g(\nu) = \frac{2(4\pi^2|m_1 m_2|)^\nu}{\pi\Gamma(2\nu)};$$

hence $M = 4\pi^2|m_1 m_2|$, $\kappa = -1/2$, and g has a simple zero at $\nu = -\rho$. Note that τ can be expressed in terms of a Bessel function. We shall not use this fact.

In case (C), $\rho = 1$ and

$$\begin{pmatrix} 1/u & 0 \\ 0 & u \end{pmatrix}^{\nu+\rho} \tau\left(\chi_1, \chi, \begin{pmatrix} 1/u & 0 \\ 0 & u \end{pmatrix}, \nu\right) = |u|^{2+2\nu} \sum_{m,n\geqslant 0} \frac{(4\pi^2 \bar{m}_1 \bar{m}_2 \bar{u}^2)^n (4\pi^2 m_1 m_2 u^2)^m}{m!n!(\nu+1)_n(\nu+1)_m}$$

$$g(\nu) = 4\frac{(4\pi^2|m_1 m_2|)^\nu}{\Gamma(\nu)\Gamma(\nu+1)},$$

so $M = 4\pi^2|m_1 m_2|$, $\kappa = 0$, and g has a double zero at $\nu = -\rho$. To check the behavior of τ, note that $\begin{pmatrix} 1/u & 0 \\ 0 & u \end{pmatrix} = ma$, with $a^\alpha = |u|^2$, so $a^{\rho+\nu} = |u|^{2(\nu+1)}$.

TABLE 1. Ingredients for the transformation function

Group	ρ	κ	$\tilde{g}(\nu)$
$SO(1,n)^0$ $(n \geq 2)$	$\frac{n-1}{2}$	$\rho - 1$	$\Gamma(\nu)^{-1}\Gamma(\nu+\rho)^{-1}$
$SU(1,n)$ $(n \geq 2)$	n	$\rho - \frac{3}{2}$	$\Gamma(\nu)^{-1}\Gamma\left(\frac{\nu+\rho}{2}\right)^{-2}$
$Sp(1,n)$ $(n \geq 2)$	$2n+1$	$\rho - \frac{5}{2}$	$(\nu+2n-1)\Gamma(\nu)^{-1}\Gamma\left(\frac{\nu+\rho}{2}\right)^{-2}$
F_4^1	11	$\rho - \frac{9}{2} = \frac{13}{2}$	$\frac{(5+\nu)(7+\nu)(9+\nu)}{\Gamma(\nu)\Gamma((\nu+\rho)/2)^2}$

In Table 1, one finds information on g for all groups of real rank 1. The quantities in the tables are the same for Lie groups G that are infinitesimally isomorphic. The function $\tilde{g}$ in the table satisfies $\tilde{g}(\nu) \approx g(\nu)$, where $A(\nu) \approx B(\nu)$ indicates that A and B are equal up to an entire nonzero factor bounded on vertical strips. As $g(\nu) \approx c(\nu)\Gamma(\nu)^{-2}$, where c is Harish-Chandra's c-function, the gamma factors can be read off from [9, Chapter IV, §6, Proposition 6.4].

The cases (R) and (C) come under $SL_2(\mathbb{R}) \cong_{loc} SO(1,2)^0$ and $SL_2(\mathbb{C}) \cong_{loc} SO(1,3)^0$. On a vertical strip, the asymptotic behavior of $\Gamma(2\nu)$ is the same as that of $\Gamma(\nu)\Gamma(\nu+1/2)$.

1.5. Discussion. In this formulation of the sum formula, the function h appears as the independent test function. The function $\tilde{h}$ in the Kloosterman term depends on it. If one wants to use the sum formula to get information on the measure $d\sigma$ or to get an estimate of a sum of Kloosterman sums $S(\gamma)$, one has to choose h in such a way that $\tilde{h}$, as well as h, have properties appropriate to the task at hand.

In both example cases $g(\nu)a^{\nu+\rho}\tau(\chi_1, \chi, ma, \nu)$ can be expressed in terms of Bessel functions. For case (R) there are several forms of the sum formula. In all of these the measure $d\sigma$ contains information on the continuous spectrum of the Casimir operator, and the discrete spectrum, as far as it corresponds to representations of G of the principal and complementary series. One may include the discrete series type spectrum, or not. If one does, the support $\mathscr{S}$ of $d\sigma$ protrudes out of the strip $|\mathrm{Re}\,\nu| \leq \sigma$, and the transformation $h \mapsto \tilde{h}$ has a comfortable image (i.e., it contains the compactly supported functions), as has been shown by Kuznetsov. If one does not include the discrete series type spectrum, then the $\tilde{h}$ are orthogonal to the Bessel functions J_k with odd positive k, and the Bessel transformation is more complicated to handle.

If the image of the Bessel transformation is large enough, $\tilde{h}$ and h play a symmetric role. Both can be viewed as the independent test function. This gives more leeway in applications. But the Bessel transformation and its inverse are complicated. Good has remarked that one does not need the precise inversion of the Bessel transformation; see [6, p. 113]. Goldfeld and Sarnak [5] use similar ideas, but do not use the sum formula at all.

We work with (P, χ) and (P', χ') fixed. Our understanding of $\tau(\chi, \chi_1, ma, \nu)$ in the general case does not suffice to estimate $\tilde{h}(ma)$ for large a^α. This would be necessary to make the influence of (P, χ), (P', χ') explicit in the estimates. So we do not go into the direction of generalizing the beautiful results of Deshouillers and Iwaniec in [2].

2. Estimation of the measure. Our first aim is to estimate the mass of $\{\nu \in \mathscr{S}: |\nu| \leqslant X\}$ for the measure $d\sigma$. From inequality (4), it follows that we can first consider the case $P = P'$ and $\chi = \chi'$. In that case the measure is nonnegative.

To study $d\sigma$ on $\nu \in i[0, \infty)$, we take the test function $h_t(\nu) = e^{t\nu^2}p(\nu)/\cos \pi\nu$ with p an even polynomial. A cosine or sine in the denominator seems to be the right thing to have; we choose a cosine. Its zeros in the strip $|\mathrm{Re}\, \nu| \leqslant \sigma$ should be cancelled, so we take the polynomial $p(\nu) = \prod_{j=0}^{l-1}((1/2 + j)^2 - \nu^2)$, with l large enough. In this way h is positive on $\mathscr{S} \subset (i\mathbb{R} \cup \mathbb{R})$. The factor $e^{t\nu^2}$ approximates 1 as $t \downarrow 0$; this will lead to an average value for the mass of $d\sigma$.

First we look at the terms in the right-hand side of the sum formula. The delta term satisfies

$$\begin{aligned}\delta(h_t) &= iA \int_{\mathrm{Re}\,\nu=0} e^{t\nu^2} \frac{\sin \pi\nu}{\cos \pi\nu} \nu p(\nu)\, d\nu \\ &= 2A \int_0^\infty e^{-ty^2}(1 + O(e^{-2\pi y}))p(iy)y\, dy \\ &= \frac{2A}{t^{l+1}} \int_0^\infty e^{-y^2} \prod_{j=0}^{l-1}\left(y^2 + t\left(\frac{1}{2} + j\right)^2\right) y\, dy + O(1) \\ &= A\, l!\, t^{-l-1} + O(t^{-l}).\end{aligned}$$

In the integral defining $\tilde{h}(ma)$, we move the line of integration to $\mathrm{Re}\, \nu = \sigma$; this is clearly allowed under our assumptions. We get, for a^α bounded from above,

$$\begin{aligned}|\tilde{h}(ma)| &\ll \int_{-\infty}^{\infty} e^{t(\sigma^2 - y^2)} \frac{M^\sigma e^{\pi|y|}|p(\sigma + iy)|}{|(1 + |y|)^{\kappa+2\sigma} \cos \pi(\sigma + iy)|}\, dy \cdot a^{\rho+\sigma} \\ &\ll t^{-1/2-l+\kappa/2+\sigma}\left(\int_0^1 \frac{(t + y^2)^l}{(\sqrt{t} + y)^{\kappa+2\sigma}}\, dy + \int_1^\infty e^{-y^2} \frac{(t + y^2)^l}{(\sqrt{t} + y)^{\kappa+2\sigma}}\, dy\right) \cdot a^{\rho+\sigma} \\ &\ll t^{-l-1/2} a^{\rho+\sigma}.\end{aligned}$$

With the convergence of $\sum_{\xi \in \Xi} |S(\xi)|(a_\xi)^{\rho+\sigma} \leqslant \sum_{\xi \in \Xi} S_0(\xi)(a_\xi)^{\rho+\sigma}$, we obtain for the Kloosterman term the estimate $O(t^{-1/2-l})$. If one would want the explicit dependence on the character χ, one would not only need a factor M^σ, but also the dependence of τ on χ.

Now we apply the sum formula and conclude that the spectral term is $A\, l!t^{-l-1} + O(t^{-l-1/2})$. The test function h_t is nonnegative on $\mathscr{S}$, and the measure $d\sigma$ is also nonnegative (assumption: $P = P'$, $\chi = \chi'$). This shows that $A \geqslant 0$.

For $X > 0$

$$\int_{\nu \in i[0,X]} \frac{|\nu|^{2l}}{\cos \pi\nu}\, d\sigma(\nu) \leqslant e^{tX^2} \int_{\nu \in i[0,X]} h_t(\nu)\, d\sigma(\nu) \leqslant e^{tX^2} \int_{\mathscr{S}} h_t(\nu)\, d\sigma(\nu).$$

Take $t = X^{-2}$. Then $\int_{y=0}^{X} (y^{2l}/\cosh \pi y)\, d\sigma(iy) \ll X^{2l+2}$ as $X \to \infty$. By partial summation, this implies $\int_{y=0}^{X} d\sigma(iy)/\cosh \pi y \ll X^2 + M^\varepsilon X$.

Indeed, put $I_m(X) = \int_{y=1}^{X} (y^m/\cosh y)\, d\sigma(iy)$. Suppose we have shown, for some $m \geqslant 1$, that $I_m(N) = O(N^{m+2})$ as $N \to \infty$, $N \in \mathbb{N}$. Then

$$I_{m-1}(N) = \sum_{n=2}^{N} \int_{n-1}^{n} \frac{y^{m-1}}{\cosh y}\, d\sigma(iy) \leqslant \sum_{n=2}^{N} \frac{1}{n-1} (I_m(n) - I_m(n-1))$$

$$\leqslant \frac{I_m(N)}{N-1} + \sum_{n=2}^{N-1} \frac{I_m(n)}{n(n-1)} \ll N^{m-1} + N^{m-1} = O(N^{m+1}).$$

For general large X the interval $[[X], X]$ does not change the estimate. The interval $[0, 1]$ always has a contribution $O(1)$.

The intersection $\mathscr{S} \cap (0, \rho)$ consists of finitely many points, so $d\sigma$ has finite mass on this set.

Use (4) to see that for a general measure $d\sigma = d\sigma_{P,P'}^{\chi,\chi'}$ that may occur in the sum formula

$$\int_{\nu \in \mathscr{S}, |\nu| \leqslant X} \min\left(1, \frac{1}{|\cos \pi\nu|}\right) |d\sigma(\nu)| \ll X^2 \qquad (X \to \infty). \tag{9}$$

3. Choice of a test function. In Sections 3 and 4 of this paper, we aim at an estimate of $\sum_{\xi, a_\xi^q \geqslant x} a_\xi^\rho S(\xi)$ as $x \downarrow 0$. We include the factor a_ξ^ρ. In case (R) this leads to sums of $S(m_1, m_2; c)/c$, that are traditionally considered in this context.

To obtain such an estimate we choose a test function h_x such that $\xi \mapsto a_\xi^{-\rho} \check{h}_x(m_\xi a_\xi)$ approximates the characteristic function of $a_\xi^q \in [x, 2x)$. We start with a compactly supported smooth function ψ_x approximating this characteristic function. The definition of $\check{h}_x$ and the properties of g and τ suggest that we could use the Mellin transform $\mathscr{M}\psi_x$ to define $h_x(\nu) = \mathscr{M}\psi_x(-\nu) g(\nu)^{-1}$. But h_x has to be even, so we have to symmetrize this. Moreover, the poles of the factor $1/g(\nu)$ may cause singularities. These have to be cancelled by zeros of $\mathscr{M}\psi_x$. At first this seems an impossible task. But the a_ξ^q occurring in the Kloosterman term all stay under a positive value B. (In the examples (R) and (C), we have $B = 1$. This depends on the discrete subgroup Γ, so other values of B are possible, even if $G = \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$.) We subtract from ψ_x another smooth function β_x with compact support in (B, ∞) without changing the Kloosterman term. Of course, the choice of β_x will influence h_x, and hence the other terms in the sum formula. So our task is to find β_x that produces the right zeros and does not add notably to the growth of h_x.

3.1. Test function on $(0, \infty)$. In our choice of the test function ψ_x we will use a large parameter Y. The larger Y, the steeper ψ_x will be near x and $2x$. The dependence on Y we leave invisible in the notation. In all estimates we tacitly assume $Y \geqslant 4$ and $0 < x \leqslant B$.

First we fix some smooth function ω on $[0, 1]$. It is increasing on $[0, 1]$, it is equal to 0 on a neighborhood of 0, and equal to 1 on a neighborhood of 1. We define $\psi \in C_c^\infty(0, \infty)$ by

$$\psi(y) = \begin{cases} 0 & \text{if } y \leqslant 1 - 1/Y \\ \omega(1 - Y + yY) & \text{if } 1 - 1/Y \leqslant y \leqslant 1 \\ 1 & \text{if } 1 \leqslant y \leqslant 2 - 2/Y \\ 1 - \omega(1 - Y + yY/2) & \text{if } 2 - 2/Y \leqslant y \leqslant 2 \\ 0 & \text{if } y \geqslant 2. \end{cases}$$

In this way $\operatorname{supp}(\psi) \subset (1 - 1/Y, 2] \subset [1/2, 2]$, $\psi = 1$ on $[1, 2 - 2/Y]$, and we get a partition of $1: \sum_{m \in \mathbb{Z}} \psi(y2^m) = 1$ for all $y > 0$. Furthermore, $\int_0^\infty |\psi^{(l)}(y)|\,dy = O(Y^{l-1})$ for $l \geqslant 1$; the constant in O depends on l.

Let $l \in \mathbb{N}$. The Mellin transform $\mathcal{M}\psi(v) = \int_0^\infty y^v \psi(y)(dy/y)$ satisfies

$$(\mathcal{M}\psi)^{(l)}(v) = O(2^{|\mathrm{Re}\, v|}(\log 2)^l)$$

(use a direct estimate $\int_0^\infty y^{\mathrm{Re}\, v} |\log y|^l \psi(y)(dy/y)$). Partial integration gives $\mathcal{M}\psi(v) = (-1)^l/(v(v+1)\cdots(v+l-1)).\mathcal{M}\psi^{(l)}(v+l)$, and hence

$$\mathcal{M}\psi(v) = O(2^{|\mathrm{Re}\, v + l|} Y^{l-1}(1 + |\mathrm{Im}\, v|)^{-l})$$

for $|\mathrm{Im}\, v| \geqslant 1$.

As approximate characteristic function of $[x, 2x)$ we take $\psi_x(y) = \psi(y/x)$. The Mellin transform satisfies $\mathcal{M}\psi_x(v) = x^v \mathcal{M}\psi(v)$. On $|\mathrm{Re}\, v| \leqslant \sigma_1$, $|\mathrm{Im}\, v| \geqslant 1$, we get

$$\mathcal{M}\psi_x(v) = O(x^{\mathrm{Re}\, v}(1 + |\mathrm{Im}\, v|)^{-l} Y^{l-1}) \tag{10}$$

for $l \geqslant 1$. The full dependence on l and σ_1 is left implicit. The derivatives are given by $(\mathcal{M}\psi_x)^{(l)}(v) = x^v \sum_{m=0}^{l} \binom{l}{m} (\log x)^m (\mathcal{M}\psi)^{(l-m)}(v)$. Hence, for $|\mathrm{Re}\, v| \leqslant \sigma_1$ and

$l \geqslant 0$

$$(\mathcal{M}\psi_x)^{(l)}(\nu) = O(x^{\operatorname{Re}\nu}(1 + |\log x|)^l). \tag{11}$$

3.2. Getting rid of the zeros of g. We shall take

$$h_x(\nu) = \frac{\mathcal{M}\psi_x(-\nu) - \mathcal{M}\beta_x(-\nu)}{g(\nu)} + \frac{\mathcal{M}\psi_x(\nu) - \mathcal{M}\beta_x(\nu)}{g(-\nu)}, \tag{12}$$

TABLE 2. Information concerning the zeros of the function g

Group	ζ_0	k_{ζ_0}
$SO(1,2)^0$	$\frac{1}{2}$	1
$SO(1,3)^0$	1	2
$SO(1,n)^0$ $n \geqslant 4$	1	1
$SU(1,n)$ $n \geqslant 2$	1	1
$Sp(1,n)$ $n \geqslant 2$	1	1
F_4	1	1

with $\beta_x \in C_c^\infty(B, \infty)$ still to be chosen. The holomorphy of h_x at $\nu = 0$ is no problem; the simple zero of g at 0 is cancelled. Let $\mathscr{Z}$ be the finite set of zeros of $\nu \mapsto g(-\nu)$ in $(0, \rho]$, and denote by k_ζ the order of the zero at $\zeta \in \mathscr{Z}$. Let ζ_0 be the minimal element of $\mathscr{Z}$. Hence $0 < \zeta_0 \leqslant \rho$; see Table 2. So $x^{\zeta_0}(1 + |\log x|)^{k_{\zeta_0}-1}$ is a small quantity; we denote it by $x^{\langle\zeta_0\rangle}$. We have to take β_x in such a way that $\mathcal{M}\beta_x - \mathcal{M}\psi_x$ has a zero of order k_ζ at each $\zeta \in \mathscr{Z}$.

The functions $y \mapsto (\log y)^l y^\zeta$, $\zeta \in \mathscr{Z}$, $0 \leqslant l < k_\zeta$, are linearly independent on any open interval contained in (B, ∞). Fix $\chi_{\zeta,0}, \ldots, \chi_{\zeta,k_\rho-1} \in C_c^\infty(B+1, B+2)$ such that $\int_0^\infty (\log y)^l y^\zeta \chi_{\eta,n}(y)(dy/y) = \delta_{\eta,\zeta}\delta_{l,n}$, for $\eta, \zeta \in \mathscr{Z}$, $0 \leqslant l < k_\zeta$, $0 \leqslant n < k_\eta$. Take

$$\mu_{\zeta,l}(x) = \sum_{m=0}^{l} \binom{l}{m} (\log x)^m x^\zeta (\mathcal{M}\psi)^{(l-m)}(\zeta).$$

Then $\beta_x = \sum_{\zeta \in \mathscr{Z}} \sum_{l=0}^{k_\zeta - 1} \mu_{\zeta,l}(x)\chi_{\zeta,l}$ satisfies $(\mathcal{M}\psi_x)^{(l)}(\zeta) = (\mathcal{M}\beta_x)^{(l)}(\zeta)$ for each $\zeta \in \mathscr{Z}$, $l = 0, \ldots, k_\zeta - 1$. This β_x leads to a test function h_x that is even and holomorphic on the strip.

The estimate $\mu_{\zeta,l}(x) = O(x^\zeta(1 + |\log x|)^l)$ leads to

$$(\mathcal{M}\beta_x)^{(m)}(\nu) = O(x^{\langle\zeta_0\rangle})$$

for $|\operatorname{Re}\nu| \leqslant \sigma$, $m \geqslant 0$. Here we have used $(\mathcal{M}\chi_{\zeta,l})^{(m)}(\nu) = O(1)$; this leaves implicit the dependence on m and B. By partial integration, we get estimates for the $\mathcal{M}\chi_{\zeta,l}$, and finally

$$\mathcal{M}\beta_x(\nu) = O(x^{\langle\zeta_0\rangle}(1 + |\mathrm{Im}\, \nu|)^{-l}) \tag{13}$$

on $|\mathrm{Re}\, \nu| \leqslant \sigma$, $|\mathrm{Im}\, \nu| \geqslant 1$, for each $l \geqslant 0$.

3.3. *Estimates of h_x.* We have arranged the holomorphy of h_x on the strip $|\mathrm{Re}\, \nu| \leqslant \sigma$. For $|\mathrm{Im}\, \nu| \geqslant 1$, $|\mathrm{Re}\, \nu| \leqslant \sigma$, we have

$$h_x(\nu) = O(e^{-\pi|\mathrm{Im}\, \nu|}(1 + |\mathrm{Im}\, \nu|)^{\kappa+2|\mathrm{Re}\, \nu|-l}x^{-|\mathrm{Re}\, \nu|}Y^{l-1}) \tag{14}$$

for each $l \geqslant 1$. To get this we have used that $x^{\langle\zeta_0\rangle} = O(1)$ and $Y \geqslant 4$. For $l > \kappa + 2\sigma + 2$, this gives the growth behavior of a test function to which the sum formula can be applied.

For $|\mathrm{Re}\, \nu| \leqslant \sigma$, $|\mathrm{Im}\, \nu| \leqslant 1$, but $\pm\nu$ staying away from the zeros of g, we obtain

$$h_x(\nu) = O(x^{-|\mathrm{Re}\, \nu|}). \tag{15}$$

Near the zeros of g, we have to look more precisely. Suppose that $g(0) = 0$. On a neighborhood $|\nu| \leqslant \varepsilon$ of 0, we can estimate $h_x(\nu)$ in terms of $|\mathcal{M}\psi_x(\nu)|$, $|\mathcal{M}\beta_x(\nu)|$, $|(\mathcal{M}\psi_x)'(\nu)|$, and $|(\mathcal{M}\beta_x)'(\nu)|$ on this neighborhood. This leads to

$$h_x(\nu) = O(x^{-|\mathrm{Re}\, \nu|}(1 + |\log x|)) \tag{16}$$

for $|\nu| \leqslant \varepsilon_0$. This ε_0 we keep fixed in the sequel. We take it smaller than the minimal element of $\mathcal{S} \cap (0, \rho)$, if this set is nonempty, and also smaller that ζ_0.

Now consider one of the finitely many points $\nu \in \mathcal{S} \cap (0, \rho)$. If $\nu \notin \mathcal{Z}$ we have $h_x(\nu) = \mathcal{M}\psi(-\nu)g(\nu)^{-1}x^{-\nu} + O(x^{\nu}) + O(x^{\langle\zeta_0\rangle})$. If ν happens to be an element of $\mathcal{Z}$, this makes no difference for the term $\mathcal{M}\psi_x(-\nu)/g(\nu) - \mathcal{M}\beta_x(-\nu)/g(\nu)$ in the definition of h_x. In the other term, we have to take into account derivatives of the Mellin transforms. This leads to

$$h_x(\nu) = \frac{\mathcal{M}\psi(-\nu)}{g(\nu)}x^{-\nu} + O(x^{\langle\zeta_0\rangle} + x^{\nu}(1 + |\log x|)^{k_\nu}), \tag{17}$$

for $\nu \in \mathcal{S} \cap (0, \rho)$. Take $k_\nu = 0$ if $\nu \notin \mathcal{Z}$.

3.4. *Comparison of $\tilde{h}_x$ and ψ_x.* Put $\alpha_x = \psi_x - \beta_x$. On $(0, B)$ the functions ψ_x and α_x coincide.

We conclude from (6), the properties of τ, and (12) that

$$\tilde{h}_x(ma) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, \nu=\sigma_2} \mathcal{M}\alpha_x(-\nu)a^{\rho+\nu}\left(1 + O\left(\frac{a^{\alpha}}{1 + |\mathrm{Im}\, \nu|}\right)\right) d\nu$$

$$+ \frac{1}{2\pi i}\int_{\mathrm{Re}\, \nu=\sigma_2} \mathcal{M}\alpha_x(\nu)\frac{g(\nu)}{g(-\nu)}a^{\rho+\nu}O(1)\, d\nu,$$

for each $\sigma_2 \in (0, \sigma]$. To estimate the latter integral, we take $\sigma_2 = \sigma$. We obtain

$$\int_{-\infty}^{\infty} (x^\sigma + x^{\langle\langle \zeta_0 \rangle\rangle})(1 + |y|)^{-4\sigma} a^{\rho+\sigma}\, dy \ll a^{\rho+\sigma} x^{\langle\langle \zeta_0 \rangle\rangle}.$$

In the other term we employ the line of integration $\mathrm{Re}\, \nu = \sigma_3$ for $\mathcal{M}\psi_x$ and $\mathrm{Re}\, \nu = \sigma_4$ for $\mathcal{M}\beta_x$. As we use the estimates (8) and (7) of τ and g, respectively, we have to take care that $\sigma_3, \sigma_4 \in [-\varepsilon, \sigma]$ for some $\varepsilon \in (0, 1/2)$. We use Mellin inversion and apply (10) with $l = 1$ and (13) with $l = 2$ to obtain

$$\begin{aligned}
\tilde{h}_x(ma) - a^\rho \psi_x(a^\alpha) &= \frac{1}{2\pi i} \int_{\mathrm{Re}\, \nu = \sigma_3} \mathcal{M}\psi_x(-\nu) O\left(\frac{a^{\rho+\sigma_3+1}}{1 + |\mathrm{Im}\, \nu|}\right) d\nu \\
&\quad + \frac{1}{2\pi i} \int_{\mathrm{Re}\, \nu = \sigma_4} \mathcal{M}\beta_x(-\nu) O(a^{\rho+\sigma_4})\, d\nu + O(x^{\langle\langle \zeta_0 \rangle\rangle} a^{\rho+\sigma}) \\
&\ll \int_{-\infty}^{\infty} x^{-\sigma_3}(1 + |y|)^{-2} a^{\rho+\sigma_3+1}\, dy \\
&\quad + \int_{-\infty}^{\infty} x^{\langle\langle \zeta_0 \rangle\rangle}(1 + |y|)^{-2} a^{\rho+\sigma_4}\, dy + O(x^{\langle\langle \zeta_0 \rangle\rangle} a^{\rho+\sigma}) \\
&\ll x^{-\sigma_3} a^{\rho+1+\sigma_3} + x^{\langle\langle \zeta_0 \rangle\rangle} a^{\rho+\sigma_4} + x^{\langle\langle \zeta_0 \rangle\rangle} a^{\rho+\sigma}.
\end{aligned}$$

The choice $\sigma_4 = \sigma$ is optimal. We want $x^{-\sigma_3}$ to be small. But in the next section we shall need that the exponent $\rho + 1 + \sigma_3$ of a_ξ be larger than 2ρ. This we arrange by taking $\sigma_3 = \sigma - 1$. (As $\sigma > 1/2$ this satisfies the condition $-\varepsilon \leqslant \sigma_3$ for some ε close to 1/2, needed to apply (8).) We have obtained

$$\tilde{h}_x(ma) = a^\rho \psi_x(a^\alpha) + a^{\rho+\sigma} O(x^{1-\sigma} + x^{\langle\langle \zeta_0 \rangle\rangle}). \tag{18}$$

4. Sums of Kloosterman sums. We estimate the quantity $\Lambda_\psi(x) = \sum_{\xi \in \Xi} a_\xi^\rho S(\xi) \psi_x(a_\xi^\alpha)$ as $x \downarrow 0$. This sum is finite for each $x > 0$.

4.1. Sum formula. From (18) it follows that

$$\Lambda_\psi(x) - \sum_{\xi \in \Xi} S(\xi) \tilde{h}_x(\xi) \ll (x^{1-\sigma} + x^{\langle\langle \zeta_0 \rangle\rangle}) \sum_{\xi \in \Xi} S_0(\xi) a_\xi^{\rho+\sigma}.$$

The convergence of the Eisenstein series implies the estimate $\sum_{\xi \in \Xi} S_0(\xi) a_\xi^{\rho+\sigma} = O(1)$. That gives

$$\Lambda_\psi(x) = \sum_{\xi \in \Xi} S(\xi) \tilde{h}_x(\xi) + O(x^{\langle\langle \zeta_0 \rangle\rangle} + x^{1-\sigma}). \tag{19}$$

The main term in the right-hand side occurs in the sum formula, it equals $\int_{\mathcal{S}} h_x(\nu)\, d\sigma(\nu) - \int_{\mathrm{Re}\, \nu = 0} h_x(\nu)\, d\delta(\nu)$.

4.2. Delta term and spectral term. If the delta term $A\int_{-\infty}^{\infty} h_x(iy)y\sinh \pi y\,dy$ is nonzero, we use

$$h_x(iy) \ll \begin{cases} 1+|\log x| & \text{for } |y|\leqslant \varepsilon_0 \quad \text{see (16)},\\ 1 & \text{for } \varepsilon_0\leqslant |y|\leqslant 1 \quad \text{see (15)},\\ e^{-\pi|y|}(1+|y|)^{\kappa-1} & \text{for } 1\leqslant |y|\leqslant Y_1 \quad \text{see (14)},\\ e^{-\pi|y|}(1+|y|)^{\kappa-l}Y^{l-1} & \text{for } |y|\geqslant Y_1 \quad \text{see (14)},\end{cases}$$

with $Y_1 \geqslant 1$ and $l > 2+\kappa$ to be determined. This leads to the estimate $\varepsilon_0^3(1+|\log x|)+1+Y_1^{\kappa+1}+Y^{l-1}Y_1^{\kappa+2-l}$ for the delta term. $Y_1 = Y$ is the best choice for Y_1; it does not matter what $l > 2+\kappa$ we take. This gives

$$\int_{\mathrm{Re}\,v=0} h_x(v)\,d\delta(v) = O(Y^{\kappa+1}+|\log x|). \tag{20}$$

To estimate $\int_{v\in i[0,\infty)} h_x(v)\,d\sigma(v)$, we use the same decomposition. The first and the second contributions are

$$\int_0^{\varepsilon_0} h_x(iy)\,d\sigma(iy) \ll \varepsilon_0(1+|\log x|),$$

$$\int_{\varepsilon_0}^{1} h_x(iy)\,d\sigma(iy) \ll 1.$$

For the region $[Y,\infty)$ we use partial summation. Let $S(u)=\int_{y=1}^{u} d\sigma(iy)/\cosh \pi y$; so $S(u)=O(u^2)$. With $l>\kappa+2$:

$$\begin{aligned}\int_Y^{\infty} h_x(iy)\,d\sigma(iy) &\ll \int_Y^{\infty}(1+y)^{\kappa-l}Y^{l-1}\frac{d\sigma(iy)}{e^{\pi y}}\\ &\ll \sum_{n=1}^{\infty}(Y+n)^{\kappa-l}Y^{l-1}(S(Y+n)-S(Y+n-1))\\ &\ll Y^{l-1}\sum_{n=1}^{\infty}((Y+n)^{\kappa-l}-(Y+n-1)^{\kappa-l})S(Y+n)\\ &\ll Y^{l-1}\sum_{n=1}^{\infty}(Y+n-1)^{\kappa-l-1}(Y+n)^2\\ &\ll Y^{l-1}\sum_{n=1}^{\infty}(Y+n-1)^{\kappa-l+1} \ll Y^{\kappa+1}.\end{aligned}$$

For $\int_1^Y h_x(iy)\, d\sigma(iy) \ll \int_1^Y ((1+y)^{\kappa-1}/\cos \pi iy)|d\sigma(iy)| \ll (1+Y)^{\kappa-1} Y^2 \ll Y^{\kappa+1}$, we also need partial summation in the case $\kappa < 1$.

For this part of the spectral term, supported on $i[0, \infty)$, we obtain the same estimate as in (20). Let $\mathscr{E} = \mathscr{E}(\chi, \chi') = \mathscr{S} \setminus i[0, \infty) = \mathscr{S} \cap (0, \rho)$. This part of the support of $d\sigma$ consists of a finite number of points. Let σ_ν denote the mass of $d\sigma$ at $\nu \in \mathscr{E}$. In view of (17), we obtain

$$\int_{\mathscr{S} \cap (0,\rho)} h_x(\nu)\, d\sigma(\nu) = \sum_{\nu \in \mathscr{E}} \left(\frac{\sigma_\nu \mathscr{M}\psi(-\nu)}{g(\nu)} x^{-\nu} + O(x^{\langle \zeta_0 \rangle} + x^\nu (1 + |\log x|)^{k_\nu}) \right). \tag{21}$$

4.3. Estimates with a smooth boundary

THEOREM 1. *Let ψ_x be the smooth approximation of the characteristic function of $[x, 2x)$ defined in Section 3.1. We have*

$$\sum_{\xi \in \Xi} a_\xi^\varrho S(\xi) \psi_x(a_\xi^\varrho) = \sum_{\nu \in \mathscr{E}(\chi, \chi')} \frac{\sigma_\nu \mathscr{M}\psi(-\nu)}{g(\nu)} x^{-\nu} + O(Y^{\kappa+1} + x^{1-\sigma} + |\log x|) \quad (x \downarrow 0). \tag{22}$$

The sum is over the $\nu \in \mathscr{E} = \mathscr{S} \cap (0, \rho)$, corresponding to exceptional eigenvalues. σ_ν is the mass of $d\sigma$ at ν; see Theorem 9 in [17] for a precise description. The quantity κ depends on the group. The parameter Y governs the steepness of ψ_x.

Proof. Collect the results in (19), (20), and (21). The term $x^{\langle \zeta_0 \rangle}$ is absorbed into $|\log x|$, and so are the terms $x^\nu (1 + |\log x|)^{k_\nu}$ if $\nu \in \mathscr{E} \neq \varnothing$.

Trivial estimate. The convergence of the Eisenstein series gives for all $q > p > 0$ the estimate $\sum_{\xi \in \Xi, p \leqslant a_\xi^\varrho < q} a_\xi^\varrho |S(\xi)| = O(p^{-\sigma})$, and hence $\Lambda_\psi(x) = O(x^{-\sigma})$ as $x \downarrow 0$. The estimate (22) is definitely better. There is a considerable cancellation in or between Kloosterman sums.

PROPOSITION 2. *Let $\chi \in C^\infty(0, \infty)$ satisfy $\chi(y) = 1$ for $y \geqslant 1$ and $\chi(y) = 0$ for $y \leqslant 3/4$. Then*

$$\sum_{\xi \in \Xi} a_\xi^\varrho S(\xi) \chi(a_\xi^\varrho / x) = \sum_{\nu \in \mathscr{E}(\chi, \chi')} \frac{\sigma_\nu \mathscr{M}\chi(-\nu)}{g(\nu)} x^{-\nu} + O(x^{1-\rho-\delta} + |\log x|^2) \tag{23}$$

as $x \downarrow 0$, for each $\delta > 0$. The implicit constant in O depends on χ.

The sum estimated here is a smoothed version of $\sum_{\xi \in \Xi, a_\xi^\varrho \geqslant x} a_\xi^\varrho S(\xi)$. The error term is $O(|\log x|^2)$ if $\rho = 1/2$, and $O(x^{1-\rho-\delta})$ in all other cases ($\rho \geqslant 1$).

Note that $\mathscr{M}\chi(-\nu)$ is well defined for $\operatorname{Re} \nu > 0$.

Proof. On the test function ψ in Theorem 1, we put the additional condition $\psi(y) = \chi(y)$ for $3/4 \leqslant y \leqslant 1$. This determines ψ completely. Moreover, we have arranged the choice of ψ such that $\chi(y) = \sum_{n=0}^\infty \psi(y/2^n)$; hence $\mathscr{M}\chi(-\nu) =$

$\mathcal{M}\psi(-\nu)/(1-2^{-\nu})$. The sum of Kloosterman sums to be estimated is equal to $\sum_{n=0}^{N_x}\sum_{\xi\in\Xi} a_\xi^\varrho S(\xi)\psi_{2^n x}(a_\xi^\sigma)$, with $N_x = [\log_2(4B/3x)]$. Indeed, if $n > \log_2(4B/3x)$, then $a_\xi^\sigma/2^n x < (B/x)(4B/3x)^{-1} \leqslant 1 - 1/Y$, so $\psi_{2^n x}(a_\xi^\sigma) = 0$.

$$\sum_{\xi\in\Xi} a_\xi^\varrho S(\xi)\chi(a_\xi^\sigma/x) = \sum_{n=0}^{N_x}\sum_{\xi\in\Xi} a_\xi^\sigma S(\xi)\psi_{2^n x}(a_\xi^\sigma)$$
$$= \sum_{\nu\in\mathscr{E}} \frac{\sigma_\nu \mathcal{M}\chi(-\nu)}{g(\nu)}(1 - 2^{-(N_x+1)\nu})x^{-\nu}$$
$$+ \sum_{n=0}^{N_x} O(1 + 2^{(1-\sigma)n}x^{1-\sigma} + n + |\log x|).$$

We have applied Theorem 1 and used that Y is fixed. We obtain the sum in the right-hand side of (23) and are left with the error term estimated by

$$\sum_{\nu\in\mathscr{E}}\left(\frac{\sigma_\nu \mathcal{M}\chi(-\nu)}{g(\nu)} x^{-\nu}2^{-(N_x+1)\nu}\right) + N_x + \frac{1-2^{(1-\sigma)(N_x+1)}}{1-2^{1-\sigma}}x^{1-\sigma} + N_x^2 + N_x|\log x|.$$

As $N_x = -\log_2 x + O(1)$, we can absorb $N_x + N_x^2 + N_x|\log x|$ into $O(|\log x|^2)$, and use for each $\nu\in\mathscr{E}$ that $2^{-(N_x+1)\nu}x^{-\nu} = O(1)$.

To handle $2^{(1-\sigma)(N_x+1)}/(1-2^{1-\sigma})$, we first consider the case $\rho \geqslant 1$. As $\sigma > \rho$, we have $2^{(1-\sigma)(N_x+1)} = O(x^{\sigma-1}) = O(1)$, and $((1-2^{(1-\sigma)(N_x+1)})/(1-2^{1-\sigma}))x^{1-\sigma} = O(x^{1-\sigma})$. Take $\sigma = \rho + \delta$.

Finally, for $\rho = 1/2$, we take $\sigma \in (1/2, 1)$. This gives $(1 - 2^{(1-\sigma)(N_x+1)})/(1-2^{1-\sigma}) = O(2^{(1-\sigma)N_x}) = O(x^{\sigma-1})$. Together with the factor $x^{1-\sigma}$, this gives $O(1) = O(|\log x|^2)$.

Smooth version of Linnik's conjecture. In case (R), introduced in 1.3, there are no exceptional eigenvalues. We obtain

$$\sum_{k=1}^{\infty}\frac{1}{k}S(m_1, m_2; k)\chi\left(\frac{1}{xk^2}\right) = O(|\log x|^2) \quad (x\downarrow 0) \tag{24}$$

for any pair (G, Γ) with G locally isomorphic to $SL_2(\mathbb{R})$, and Γ a discrete subgroup for which $\mathscr{E} = \varnothing$.

Linnik's conjecture for the classical Kloosterman sums is

$$\sum_{1\leqslant k\leqslant 1/\sqrt{x}} S(1, m; k) = O(x^{-1/2-\delta}) \qquad \text{for each } \delta > 0;$$

see [15, p. 277]. This is equivalent to $\sum_{1\leqslant k\leqslant 1/\sqrt{x}} S(1, m; k)/k = O(x^{-\delta})$ for each $\delta > 0$. Up till now this conjecture has not been proved. Proposition 2 can be viewed as a smooth version of it.

In case (C) we have $\rho = 1$. We get

$$\sum_{u \in \mathcal{O}, u \neq 0} |u|^{-2} S(u) \chi\left(\frac{1}{x|u|^2}\right) = \sum_{v \in \mathscr{E}} x^{-v} \sigma_v \frac{\mathscr{M}\chi(-v)\Gamma(v)\Gamma(v+1)}{4(4\pi^2 |m_1 m_2|)^v} + O(x^{-\delta}) \quad (x \downarrow 0), \tag{25}$$

for each $\delta > 0$.

4.4. Estimate with a sharp boundary—discussion. Equation (22) gives an estimate for the sum of the $a_\xi^\rho S(\xi)$ with a_ξ approximately in the interval $[x, 2x)$; the contribution near the end points is weighted by the smooth function ψ. We now would like to use a sharp cut-off function.

Good's argument on p. 119 of [6] is not clear to us. He puts the characteristic function of the desired interval between two functions with the role of our ψ_x. This would work if the $S(\xi)$ were nonnegative.

Denote $\Lambda(x) = \sum_{\xi \in \Xi, x \leqslant a_\xi < 2x} a_\xi^\rho S(\xi)$. We do not see another way of comparing $\Lambda(x)$ and $\Lambda_\psi(x)$ other than by trivially estimating the difference. Let us define

$$\begin{aligned} s(v) &= \sum_{\xi \in \Xi, a_\xi^a = v} S(\xi) \\ T_Y(u) &= \sum_{v > 0, u(1-1/Y) \leqslant v < u} v^\rho |s(v)|, \end{aligned} \tag{26}$$

where v runs over the values of a_ξ^a for $\xi \in \Xi$. Clearly $|\Lambda(x) - \Lambda_\psi(x)| \leqslant T_Y(x) + T_Y(2x)$. The problem is to get a reasonable estimate of $T_Y(x)$ as $x \downarrow 0$.

In Example (R), Weil's estimate of Kloosterman sums [22] gives $S(n, m; k) = O(k^{1/2} d(k))$, with $d(k)$ the number of divisors of k. The dependence on n and m of the implicit constant does not matter here. Now we can proceed as indicated by Kuznetsov in [11, (7.7)]. We use $\sum_{n=1}^k d(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n})$ as $k \to \infty$; see [7, §18.2]. Let $a + a^{1/2} \ll b \ll a + a^{2/3}$. Then

$$\begin{aligned} \sum_{a \leqslant k \leqslant b} \frac{1}{k} |S(n, m; k)| &\ll \frac{1}{\sqrt{a}} \sum_{a \leqslant k \leqslant b} d(k) \\ &\ll \frac{1}{\sqrt{a}}\left((b-a)(\log a + 2\gamma - 1) + b \log\frac{b}{a} + \sqrt{b} + \sqrt{a}\right) \\ &\ll \frac{(b-a)(\log a + 2\gamma - 1) + b(b-a)/a + a^{1/2}}{\sqrt{a}} \\ &\ll \frac{(b-a)(\log a + 1)}{\sqrt{a}} \ll \frac{(b-a)\log a}{\sqrt{a}}. \end{aligned}$$

(Kuznetsov states this estimate with $a^{1/3} \ll b - a \ll a^{2/3}$.)

This amounts to $T_Y(u) = O(u^{-1/4}Y^{-1}|\log u|)$ for $u^{-1/6} \ll Y \ll u^{-1/4}$. Theorem 1 and this estimate of T_Y give

$$\sum_{1/\sqrt{2x}<k\leqslant 1/\sqrt{x}} \frac{1}{k} S(n, m; k) = O(Y^{1/2} + Y^{-1}x^{-1/4}|\log x|).$$

The estimate $\mathcal{O}(x^{-1/12}|\log x|^{1/3})$ is obtained from the optimal choice $Y = x^{-1/6}|\log x|^{2/3}$. This leads to Kuznetsov's result (1).

4.5. Assumptions. In general it seems impossible to get an estimate of $T_Y(x)$ that does not overwhelm the other terms in (22). We need some assumption to be able to proceed.

The *Linnik-Selberg series* $Z(s) = \sum_{\xi\in\Xi} S(\xi)a_\xi^{\rho+s}$ converges absolutely on the halfplane $\mathrm{Re}\, s > \rho$. In several cases it has been shown that there is a larger halfplane of convergence. So it is worthwhile to discuss the consquences of the following *assumption*:

(LSC) *The Linnik-Selberg series converges absolutely on the halfplane* $\mathrm{Re}\, s > \sigma_1$, *with* $\sigma_1 < \rho$.

This is well known to hold in case (R) for $\sigma_1 = 1/4$. It holds in case (C), with $\sigma_1 = 1/2$; see Proposition 3.4 of [20]. For congruence subgroups of $SO(1, n)^0$, $n \geqslant 3$, and infinitesimally isomorphic groups, one finds Assumption (LSC) valid with $\sigma_1 = n/2 - 1$ in Theorem 5.1 of [1], and Theorem 7.17 of [3]. This case, with $n \geqslant 4$ and Γ a congruence subgroup of the Vahlen group considered in [3], we shall denote by (V). The Vahlen groups are covering groups of $SO(1, n)^0$.

In 4.6 we shall see that Assumption (LSC) itself does not suffice to get a nontrivial estimate of sums of Kloosterman sums.

In all cases in which Assumption (LSC) has been proved, it is based on two facts: the set $\{a_\xi : \xi \in \Xi\}$ is parametrized by $\mathbb{N}$ in a natural way, and many or all Kloosterman sums can be nontrivially estimated. A strong assumption of this type is:

(KLE) *There are positive numbers* γ *and* β, $\beta < 2\rho/\gamma - 1$, *such that the* $a_\xi^{-\gamma}$, $\xi \in \Xi$, *are positive integers, and* $s(n^{-1/\gamma}) = O(n^\beta)$.

The quantity $s(v)$ has been defined in (26).

Assumption (KLE) has two parts. The integrality assumption seems sensible if Γ is an arithmetic group. It is satisfied in case (R) with $\gamma = 1/2$ and in case (C) with $\gamma = 1$. For case (V), we also have integrality with $\gamma = 1$.

The estimate of $s(n^{-1/\gamma})$ is more delicate. Weil's estimate gives it in case (R) for each $\beta > 1/2$. In Section 5 we shall prove that Assumption (KLE) holds in case (C) for any $\beta > 1/2$.

Assumption (KLE) implies Assumption (LSC) with $\sigma_1 = \gamma(\beta + 1) - \rho$. In the proof of the absolute convergence of the Linnik-Selberg series in [20] and [3], not all Kloosterman sums are estimated nontrivially, but only those corresponding to prime numbers. If one extends the estimates in [3] to squares of primes, one can prove a weaker version of Assumption (KLE) that can be used to estimate sums of Kloosterman sums in case (V) with a sharp boundary.

Table 3 gives a summary of what we know for the example cases.

TABLE 3. Values of the parameters in the assumptions for the example cases

case	κ	ρ	γ	σ_1	β	$2\rho/\gamma - 1$
(R)	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$>\frac{1}{2}$	1
(C)	0	1	1	$\frac{1}{2}$	$>\frac{1}{2}$	1
(V)	$\frac{n-3}{2}$	$\frac{n-1}{2}$	1	$\frac{n}{2}-1$		$2n-2$

4.6. Results based on Assumption (LSC). Assumption (LSC) seems realistic only if σ_1 is not too far below ρ. In the sequel we assume $\sigma_1 > 0$ and $\sigma_1 > \rho - 1$.

Take σ_2 slightly larger than σ_1. Under Assumption (LSC) we have the estimate

$$\sum_{\xi \in \Xi, v \le a_\xi^2 < u} a_\xi^\rho S(\xi) \ll v^{-\sigma_2}.$$

In particular, it gives the estimate $\Lambda(x) = O(x^{-\sigma_2})$ for each $\sigma_2 > \sigma_1$. If we combine the resulting estimate $T_Y(u) \ll u^{-\sigma_2}(1 - 1/Y)^{-\sigma_2} \ll u^{-\sigma_2}$ with the error term in (22), we find $Y^{\kappa+1} + x^{1-\sigma} + x^{-\sigma_2}$. Take $Y = x^{-p}$ with $p = \sigma_2/(\kappa + 1)$, and $\sigma \in (\rho, \rho + \sigma_2 - \sigma_1)$ to obtain an error term that satisfies $O(x^{-\sigma_2})$.

Consider $\nu \in \mathscr{E}$. The difference between $\mathscr{M}\psi(-\nu)$ and its limiting value $(1 - 2^{-\nu})/\nu$ is $O(Y^{-1})$. It is easily absorbed into $O(x^{-\sigma_2})$. Thus we have obtained

$$\Lambda(x) = \sum_{\nu \in \mathscr{E}} \sigma_\nu \frac{(1 - 2^{\nu})}{\nu g(\nu)} x^{-\nu} + O(x^{-\sigma_2}) \qquad (x \downarrow 0)$$

for each $\sigma_2 > \sigma_1$. The error term here is the same one as obtained by the trivial estimate of $\Lambda(x)$ based on assumption (LSC). This gives the following.

PROPOSITION 3. *Under Assumption (LSC) with $\sigma_1 > 0$ and $\sigma_1 > \rho - 1$, the support of the measure $d\sigma$ is contained in $i\mathbb{R} \cup (0, \sigma_1]$.*

This does not mean that there are no exceptional eigenvalues smaller than the bound corresponding to σ_1, but only that the corresponding eigenfunctions cannot be detected by looking at the Fourier coefficients corresponding to (P, χ) and (P', χ'). In Theorem 10 of [3], it is shown that for case (V) the absolute convergence of *all* Linnik-Selberg series on a halfplane Re $s > \sigma_1$ implies the absence of small eigenvalues.

We conclude that under our choice of test functions, the sum formula does not give better estimates of sums of Kloosterman sums than can be obtained directly.

4.7. Results based on Assumption (KLE). As in 4.6 we restrict our discussion to values of β that seem realistic. We assume $\beta > \rho/\gamma - 1$.

Assumption (KLE) gives directly $T_Y(u) \ll \sum_n n^{\beta-\rho/\gamma}$, where the integer n runs between $u^{-\gamma}$ and $u^{-\gamma}(1 - 1/Y)^{-\gamma}$. This gives the estimate

$$T_Y(u) \ll u^{\rho-\gamma\beta}\left(u^{-\gamma}\left(1-\frac{1}{Y}\right)^{-\gamma} - u^{-\gamma}+1\right) \ll u^{\rho-(\beta+1)\gamma}\left(\frac{1}{Y}+u^{\gamma}\right).$$

For $T_Y(x) + Y_Y(2x)$ plus the error term in (22), we get the estimate

$$O(Y^{\kappa+1} + x^{1-\sigma} + |\log x| + x^{\rho-(\beta+1)\gamma}Y^{-1} + x^{\rho-\beta\gamma}).$$

The difference between $\mathcal{M}\psi(-\nu)$ and $(1-2^{-\nu})/\nu$ is $O(1/Y) = O(1)$. This can be absorbed into $O(|\log x|)$. The optimal choice for $p > 0$ in $Y = x^{-p}$ is $p = ((\beta+1)\gamma - \rho)/(\kappa+2)$. This leads to the estimate

$$O(x^{1-\sigma} + x^{\rho-\beta\gamma} + x^{-((\beta+1)\gamma-\rho)(\kappa+1)/(\kappa+2)} + |\log x|)$$

for the error term. The assumption $\beta > \rho/\gamma - 1$ implies that $((\beta+1)\gamma - r)(\kappa+1)/(\kappa+2) > 0$, so we can absorb $|\log x|$ into the third term.

PROPOSITION 4. *Suppose Assumption (KLE) holds with $\beta > \rho/\gamma - 1$, then*

$$\sum_{\xi\in\Xi,\, x\leqslant a_\xi^2<2x} a_\xi^g S(\xi) = \sum_{\nu\in\mathcal{E}} \frac{\sigma_\nu(1-2^{-\nu})}{\nu g(\nu)} x^{-\nu} + O(x^{-\lambda}) \qquad (x\downarrow 0) \tag{27}$$

for each $\lambda > 0$ satisfying the inequalities

$$\lambda \geqslant ((\beta+1)\gamma - \rho)\frac{\kappa+1}{\kappa+2}, \qquad \lambda > \rho - 1, \qquad \lambda \geqslant \beta\gamma - \rho.$$

Under the additional condition $\beta > (2\rho-1)/\gamma - 1$, all $\nu \in \mathcal{E}(\chi, \chi')$ satisfy $\nu \leqslant \gamma(\beta+1) - \rho$.

Proof. We can choose $\sigma > \rho$ as near to ρ as we want. So the condition $\lambda > \rho - 1$ takes care of $x^{1-\sigma}$ in the estimate. So the estimate of the error term obtained above leads to the first part of the proposition.

We have noted that Assumption (KLE) implies Assumption (LSC) with $\sigma_1 = \gamma(\beta+1) - \rho$. The conditions $\beta > \rho/\gamma - 1$ and $\beta > (2\rho-1)/\gamma - 1$ suffice to apply Proposition 3.

Remarks. The $\nu \in \mathcal{E}$ with $\nu \leqslant \lambda$ can be removed from the sum.

Let $\beta > \rho/\gamma - 1$ and $\beta > (2\rho - 1)/\gamma - 1$. Then $\rho - 1 < (\beta + 1)\gamma - \rho$, and $\kappa \geqslant -1/2$ gives $((\beta + 1)\gamma - \rho)(\kappa + 1)/(\kappa + 2) < (\beta + 1)\gamma - \rho$. This means that the error term in the proposition is better than can be obtained from the trivial estimate based on Assumption (LSC) with $\sigma_1 = (\beta + 1)\gamma - \rho$.

In case (R) we have Assumption (KLE) with $\beta = (1/2) + \varepsilon$ (Weil estimate). The conditions on λ are $\lambda > 1/12$, $\lambda > -1/2$, and $\lambda > -1/4$. Thus we get the error term $O(x^{-1/12-\varepsilon})$. We have seen in Section 4.4 that a slightly more complicated form of Assumption (KLE) holds in case (R), yielding a slightly stronger result, namely Kuznetsov's estimate (1). For more general discrete subgroups $\Gamma \subset SL_2(\mathbb{R})$ we get under Assumption (KLE) with $0 < \beta < 1$ an error term $O(x^{-\beta/6})$ and an upper bound $\beta/2$ for the ν corresponding to exceptional eigenvalues. Goldfeld and Sarnak obtain in Theorem 2 of [5] the error term $O(x^{-\beta/6-\varepsilon})$, for each $\varepsilon > 0$. Hejhal refines their method in Theorem I in Appendix E of [8, p. 694] to arrive at the error term $O(x^{-\beta/6}|\log x|)$; he takes into account the dependence on the characters χ and χ'.

In case (C) we have Assumption (KLE) for each $\beta > 1/2$; see Theorem 10 and Section 5.3. The conditions in the proposition are $\lambda > 1/4$, $\lambda > 0$, and $\lambda > -1/2$. This gives the error term $O(x^{-1/4-\varepsilon})$ and a bound $\nu \leqslant 1/2$ for the elements of $\mathscr{E}(\chi, \chi')$. In the theorem on p. 308 of [21], Sarnak gives this error term for the number field $\mathbb{Q}\sqrt{-2})$.

In comparing our result for the cases (R) and (C) with those in the literature, we have carried out a computation like that in the proof of Proposition 2, and we relate the small variable x to the large variable $X = x^{-\gamma}$.

5. Estimates for certain Kloosterman sums. The goal of this section is to obtain some estimates for certain Kloosterman sums naturally associated to a number field F. We shall only use the results in the case when F is imaginary quadratic. However, since the proof is not much more complicated, we let F be an arbitrary number field. The estimates to be proved are generalizations of those of Salié [19], Weil [22], and Estermann [4] for classical Kloosterman sums. Our approach will be a variation of the treatment given by Estermann in the classical case.

We first set some notation. Let F be a number field with ring of integers $\mathcal{O} = \mathcal{O}_F$. If F is an ideal in $\mathcal{O}$, let $N(I)$ be the norm of I. Let ψ, φ be unitary characters of the finite abelian group $\mathcal{O}/I$. Define the generalized Kloosterman sum (as in [20])

$$S(\varphi, \psi, I) = \sum_{x \in (\mathcal{O}/I)^*} \varphi(x)\psi(x^{-1}), \tag{28}$$

where x^{-1} denotes the inverse of x mod I.

If $I = \prod_{j=1}^r P_j^{m_j}$ with prime ideals P_j, $1 \leqslant j \leqslant r$, then $\mathcal{O}/I \cong \prod_{j=1}^r \mathcal{O}/P_j^{m_j}$, and, correspondingly, $\varphi = \times_1^r \varphi_j$, $\psi = \times_1^r \psi_j$, with $\varphi_j, \psi_j \in (\mathcal{O}/P_j^{m_j})^\wedge$. It is clear that we have the multiplicative property

$$S(\varphi, \psi, I) = \prod_{j=1}^{r} S(\varphi_j, \psi_j, P_j^{m_j}). \tag{29}$$

Our main goal is to obtain estimates for sums of the form

$$S[r, r_1; c] = \sum_{x \in (\mathcal{O}/(c))^*} e^{2\pi i \operatorname{Tr}((rx + r_1 x^{-1})/c)}, \tag{30}$$

where $\operatorname{Tr} = \operatorname{Tr}_{F/\mathbb{Q}}$, and $r, r_1 \in F \setminus \{0\}$ satisfy $\operatorname{Tr}(rx), \operatorname{Tr}(r_1 x) \in \mathbb{Z}$ for all $x \in \mathcal{O}$. So $S[r, r_1; c] = S(\psi_{r/c}, \psi_{r_1/c}, (c))$, with $\psi_q(y) = e^{2\pi i \operatorname{Tr}(qy)}$. We want to use the multiplicativity (29). That forces us to estimate the more general Kloosterman sums (28).

5.1. *Local estimates.* We consider a prime ideal P and estimate $S(\varphi, \psi, P^m)$ for $\varphi, \psi \in (\mathcal{O}/P^m)^\wedge$. By $\mathcal{O}_P$ we denote the localization of $\mathcal{O}$ with respect to P. Then $M_P = P\mathcal{O}_P$ is its unique maximal ideal. We have $\mathcal{O}/P^m \cong \mathcal{O}_P/M_P^m$. So we can work with the localized ring in this subsection. It has the advantage of being a principal ideal ring.

We first prove the following lemma.

LEMMA 5. *Let $s \in \mathcal{O}_P \setminus M_P$, $t \in M_P^n$, for $1 \leqslant n < m$. Then*

$$(s + t)^{-1} \equiv \sum_{j=1}^{m} (-1)^{j+1} s^{-j} t^{j-1} \pmod{M_P^m}. \tag{31}$$

Proof. Indeed,

$$\left(\sum_{j=1}^{m} (-1)^{j+1} s^{-j} t^{j-1}\right)(s + t) = \sum_{j=1}^{m} (-1)^{j+1} s^{-j+1} t^{j-1} + \sum_{j=1}^{m} (-1)^{j+1} s^{-j} t^{j}$$
$$= 1 + (-1)^{m+1} s^{-m} t^m \equiv 1 \pmod{M_P^m}.$$

We note that if $0 \leqslant n \leqslant m$ and if s (respectively t) runs through a complete system of representatives of $\mathcal{O}_P/M_P^n$ (respectively M_P^n/M_P^m), then $s + t$ runs through a complete system of representatives of $\mathcal{O}_P/M_P^m$. Furthermore, $s + t \in (\mathcal{O}_P/M_P^m)^*$, if and only if $s \in (\mathcal{O}_P/M_P^n)^*$. We may thus write

$$S(\varphi, \psi, M_P^m) = \sum_{s \in (\mathcal{O}_P/M_P^n)^*} \varphi(s)\psi(s^{-1}) \sum_{t \in M_P^n/M_P^m} \varphi(t)\psi(-s^{-2}t + s^{-3}t^2 - \cdots). \tag{32}$$

By the conductor of a character $\varphi \in (\mathcal{O}_P/M_P^m)^\wedge$, we mean the largest ideal in $\mathcal{O}_P$ on which φ is trivial. We have assumed that M_P^m is contained in the conductors of both φ and ψ. We write $M_P^{N_\varphi}$ for the conductor of φ, and similarly for ψ.

Suppose that $N_\varphi \leqslant N$ and $N_\psi \leqslant N$, for $0 \leqslant N < m$. Letting $n = N$ in (32), we get

$$S(\varphi, \psi, M_P^m) = S(\varphi, \psi, M_P^N) \cdot N(P)^{m-N}. \tag{33}$$

We claim that $N_\varphi \neq N_\psi$ implies $S(\varphi, \psi, M_P^m) = 0$. Indeed, if $N_\varphi > N_\psi$, let $n = N_\varphi$ in (32). Then ψ disappears in the inner sum, so we get $\sum_{t \in M_P^n/M_P^m} \varphi(t) = 0$, since φ

is a nontrivial character. If $N_\varphi < N_\psi$, we choose $n = N_\psi$ in (32), and φ goes away in the inner sum, and also the terms $s^{-3}t^2$ and higher powers of t. Again the inner sum yields 0.

So we fix our attention on the case $N_\varphi = N_\psi = m$. We shall use the sets

$$\mathscr{C}(\varphi, \psi; k, n) = \{s \in (\mathcal{O}_P/M_P^k)^*: \psi(s^2 t) = \varphi(t)\ \forall t \in M_P^n/M_P^m\}$$

for $k, n \in [1, m-1]$ with $k + n = m$.

LEMMA 6. *Let $m \geqslant 2$ and $N_\varphi = N_\psi = m$. Then*

$$|S(\varphi, \psi, M_P^m)| \leqslant \begin{cases} |\mathscr{C}(\varphi, \psi; m/2, m/2)| \cdot N(P)^{m/2} & \text{if } m \text{ is even}, \\ |\mathscr{C}(\varphi, \psi; (m-1)/2, (m+1)/2)| \cdot N(P)^{(m+1)/2} & \text{if } m \text{ is odd}, \\ |\mathscr{C}(\varphi, \psi; (m-1)/2, (m+1)/2)| \cdot N(P)^{m/2} & \text{if } m \text{ is odd and } 2 \nmid N(P). \end{cases}$$

Proof. Take $n = m/2$, if m is even, and $n = (m+1)/2$, if m is odd. By (32) we have

$$S(\varphi, \psi, M_P^m) = \sum_{s \in (\mathcal{O}_P/M_P^n)^*} \varphi(s^{-1})\psi(s) \sum_{t \in M_P^n/M_P^m} \varphi(t)\psi(-s^2 t).$$

Thus

$$\begin{aligned} &|S(\varphi, \psi, M_P^m)| \\ &\quad = |\{s \in (\mathcal{O}_P/M_P^n)^*: \psi(-s^2 t) = \varphi(-t)\ \forall t \in M_P^n/M_P^m)\}| \cdot N(P)^{m-n} \\ &\quad \leqslant |\{s \in (\mathcal{O}_P/M_P^{m-n})^*: \psi(s^2 t) = \varphi(t)\ \forall t \in M_P^n/M_P^m\}| \cdot N(P)^{2n-m} N(P)^{m-n} \\ &\quad = |\mathscr{C}(\varphi, \psi; m-n, n)| \cdot N(P)^n. \end{aligned}$$

Thus we get the first two statements.

For odd m we can take $n = (m-1)/2$, and write

$$S(\varphi, \psi, M_P^m) = \sum_{s \in (\mathcal{O}_P/M_P^n)^*} \varphi(s^{-1})\psi(s) S_s,$$

with $S_s = \sum_{t\, M_P^n/M_P^m} \varphi(t)\psi(-s^2 t + s^3 t^2)$. We compute

$$\begin{aligned} |S_s|^2 &= \left(\sum_t \varphi(t)\psi(-s^2 t)\psi(s^3 t^2)\right)\left(\sum_u \varphi(-u)\psi(s^2 u)\psi(-s^3 u^2)\right) \\ &= \sum_u \sum_t \varphi(t-u)\psi(-s^2(t-u))\psi(s^3(t^2-u^2)) \end{aligned}$$

$$(t \mapsto t + u) = \sum_u \sum_t \varphi(t)\psi(-s^2 t)\psi(s^3(t^2 + 2tu))$$

$$= \sum_{t \in M_P^n/M_P^m} \varphi(t)\psi(-s^2 t)\psi(s^3 t^2)\left(\sum_{u \in M_P^n/M_P^m} \psi(2tus^3)\right).$$

Now we assume $2 \nmid N(P)$. Since ψ is nontrivial on M_P^{m-1}/M_P^m, it is so on M_P^n/M_P^m, hence the inner sum is zero, unless $t \in M_P^{n+1}$. In this case $\psi(s^3 t^2) = 1$, so we get

$$|S_s|^2 = \left(\sum_{t \in M_P^{n+1}/M_P^m} \varphi(t)\psi(-s^2 t)\right) \cdot N(P)^{m-n}.$$

Here the sum is zero, unless $\psi(s^2 t) = \varphi(t)$ for all $t \in M_P^{n+1}/M_P^m$. If this is the case, we get $N(P)^{m-n-1} N(P)^{m-n} = N(P)^m$. So we get $|S_s| = 0$ if $s \notin \mathscr{C}(\varphi, \psi, n, n+1)$, and $|S_s| = N(P)^{m/2}$ if $s \in \mathscr{C}(\varphi, \psi, n, n+1)$. For odd $m \geqslant 3$, we conclude that $|S(\varphi, \psi, M_P^m)| \leqslant |\mathscr{C}(\varphi, \psi, n, n+1)| \cdot N(P)^{m/2}$.

LEMMA 7. *Suppose $N_\varphi = N_\psi = m \geqslant 2$. Let k, $n \geqslant 1$ and $k + n = m$, and let $v = v_P(2)$ the valuation of 2 with respect to P, i.e., $v \in \mathbb{N} \cup \{0\}$ is such that $2 \in P^v \backslash P^{v+1}$. Then*

$$|\mathscr{C}(\varphi, \psi; k, n)| \leqslant 2N(P)^v.$$

Proof. Put $\mathscr{C} = \mathscr{C}(\varphi, \psi; k, n)$, and suppose that $\mathscr{C}$ contains at least one element s_1. Then $-s_1 \in \mathscr{C}$ as well.

Consider $s_2 \in \mathscr{C}$. We have $s_2^2 - s_1^2 = (s_2 - 1)(s_2 + s_1) \in M_P^k$, i.e., one of $s_2 \pm s_1 \in M_P^h$ for some $h \geqslant k/2$. If $t = s_2 \mp s_1 \in M_P^h$, then

$$s_2^2 = (\pm s_1 + t)^2 = s_1^2 \pm 2s_1 t + t^2, \qquad t^2 \in M_P^k,$$

forces $2s_1 t \in M_P^k$ or, equivalently, $t \in M_P^{k-v}$, if $k \geqslant v$, and imposes no condition, i.e., $t \in \mathcal{O}_P$, if $k < v$. (This works since we are in the localized $\mathcal{O}_P$.) Hence $t \in M_P^r$ with $r = \max(k - v, 0)$. This proves that $\mathscr{C} \subset \mathscr{C}' = \{\pm s_1 + t \colon t \in M_P^r/M_P^k\}$. Now

$$|\mathscr{C}'| \leqslant 2|M_P^r/M_P^k| = 2 \cdot \begin{cases} N(P)^v & \text{if } k \geqslant v \\ N(P)^k & \text{if } k < v. \end{cases}$$

In all cases $|\mathscr{C}| \leqslant 2N(P)^v$ as asserted.

Remarks. (i) The special case $v_P(2) = 0$, i.e., $2 \nmid N(P)$, yields $|\mathscr{C}| \leqslant 2$.

(ii) If m is large compared to $v = v_P(2)$ and to $k = [m/2]$, then $|\mathscr{C}| = 2N(P)^v$ or 0.

More precisely, assume $m \geqslant 4v + 2$, so $k = [m/2] \geqslant 2v + 1$. Let $s_1 \in \mathscr{C}$. Then $-s_1 \in \mathscr{C}$, and $-s_1 \neq s_1 \bmod M_P^k$, since otherwise $2s_1 \in M_P^k$, or $2 \in M_P^k$, and this is not true since $2 \notin M_P^{v+1}$ and $v + 1 \leqslant k$. Put $\mathscr{C}' = \{\pm s_1 + t \colon t \in M_P^{k-v}/M_P^k\}$. Now

$\mathscr{C}' \subset \mathscr{C}$ since

$$\begin{cases} (\pm s_1 + t)^2 = s_1^2 \pm 2s_1 t + t^2 \\ t^2 \in M_P^{2k-2v} \subset M_P^k \quad (2k = k + [m/2] \geqslant k + 2v + 1) \\ 2s_1 t \in M_P^k. \end{cases}$$

We claim $|\mathscr{C}'| = 2|M_P^{k-v}/M_P^k| = 2N(P)^v$, i.e., if $\{t_i\}$ is a complete set of representatives of M_P^{k-v}/M_P^k, then $\{\pm s_1 + t_i\}$ are all distinct:

$$s_1 + t_i = s_1 + t_j \Leftrightarrow t_i - t_j \equiv 0 \bmod M_P^k$$

$$s_1 + t_i = -s_1 + t_j \Leftrightarrow 2s_1 + t_i - t_j \equiv 0 \bmod M_P^k.$$

In the latter case, we get a contradiction: If $i \neq j$, then $2 \in M_P^{k-v} \subset M_P^{v+1}$. If $i = j$ we get even $2 \in M_P^k$.

Thus we have obtained $|\mathscr{C}| = 2N(P)^v$ or 0.

(iii) If we specialize (ii) to the simplest case $F = \mathbb{Q}$, $P = (2)$, then $v = 1$, and if $m \geqslant 6$, we get, for $k = [m/2] \geqslant 3$, $|\mathscr{C}| = 4$, the number of solutions of $s^2 \equiv 1 \bmod 2^k$.

LEMMA 8. *Let $m \geqslant 2$ and $N_\varphi = N_\psi = m$. Define $c_P = 2$ if $2 \nmid N(P)$ and $c_P = 2N(P)^{v+1/2}$ if $v = v_P(2) \geqslant 1$. Then $|S(\varphi, \psi, P^m)| \leqslant c_P N(P)^{m/2}$.*

Proof. See the previous lemmas.

Case $m = 1$. From Weil's result [22] we have $|S(\varphi, \psi, M_P) \leqslant 2N(P)^{1/2}$ if $N_\varphi = N_\psi = 1$ and $2 \nmid N(P)$. If $2|N(P)$ we use the trivial estimate by $|N(P)|$. This gives the bound $c_P N(P)^{1/2}$ in all cases, with c_P as in Lemma 8.

Thus we have obtained the following.

PROPOSITION 9. *Let P be a prime ideal in $\mathcal{O}$. If $m \geqslant 1$, let $\varphi, \psi \in (\mathcal{O}/P^m)^\wedge$. Then we have*

$$|S(\varphi, \psi, P^m)| \leqslant c_P N(P)^{m-N/2},$$

where N is minimal such that P^N is contained in both $\ker(\varphi)$ *and* $\ker(\psi)$, *and c_P is as in Lemma 8.*

Note that c_P depends only on F and P, not on m.

5.2. Global results. Now we put the local results together and obtain an estimate for the Kloosterman sums defined in (28).

Consider for $q \in F$, $q \neq 0$, the character $\psi_q\colon x \mapsto e^{2\pi i \operatorname{Tr}(qx)}$ of F. It is an extension of the character of $\mathcal{O}$ considered above. The conductor C_q we define as the largest fractional ideal contained in the kernel of ψ_q. In the terminology of complementary modules of Chapter III, §1 of [12], we have $C_q = (\mathcal{O}q)'$. So the conductor is a fractional ideal of F. One can check that $C_q = \mathcal{O}'q^{-1}$, where $\mathcal{O}'$ is the inverse ideal of the different $\mathfrak{D}_{F/\mathbb{Q}}$.

To write ψ_q as a product, consider first the character $x \mapsto e^{2\pi i\lambda(x)}$ of the adele group A_F of F defined in [12, Chapter XIV, §6]. Let $\hat{\psi}$ denote its restriction to the finite adele group A_F^f (i.e., A_F^f is the restricted direct product of the completions $\hat{F}_P$ at all *finite* primes P of $\mathcal{O}$). Then $\hat{\psi}(\xi) = e^{2\pi i \operatorname{Tr} \xi}$ for $\xi \in F \subset A_F^f$. Furthermore, $\hat{\psi} = \times_P \hat{\psi}_P$, and if p is the rational prime number dividing $N(P)$, then $\hat{\psi}_P(x) = e^{2\pi i \lambda_P(x)}$ with $\lambda_P(x) = \lambda_p \operatorname{Tr}_{\mathbb{Q}_p}^{\hat{F}_P} x$ and λ_p the composition of the natural maps $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$; see [12, Chapter XIV, §1]. This shows that the conductor of $\hat{\psi}_P$ is the inverse of the local different $\mathfrak{D}_{\hat{F}_P/\mathbb{Q}_p}$. If $\mathfrak{D}_{F/\mathbb{Q}} = \prod_P P^{d_P}$, then the local differents are $\hat{M}_P^{d_P}$. The embedding $F \mapsto \hat{F}_P$ sends the localized ring $\mathcal{O}_P$ into the completion $\hat{\mathcal{O}}_P$, and the different for $\mathcal{O}_P$ is $M_P^{d_P}$. See [12, Chapter III, §1, p. 61].

Let us restrict ψ_q to $\mathcal{O}$. If the ideal $I = \prod_P P^{n_P}$ is contained in the kernel of this restriction, then $\psi_q(x + I) = \prod_P \hat{\psi}_P(qx)$ gives the decomposition of ψ_q corresponding to $\mathcal{O}/I \cong \prod_P \mathcal{O}/P^{n_P}$.

In the definition of Kloosterman sums (30), we take $q = r/c$, with $c \in \mathcal{O}$, $c \neq 0$, $r \in \mathcal{O}'$, $r \neq 0$. The decomposition of $\psi_{r/c}$ used in (29) corresponds to the decomposition $\psi_{r/c}(x) = \prod_P \hat{\psi}_P(rx/c)$. Denote by v_P the valuation corresponding to the prime P of F. For each factor the minimal a_P such that P^{a_P} is contained in the kernel is given by $a_P = \max(0, -v_P(r) - d_P + v_P(c))$.

For $r, r_1 \in \mathcal{O}'$, both nonzero, we find

$$S[r, r_1; c] = \prod_P S(\hat{\psi}_P(r \cdot /c), \hat{\psi}_P(r_1 \cdot /c), P^{v_P(c)}), \tag{34}$$

with almost all factors equal to 1. From Proposition 9, we obtain the estimate

$$|S[r, r_1; c]| \leqslant \prod_{P, v_P(c) > 0} c_P N(P)^{v_P(c) - N_P/2}, \tag{35}$$

with $N_P = \max(0, -v_P(r) - d_P + v_P(c), -v_P(r_1) - d_P + v_P(c))$. This gives

$$|S[r, r_1; c]| \leqslant \sqrt{N((c))N(\mathfrak{D})} \prod_{P, v_P(c) > 0} c_P N(P)^{\min(v_P(c) - d_P, v_P(r), v_P(r_1))/2}.$$

The norm of the different is an ideal in $\mathbb{Z}$. This ideal is generated by the discriminant D_F of the number field. The norm $N((c))$ of the ideal (c) is equal to the ideal in $\mathbb{Z}$ generated by the norm $N_{F/\mathbb{Q}}(c)$. We have obtained the next global estimate.

THEOREM 10. *Let $r, r_1 \in \mathcal{O}'$ be nonzero. For all $c \in \mathcal{O} \backslash \{0\}$:*

$$|S[r, r_1; c]| \leqslant 2^{d + f/2} \sqrt{|D_F|} \sqrt{N_{r, r_1}(c)} 2^{p r(c)} \sqrt{|N_{F/\mathbb{Q}}(c)|},$$

where

> $d = [F : \mathbb{Q}]$ *is the degree of* F, *and* $f = \sum_{P,\, 2|N(P)} f_P$ *the sum of the residue class degrees of the primes of* F *above* (2),
>
> D_F *is the discriminant of* F,
>
> $\mathrm{pr}(c)$ *is the number of prime ideals dividing the ideal* (c),
>
> $N_{r,r_1}(c) = \prod_{P,\, v_P(c)>0} N(P)^{\min(v_P(r),\, v_P(r_1),\, v_P(c)-d_P)}$, *with* v_P *the valuation associated to the prime ideal* P; *the* d_P *describe the different:* $\mathfrak{D}_{F/\mathbb{Q}} = \prod_P P^{d_P}$.

Remarks. (i) $N_{r,r_1}(c)$ can be viewed as the norm of a kind of largest common divisor of r, r_1 and c.

(ii) In the case $F = \mathbb{Q}$ we have $r,\ r_1 \in \mathbb{Z}$ and $N_{r,r_1}(c) = (r, r_1, c)$. The norm $N_{\mathbb{Q}/\mathbb{Q}}(c)$ is c itself. Thus we get

$$|S(r, r_1; c)| \leqslant 2^{\mathrm{pr}(c)} c^{1/2} (r, r_1, c)^{1/2}.$$

The result of Estermann, [4], has the number $d(c)$ of divisors of c instead of $2^{\mathrm{pr}(c)}$. So Theorem 10 is slightly stronger. Estermann remarks that $d(c)$ can be replaced by $dc/(r, r_1, c)$.

(iii) This refinement is possible in the general situation. Note first that if $N = N_P = 0$ in Proposition 9, then $S(\varphi, \psi, P^m) = N(P)^m(1 - (1/N(P))$. So in (34) and (35) we can take $c_P = 1$ if $N_P = 0$. In the statement of Theorem 10, we can replace $\mathrm{pr}(c)$ by the number of primes for which $v_P(c) > d_P + \min(v_P(r), v_P(r_1))$. That extends Estermann's refinement.

(iv) $2^{\mathrm{pr}(c)} = O(|N(c)|^{\varepsilon})$ for each $\varepsilon > 0$.

To see this, put $a = \mathrm{pr}(c)$ and $x = |N(c)|$. Then $x \geqslant N(P_1) \cdots N(P_a)$ for different prime ideals $P_1, \ldots, P_a$. Take a rational prime number q_j dividing $N(P_j)$. Then each prime p can occur at most d times among the q_j, where $d = [F : \mathbb{Q}]$. Let $p_1 = 2$, $p_2 = 3$, $\ldots$ be the rational prime numbers in increasing order. We conclude that $\log x \geqslant d \log p_1 + d \log p_2 + \cdots + d \log p_l + b \log p_{l+1}$ with $l = [a/d]$ and $b = a - ld$. From $p_j > j$ it follows that $\log x \geqslant d \int_1^{l+1} \log t \, dt \geqslant d(l+1)(\log(l+1) - 1)$. So for all a that are sufficiently large, we have $a \leqslant d(l+1) \leqslant \varepsilon \log x$; hence $2^a \leqslant x^{\varepsilon \log 2}$.

5.3. Assumption (KLE) for case (C). In case (C) the number field F is imaginary quadratic. The set Ξ corresponds to the $c \in \mathcal{O} \backslash \{0\}$. The Kloosterman sums $S(\xi)$ are the $S[r, r_1; c]$ for a fixed choice of $r,\ r_1 \in \mathcal{O}' \backslash \{0\}$. We have $a_\xi^a = |c|^{-2} = N(c)^{-1}$. So the integrality in Assumption (KLE) holds with $\gamma = 1$. Furthermore,

$$s(n^{-1}) = \sum_{c \in \mathcal{O},\, N(c) = n} S[r, r_1; c].$$

For fixed r and r_1 Theorem 10 gives $S[r, r_1; c] = O_{F,r,r_1,\varepsilon}(|N(c)|^{1/2+\varepsilon/2})$ for each $\varepsilon > 0$. The number N_n of elements $c \in \mathcal{O}\backslash\{0\}$ with $N(c) = n$ is $O_\varepsilon(n^{\varepsilon/2})$. Indeed, the unit group $\mathcal{O}^*$ is finite; hence N_n is bounded by a multiple of the number of ideals in $\mathcal{O}$ with norm n. Now apply Lemma 4.2 in [18] to get the desired result. Thus we obtain

$$s(n^{-1}) = O(n^{1/2+\varepsilon})$$

for all $\varepsilon > 0$. This gives Assumption (KLE) for case (C), with $\gamma = 1$ and $\beta = 1/2 + \varepsilon$. A trivial estimate of the Kloosterman sums would lead to $\beta = 1 + \varepsilon$.

6. Estimate for the function τ. The goal of this section is to prove estimate (8) on the function $\tau(\nu)$ (see Lemma 12). Although the methods and many of the results are already present in Appendix 1 of [16], we give a rather detailed exposition for the benefit of the reader.

6.1. Notations and definitions. Let G be as in the rest of this paper, a real semisimple Lie group of real rank one and finite center. Let K be a maximal compact subgroup of G, with corresponding Cartan involution θ. Let $G = NAK$ be an Iwasawa decomposition of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding decomposition at the Lie algebra level. Denote by M the centralizer of A in K, let $P = MAN$ and let $\mathfrak{m}$, $\mathfrak{p}$ be the Lie algebras of M and P respectively. If α is the simple root of (P, A) then $\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha}$ where $\mathfrak{n}_\alpha$ (resp. $\mathfrak{n}_{2\alpha}$) is the root space associated to α (resp. 2α). Set $r = \dim(\mathfrak{n}_\alpha)$, $q = \dim(\mathfrak{n}_{2\alpha})$, $\rho = (r/2 + q)\alpha$.

Fix $H_0 \in \mathfrak{a}$ such that $\alpha(H_0) = 1$, and let B be the multiple of the Killing form of $\mathfrak{g}$ such that $B(H_0, H_0) = 1$. With this choice $H_\alpha = H_0$. If $X, Y \in \mathfrak{g}$, let $\langle X, Y\rangle = -B(X, \theta Y)$. Then $\langle\ ,\ \rangle$ defines an inner product on $\mathfrak{g}$ which coincides with B on $\mathfrak{a}$. We will also denote by $\langle\ ,\ \rangle$ the bilinear extension of B to $\mathfrak{a}_c = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$, and its dual form on $\mathfrak{a}_c^*$.

Choose $\{X_j\colon 1 \leqslant j \leqslant n\}$ an orthonormal basis of $\mathfrak{n}$ such that $[H_0, X_j] = a_j X_j$ with a_j with $a_j = 1$, $(1 \leqslant j \leqslant r)$ and $a_j = 2$, $(j > r)$. Set $Y_j = -\theta X_j$. Then $\{Y_j\colon 1 \leqslant j \leqslant n\}$ is an orthonormal basis of $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. If χ is a unitary character of N, let $Y_\chi \in \bar{\mathfrak{n}}$ be such that $d\chi = iB(\cdot, Y_\chi)$. The basis $\{X_j\}$ above may then be chosen so that one furthermore has $Y_\chi = cY_1$ with $c \in \mathbb{R}$. Then $d\chi(X_1) = ic$ and $\ker(d\chi) = \operatorname{span}\{X_2, \ldots, X_n\}$.

6.2. Verma module and Whittaker vector. If $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$), denote by $I! = \prod_{j=1}^n i_j!$, $|I| = \sum_1^n i_j$, $w(I) = \sum_1^r i_j + \sum_{r+1}^n 2i_j$ and $Y(I) = \operatorname{Symm}(Y_1^{i_1} \ldots Y_n^{i_n})$. By the Poincaré-Birkhoff-Witt theorem $\{Y(I)\colon I \in \mathbb{N}^n\}$ is a basis of $\mathcal{U}(\bar{\mathfrak{n}})$. Under the adjoint action of $\mathfrak{a}$, the space $\mathcal{U}(\bar{\mathfrak{n}})$ decomposes into weight spaces $\mathcal{U}(\bar{\mathfrak{n}}) = \bigoplus_{j=0}^\infty \mathcal{U}(\bar{\mathfrak{n}})_{-j}$, where $\mathcal{U}(\bar{\mathfrak{n}})_{-j}$ is the weight space associated to $-j\alpha$. Clearly $\mathcal{U}(\bar{\mathfrak{n}})_{-j} = \operatorname{span}\{Y(I)\colon w(I) = j\}$. We extend the inner product on $\bar{\mathfrak{n}}$ to an inner product on $\mathcal{U}(\bar{\mathfrak{n}})$ by setting $\langle Y(I), Y(J)\rangle = \delta_{I,J} I!$. Then $\{Y(I)/\sqrt{I!}\colon I \in \mathbb{N}^n\}$ is an orthonormal basis of $\mathcal{U}(\bar{\mathfrak{n}})$. Moreover, $\langle\ ,\ \rangle$ is $\operatorname{Ad}(M)$-invariant.

Now consider the *Verma module* $M(-\nu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_\nu$, where $\mathbb{C}_\nu$ denotes the $\mathfrak{p}$-module $\mathbb{C}$ with $\mathfrak{m}$, $\mathfrak{n}$ acting by 0 and $\mathfrak{a}$ acting by $-\nu - \rho$, $\nu \in \mathfrak{a}_c^*$. Then $M(-\nu)$ is an $\mathcal{U}(\mathfrak{g})$-module with a basis given by $\{Y(I) \otimes 1 \colon I \in \mathbb{N}^n\}$. There is a decomposition of $M(\nu)$ into weight spaces, corresponding to that of $\mathcal{U}(\bar{\mathfrak{n}})$,

$$M(-\nu) = \sum_{j=0}^{\infty} M(-\nu)_{-j} = \operatorname{span}\{Y(I) \otimes 1 \colon w(I) = j\},$$

with $\mathfrak{a}$ acting by $-\nu - \rho - j\alpha$ on $M(-\nu)_{-j}$.

Let $M(-\nu)_{[\bar{\mathfrak{n}}]}$ denote the $\bar{\mathfrak{n}}$-adic completion of $M(-\nu)$ (see [GW]). This completion allows us to work with infinite sums of the type $\sum_{j \geq 0} v_j$ with $v_j \in M(-\nu)_j$. Let

$$\mathrm{Wh}_\chi(M(-\nu)_{[\bar{\mathfrak{n}}]}) = \{v \in M(-\nu)_{[\bar{\mathfrak{n}}]} \colon Xv = d\chi(X)v, \quad \forall X \in \bar{\mathfrak{n}}\},$$

the subspace of χ-Whittaker vectors. The function $\tau(\nu)$ to be studied is closely related to the canonical Whittaker vector $u(\nu)$ on $M(\nu)_{[\bar{\mathfrak{n}}]}$ discussed in the following lemma.

LEMMA 11. *There is a unique* $u(\nu) \in \mathrm{Wh}_\chi(M(-\nu)_{[\bar{\mathfrak{n}}]})$ *that has an expansion* $u(\nu) = \sum_{j=0}^{\infty} u_j(\nu)$ *such that* $u_j(\nu) \in M(-\nu)_{-j}$ $(j \in \mathbb{N})$ *and* $u_0(\nu) = 1 \otimes 1$. *If* $u_j(\nu) = \sum_{w(I)=j} a_I(\nu) Y(I) \otimes 1$, *then each* $a_I(\nu)$ *is a rational function on* $\mathfrak{a}_c^*$ *with poles, for all* $I \in \mathbb{N}^n$, *lying in* Z_0, *a closed discrete subset of* $(-\infty, -1/2] \cdot \alpha$. *If* $b_1, b_2 \in \mathbb{R}$, $t > 0$, *let* $S_{b_1,b_2} = \{\nu \in \mathfrak{a}_c^* \colon b_1 \leqslant \langle \operatorname{Re} \nu, \alpha \rangle \leqslant b_2\}$, *and let* $S^t_{b_1,b_2} = S_{b_1,b_2} \cap \{\nu \in \mathfrak{a}_c^* \colon |\operatorname{Im}(\nu)| \geqslant t\}$.

(i) *If* $[b_1, b_2] \cap Z_0 = \varnothing$, *there exist positive constants* $\delta = \delta(b_1, b_2)$, $C = C(\chi, b_1, b_2)$, *such that for all* $j, k \in \mathbb{N}, j \geqslant k$,

$$\|u_j(\nu)\| \leqslant \frac{C^j k!}{j! 2^{k/2} (\delta + |\operatorname{Im}(\nu)|)^k} \qquad \nu \in S_{b_1,b_2}.$$

(ii) *If* $[b_1, b_2] \cap Z_0 \neq \varnothing$, *there exists* $C = C(\chi, b_1, b_2, t)$ *such that for all* j, $k \in \mathbb{N}, j \geqslant k$,

$$\|u_j(\nu)\| \leqslant \frac{C^j C_t^k k!}{j! (1 + |\operatorname{Im}(\nu)|)^k} \qquad \nu \in S^t_{b_1,b_2},$$

with $C_t = \sqrt{(t+1)/2t}$.

Remark. In the case that $\mathfrak{n}$ is abelian, i.e., if G is locally isomorphic to $\mathrm{SO}(1, n)^0$, the factor $j!$ in the denominators in (i) and (ii) can be replaced by $j!^{3/2}$. In the proof we shall indicate at which point this better estimate originates.

Proof. We will need to recall the main steps in the proof of Lemma A.1.5 in [16].

Assume $u(\nu) = \sum_{j=0}^{\infty} u_j(\nu)$ is a χ-Whittaker vector with $u_j(\nu) \in M(-\nu)_{-j}$, $(j \in \mathbb{N})$. We now compute $Cu(\nu)$ in two different ways, C the Casimir element of $\mathfrak{g}$. First,

$$Cu(\nu) = (\langle \nu, \nu \rangle - \langle \rho, \rho \rangle) u(\nu).$$

On the other hand, $C = C_M + H_0^2 + 2H_\rho + 2(\sum_1^n Y_i X_i)$, C_M the Casimir of M. We now use this expression to compute $Cu(\nu)$. If $\gamma \in \hat{M}$ is an M-type, let p_γ: $M(-\nu) \to M(-\nu)_\gamma$ denote the orthogonal projection onto the γ-isotypic component, and let λ_γ be the eigenvalue of C_M on γ (recall that $\lambda_\gamma \geqslant 0$). If $j \in \mathbb{N}$, set $\hat{M}_j = \{\gamma \in \hat{M} : p_\gamma(M(-\nu)_j) \neq 0\}$. Write $u_j(\nu) = \sum_{\gamma \in \hat{M}_j} u_{j,\gamma}(\nu)$ with $u_{j,\gamma}(\nu) \in p_\gamma(M(-\nu)_{-j})$. A calculation shows that

$$(C_M + H_0^2 + 2H_\rho) u_j(\nu) = \sum_{\gamma \in \hat{M}_j} (\lambda_\gamma + \langle \nu + j\alpha, \nu + j\alpha \rangle - \langle \rho, \rho \rangle) u_{j,\gamma}(\nu).$$

On the other hand, $Xu(\nu) = d\chi(X) u(\nu)$ implies that $2(\sum Y_i X_i) u_j(\nu) = 2ic Y_1 u_{j-1}(\nu)$. Hence,

$$u_j(\nu) = \sum_{\gamma \in \hat{M}_j} \frac{-2ic}{C(\nu, j, \gamma)} p_\gamma \left(\sum_{\mu \in \hat{M}_{j-1}} Y_1(p_\mu(u_{j-1}(\nu))) \right),$$

with $C(\nu, j, \gamma) = \lambda_\gamma + 2j\langle \nu, \alpha \rangle + j^2$, for $\gamma \in \hat{M}_j, j \geqslant 1$.

This formula gives a recursive way to compute $u_j(\nu)$, starting from $u_0(\nu) = 1 \otimes 1$. By iteration of the above, we obtain

$$u_j(\nu) = (-2ic)^j \sum_{\{\gamma_s \in \hat{M}_s;\, 1 \leqslant s \leqslant j\}} \frac{1}{\prod_{s=1}^{j} C(\nu, s, \gamma_s)} p_{\gamma_j}(Y_1(p_{\gamma_{j-1}} Y_1 \cdots (p_{\gamma_1}(Y_1 \otimes 1) \cdots). \quad (36)$$

In order to estimate (36), we consider the function

$$\varphi(\nu) = \inf \left\{ \frac{|C(\nu, j, \gamma)|}{j^2} : \gamma \in \hat{M}_j, j \geqslant 1 \right\}.$$

In [16, A.1.4], one proves that $\varphi(\nu)$ is continuous. We shall check that the zero set of φ is equal to

$$Z_0 = \left\{ z\alpha : z = -\frac{\lambda_\gamma}{2j} - \frac{1}{2}, \gamma \in \hat{M}_j, j \geqslant 1 \right\}.$$

As each $\hat{M}_j$ is a finite set, and as all λ_γ are nonnegative, it is clear that Z_0 is a closed discrete subset of $(-\infty, -1/2] \cdot \alpha$.

Suppose that $\varphi(z\alpha) = 0$. Then there are sequences (j_n) and (γ_{j_n}), with $\gamma_{j_n} \in \hat{M}_{j_n}$, such that $\lambda_{\gamma_n} j_n^{-2} + 2z j_n^{-1} + 1 \to 0$. From $\lambda_{\gamma_n} j_n^{-2} + 1 \geqslant 1$, it follows that $j_n \to \infty$ is impossible. So the j_n stay in a finite set, and so do the γ_{j_n}. So the infimum defining $\varphi(z\alpha)$ is a minimum, and $z\alpha = Z_0$.

Note also that if $|t_1| \leqslant |t_2|$, then $\varphi(x+it_1) \leqslant \varphi(x+it_2)$. Now let b_1, b_2, t be as in the statement.

Assume first that $Z_0 \cap [b_1, b_2] = \emptyset$. Then, if $1 \leqslant s \leqslant k$, we use

$$|C(\nu, s, \gamma_s)|^2 \geqslant 4s^2(\varphi(\mathrm{Re}\,\nu)^2/4 + |\mathrm{Im}(\nu)|^2) \geqslant 2s^2(\varphi(\mathrm{Re}\,\nu)/2 + |\mathrm{Im}(\nu)|)^2.$$

If $k+1 \leqslant s \leqslant j$ we use

$$|C(\nu, s, \gamma_s)|^2 \geqslant s^4 \varphi(\mathrm{Re}\,\nu)^2.$$

Thus

$$\frac{1}{|\prod_1^j C(\nu, s, \gamma - s)|^2} \leqslant \frac{k!^2}{2^k j!^4 \varphi(\mathrm{Re}\,\nu)^{2(j-k)}(\varphi(\mathrm{Re}\,\nu)/2 + |\mathrm{Im}\,\nu|)^{2k}}.$$

On the other hand,

$$\begin{aligned} &\sum_{\{\gamma_s \in \hat{M}_s;\, 1 \leqslant s \leqslant j\}} \|p_{\gamma_j}(Y_1(p_{\gamma_{j-1}} Y_1 \cdots (p_{\gamma_1}(Y_1 \otimes 1)\cdots)\|^2 \\ &\leqslant C_1^2 j^2 \sum_{\{\gamma_s \in \hat{M}_s;\, 1 \leqslant s \leqslant j-2\}} \|Y_1 p_{\gamma_{j-2}}(Y_1(p_{\gamma_{j-3}} Y_1 \cdots (p_{\gamma_1}(Y_1 \otimes 1)\cdots)\|^2 \\ &\leqslant C_1^{2j} j!^2. \end{aligned}$$

Here we have used the inequality $\|Y_1 u\| \leqslant C_1(j+1)\|u\|$, for some $C_1 > 0$, valid for any $u \in \mathscr{U}(\bar{\mathfrak{n}})_{-j}$, $j \geqslant 1$, (see [16, A.1.1]). In the abelian case, one has $\|Y_1 u\| \leqslant C_1\sqrt{j+1}\,\|u\|$ (loc. cit.). That gives $j!$ instead of $j!^2$ in the estimate above. This yields the estimate indicated in the remark to the lemma.

Thus (36) implies that

$$\|u_j(\nu)\|^2 \leqslant \frac{(2cC_1)^{2j} k!^2}{2^k j!^2 \varphi(\mathrm{Re}\,\nu)^{2(j-k)}(\varphi(\mathrm{Re}\,\nu)/2 + |\mathrm{Im}\,\nu|)^{2k}}$$

uniformly for $\nu \in S_{b_1, b_2}$.

Now let $\delta = \inf\{\varphi(x): b_1 \leqslant x \leqslant b_2\}/2$; $\delta > 0$, by assumption. Thus

$$\|u_j(\nu)\|^2 \leqslant \frac{k!^2 C^{2j}}{j!^2 2^k(\delta + |\mathrm{Im}\,\nu|)^{2k}}$$

for $\nu \in S_{b_1, b_2}$. Note that we can ignore $\delta^{2(j-k)}$ in the denominator if $\delta \geqslant 1$. If $0 < \delta < 1$, we take it into the definition of C, and ignore δ^{-2k}.

If $Z_0 \cap [b_1, b_2] \neq \emptyset$ and $1 \leqslant s \leqslant k$, we have for $|\mathrm{Im}\,\nu| \geqslant t$,

$$|C(\nu, s, \gamma_s)|^2 \geqslant 4s^2 |\mathrm{Im}\,\nu|^2 \geqslant 4s^2 C_t'(1 + |\mathrm{Im}\,\nu|)^2$$

(with $C_t' = t^2/(t+1)^2$).

If $k+1 \leqslant s \leqslant j$, we use

$$|C(\nu, s, \gamma_s)|^2 \geqslant s^4 \varphi(\nu)^2 \geqslant s^4 \varphi(\mathrm{Re}\, \nu + it)^2$$

for $\nu \in S^t_{b_1, b_2}$. We get in this case,

$$\|u_j(\nu)\|^2 \leqslant \frac{(C_1')^{-k} C_1^{2j} k!^2}{j!^2 \varphi(\mathrm{Re}\, \nu + it)^{2(j-k)} 2^{2k} (1 + |\mathrm{Im}\, \nu|)^{2k}}$$

$$\leqslant \frac{((t+1)/2t)^{2k} C^{2j} k!^2}{j!^2 (1 + |\mathrm{Im}\, \nu|)^{2k}}$$

for $\nu \in S^t_{b_1, b_2}$, as asserted. (We handle $\varphi(\mathrm{Re}\, \nu + it)^{2(j-k)}$ in the same way as we treated $\delta^{2(j-k)}$ in the previous case.)

6.3. *Asymptotic expansion of* τ. Lemma 11 implies the following result on $\tau(\nu)$. The case $k = 1$ gives (8).

PROPOSITION 12. *Let* $\chi, \chi' \in \hat{N}$, $a \in A$, $m \in M$, *and let* Z_0, S_{b_1, b_2}, $S^t_{b_1, b_2}$ *be as in Lemma 11. The series*

$$\sum_{I \in \mathbb{N}^n} a_I(\nu)\, d\chi'(\mathrm{Ad}(mas^*)^{-1} Y(I)^T)$$

converges absolutely and uniformly to $\tau(\chi, \chi', \nu, ma)$ *on* $S^t_{b_1, b_2}$, *and on* S_{b_1, b_2}, *in case* $Z_0 \cap [b_1, b_2] = \varnothing$. *If* $Z_0 \cap [b_1, b_2] = \varnothing$, *there exists a positive constant* $\delta = \delta(b_1, b_2)$ *such that for each* $k \geqslant 0$,

$$\left| \tau(\chi, \chi', \nu, ma) - \sum_{w(I) < k} a_I(\nu)\, d\chi'(\mathrm{Ad}(ms^*)^{-1} Y(I)^T) a^{w(I)\alpha} \right|$$

$$\leqslant \frac{a^{k\alpha} k!}{2^{k/2} (\delta + |\mathrm{Im}\, \nu|)^k} \psi_k(a^\alpha), \qquad (37)$$

uniformly for $\nu \in S_{b_1, b_2}$, *and uniformly for* $a^\alpha \leqslant B$ *for each* $B > 0$. *Here* ψ_k *is an everywhere analytic function, depending on* χ, χ', b_1, b_2, *and* k.

In the case when $Z_0 \cap [b_1, b_2] \neq \varnothing$, *we have a uniform estimate on* $S^t_{b_1, b_2}$ *by*

$$\frac{((t+1)/2t)^k k!\, a^{k\alpha}}{(1 + |\mathrm{Im}\, \nu|)^k} \tilde{\psi}_k(a^\alpha)$$

with $C = C(b_1, b_2, t, \chi, \chi')$ *and* $\tilde{\psi}_k$ *similar to* ψ_k, *but depending on* t *as well.*

Remarks. $y \mapsto y^T$ is the transpose in the universal enveloping algebra. It is the antiautomorphism that acts by $X^T = -X$ on $X \in \mathfrak{g}$. Note that $Y(I)^T = (-1)^{|I|} Y(I)$.

The statement still holds with the factor $k!$ in the numerator of the bounds replaced by $k!^{\varepsilon}$ for any $\varepsilon > 0$. In the abelian case, the factor $k!$ in the numerator of the bounds can even be replaced by $k!^{-1/2+\varepsilon}$.

Proof. We consider the case $Z_0 \cap [b_1, b_2] = \emptyset$, and leave the other case, which is completely analogous, to the reader.

Take one term of the series

$$a_I(\nu)\, d\chi'(\mathrm{Ad}(mas^*)^{-1} Y(I)^T) = a^{j\alpha} a_I(\nu)\, d\chi'(\mathrm{Ad}(ms^*)^{-1} Y(I)^T),$$

with $I \in \mathbb{N}^n$, $j = w(I)$. Note that $\mathrm{Ad}(m)$ and $\mathrm{Ad}(s^*)$ do not change the norm, so $\mathrm{Ad}(ms^*)^{-1} Y(I)^T$ is an element of $\mathscr{U}(\mathfrak{n})_j$ of norm $\| Y(I)^T \| = \| Y(I) \|$. As $d\chi'$ is a linear form on $\mathfrak{n}$, its multilinear extension to $\mathscr{U}(\mathfrak{n})_j$ has norm bounded by v^j for some constant v depending on χ'. Thus we have

$$|a_I(\nu)\, d\chi'(\mathrm{Ad}(mas^*)^{-1} Y(I)^T)| \leqslant |a_I(\nu)|\, a^{j\alpha} v^j \| Y(I) \| \leqslant v^j \| u_j(\nu) \| a^{j\alpha}.$$

We apply Lemma 11 with $k = 0$ to obtain

$$\sum_{I \in \mathbb{N}^n} |a_I(\nu)(d\chi')(\mathrm{Ad}(mas^*)^{-1} Y(I)^T|$$
$$\leqslant \sum_{j=0}^{\infty} (j+1)^n v^j \| u_j(\nu) \| a^{j\alpha} \leqslant \sum_{j=0}^{\infty} \frac{(vC)^j (j+1)^n}{j!} a^{j\alpha} < \infty.$$

(We have used here that $|\{I : w(I) = j\}| \leqslant (j+1)^n$.) This gives the absolute convergence of the series, uniformly on S_{b_1, b_2}.

$\tau(\chi, \chi', \nu, ma)$ has been defined in [16] as the sum of this series. We call $\eta_k(ma)$ the left-hand side of (37). The estimates in Lemma 11 imply

$$\eta_k(ma) \leqslant \sum_{j=k}^{\infty} (j+1)^n v^j \| u_j(\nu) \| a^{j\alpha}$$
$$\leqslant \left(\sum_{j=k}^{\infty} \frac{(j+1)^n (vC)^j a^{(j-k)\alpha}}{j!} \right) \frac{a^{k\alpha} k!}{(\delta + |\mathrm{Im}\, \nu|)^k 2^{k/2}},$$

uniformly for $\nu \in S_{b_1, b_2}$.

Thus (37) holds as asserted with $\psi_k(x) = \sum_{l \geqslant 0} (vC)^{l+k} (l+k+1)^n x^l / (l+k)!$, which converges uniformly for x in bounded sets.

For the convergence of the series defining ψ_k, it is sufficient to have $(l+k)!^{\varepsilon}$ in the denominator instead of $(l+k)!$. The remaining $(l+k)^{1-\varepsilon}$ can be used to replace $k!$ by $k!^{\varepsilon}$. The better estimate in the abelian case (see the remark to Lemma 11) gives an additional factor $k!^{-1/2}$.

Remark. Since $Z_0 \subset (-\infty, -1/2] \cdot \alpha$, this implies that $\tau(\nu)$ is holomorphic on strips of the form $S_{-\varepsilon, b}$, with $b > 0$ arbitrary and $\varepsilon \in (0, 1/2)$, and satisfies an estimate

$$\tau(\nu, ma) = 1 + \frac{a^\alpha}{\delta + |\nu|}\psi_1(a) \tag{38}$$

in any such strip.

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