Differentiable manifolds – hand-in sheet 1

hand in by: 19/Oct

Čech cohomology and lines bundles

Exercise 1. Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be a locally finite cover of a manifold M by open sets such that each $U_{\rm a}$ is either empty or homeomorphic to a disc for every multi-index ${\rm a} \subset A$, that is, each U_α is a disc, and each double intersection is a disc (or emtpy) and each triple intersection is a disc, or empty, etc.

For the rest of the exercise, we let $\pi : E \longrightarrow M$ be a rank 1 real vector bundle (i.e. a line bundle) over M.

1. Show that a choice of local nonvanishing sections s_α over U_α (for each α) gives isomorphisms

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}.$$

2. Define the transition functions g_α^β for this collection of Φ_α by

$$\Phi_\beta \circ \Phi_\alpha^{-1} : U_{\alpha\beta} \times \mathbb{R} \longrightarrow U_{\alpha\beta} \times \mathbb{R}$$

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, g_\beta^\alpha(x)v) \qquad g_\beta^\alpha(x) \in Gl(1;\mathbb{R}) = \mathbb{R}^*.$$

   Show that the collection $\check{g} = \{g_\beta^\alpha : \alpha, \beta \in A\}$ forms a degree 1 Čech cochain with coefficients in the smooth functions with values in the abelian group $\mathbb{R}^*$.

3. Show that $\delta\check{g} = 0$.

4. Show that if we choose different nonvanishing sections σ_α of E over U_α and run the same argument above with s_α replaced by σ_α, the Čech cocyle $\check{g}$ changes by a coboundary: $\check{g} + \delta\check{f}$, with $\check{f} \in \check{C}^0(M, \mathbb{R}^*)$, hence the cohomology class $[\check{g}] \in \check{H}^1(M; C^\infty(M;\mathbb{R}^*); \mathfrak{U})$ does not depend on the choices made.

5. Conversely, argue that given a cohomology class $[\check{g}] \in \check{H}^1(M; C^\infty(M;\mathbb{R}^*); \mathfrak{U})$, any representative $\check{g} = \{g_\beta^\alpha : \alpha, \beta \in A\}$ of that class can be used to construct a line bundle for which the procedure above associates to the class $[\check{g}]$.