# Differentiable manifolds – homework 7

Exercises from the book: Chapter 1: 9, 10, 16, 20 and 21.

**Definition**: Given a vector space $V$, we let $V^*$ be its dual, i.e.,

$$V^* = \{f : V \longrightarrow \mathbb{R} : f \text{ is linear}\}.$$

Given a linear map $A : V \longrightarrow W$, we define a linear map $A^* : W^* \longrightarrow V^*$ by

$$f \overset{A^*}{\mapsto} A^* f \qquad A^* f(v) = f(A(v)) \qquad \forall v \in V.$$

1 a) Show that $V^*$ is a vector space and that if $V$ is finite dimensional, then $V^{**} = V$.
1 b) Show that if $A : V \longrightarrow W$ is linear, then $A^* : W^* \longrightarrow V^*$ is linear. Further if $A$ is surjective, then $A^*$ is injective and if $A$ is injective, then $A^*$ is surjective.
1 c) Using the natural identifications $V^{**} = V$ and $W^{**} = W$, show that $A^{**} : V^{**} \longrightarrow W^{**}$ is just $A : V \longrightarrow W$.

2) Let $M \overset{\varphi}{\longrightarrow} N$ be an embedded submanifold. Show that if $X \in \mathfrak{X}(M)$, then there exists a vector field $\tilde{X} \in \mathfrak{X}(N)$ which is $\varphi$-related to $X$. Such $\tilde{X}$ is normally called an *extension* of $X$ to $N$. Given $X, Y \in \mathfrak{X}(M)$, let $\tilde{X}, \tilde{Y}$ be extensions of $X$ and $Y$ to $N$. Show that for $p \in \varphi(M)$, $[\tilde{X}, \tilde{Y}](p)$ is tangent to $\varphi(M)$ and depends only on $X$ and $Y$ and not on the particular extensions $\tilde{X}$ and $\tilde{Y}$ chosen.