Differentiable manifolds – homework 9

Definition 1. A *complex structure* on a manifold is a choice of atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ such that $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ and the change of coordinates $\varphi_\beta \circ \varphi_\alpha^{-1} : V \subset \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are holomorphic maps.

An *almost complex structure* on a manifold M is bundle map $I : TM \longrightarrow TM$ such that $I^2 = -\mathrm{Id}$.

1) (**Complex structures on vector spaces**). A complex vector space is a vector space V over the field of the complex numbers, i.e., the scalars are taken to be complex numbers. Now let V be a real vector space and let $I : V \longrightarrow V$ be a linear transformation such that $I^2 = -\mathrm{Id}$. Show that the linear map I allows to make V into a complex vector space by declaring

$$(x + iy) \cdot v = xv + yIv, \qquad \text{for all } x, y \in \mathbb{R} \text{ and } v \in V.$$

Conversely, prove that if V is a complex vector space, there is a real-linear transformation $I : V \longrightarrow V$ such that $I^2 = -\mathrm{Id}$.

2) Show that a complex manifold has an almost complex structure.

3) The sphere S^2 can be parametrized with two charts using stereographic projection, namely, we let $U_1 = S^2 \backslash \{(0, 0, 1)\}$, $U_2 = S^2 \backslash \{(0, 0, -1)\}$ and let

$$\varphi_1 : U_1 \longrightarrow \mathbb{C} = \mathbb{R}^2 \qquad \varphi_1(x, y, z) = \frac{x + iy}{1 - z}$$

$$\varphi_2 : U_2 \longrightarrow \mathbb{C} = \mathbb{R}^2 \qquad \varphi_2(x, y, z) = \frac{x - iy}{1 + z}$$

Show that this atlas makes S^2 into a complex manifold.

4) (**Complex projective space**). We define the complex projective space $\mathbb{C}P^1$ to be the set of (complex) lines through the origin in $\mathbb{C}^2$. That is, $\mathbb{C}P^1$ is the set of equivalence class on $\mathbb{C}^2 \backslash \{(0, 0)\}$ where (z_1, z_2) is equivalent (w_1, w_2) if and only if there is $\lambda \in \mathbb{C}^*$ such that $(z_1, z_2) = \lambda(w_1, w_2)$. Show that $\mathbb{C}P^1$ can be made into a complex manifold. Better, show that $\mathbb{C}P^1$ is diffeomorphic to the sphere S^2.

5) In an almost complex manifold (M, I) we define the Nijenhuis operator as the following map

$$N : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$
$$N(X, Y) = [X, Y] + I([IX, Y] + [X, IY]) - [IX, IY] \qquad \forall X, Y \in \mathfrak{X}(M).$$

Show that the $+i$-eigenspace of I is involutive if and only if $N \equiv 0$.

6) Let D be the generalized distribution of $\mathbb{R}^2$ generated by $X = x\partial/\partial x$ and $Y = x\partial/\partial y$. Show that D is an integrable generalized distribution and describe its leaves.

7) (**...oids**) A *Lie algebroid* over a manifold M is a vector bundle L over M together with a bundle map $\pi : L \longrightarrow TM$ (that is, if we let L_p be the fiber of L over $p \in M$, $\pi(L_p) \subset T_pM$ and $\pi : L_p \longrightarrow T_pM$ is linear) and a Lie bracket on the space of sections of L such that $\pi : \Gamma(L) \longrightarrow \Gamma(TM)$ is a map of Lie algebras (i.e., it sends the bracket in $\Gamma(L)$ to the Lie brackets in $\Gamma(TM)$).

Given a Lie algebroid L over M, let $D = \pi(L) \subset TM$. Show that D is an integrable generalized distribution.