Differentiable manifolds – Mock Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **allowed** to consult any text book and class notes but **not allowed** to consult colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Some definitions you should know, but may have forgotten.

- An n dimensional complex manifold is a manifold whose charts take values in $\mathbb{C}^n$ and for which the change of coordinates are holomorphic maps.

Questions

1) Show that $\mathbb{CP}^1$, the set of complex lines through the origin in $\mathbb{C}^2$, can be given the structure of a complex manifold. Show that $\mathbb{CP}^1$ is isomorphic to the 2-dimensional sphere S^2.

Solution: A line l through the origin on $\mathbb{C}^2$ is determined by any point $p \in l$ different of $(0,0)$ and two points p and q represent the same line if there is a complex number $\lambda \in \mathbb{C}^*$ such that $\lambda p = q$. Given a point $p \in \mathbb{C}^2 \backslash \{0\}$, we denote by $[p]$ the corresponding point in $\mathbb{CP}^1$.

This means that if a line can be represented by a point (z_1, z_2) with $z_1 \neq 0$, then the same point can be represented by a the point $(1, \frac{z_2}{z_1})$ and there is a unique such point representing a line with nonzero first coordinate. That is, we have a map

$$\varphi_1 : \mathbb{CP}^1 \backslash \{[1,0]\} \longrightarrow \mathbb{C} \qquad \varphi_1([z_1, z_2]) = \frac{z_2}{z_1}.$$

Similarly, we have a map

$$\varphi_2 : \mathbb{CP}^1 \backslash \{[0,1]\} \longrightarrow \mathbb{C} \qquad \varphi_1([z_1, z_2]) = \frac{z_1}{z_2}.$$

The claims made above mean that φ_1 and φ_2 are bijections, but we check that rigorously as well.

1. φ_1 *is well defined*: Indeed, if $[z_1, z_2] = [u_1, u_2]$ then there is $\lambda \in \mathbb{C}^*$ such that $(z_1, z_2) = \lambda(u_1, u_2)$ and hence $\frac{z_2}{z_1} = \frac{u_2}{u_1}$.
2. φ_1 *is an injection*: Indeed, if $\frac{z_2}{z_1} = \frac{u_2}{u_1}$, then we let $\lambda = \frac{u_1}{z_1}$ and we get

$$[z_1, z_2] = [\lambda z_1, \lambda z_2] = [u_1, \frac{u_1 z_2}{z_1}] = [u_1, u_2].$$

3. φ_1 *is onto*: Given $z \in \mathbb{C}$, $\varphi_1([1, z]) = z$.

4. φ_1 *is a homeomorphism*: Indeed, since, as a topological space, $\mathbb{CP}^1$ is endowed with the quotient topology, to check that φ_1 is a homeomorphism, we observe that $\mathbb{C}^2 \backslash \{(0, z_2) : z_2 \in \mathbb{C}\}$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}$ via the identification
$$(z_1, z_2) \overset{\Psi}{\mapsto} (z_1, \frac{z_2}{z_1})$$
and the orbits of the $\mathbb{C}^*$ action are the level sets of the projection onto the second factor $p : \mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C}$, $p(u, v) = v$, hence the composition $p \circ \Psi$ induces an homemorphism between the quotient of $\mathbb{C}^2 \backslash \{(0, z_2) : z_2 \in \mathbb{C}\}$ by the $\mathbb{C}^*$ action and the image of $p \circ \Psi$.

Now we notice that the change of coordinates is given by

$$\varphi_1 \circ \varphi_2^{-1} : \mathbb{C}^* \longrightarrow \mathbb{C}^* \qquad \varphi_1 \circ \varphi_2^{-1}(z) = \frac{1}{z},$$

which is a holomorphic map on its domain of definition, hence $\mathbb{CP}^1$ is a complex manifold.

As we have seen earlier, the sphere can be parametrized by two charts using (a small variation on) stereographic projections:

$$\psi_1 : S^2 \backslash \{(0, 0, 1)\} \longrightarrow \mathbb{C} \qquad \psi_1(x, y, z) = \frac{x + iy}{1 - z}$$

$$\psi_2 : S^2 \backslash \{(0, 0, -1)\} \longrightarrow \mathbb{C} \qquad \psi_2(x, y, z) = \frac{x - iy}{1 + z}$$

And for these maps we have that for $(x, y, z) \in S^2 \backslash \{(0, 0, \pm 1)\}$:

$$\frac{x + iy}{1 - z} \frac{x - iy}{1 + z} = \frac{x^2 + y^2}{1 - z^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1.$$

that is $\psi_1(x, y, z) = (\psi_2(x, y, z))^{-1}$.

Therefore, with these parametrizations, S^2 and $\mathbb{CP}^1$ have the same charts and same transition functions, hence are the same manifold.

2) Compute the integral of the form

$$\rho = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

over the 2-torus in $\mathbb{R}^3$ parametrized by

$$S^1 \times S^1 \longrightarrow \mathbb{R}^3 \qquad (\theta, \varphi) \mapsto ((\cos\theta + 4)\cos\varphi, (\cos\theta + 4)\cos\varphi, \sin\theta)$$

Hint: First compute $d\rho$.

Solution: We start computing $d\rho$:

$$\begin{aligned} d\rho &= \frac{3dx \wedge dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3}{2} \frac{(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)(2xdx + 2ydy + 2zdz)}{(x^2 + y^2 + z^2)^{5/2}} \\ &= \frac{3(x^2 + y^2 + z^2)dx \wedge dy \wedge dz - 3((x^2 + y^2 + z^2)dx \wedge dy \wedge dz)}{(x^2 + y^2 + z^2)^{5/2}} \\ &= 0. \end{aligned}$$

And ρ is defined on $\mathbb{R}^3\backslash\{0\}$. Now notice that the 2-torus, T^2, is the boundary of the solid torus M^3 embedded in $\mathbb{R}^3\backslash\{0\}$, hence by Stoke's theorem,

$$\int_{T^2} \rho = \int_{\partial M} \rho = \int_M d\rho = 0.$$

3) Given a manifold M, the bundle $TM \oplus T^*M$ is endowed with the natural pairing

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\eta(X) + \xi(Y))$$

and its space of sections is endowed with a bracket (the *Courant bracket*):

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi, \qquad X, Y \in \Gamma(TM);\ \xi, \eta \in \Gamma(T^*M).$$

- Show that for $f \in C^\infty(M)$ we have
$$[\![X + \xi, f(Y + \eta)]\!] = f[\![X + \xi, Y + \eta]\!] + (\mathcal{L}_X f)(Y + \eta).$$
- $$[\![X + \xi, X + \xi]\!] = d(\xi(X)).$$

Solution: Both of these are simple computations using the formula for the Courant bracket and the Leibniz rule for the Lie derivative:

$$\begin{aligned}[\![X + \xi, f(Y + \eta)]\!] &= [X, fY] + \mathcal{L}_X(f\eta) - i_{fY} d\xi \\ &= (\mathcal{L}_X f)Y + f[X, Y] + (\mathcal{L}_X f)\eta + f(\mathcal{L}_X \eta) - f i_Y d\xi \\ &= (\mathcal{L}_X f)(Y + \eta) + f[\![X + \xi, Y + \eta]\!].\end{aligned}$$

And computing directly we have

$$\begin{aligned}[\![X + \xi, X + \xi]\!] &= [X, X] + \mathcal{L}_X(\xi) - i_X d\xi \\ &= d(\xi(X)) + i_X d\xi - i_X d\xi \\ &= d(\xi(X))\end{aligned}$$

4*) Given a metric g on a manifold M, let V_+ and V_- be the subbundles of $TM \oplus T^*M$ given by

$$V_+ = \{X + g(X) : X \in TM\} \qquad V_- = \{X - g(X) : X \in TM\}.$$

1. Show that for every point $p \in M$ we have
$$V_+|_p \cap V_-|_p = \{0\} \qquad \text{and} \qquad V_+|_p + V_-|_p = (TM \oplus T^*M)|_p.$$
This allows us to define projections $\pi_\pm : TM \oplus T^*M \longrightarrow V_\pm$. For $X \in TM$ we let $X_+ = X + g(X) \in V_+$ and $X_- = X - g(X) \in V_-$. Finally, there is a projection $\pi_T : TM \oplus T^*M \longrightarrow TM$, $\pi_T(X + \xi) = X$.
We define $\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ by
$$\nabla_X Y = \pi_T(\pi_-([\![X_+, Y_-]\!])).$$
Show that the following hold
2. $$\nabla_{fX} Y = f \nabla_X Y,$$

3.
$$\nabla_X fY = f\nabla_X Y + \mathcal{L}_X fY,$$

4.
$$\mathcal{L}_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

5.
$$\nabla_X Y - \nabla_Y X = [X,Y].$$

Solution: We check that $V_+ \cap V_- = \{0\}$. If $X + \xi \in V_+ \cap V_-$, then $\xi = g(X)$, since this is an element in V_+ and $\xi = -g(X)$, since this is an element in V_-, hence $g(X) = 0$. Since the metric is nondegenerate, $g(X) = 0$ implies $X = 0$ and hence $X + \xi = 0$.

Since the intersection $V_+ \cap V_-$ is trivial and V_+ and V_- have complementary dimension, we conclude that $V_+ + V_- = TM \oplus T^*M$.

Next we check that ∇ has the stated properties. We use the results of the previous exercise. Firstly we notice that

$$d\langle v+w, v+w\rangle = [\![v+w, v+w]\!] = [\![v,v]\!] + [\![v,w]\!] + [\![w,v]\!] + [\![w,w]\!] = d\langle v,v\rangle + [\![v,w]\!] + [\![w,v]\!] + d\langle w,w\rangle$$

hence
$$[\![v,w]\!] + [\![w,v]\!] = 2\langle v,w\rangle$$
or equivalently
$$[\![v,w]\!] = -[\![w,v]\!] + 2\langle v,w\rangle.$$
And also notice that if $X + g(X) \in V_+$ and $Y - g(Y) \in V_-$ then $\langle X + g(X), Y - g(Y)\rangle = \frac{1}{2}(g(X,Y) - g(X,Y) = 0$, hence $[\![X_+, Y_-]\!] = -[\![Y_-, X_+]\!]$.

Now we compute

$$\begin{aligned}
\nabla_{fX} Y &= \pi_T(\pi_-([\![fX_+, Y_-]\!])) \\
&= \pi_T(\pi_-(-[\![Y_-, fX_+]\!]) \\
&= \pi_T(\pi_-(-(\mathcal{L}_Y f)X_+ - f[\![Y_-, X_+]\!])) \\
&= \pi_T((-(\mathcal{L}_Y f)\pi_-(X_+) - \pi_-(f[\![Y_-, X_+]\!]))) \\
&= -f\pi_T(\pi_-([\![Y_-, X_+]\!]) \\
&= f\pi_T(\pi_-([\![X_+, Y_-]\!])) \\
&= f\nabla_X Y,
\end{aligned}$$

proving the first property.

Next we have

$$\begin{aligned}
\nabla_X fY &= \pi_T(\pi_-([\![X_+, fY_-]\!])) \\
&= \pi_T(\pi_-(f[\![X_+, Y_-]\!] + (\mathcal{L}_X f)Y_-)) \\
&= f\pi_T(\pi_-([\![X_+, Y_-]\!])) + (\mathcal{L}_X f)\pi_T(\pi_- Y_-) \\
&= f\nabla_X Y + (\mathcal{L}_X f)Y.
\end{aligned}$$

Next we compute both sides of

$$\mathcal{L}_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

We start with the right hand side and notice that $g(v,w) = -\langle v_-, w_-\rangle$ and use again that $\langle v_+, w_-\rangle = 0$, hence

$$\begin{aligned}
g(\nabla_X Y, Z) + g(Y, \nabla_X Z) &= \langle \pi_- [\![X_+, Y_-]\!], Z_-\rangle + \langle Y_-, \pi_-[\![X_+, Z_-]\!]\rangle \\
&= \langle [\![X_+, Y_-]\!], Z_-\rangle + \langle Y_-, [\![X_+, Z_-]\!]\rangle \\
&= \langle [X,Y] - \mathcal{L}_X g(Y) - i_Y d(g(X)), Z - g(Z)\rangle + \langle Y - g(Y), [X,Z] - \mathcal{L}_X g(Z) - i_Z d(g(X))\rangle \\
&= -\tfrac{1}{2}(g([X,Y],Z) + i_Z(\mathcal{L}_X g(Y) - i_Y d(g(X))) + g(Y,[X,Z]) + i_Y(\mathcal{L}_X g(Z) - i_Z d(g(X)))) \\
&= -\tfrac{1}{2}(g([X,Y],Z) + g(Y,[X,Z]) + i_Z \mathcal{L}_X g(Y) + i_Y \mathcal{L}_X g(Z))
\end{aligned}$$

And the left hand side gives us

$$\begin{aligned}
\mathcal{L}_X g(Y,Z) &= \mathcal{L}_X \langle Y - g(Y), Z - g(Z)\rangle \\
&= \langle \mathcal{L}_X Y - \mathcal{L}_X g(Y), Z - g(Z)\rangle + \langle Y - g(Y), \mathcal{L}_X Z - \mathcal{L}_X g(Z)\rangle \\
&= \langle [X,Y] - \mathcal{L}_X g(Y), Z - g(Z)\rangle + \langle Y - g(Y), [X,Z] - \mathcal{L}_X g(Z)\rangle \\
&= -\tfrac{1}{2}((g(Z,[X,Y]) + g(Y,[X,Z]) + i_Z \mathcal{L}_X g(Y) + i_Y \mathcal{L}_X g(Z)).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \pi_T \circ \pi_-([\![X_+, Y_-]\!] - [\![Y_+, X_-]\!]) \\
&= \pi_T \circ \pi_-([\![X + g(X), Y - g(Y)]\!] - [\![Y + g(Y), X - g(X)]\!]) \\
&= \pi_T \circ \pi_-([\![X,Y]\!] - [\![X, g(Y)]\!] + [\![g(X), Y]\!] - ([\![Y,X]\!] - [\![Y, g(X)]\!] + [\![g(Y), X]\!])) \\
&= \pi_T \circ \pi_-(2[X,Y] - 2d\langle X, g(Y)\rangle + 2d\langle g(X), Y\rangle) \\
&= \pi_T \circ \pi_-(2[X,Y]) \\
&= \pi_T([X,Y] - g([X,Y]) \\
&= [X,Y].
\end{aligned}$$

Remark: There is only one operator ∇ with the properties 2–5 listed above and it goes by name "Levi–Civita connection". This operator is to Riemannian geometry what the exterior derivative is for differentiable manifolds. For sake on completeness, I asked the proof of all 4 properties of ∇, but in an exam you can expect a smaller question.

5) Compute the de Rham cohomology of $S^1 \times S^1$.

Solution: We know that $H^0(M) = \mathbb{R}^d$, where d is the number of connected components of M. Hence $H^0(S^1 \times S^1) = \mathbb{R}$. Similarly, we have been told in lectures that in a compact, orientable manifold, M^n, $H^k(M) = H^{n-k}(M)$ (Poincaré duality), hence $H^2(S^1 \times S^1) = \mathbb{R}$ and the only cohomology group which we still must compute is $H^1(S^1 \times S^1)$.

Let θ be a volume form on S^1 such that $\int_{S^1} \theta = 1$. We have seen explicit formulas for θ in the exercises, e.g.,

$$\theta = \frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}.$$

But it is not necessary to have an explicit formula, since, we have seen in lectures that for a compact orientable manifold M^n, $H^n(M) = \mathbb{R}$, hence we can simply let θ be a representative of the class $1 \in H^1(S^1)$.

For $S^1 \times S^1$,we have two projections

$$\pi_1 : S^1 \times S^1 \longrightarrow S^1 \qquad \pi_1(\varphi, \psi) = \varphi,$$

$$\pi_2 : S^1 \times S^1 \longrightarrow S^1 \qquad \pi_2(\varphi, \psi) = \psi,$$

Let $\theta_i = \pi_i^*\theta$. Then $d\theta_i = d\pi_i^*\theta = \pi_i^* d\theta = 0$, so the forms θ_i is closed.

Also, if we let $\iota_1 : S^1 \longrightarrow S^1 \times S^1$ be given by $i_1(\varphi) = (\varphi, \psi_0)$, for some fixed ψ_0 and similarly $i_2 : S^1 \longrightarrow S^1 \times S^1$ be given by $\iota_2(\psi) = (\varphi_0, \psi)$, then, since $\pi_i \circ \iota_i = \mathrm{Id}$, we have

$$\int_{S^1} \iota_i^*\theta_i = \int_{S^1} \iota_i^*\pi_i^*\theta = \int_{S^1} (\pi_i \circ \iota_i)^*\theta = \int_{S^1} \theta = 1,$$

hence θ_1 and θ_2 represent linearly independent elements in $H^1(S^1 \times S^1)$ and we define a map

$$\Psi : H^1(S^1 \times S^1) \longrightarrow \mathbb{R}^2;$$

$$\Psi(a) = (\int_{S^1} \iota_1^* a, \int_{S^1} \iota_2^* a).$$

We want to prove that Ψ is a bijection. By our previous comments, we know that Ψ is onto, so now we must prove it is an injection, i.e., if $\Psi(a) = 0$ then a is the trivial class.

We can write $S^1 \times S^1$ as a union of nine squares, such that the intersections of any two such squares is connected:

$$U_1 = \{(\varphi, \psi) : 0 < \varphi < \pi, 0 < \psi < \pi\}$$

$$U_2 = \{(\varphi, \psi) : \frac{2}{3}\pi < \varphi < \frac{5}{3}\pi, 0 < \psi < \pi\}$$

$$U_3 = \{(\varphi, \psi) : \frac{4}{3}\pi < \varphi < \frac{7}{3}\pi, 0 < \psi < \pi\}$$

$$U_4 = \{(\varphi, \psi) : 0 < \varphi < \pi, \frac{2}{3}\pi < \psi < \frac{5}{3}\pi\}$$

$$U_5 = \{(\varphi, \psi) : \frac{2}{3}\pi < \varphi < \frac{5}{3}\pi, \frac{2}{3}\pi < \psi < \frac{5}{3}\pi\}$$

$$U_6 = \{(\varphi, \psi) : \frac{4}{3}\pi < \varphi < \frac{7}{3}\pi, \frac{2}{3}\pi < \psi < \frac{5}{3}\pi\}$$

$$U_7 = \{(\varphi, \psi) : \frac{4}{3}\pi < \varphi < \frac{7}{3}\pi, \frac{4}{3}\pi < \psi < \frac{7}{3}\pi\}$$

$$U_8 = \{(\varphi, \psi) : \frac{4}{3}\pi < \varphi < \frac{7}{3}\pi, \frac{4}{3}\pi < \psi < \frac{7}{3}\pi\}$$

$$U_9 = \{(\varphi, \psi) : \frac{4}{3}\pi < \varphi < \frac{7}{3}\pi, \frac{4}{3}\pi < \psi < \frac{7}{3}\pi\}$$

Given a representative α for a class $a \in \ker(\Psi)$, by the Poincaré lemma there is a function $f_i \in \Omega^0(U_i)$ such that

$$\alpha = df_i \qquad \text{on } U_i$$

and hence, on the overlaps $U_{ij} = U_i \cap U_j$ with $i < j$we have $d(f_i - f_j) = 0$ and $f_i - f_j = c_{ij}$, with $c_{ij} \in \mathbb{R}$. To keep symmetry, we can declare $c_{ij} = -c_{ji}$ and then we have $f_i - f_j = c_{ij}$ for all pairs (i, j).

Integrating α over circles of the form (φ, ψ_0) for $\psi_0 = 0$, $\frac{\pi}{2}$ and π, we get

$$c_{12} + c_{23} + c_{31} = 0; \qquad c_{45} + c_{56} + c_{64} = 0; \qquad c_{78} + c_{89} + c_{97} = 0.$$

And we can define new functions

$$\hat{f}_1 = f_1 - c_{13} \qquad \hat{f}_2 = f_2 - c_{23} \qquad \hat{f}_3 = f_3$$

then $\hat{f}_1 - \hat{f}_2 = f_1 - f_2 - c_{13} + c_{23} = c_{12} - -c_{13} + c_{23} = 0$.

Also, $\hat{f}_1 - \hat{f}_3 = f_1 - f_3 - c_{13} = 0$ and $\hat{f}_2 - \hat{f}_3 = 0$, hence we have a function g_1 defined on $U_1 \cup U_2 \cup U_3$ such that $f|_{U_i} = \hat{f}_i$ and $dg_1 = \alpha|_{U_1 \cup U_2 \cup U_3}$ Repeating the same argument we get a function g_2 defined on $U_4 \cup U_5 \cup U_6$ such that $dg_2 = \alpha|_{U_4 \cup U_5 \cup U_6}$ and a function g_3 defined on $U_7 \cup U_8 \cup U_9$ such that $dg_2 = \alpha_{U_7 \cup U_8 \cup U_9}$.

Let $V_1 = U_1 \cup U_2 \cup U_3$, $V_2 = U_4 \cup U_5 \cup U_6$ and $V_3 = U_7 \cup U_8 \cup U_9$. Then the intersections V_{ij} are connected and there we have $dg_{ij} = 0$, hence $g_{ij} = k_{ij}$ for some constant k_{ij} and once again we can assume that $k_{ij} = -k_{ji}$.

Integrating α over (φ_0, ψ) we get

$$k_{12} + k_{23} + k_{31} = 0,$$

and we define $\hat{g}_1 = g_1 - k_{13}$, $\hat{g}_2 = g_2 - k_{23}$ and $\hat{g}_3 = g_3$. Then, once again we see that on $V_i \cap V_j$ $\hat{g}_i = \hat{g}_j$ and hence there is a globally defined function h such that $h|V_i = \hat{g}_i$ and therefore $dh = \alpha$, showing that α represents the trivial class.