Differentiable manifolds – homework 5

Definition: An *algebra* over the real numbers is a real vector space, A, endowed with a bilinear operation

$$A \times A \xrightarrow{[\cdot,\cdot]} A \qquad (X,Y) \mapsto [X,Y].$$

A *Lie algebra* is an algebra over the real numbers for which the algebra operation satisfies the following two conditions:

1. $[X,Y] = -[X,Y]$ (skew symmetry);
2. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity).

1) Solve exercises 15 and 24 of Chapter 1 from Warner.

2) Let $\mathfrak{gl}(n;\mathbb{R})$ be the set of $n \times n$ real matrices and define a bracket on $\mathfrak{gl}(n;\mathbb{R})$ by

$$[A,B] = AB - BA.$$

Show that with this operation, $\mathfrak{gl}(n;\mathbb{R})$ is a Lie algebra.

3a) Let $\mathfrak{sl}(n;\mathbb{R})$ be the subset of $\mathfrak{gl}(n;\mathbb{R})$ of matrices with zero trace. Show that $\mathfrak{sl}(n;\mathbb{R})$ is a Lie algebra if endowed with the bracket from exercise 2.
3b) Let $\mathfrak{so}(n)$ be the subset of $\mathfrak{gl}(n;\mathbb{R})$ of skew symmetric matrices. Show that $\mathfrak{so}(n)$ is a Lie algebra if endowed with the bracket from exercise 2.

4) Let $E \xrightarrow{\pi} M$ be a vector bundle over M of rank k. Show that if there are k sections of E, $\sigma_1, \sigma_2, \cdots, \sigma_k$ for which $\{\sigma_1(p), \sigma_2,(p) \cdots, \sigma_k(p)\}$ is a linearly independed set of E_p, the fiber of E over p, for all $p \in M$, then E is isomorphic to $M \times \mathbb{R}^k$.

5) Let $E \xrightarrow{\pi} M$ be a vector bundle over M of rank k and let (U_α) be an open cover of M over which E has trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$. On double overlaps, $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we can write

$$\varphi_\beta \circ \varphi_\alpha^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \longrightarrow U_{\alpha\beta} \times \mathbb{R}^k.$$

Show that this map is the identity map on the first factor and write it as

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x,v) = (x, g_\beta^\alpha(x)v),$$

where $g_\beta^\alpha(x) \in Gl(k,\mathbb{R})$ for all $x \in U_{\alpha\beta}$.

Show that the functions $g_\beta^\alpha : U_{\alpha\beta} \longrightarrow Gl(k,\mathbb{R})$ satisfy

1. $g_\alpha^\alpha = \mathrm{Id}$,
2. $g^\alpha \beta g_\alpha^\beta = \mathrm{Id}$,
3. $g_\alpha^\gamma g_\gamma^\beta g_\beta^\alpha = \mathrm{Id}$.

6) Let G be a group. Show that if a collection of functions $g_\beta^\alpha : U_{\alpha\beta} \longrightarrow G$ satisfies conditions 1, 2 and 3 of the previous exercise then

$$g_{\alpha_1}^{\alpha_n} g_{\alpha_n}^{\alpha_{n-1}} \cdots g_{\alpha_3}^{\alpha_2} g_{\alpha_2}^{\alpha_1} = \mathrm{Id}$$

whenever it is defined.